Discrete Applied Mathematics 227 (2017) 13–20

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Communication

The computational complexity of three graph problems for instances with bounded minors of constraint matrices

D.V. Gribanov [a,b], D.S. Malyshev [b,*]

[a] Lobachevsky State University, 23 Gagarina av., Nizhny Novgorod, 603950, Russia
[b] National Research University Higher School of Economics, 25/12 Bolshaja Pecherskaja Ulitsa, Nizhny Novgorod, 603155, Russia

ARTICLE INFO

Article history:
Received 18 April 2017
Accepted 20 April 2017
Available online 26 May 2017
Communicated by Vadim Lozin

Keywords:
Boolean linear programming
Independent set problem
Dominating set problem
Matrix minor
Efficient algorithm

ABSTRACT

We consider boolean linear programming formulations of the independent set, the vertex and the edge dominating set problems and prove their polynomial-time solvability for classes of graphs with (augmented) constraint matrices having bounded minors in the absolute value.

1. Introduction

For given $\mathbf{A} \in \mathbb{Z}^{m\times n}$, $\mathbf{b} \in \mathbb{Z}^m$, $\mathbf{c} \in \mathbb{Z}^n$, the *primal linear programming problem* (abbreviated as the PLPP, for short) is to solve the following primal linear program:

$$\begin{aligned} \max \quad & \mathbf{c}^{\mathsf{T}}\mathbf{x} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} \le \mathbf{b}, \\ \text{and} \quad & \mathbf{x} \ge \mathbf{o}_n, \end{aligned}$$

where $\mathbf{o}_n$ denotes the all-zeros vector with n components and $\mathbf{x}$ is a vector of n variables to be determined. The *primal integer linear programming problem* (the PILPP, for short) differs from the PLPP by the requirement that all variables must have integer values. In the *primal boolean linear programming problem* (the PBLPP, for short), every entry of $\mathbf{A}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{x}$ is boolean.

For given $\mathbf{A} \in \mathbb{Z}^{m\times n}$, $\mathbf{b} \in \mathbb{Z}^m$, $\mathbf{c} \in \mathbb{Z}^n$, the *dual linear programming problem* (the DLPP, for short) is to solve the following program, which is dual to the primal above:

$$\begin{aligned} \min \quad & \mathbf{b}^{\mathsf{T}}\mathbf{y} \\ \text{s.t.} \quad & \mathbf{A}^{\mathsf{T}}\mathbf{y} \ge \mathbf{c}, \\ \text{and} \quad & \mathbf{y} \ge \mathbf{o}_m. \end{aligned}$$

In the *dual integer linear programming problem* and *the dual boolean linear programming problem* (the DILPP and the DBLPP, for short) we additionally impose the restriction of integrality to variables and the restriction of booleanity to all data and variables, respectively.

* Corresponding author.
E-mail addresses: dimitry.gribanov@gmail.com (D.V. Gribanov), dsmalyshev@rambler.ru, dmalishev@hse.ru (D.S. Malyshev).

http://dx.doi.org/10.1016/j.dam.2017.04.025
0166-218X/

There are several polynomial-time algorithms for solving the PLPP and the DLPP. We mention Khachiyan's algorithm [7], Karmarkar's algorithm [6], and Nesterov's algorithm [10,11]. Unfortunately, it is well known that the PBLPP and the DBLPP are NP-hard problems. Hence, polynomial-time algorithms to solve the PBLPP and the DBLPP are unlikely to exist. Therefore, it would be interesting to reveal polynomially solvable cases of the PILPP and the DILPP.

Recall that an integer matrix is called *totally unimodular* if any of its minor is equal to $+1$ or -1 or 0. It is well known that all optimal solutions of any primal or dual linear program with a totally unimodular constraint matrix are integer. Hence, for any primal linear program and the corresponding primal integer linear program with a totally unimodular constraint matrix, the sets of their optimal solutions coincide. Therefore, any polynomial-time linear optimization algorithm (like algorithms in [6,7,10,11]) is also an efficient algorithm for the PILPP and the DILPP with totally unimodular constraint matrices.

The next natural step is to consider the *bimodular case*, i.e. the PILPP and the DILPP having constraint matrices with the absolute values of all minors in the set $\{0, 1, 2\}$. More generally, it would be interesting to investigate the complexity of the problems with constraint matrices having bounded minors. The maximum absolute value of all minors of an integer matrix can be interpreted as a proximity measure to the class of totally unimodular matrices. A conjecture arises that for each fixed natural number c the PILPP and the DILPP can be solved in polynomial time in any class of linear programs with constraint matrices each minor of which has the absolute value at most c [13]. There are variants of this conjecture, where the augmented matrices $\binom{\mathbf{c}^{\mathrm{T}}}{\mathbf{A}}$ and $\begin{pmatrix}\mathbf{A} & \mathbf{b}\end{pmatrix}$ are considered [13]. We call any variant of this conjecture the *conjecture of bounded minors*.

Unfortunately, not much is known about the complexity of the PILPP and the DILPP for classes of linear programs with bounded minors. For example, the complexity statuses of the PILPP and the DILPP with bimodular constraint matrices are still unknown. A step towards a clarification of the complexity in the bimodular case was done in [14]. Namely, it has been shown that if the rank of a bimodular $m \times n$ matrix $\mathbf{A}$ equals n and every $n \times n$ sub-matrix of $\mathbf{A}$ is not singular, then the PILPP can be solved in polynomial time. A more general result was obtained in [2]. Namely, the PILPP can be solved in polynomial time whenever the absolute values of all maximal sub-determinants of constraint matrices lie between 1 and a constant.

