The Same Upper Bound for Both: The 2-Page and the Rectilinear Crossing Numbers of the n-Cube

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Abstract. We present two main results: a 2-page drawing and a rectilinear drawing of the n-dimensional cube Q_n. Both drawings have the same number $\frac{125}{768}4^n - \frac{2^{n-3}}{3}\left(3n^2 + \frac{9+(-1)^{n+1}}{2}\right)$ of crossings, even though they are given by different constructions. The first improves the current best general 2-page drawing, while the second is the first non-trivial rectilinear drawing of Q_n.

Keywords: two-page crossing number, rectilinear crossing number, n-cube, topological graph theory.

1 Introduction

A *drawing* $D(G)$ of a graph $G = (V, E)$ is a mapping of G to a topological space (usually the plane, but not always). The vertices go into distinct points called *nodes*, and an edge maps into an *arc* – a homeomorphic image of the closed interval $[0, 1]$ – such that its interior contains no node and the endpoints of the arc associated to an edge $e = uv \in E$ are the nodes associated to the end vertices of e: u and v.

A *good drawing* $D(G)$ of a graph G is a drawing where each pair of arcs have at most one point in common which is either a node or a *crossing*. All drawings considered in this paper are good.

An *embedding* of a graph $G = (V, E)$ in a topological space S is a drawing of G in S without crossings.

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Given a graph $G = (V, E)$, the *crossing number* $\nu(G)$ of G is the minimum number of crossings in a drawing of G in the plane [10].

For a positive integer k, and a graph G, a *k-page drawing of* G is a drawing of G in the union H_k of k closed half-planes, all having their boundary lines in common, but otherwise disjoint, so that $V(G)$ is contained in the common boundary line and, for each edge e of G, the arc representing e is contained in one of the half-planes [4]. The *page number* of a graph G is the smallest k such that G embeds in H_k.

The *k-page crossing number* $\nu_k(G)$ of a graph G is the minimum number of crossings of edges in a k-page drawing of G.

Since every 1-page drawing is a 2-page drawing, and every 2-page drawing is a planar drawing, $\nu(G) \leq \nu_2(G) \leq \nu_1(G)$. Yannakakis [13] proved that every planar graph has a 4-page embedding and announced the existence of a planar graph that has no 3-page embedding.

The 2-page crossing number of K_n has been recently determined by Ábrego et al. [1]. de Klerk and Pasechnik [6] used semidefinite programming to find estimates for $\nu_2(K_{m,n})$ and $\nu(K_n)$. Masuda et al. [12] proved that it is NP-complete to determine if there is a 2-page drawing of G having at most k crossings using a given linear order. This problem is called *Fixed Linear Crossing Number.*

A *rectilinear drawing* of a graph $G = (V, E)$ is a drawing of G in the plane such that edges are drawn as straight line segments. The *rectilinear crossing number* $\overline{cr}(G)$ of a graph G is the smallest number of crossings in a rectilinear drawing of G.

For $n \geq 0$, the *n-cube* Q_n has as its vertex set $V(Q_n)$ all binary strings of length n and two vertices are adjacent if and only if the corresponding strings differ in precisely one position. (Note that Q_0 has one vertex and no edges.) Thus, each edge e determines the position at which its incident vertices differ; if this is position i, then *e is in the i-th dimension.*

Very little is known about exact values of $\nu(Q_n)$, $\nu_2(Q_n)$, and $\overline{cr}(Q_n)$, respectively. It is known that if $n \leq 3$, then $\nu(Q_n) = \nu_2(Q_n) = \overline{cr}(Q_n) = 0$. Dean and Richter [5] showed that $\nu(Q_4) = 8$; now appropriate drawings imply that $\nu_2(Q_4) = \overline{cr}(Q_4) = 8$. Buchheim and Zheng [3] used a brute force computer approach with MAX CUT on an auxiliary graph to prove that $\nu_2(Q_5) = 60$, $\nu_2(Q_6) = 368$, and $\nu_2(Q_7) \leq 1874$.

Madej [11] exhibited a 2-page drawing for Q_n, thereby showing that $\nu_2(Q_n) \leq \frac{4^n}{6} - 2^{n-3}n^2 - 2^{n-4}3 + \frac{(-2)^n}{48}$. In these 2-page drawings, Q_5 has 64 crossings, Q_6 has 384 crossings, and Q_7 has 1920 crossings. Madej also exhibited a drawing for Q_5 with 56 crossings; this is not a 2-page drawing. At WG'2003 [8], Faria et al. [9] exhibited drawings that show $\nu(Q_5) \leq 56$, $\nu(Q_6) \leq 352$, and $\nu(Q_7) \leq 1760$. These are in line with the conjecture of Erdős and Guy [7]: $\nu(Q_n) \leq 4^n \frac{5}{32} - \lfloor \frac{n^2+1}{2} \rfloor 2^{n-2}$ [9].

In Section 3, we give a 2-page drawing of Q_n having

$$\frac{125}{768}4^n - \frac{2^{n-3}}{3}\left(3n^2 + \frac{9 + (-1)^{n+1}}{2}\right)$$