Positivity Preserving Numerical Method for Non-linear Black-Scholes Models

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Abstract. A motivation for studying the nonlinear Black- Scholes equation with a nonlinear volatility arises from option pricing models taking into account e.g. nontrivial transaction costs, investors preferences, feedback and illiquid markets effects and risk from a volatile (unprotected) portfolio. In this work we develop positivity preserving algorithm for solving a large class of non-linear models in mathematical finance on the original (infinite) domain. Numerical examples are discussed.

1 Introduction and Model Formulation

The solution of the (linear) Black-Scholes equation (Black and Scholes, 1973) has been derived under several restrictive assumptions like e.g. frictionless, liquid and complete markets, etc. We also recall that the linear Black-Scholes equation provides a perfectly replicated hedging portfolio. In the last decades some of these assumptions have been relaxed in order to model, for instance, the presence of transaction costs (see e.g. Leland [19], Avellaneda and Parás [5]), feedback and illiquid market effects due to large traders choosing given stock-trading strategies (Frey and Patie [10]), imperfect replication and investors preferences (Barles and Soner [6]), risk from unprotected portfolio (Jandačka-Ševčovič, [15]).

This models defer from the classical Black-Scholes equation by a non-constant volatility term σ, which depends on time t, spot price S of the underlying and the second derivative (Greek $\varGamma$) of the option price $V(S,t)$. Hence, the model equation is the following nonlinear partial differential equation

$$V_t + \frac{1}{2}\sigma^2(t,S,V_{SS})S^2V_{SS} + (r-q)SV_S - rV = 0, \quad 0 \le S < \infty, \quad 0 \le t \le T, \tag{1}$$

with constant short rate r, dividend yield q, maturity T and volatility $\sigma^2(t,S,V_{SS})$ depending on the particular model.

We will study (1) for European Call option, i.e. the value $V(S,t)$ is the solution to (1), $q=0$ on $0 \le S < \infty$, $0 \le t \le T$ with the following terminal and boundary conditions ($E>0$ is the exercise price):

$$\begin{aligned} &V(S,T) = \max\{0, S-E\}, \quad 0 \le S < \infty, \\ &V(0,t) = 0, \quad 0 \le t \le T, \\ &V(S,t) = S - Ee^{-r(T-t)}, \quad S \to \infty. \end{aligned} \tag{2}$$

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The model (1) can be written as the backward parabolic fully nonlinear PDE $V_t + S^2F(S, V_S, V_{SS}) = 0$. At some conditions on F, the most used of which are $F(S, p, r) \in C^2$, $V(S, T) = V_T(S)$, $F_r(S, V_T'(S), V_T''(S)) > 0$, in [1,2,6,15] was obtained results for existence and uniqueness of solutions (classical or viscosity). It was checked that the model described above satisfies these conditions.

Let $G(S, V_S, V_{SS}) = S^2F(S, V_S, V_{SS})$, $G \in C^k((0,T) \times R^3)$. We briefly discuss the maximum principle (MP) for (1). Let Π_T be the rectangle $\Pi_T = \{(S,t) : 0 < t < T, -\infty < a < b < +\infty\}$ and $G_r(S, p, r) \geq 0$ everywhere and $G_r(S, 0, 0) = 0$. Then, if V is a classical solution of $V_t + G(S, V_S, V_{SS}) = 0$ in Π_T we have: $\max\limits_{\overline{\Pi}_T} V = \max\limits_{\Gamma_T} u$ and $\min\limits_{\overline{\Pi}_T} V = \min\limits_{\Gamma_T} V$, where $\Gamma_T = I \cup II \cup III$, $I = \{0 < t < T,\ x = a\}$, $II = \{a < x < b,\ t = 0\}$, $III = \{0 < t < T,\ x = b\}$ is the parabolic part of the boundary. For the proof, see [2].

There exists many discretizations, algorithms and some numerical methods for different versions of the non-linear Black-Scholes equation [4,7,8,12,14]. In [20], authors develop positivity-preserving (i.e. the non-negativity of the numerical solution to be guaranteed) first-order fully implicit scheme for models arising from pricing European options under transaction costs. In our previous work [17] we developed a *fast*, second order both in space and time a kernel-based method for solving a large class of non-linear models in mathematical finance, computed on large enough truncated region. But the non-negativity of the numerical solution is not guaranteed. In this work, having in mind MP discussed above, we will present efficient, *positivity preserving* algorithm for solving the same non-linear Black-Scholes models on the original (infinite) interval. We develop implicit-explicit methods on quasi-uniform mesh (QUM), implementing the idea of van Leer flux limiter [11,13,21].

An often used approach to overcome the *degeneration* at $S = 0$ and to obtain a *forward* parabolic problem, is the variable transformation [1,8,12]

$$x(S) = \log\left(\frac{S}{E}\right), \quad \tau(t) = \frac{1}{2}\sigma_0^2(T - t), \quad u(x, \tau) = e^{-x}\frac{V}{E}.$$

Now, denoting $K = 2r/\sigma_0^2$ (σ_0 is the volatility of the underlying asset), the equation (1) transforms into

$$u_\tau - \widetilde{\sigma}^2(\tau, x, u_x, u_{xx})(u_x + u_{xx}) - Ku_x = 0, \quad x \in \mathbb{R}, \quad 0 \leq \tau \leq \frac{\sigma_0^2 T}{2}, \qquad (3)$$

$$\begin{aligned}
&\widetilde{\sigma}_{\mathrm{L}}^2 = 1 + f(u_x, u_{xx}), && f(u_x, u_{xx}) = Le \cdot \mathrm{sign}(u_x + u_{xx}),\\
&\widetilde{\sigma}_{\mathrm{BS}}^2 = 1 + f(x, \tau, u_x, u_{xx}), && f(x, \tau, u_x, u_{xx}) = \Psi[a^2Ee^{K\tau + x}(u_x + u_{xx})],\\
&\widetilde{\sigma}_{\mathrm{JS}}^2 = 1 + f(x, u_x, u_{xx}), && f(x, u_x, u_{xx}) = \mu[Ee^x(u_x + u_{xx})]^{1/3},\\
&\widetilde{\sigma}_{\mathrm{AP}}^2 = f(u_x, u_{xx}), && f(u_x, u_{xx}) = \begin{cases} \sigma_{\max}^2, \ u_x + u_{xx} \leq 0,\\ \sigma_{\min}^2, \ u_x + u_{xx} > 0,\end{cases}\\
&\widetilde{\sigma}_{\mathrm{FP}}^2 = f(x, u_x, u_{xx}), && f(x, u_x, u_{xx}) = [1 - \rho \cdot \lambda(Ee^x)(u_x + u_{xx})]^{-2},
\end{aligned}$$

where $\widetilde{\sigma}^2 = \widetilde{\sigma}^2_{\mathrm{L,BS,JS,AP,FP}}$ corresponds to Leland, Barles-Soner, Jandačka-Ševčovič, Avellaneda-Parás and Frey-Patie models, respectively. Here $0 < Le <$