The PBLPP was considered in [1]. It has been shown that if $\mathbf{A}$ is a boolean matrix with at most two 1s per row, $\mathbf{b}$ and $\mathbf{c}$ are boolean vectors, and the absolute values of all minors of $\binom{\mathbf{c}^{\mathrm{T}}}{\mathbf{A}}$ are at most C', then the PBLPP can be solved in polynomial time for any fixed C'. This result has a graph-theoretical nature, since a linear program for the independent set problem of finding a maximum subset of pairwise non-adjacent vertices in a given graph G has the transposed incidence matrix $\mathbf{I}^{\mathrm{T}}(G)$ of G as the constraint matrix. For this problem, 1s are the only components of the objective function vector and the right-hand vector.

Despite the fact that advances in the theory of integer linear programming with bounded minors are not substantial, we believe that at least some variants of the conjecture of bounded minors are true. The aim of this article is to prove the conjecture for some types of instances. In this paper, we consider boolean linear programming formulations of the independent set, the vertex and the edge dominating set problems and prove their polynomial-time solvability for classes of graphs with (augmented) constraint matrices having bounded minors in the absolute value. Namely, we prove that for each fixed c the independent set problem can be solved in polynomial time in the class $\{G|$ all minors of $\mathbf{I}^{\mathrm{T}}(G)$ augmented with a row consisting of 1s only are at most c in the absolute value$\}$. Let $\mathbf{A}_v(G)$ and $\mathbf{A}_e(G)$ be the vertex and the edge adjacency matrices of a graph G, respectively. We also prove that for each fixed c the vertex (edge) dominating set problem can be solved in polynomial time in the class of graphs $\{G|$ all absolute values of minors of $\mathbf{A}_v(G)$ $(\mathbf{A}_e(G))$ are at most $c\}$.

2. Definitions and notation

A graph H is called a *subgraph* of a graph G if H is obtained from G by deletion of vertices and edges assuming that deletion of a vertex implies deletion of all its incident edges. A graph H is called an *induced subgraph* of a graph G if H is obtained from G by deletion of vertices.

A class of graphs is called *hereditary* if it is closed under deletion of vertices. It is well known that any hereditary class $\mathcal{X}$ can be defined by a set of its *forbidden induced subgraphs* $\mathcal{Y}$, i.e. graphs not belonging to $\mathcal{X}$ and minimal under deletion of vertices. We write $\mathcal{X} = Free(\mathcal{Y})$. A *strongly hereditary* graph class is a hereditary class closed under deletion of edges. Any strongly hereditary class $\mathcal{X}$ can be defined by a set of its forbidden subgraphs $\mathcal{Y}$ denoted $\mathcal{X} = Free_s(\mathcal{Y})$.

A graph is *bipartite* if its vertex set can be partitioned into at most two sets inducing empty graphs. The edge adjacency graph of another graph is called a *line graph*.

We use the following notation for matrices:

- $\mathbf{J}_n$ — the all-ones square matrix of order n,
- $\mathbf{O}_n$ — the all-zeros square matrix of order n,
- $\mathbf{I}_n$ — the identity matrix of order n,
- $\mathbf{j}_n$ — the all-ones vector with n components,
- $\mathbf{o}_n$ — the all-zeros vector with n components,
- $\mathbf{A}^{\mathrm{T}}$ — the matrix transposed to $\mathbf{A}$,
- $\mathbf{I}(G)$ — the incidence matrix of a graph G,

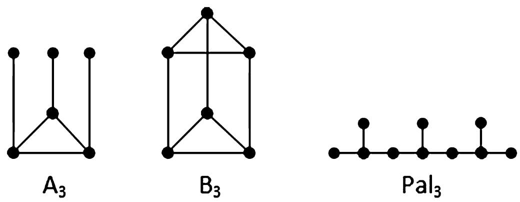

Fig. 1. The graphs A_3, B_3, Pal_3.

- $\mathbf{A}_v(G)$ — the vertex adjacency matrix of a graph G,
- $\mathbf{A}_e(G)$ — the edge adjacency matrix of a graph G.

We use the following notation associated with graphs:

- $K_{p,q}$ — the complete bipartite graph with p vertices in the first part and q vertices in the second one,
- $K'_{1,p}$ — the graph obtained from the graph $K_{1,p}$ by subdividing each of its edges exactly once,
- K_n — the complete graph with n vertices,
- O_n — the empty graph with n vertices,
- A_n — the graph with vertex set $\{v_1, \ldots, v_n, u_1, \ldots, u_n\}$ and edge set $\{v_i v_j |\ i \neq j\} \cup \{v_1u_1, v_2u_2, \ldots, v_nu_n\}$,
- B_n — the graph with vertex set $\{v_1, \ldots, v_n, u_1, \ldots, u_n\}$ and edge set $\{v_i v_j |\ i \neq j\} \cup \{u_i u_j |\ i \neq j\} \cup \{v_1u_1, v_2u_2, \ldots, v_nu_n\}$,
- Pal_n — the graph with vertex set $\{v_1, \ldots, v_{2n+1}, u_1, \ldots, u_n\}$ and edge set $\{v_i v_{i+1} |\ 1 \leq i \leq 2n\} \cup \{v_{2i}u_i |\ 1 \leq i \leq n\}$,
- kG — the disjoint union of k copies of a graph G,
- for a graph G and a subset $V' \subseteq V(G)$, $G[V']$ denotes the subgraph of G induced by V' and $G \setminus V'$ denotes the subgraph of G obtained by deleting every element of V',
- $N(x)$ — the neighborhood of a vertex x, $N[x] \triangleq N(x) \cup \{x\}$, where $\triangleq$ means the equality by definition.

The graphs A_3, B_3, Pal_3 are depicted in Fig. 1.
By $\overline{1,k}$ we denote the set $\{1, \ldots, k\}$.

3. Some classical graph problems and their boolean linear programming formulations

An *independent set* in a graph is a subset of its pairwise non-adjacent vertices. The size of a maximum independent set in a graph G is called the *independence number* of G and is denoted by $\alpha(G)$. The *independent set problem* (briefly, the ISP) is to determine, for a given graph G and a natural number k, whether $\alpha(G) \geq k$ or not. It is a classical NP-complete graph problem.

For a given graph G with n vertices and m edges, the ISP can be formulated as the following linear program:

$$\begin{aligned} &\max\ \mathbf{j}_n^{\mathrm{T}}\mathbf{x} \\ &\text{s.t. } \mathbf{I}^{\mathrm{T}}(G)\mathbf{x} \leq \mathbf{j}_m, \\ &\text{and } \ \mathbf{x} \in \{0, 1\}^n. \end{aligned}$$

Indeed, a variable x_v is an indicator that the corresponding vertex v belongs to an optimal solution of the ISP. The inequality $x_v + x_u \leq 1$ ensures that u and v do not simultaneously belong to any feasible solution of the program, i.e. its every feasible solution is an independent set.

Let $\mathcal{ISP}(c)$ be the set of all graphs G such that the absolute values of all minors of $\binom{\mathbf{j}_n^{\mathrm{T}}}{\mathbf{I}^{\mathrm{T}}(G)}$ are at most c. In this paper, we will show that for each fixed c the ISP can be solved for graphs in $\mathcal{ISP}(c)$ in polynomial time. This result was originally obtained in [1], but our proof is simpler and shorter.

A *vertex dominating set* in a graph G is a subset $D \subseteq V(G)$ such that any element of $V(G) \setminus D$ has a neighbor in D. The size of a minimum vertex dominating set in a graph G is called the *vertex domination number* of G and is denoted by $\gamma(G)$. The *vertex dominating set problem* (briefly, the VDSP) is to determine, for a given graph G and a natural number k, whether $\gamma(G) \leq k$ or not. The *edge dominating set problem* (briefly, the EDSP) is defined in a similar way. The VDSP and the EDSP are classical NP-complete graph problems.

For a given graph G with n vertices and m edges, the VDSP and the EDSP can be formulated as the following linear programs:

$$\begin{aligned} &\min\ \ \mathbf{j}_n^{\mathrm{T}}\mathbf{y} \\ &\text{s.t. } (\mathbf{A}_v(G) + \mathbf{I}_n)\mathbf{y} \geq \mathbf{j}_n, \\ &\text{and } \mathbf{y} \in \{0, 1\}^n \end{aligned} \qquad \begin{aligned} &\min\ \ \mathbf{j}_m^{\mathrm{T}}\mathbf{y} \\ &\text{s.t. } (\mathbf{A}_e(G) + \mathbf{I}_m)\mathbf{y} \geq \mathbf{j}_m, \\ &\text{and } \mathbf{y} \in \{0, 1\}^m. \end{aligned}$$

To justify this, let us consider the VDSP. A variable y_v is an indicator that the corresponding vertex v belongs to an optimal solution of the VDSP. The inequality $x_v + \sum_{u \in N(v)} x_u \geq 1$ ensures that the set $N[v]$ contains an element of any feasible solution, i.e. every feasible solution of the program is a vertex dominating set.

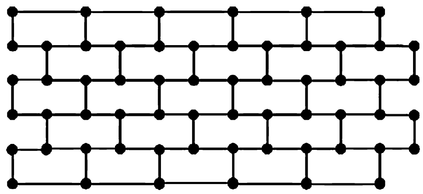

Fig. 2. The elementary wall of height 5.

Let $\mathcal{V}DSP(c)$ and $\mathcal{E}DSP(c)$ be the sets of all graphs G such that the absolute values of all minors of $\mathbf{A}_v(G)+\mathbf{I}_n$ and $\mathbf{A}_e(G)+\mathbf{I}_m$ are at most c, respectively. In this paper, we will show that for each fixed c the VDSP can be solved for graphs in $\mathcal{V}DSP(c)$ in polynomial time. We also show a similar result for the EDSP and $\mathcal{E}DSP(c)$.

For each c, the classes $\mathcal{I}SP(c)$ and $\mathcal{E}DSP(c)$ are strongly hereditary. For each c, the class $\mathcal{V}DSP(c)$ is hereditary.

4. The independent set problem

4.1. An inclusion

Lemma 1. *For every $c \geq 2$, the inclusion $\mathcal{I}SP(c) \subseteq Free_s(\{Pal_c\})$ is true.*

Proof. Let $\mathbf{M}(k, a)$ be the matrix obtained from $\binom{\mathbf{j}^{\mathrm{T}}_{3k+1}}{\mathbf{I}^{\mathrm{T}}(Pal_k)}$ by changing 1 to a in the entry corresponding to u_1 in the first row. Let us consider the sub-matrix $\mathbf{M}'$ of $\mathbf{M}(k, a)$ induced by the columns corresponding to v_1, v_2, v_3, u_1 and the first row and the rows corresponding to the edges v_1v_2, v_2v_3, v_2u_1. The matrix $\mathbf{M}'$ has the form $\begin{pmatrix} a & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$ assuming that its first column corresponds to u_1, $(i+1)$th column corresponds to v_i for any $1 \leq i \leq 3$, the second, third, and fourth rows correspond to u_1v_2, v_1v_2, v_2v_3, respectively. The following diagram shows a sequence of elementary row and column operations to $\mathbf{M}'$:

$$\begin{pmatrix} a & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} a & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} a & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 0 & -a & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 0 & 0 & -a & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 0 & -a & 1+a \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 0 & 0 & 1+a \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Therefore, elementary operations transform the matrix $\mathbf{M}(k, a)$ to the matrix

$$\begin{pmatrix} 1 & 0 & 0 & \mathbf{o}^{\mathrm{T}}_{3k-2} \\ 0 & 1 & 0 & \mathbf{o}^{\mathrm{T}}_{3k-2} \\ 0 & 0 & 1 & \mathbf{o}^{\mathrm{T}}_{3k-2} \\ \mathbf{o}_{3k-2} & \mathbf{o}_{3k-2} & \mathbf{o}_{3k-2} & \mathbf{M}(k-1, a+1) \end{pmatrix}.$$

Hence, $|\det(\mathbf{M}(k, a))| = |\det(\mathbf{M}(k-1, a+1))|$, i.e. $|\det(\mathbf{M}(k, a))| = |\det(\mathbf{M}(1, a+k-1))|$. Clearly, $\det(\mathbf{M}(1, a)) = -1 - a$. Hence, $|\det(\mathbf{M}(k, a))| = |a + k|$. As $\mathbf{M}(k, 1) = \binom{\mathbf{j}^{\mathrm{T}}_{3k+1}}{\mathbf{I}^{\mathrm{T}}(Pal_k)}$, $|\det \binom{\mathbf{j}^{\mathrm{T}}_{3k+1}}{\mathbf{I}^{\mathrm{T}}(Pal_k)}| = k + 1$.

The matrix $\binom{\mathbf{j}^{\mathrm{T}}_{3k+1}}{\mathbf{I}^{\mathrm{T}}(Pal_k)}$ is the augmented constraint matrix of the ISP for the graph Pal_k. Hence, $\mathcal{I}SP(c)$ does not contain the graph Pal_c. Recall that $\mathcal{I}SP(c)$ is strongly hereditary. Hence, the inclusion $\mathcal{I}SP(c) \subseteq Free_s(\{Pal_c\})$ holds. ■

4.2. Reed's theorem

An *odd cycle cover* in a graph G is a subset $X \subseteq V(G)$ such that $G \setminus X$ is a bipartite graph.

The *elementary wall of height h* is a graph consisting of h levels each containing h bricks, where a brick is a cycle of length 6 if the level is not top and bottom, otherwise a brick is a cycle of length 5. The elementary wall of height 5 is depicted in Fig. 2.

The *Escher Wall of height h* can be obtained from the elementary wall of height h as follows. Let $(v_1, \ldots, v_{h+1})$ and $(u_1, \ldots, u_{h+1})$ be the top and bottom paths of the elementary wall, respectively. We replace every edge (v_i, v_{i+1}) with a

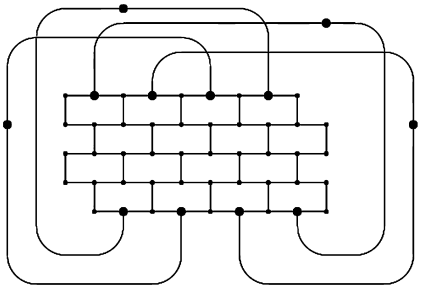

Fig. 3. The Escher Wall of height 4.

path (v_i, w'_i, v_{i+1}) and every edge (u_i, u_{i+1}) with a path (u_i, w''_i, u_{i+1}) for every i. Next, for every i, we add an edge (w'_i, w''_{h+1-i}) and subdivide it. The Escher Wall of height 4 is shown in Fig. 3.

B. Reed has proved the following result in the paper [12].

Theorem 1. *For any k and w, there is a number $t(k, w)$ such that if G is a graph with neither k vertex-disjoint odd cycles nor the Escher Wall of height w as a subgraph, then G contains an odd cycle cover X with $|X| \le t(k, w)$.*

4.3. Main result of this section

Theorem 2. *For each fixed c, the* ISP *can be solved for graphs in $\mathcal{ISP}(c)$ in polynomial time.*

Proof. Let G be an arbitrary graph in $\mathcal{ISP}(c)$ and $c^* \triangleq \lceil \log_2(c) \rceil + 1$. If G contains c^* vertex-disjoint odd cycles, then $\mathbf{I}(G)$ contains a sub-matrix with c^* blocks each having the absolute value of the determinant equal to 2. Hence, it contains a minor with the absolute value 2^{c^*}, which is more than c. Therefore, G does not contain c^* vertex-disjoint odd cycles.

Clearly, the graph Pal_c is an induced subgraph of the Escher Wall of height c. By Lemma 1 and Theorem 1, G has an odd cycle cover X of a cardinality at most $t(c^*, c)$. This cover can be found in polynomial time. Clearly, $\alpha(G) = \max_{X' \subseteq X, X' \text{ is independent}}(|X'| + \alpha(G \setminus (X \cup \bigcup_{v \in X'} N(v))))$ and for any $X' \subseteq X$ a graph $G \setminus (X \cup \bigcup_{v \in X'} N(v))$ is bipartite. The ISP can be solved for bipartite graphs in polynomial time [15]. Hence, for each fixed c, the ISP for graphs in $\mathcal{ISP}(c)$ can be polynomially reduced to the ISP for bipartite graphs. Therefore, for each fixed c, the ISP can be polynomially solved for graphs in $\mathcal{ISP}(c)$. ■

Recall that the result of Theorem 2 was initially obtained by V.E. Alekseev and D.V. Zakharova in the paper [1]. However, our proof is much shorter than those in [1]. The reason for reducing the amount of our proof in comparison with the Alekseev–Zakharova proof is that we use Reed's theorem and another types of obstructions for graphs in $\mathcal{ISP}(c)$. The authors of [1] use odd cycles and the so-called burs (i.e. cycles with edges sticking out from the cycles) as the obstructions, and they do not use Reed's theorem. Theorem 2 certifies that the variant of the conjecture of bounded minors with the extended matrix $\binom{\mathbf{c}^\top}{\mathbf{A}}$ holds for some instances $\mathbf{A}$, $\mathbf{b}$, $\mathbf{c}$. This gives hope that the conjecture of bounded minors with the extended matrix is really true.

5. The vertex dominating set problem

5.1. Auxiliary results

Lemma 2. *Let c be a natural number and $c^* \triangleq \lceil \log_2(c) \rceil + 1$. Then $\mathcal{VDSP}(c) \subseteq Free(\{K_{1,c+2}, A_{c+2}, B_{c+1}, c^*K_{1,3}, c^*A_3\})$.*

Proof. The constraint matrix $\mathbf{A}_v(K_{1,c+2}) + \mathbf{I}_{c+3}$ of the VDSP for the graph $K_{1,c+2}$ is the matrix $\begin{pmatrix} 1 & \mathbf{j}^\top_{c+2} \\ \mathbf{j}_{c+2} & \mathbf{I}_{c+2} \end{pmatrix}$. Its determinant is equal to $-c - 1$, as the matrix can be transformed to the matrix $\begin{pmatrix} -c-1 & \mathbf{o}^\top_{c+2} \\ \mathbf{o}_{c+2} & \mathbf{I}_{c+1} \end{pmatrix}$ with elementary row and column operations. The constraint matrix of the VDSP for the graph $c^*K_{1,3}$ is the block matrix having c^* blocks each of which has the determinant equal to -2. Hence, the determinant of the whole matrix is $(-2)^{c^*}$, whose absolute value is more than c. Hence, $K_{1,c+2} \notin \mathcal{VDSP}(c)$ and $c^*K_{1,3} \notin \mathcal{VDSP}(c)$. As $\mathcal{VDSP}(c)$ is hereditary, the inclusion $\mathcal{VDSP}(c) \subseteq Free(\{K_{1,c+2}, c^*K_{1,3}\})$ holds.

It is not hard to see that the VDSP for the graphs A_{c+2} and B_{c+1} has the constraint matrices $\begin{pmatrix} \mathbf{J}_{c+2} & \mathbf{I}_{c+2} \\ \mathbf{I}_{c+2} & \mathbf{I}_{c+2} \end{pmatrix}$ and $\begin{pmatrix} \mathbf{J}_{c+1} & \mathbf{I}_{c+1} \\ \mathbf{I}_{c+1} & \mathbf{J}_{c+1} \end{pmatrix}$, respectively. The first matrix can be transformed to the matrix $\begin{pmatrix} \mathbf{J}_{c+2} - \mathbf{I}_{c+2} & \mathbf{O}_{c+2} \\ \mathbf{O}_{c+2} & \mathbf{I}_{c+2} \end{pmatrix}$ with elementary row and column operations. The matrix $\mathbf{J}_{c+2} - \mathbf{I}_{c+2}$ is a circulant matrix, whose determinant is equal to $\prod_{j=0}^{c+1} p(w_j)$, where $p(x) \triangleq x + x^2 +$

$\cdots + x^{c+1}$ and $w_j \triangleq e^{2\pi i \cdot \frac{j}{c+2}}$ [5]. Clearly, $p(w_0) = c + 1$ and $p(x) = x + x^2 + \cdots + x^{c+1} = \frac{x^{c+2}-1}{x-1} - 1$ for any real number $x \neq 1$. Hence, $p(w_j) = -1$ for any $j \in \overline{1, c+1}$. Therefore, $|\det(\mathbf{J}_{c+2} - \mathbf{I}_{c+2})| = c + 1$. Thus, the graphs A_{c+2} and c^*A_3 do not belong to $\mathcal{V}DSP(c)$, i.e. $\mathcal{V}DSP(c) \subseteq Free(\{A_{c+2}, c^*A_3\})$. The sub-matrix of the matrix $\begin{pmatrix} \mathbf{J}_{c+1} & \mathbf{I}_{c+1} \\ \mathbf{I}_{c+1} & \mathbf{J}_{c+1} \end{pmatrix}$ induced by the first $c+2$ rows and the last $c+2$ columns is the matrix $\begin{pmatrix} \mathbf{j}_{c+1} & \mathbf{I}_{c+1} \\ 0 & \mathbf{j}^{\mathrm{T}}_{c+1} \end{pmatrix}$. The absolute value of its determinant is $c + 1$. Therefore, $\mathcal{V}DSP(c) \subseteq Free(\{B_{c+1}\})$. ■

By $R(a, b)$ we denote the corresponding *Ramsey number*, i.e. the minimal number n such that any graph with n vertices contains K_a or O_b as an induced subgraph.

Lemma 3. *Let G be an arbitrary graph in $\mathcal{V}DSP(c)$ and D be an arbitrary minimal dominating set. Then $G[D]$ is $K_{R(c+1,c+2)}$-free.*

Proof. Assume that $G[D]$ contains a clique with $k \geq R(c+1, c+2)$ vertices. Let vertices $v_1, \ldots, v_k$ form the clique. As D is a minimal dominating set of G, for every $i \in \overline{1, k}$, there is a vertex $u_i \in N(v_i) \setminus \bigcup_{j=1, j\neq i}^{k} N(v_j)$. By Ramsey's theorem, the induced subgraph $G[\{u_1, \ldots, u_k\}]$ of G contains K_{c+1} or O_{c+2} as an induced subgraph. Hence, G contains either A_{c+2} or B_{c+1} as an induced subgraph. We have a contradiction with the previous lemma. Hence, our initial assumption was false. ■

Lemma 4. *Let G be an arbitrary graph in $\mathcal{V}DSP(c)$, r be a vertex of G, and $V_k(r)$ be the set of all vertices of G lying at the distance k from r. There is a function $f_c(\cdot) : \mathbb{N} \cup \{0\} \longrightarrow \mathbb{N}$ such that for every k the inequality $\alpha(G[V_k(r)]) \leq f_c(k)$ holds.*

Proof. By Lemma 2, one can put $f_c(0) = 1$ and $f_c(1) = c + 1$. Let $k \geq 2$. Assume that $f_c(0), f_c(1), \ldots, f_c(k-1)$ have already been defined. Let us define $f_c(k)$. Let S_k be a maximum independent set in $G[V_k(r)]$. Let D_{k-1} be a minimum subset of $\bigcup_{x\in S_k} N(x) \cap V_{k-1}(r)$ dominating S_k. By Lemma 2, none of the vertices of D_{k-1} can be adjacent to $c + 2$ vertices of S_k. Hence, $|D_{k-1}| \geq \frac{|S_k|}{c+1}$, by the pigeonhole principle. As $\mathcal{V}DSP(c)$ is hereditary and $G \in \mathcal{V}DSP(c)$, the induced subgraph $G[D_{k-1} \cup S_k]$ of G belongs to $\mathcal{V}DSP(c)$. By our assumption, $G[D_{k-1}]$ is $O_{f_c(k-1)+1}$-free. By Lemma 3, $G[D_{k-1}]$ is $K_{R(c+1,c+2)}$-free. Hence, by Ramsey's theorem, $|D_{k-1}| \leq R(R(c+1, c+2), f_c(k-1)+1)$. Therefore, $|S_k| \leq (c+1)R(R(c+1, c+2), f_c(k-1)+1)$. So, we can put $f_c(k) = (c+1)R(R(c+1, c+2), f_c(k-1)+1) + 1$. ■

A $(K_{1,3}, A_3)$-*packing* in a graph G is an arbitrary set $\{G_1, G_2, \ldots, G_s\}$ of graphs such that:

1. for every i, a graph G_i is an induced subgraph of G isomorphic to $K_{1,3}$ or to A_3,
2. for any distinct i and j, vertex sets of G_i and G_j do not intersect and there are no two adjacent vertices $u \in V(G_i)$ and $v \in V(G_j)$.

A $(K_{1,3}, A_3)$-packing is called *optimal* if it contains the maximum possible number of elements. By Lemma 2, any $(K_{1,3}, A_3)$-packing in a graph in $\mathcal{V}DSP(c)$ has at most $\lceil \log_2(c) \rceil$ elements each isomorphic to $K_{1,3}$ and at most $\lceil \log_2(c) \rceil$ elements each isomorphic to A_3. Hence, an optimal $(K_{1,3}, A_3)$-packing in any graph in $\mathcal{V}DSP(c)$ can be computed in polynomial time, as it can be found by enumeration of all subsets of vertices with at most $(4+6)\lceil \log_2(c) \rceil$ elements.

Let G be an arbitrary connected graph in $\mathcal{V}DSP(c)$ and $P \triangleq \{G_1, \ldots, G_s\}$ be its optimal $(K_{1,3}, A_3)$-packing. Let $N_d(P) \triangleq \{x \in V(G) | \exists i \in \overline{1, s}\ \exists y \in V(G_i)$ such that the distance between x and y is at most $d\}$. Let $\mathfrak{D}_G$ be the set $\{D^* | D^*$ is a subset of $N_2(P)$ dominating $N_1(P)\}$. For any element $D^* \in \mathfrak{D}_G$, we delete every vertex of G dominated by D^* assuming that any element of D^* dominates itself. The resultant graph is denoted by $G(D^*)$.

Lemma 5. *For any $D^* \in \mathfrak{D}_G$, the graph $G(D^*)$ is $\{K_{1,3}, A_3\}$-free. If D is a minimum dominating set of G, then $\gamma(G) \geq \gamma(G(D \cap N_2(P))) + |D \cap N_2(P)|$.*

Proof. Clearly, $V(G(D^*)) \cap N_1(P) = \emptyset$, by the definition of $G(D^*)$. By this fact and the optimality of P, the graph $G(D^*)$ cannot contain $K_{1,3}$ or A_3 as an induced subgraph. In other words, $G(D^*)$ is $\{K_{1,3}, A_3\}$-free.

Let $\tilde{D} \triangleq D \cap N_2(P)$ and $D' \triangleq D \setminus \tilde{D}$. Clearly, $\tilde{D} \in \mathfrak{D}_G$. Let us show that there is a dominating set of $G(\tilde{D})$ having at most $|D'|$ elements. This is clear if $D' \subseteq V(G(\tilde{D}))$. Assume that there is a vertex $x \in D' \setminus V(G(\tilde{D}))$. Notice that $x \notin N_2(P)$. By the construction of $G(\tilde{D})$, there is a vertex $y \in \tilde{D}$ such that $xy \in E(G)$. Clearly, $y \in N_2(P) \setminus N_1(P)$. As D is a minimum dominating set of G, there is a vertex z in $N(x) \setminus \bigcup_{v\in D, v\neq x} N[v]$. Hence, z is not dominated by $\tilde{D}$. Therefore, z belongs to $G(\tilde{D})$. The set $N(x) \cap V(G(\tilde{D}))$ is a clique. Indeed, if it contains two non-adjacent vertices v and u, then $N(y) \cap \{v, u\} = \emptyset$ and x, y, v, u induce $K_{1,3}$. This is impossible, as P is an optimal $(K_{1,3}, A_3)$-packing. Therefore, $(D' \setminus \{x\}) \cup \{z\}$ dominates $V(G(\tilde{D}))$. Thus, there is a dominating set D'' of $G(\tilde{D})$ with at most $|D'|$ elements. The set $\tilde{D} \cup D''$ is a dominating set of G. Moreover, $|\tilde{D} \cup D''| = |\tilde{D}| + |D''| \leq |\tilde{D}| + |D'| = |D| = \gamma(G)$. As $\gamma(G(\tilde{D})) \leq |D''|$, the inequality $\gamma(G) \geq \gamma(G(\tilde{D})) + |\tilde{D}|$ holds. ■

5.2. Main result of this section

Theorem 3. *For each fixed c, the* VDSP *for graphs in $\mathcal{V}DSP(c)$ can be solved in polynomial time.*

Proof. Let G be a connected graph in $\mathcal{V}DSP(c)$. An optimal $(K_{1,3}, A_3)$-packing P in G can be computed in polynomial time. Let D_{opt} be a minimum dominating set in the graph G. By Lemmas 3 and 4, there is a function $g(\cdot) : \mathbb{N} \longrightarrow \mathbb{N}$ such that $|D_{opt} \cap N_2(P)|$ is at most the value of the function at the point c. Let $\mathfrak{D}^*_G \triangleq \{D \in \mathfrak{D}_G | \, |D| \leq g(c)\}$. Hence, the set $\mathfrak{D}^*_G$ can be computed in polynomial time. Let $D \in \mathfrak{D}^*_G$. The union of D and any minimum dominating set of $G(D)$ is a dominating set of G. Hence, the inequality $\gamma(G) \leq |D| + \gamma(G(D))$ is true. By this fact and by the second part of Lemma 5, $\gamma(G) = \min_{D \in \mathfrak{D}^*_G}(\gamma(G(D)) + |D|)$. As $\mathfrak{D}^*_G \subseteq \mathfrak{D}_G$, by the first part of Lemma 5, the graph $G(D)$ is $\{K_{1,3}, A_3\}$-free for any $D \in \mathfrak{D}^*_G$. The VDSP can be solved in polynomial time for $\{K_{1,3}, A_3\}$-free graphs [3]. Therefore, for each fixed c, the VDSP for graphs in $\mathcal{V}DSP(c)$ can be solved in polynomial time. ■

Theorem 3 certifies that the conjecture of bounded minors holds for some instances **A**, **b**, **c**. This gives hope that the conjecture is really true.

6. The edge dominating set problem

6.1. Clique-width of graphs and its importance

Clique-width is an important parameter of graphs. This is explained by the fact that many graph problems can be solved in polynomial time for any class of graphs of bounded clique-width (see [4] for more information). More precisely, for each fixed number C, many problems that are NP-complete for the set of all graphs become polynomial-time solvable for any class of graphs having clique-width at most C. In particular, this category includes the independent set and the vertex dominating set problems [4].

The class $\mathcal{S}$ is the set of all forests having at most three leaves in each connected component. The following result is a sufficient condition for boundedness of clique-width in strongly hereditary classes. It was proved in [9].

Lemma 6. *If $\mathcal{X}$ is a strongly hereditary class and $\mathcal{S} \not\subseteq \mathcal{X}$, then there is a constant $C(\mathcal{X})$ such that any graph in $\mathcal{X}$ has clique-width at most $C(\mathcal{X})$.*

Lemma 7. *Let c be a natural number and $c^* \triangleq \lceil \log_2(c) \rceil + 1$. Then the inclusion $\mathcal{E}DSP(c) \subseteq Free_s(\{c^* K'_{1,3}\})$ holds.*

Proof. The constraint matrix $\mathbf{A}_e(K'_{1,3}) + \mathbf{I}_6$ of the EDSP for the graph $K'_{1,3}$ is the matrix $\mathbf{M} \triangleq \begin{pmatrix} 1&1&1&1&0&0\\ 1&1&1&0&1&0\\ 1&1&1&0&0&1\\ 1&0&0&1&0&0\\ 0&1&0&0&1&0\\ 0&0&1&0&0&1 \end{pmatrix}$ up to permutations of rows and columns. It is not hard to see that $\det(\mathbf{M}) = -2$. If a graph G in $\mathcal{E}DSP(c)$ contains c^* vertex-disjoint copies of $K'_{1,3}$, then the constraint matrix of the problem for G contains a block sub-matrix having c^* blocks each of which is $\mathbf{M}$. Hence, the constraint matrix has a sub-matrix, whose determinant is $(-2)^{c^*}$. This is impossible, as $2^{c^*} > c$. Therefore, $\mathcal{E}DSP(c) \subseteq Free_s(\{c^* K'_{1,3}\})$. ■

For any c and p, $pK'_{1,3} \in \mathcal{S}$ and $\mathcal{E}DSP(c)$ is strongly hereditary. Hence, by the previous lemmas, clique-width of all graphs in $\mathcal{E}DSP(c)$ is bounded for every c.

6.2. Main result of this section

Theorem 4. *For each fixed c, the* EDSP *can be solved for graphs in $\mathcal{E}DSP(c)$ in polynomial time.*

Proof. The EDSP can be solved in polynomial time in any class of graphs of bounded clique-width [8]. By this fact and Lemmas 6–7, for each fixed c, the EDSP can be solved for graphs in $\mathcal{E}DSP(c)$ in polynomial time. ■

Theorem 4 certifies that the conjecture of bounded minors holds for some instances **A**, **b**, **c**. This gives hope that the conjecture is really true.

Acknowledgment

This work was supported by the Russian Science Foundation Grant No. 17-11-01336.

References

[1] V.E. Alekseev, D.V. Zakharova, Independent sets in the graphs with bounded minors of the extended incidence matrix, J. Appl. Ind. Math. 5 (1) (2011) 14–18.

[2] S. Artmann, F. Eisenbrand, C. Glanzer, T. Oertel, S. Vempala, R. Weismantel, A note on non-degenerate integer programs with small sub-determinants, Oper. Res. Lett. 44 (5) (2016) 635–639.

[3] A. Brandstädt, F.F. Dragan, On linear and circular structure of (claw, net)-free graphs, Discrete Appl. Math. 129 (2–3) (2003) 285–303.

[4] B. Courcelle, J. Makowsky, U. Rotics, Linear time solvable optimization problems on graphs of bounded clique-width, Theory Comput. Syst. 33 (2) (2000) 125–150.
[5] R.M. Gray, Toeplitz and circulant matrices: a review, Found. Trends Commun. Inf. Theory 2 (3) (2006) 155–239.
[6] N. Karmarkar, A new polynomial time algorithm for linear programming, Combinatorica 4 (4) (1984) 373–395.
[7] L.G. Khachiyan, Polynomial algorithms in linear programming, USSR Comput. Math. Math. Phys. 20 (1) (1980) 53–72.
[8] D. Kobler, U. Rotics, Edge dominating set and colorings on graphs with fixed clique-width, Discrete Appl. Math. 126 (2–3) (2003) 197–221.
[9] V. Lozin, M. Millanic, Critical properties of graphs of bounded clique-width, Discrete Math. 313 (9) (2013) 1035–1044.
[10] Y.E. Nesterov, A.S. Nemirovsky, Interior Point Polynomial Methods in Convex Programming, Society for Industrial and Applied Math, USA, 1994.
[11] P.M. Pardalos, C.G. Han, Y. Ye, Interior point algorithms for solving nonlinear optimization problems, COAL Newsl. 19 (1991) 45–54.
[12] B. Reed, Mangoes and blueberries, Combinatorica 19 (2) (1999) 267–296.
[13] V.N. Shevchenko, Qualitative Topics in Integer Linear Programming, American Mathematical Society, Providence, 1997.
[14] S.I. Veselov, A.J. Chirkov, Integer program with bimodular matrix, Discrete Optim. 6 (2) (2009) 220–222.
[15] S.H. Whitesides, A method for solving certain graph recognition and optimization problems, with applications to perfect graphs, Ann. Discrete Math. 88 (1984) 281–297.