Flip Distance between Triangulations of a Simple Polygon is NP-Complete⋆

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Abstract. Let T be a triangulation of a simple polygon. A *flip* in T is the operation of removing one diagonal of T and adding a different one such that the resulting graph is again a triangulation. The *flip distance* between two triangulations is the smallest number of flips required to transform one triangulation into the other. For the special case of convex polygons, the problem of determining the shortest flip distance between two triangulations is equivalent to determining the rotation distance between two binary trees, a central problem which is still open after over 25 years of intensive study.

We show that computing the flip distance between two triangulations of a simple polygon is NP-complete. This complements a recent result that shows APX-hardness of determining the flip distance between two triangulations of a planar point set.

1 Introduction

Let P be a simple polygon in the plane, that is, a closed region bounded by a piece-wise linear, simple cycle. A *triangulation* T of P is a geometric (straight-line) maximal outerplanar graph whose outer face is the complement of P and whose vertex set consists of the vertices of P. The edges of T that are not on the outer face are called *diagonals*. Let d be a diagonal whose removal creates a convex quadrilateral f. Replacing d with the other diagonal of f yields another triangulation of P. This operation is called a *flip*. The *flip graph* of P is the abstract graph whose vertices are the triangulations of P and in which two triangulations are adjacent if and only if they differ by a single flip. We study the *flip distance*, i.e., the minimum number of flips required to transform a given source triangulation into a target triangulation.

Edge flips became popular in the context of Delaunay triangulations. Lawson [9] proved that any triangulation of a planar n-point set can be transformed into any other by $O(n^2)$ flips. Hence, for every planar n-point set the flip graph is connected with diameter $O(n^2)$. Later, he showed that in fact every triangulation can be transformed to the Delaunay triangulation by $O(n^2)$ flips that locally

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fix the Delaunay property [10]. Hurtado, Noy, and Urrutia [7] gave an example where the flip distance is $\Omega(n^2)$, and they showed that the same bounds hold for triangulations of simple polygons. They also proved that if the polygon has k reflex vertices, then the flip graph has diameter $O(n + k^2)$. In particular, the flip graph of any planar polygon has diameter $O(n^2)$. Their result also generalizes the well-known fact that the flip distance between any two triangulations of a convex polygon is at most $2n - 10$, for $n > 12$, as shown by Sleator, Tarjan, and Thurston [15] in their work on the flip distance in convex polygons. The latter case is particularly interesting due to the correspondence between flips in triangulations of convex polygons and rotations in binary trees: The dual graph of such a triangulation is a binary tree, and a flip corresponds to a rotation in that tree; also, for every binary tree, a triangulation can be constructed.

We mention two further remarkable results on flip graphs for point sets. Hanke, Ottmann, and Schuierer [6] showed that the flip distance between two triangulations is bounded by the number of crossings in their overlay. Eppstein [5] gave a polynomial-time algorithm for calculating a lower bound on the flip distance. His bound is tight for point sets with no empty 5-gons; however, except for small instances, such point sets are not in general position (i.e., they must contain collinear triples) [1]. For a recent survey on flips see Bose and Hurtado [3].

Very recently, the problem of finding the flip distance between two triangulations of a point set was shown to be NP-hard by Lubiw and Pathak [11] and, independently, by Pilz [12], and the latter proof was later improved to show APX-hardness of the problem. Here, we show that the corresponding problem remains NP-hard even for simple polygons. This can be seen as a further step towards settling the complexity of deciding the flip distance between triangulations of convex polygons or, equivalently, the rotation distance between binary trees. This variant of the problem was probably first addressed by Culik and Wood [4] in 1982 (showing a flip distance of $2n - 6$).

The formal problem definition is as follows: given a simple polygon P, two triangulations T_1 and T_2 of P, and an integer l, decide whether T_1 can be transformed into T_2 by at most l flips. We call this decision problem PolyFlip. To show NP-hardness, we give a polynomial-time reduction from Rectilinear Steiner Arborescence to PolyFlip. Rectilinear Steiner Arborescence was shown to be NP-hard by Shi and Su [14]. In Section 2, we describe the problem in detail. We present the well-known *double chain* (used by Hurtado, Noy, and Urrutia [7] for giving their lower bound), a major building block in our reduction, in Section 3. Finally, in Section 4, we describe our reduction and prove that it is correct. An extended abstract of this work was presented at the 29$^{\text{th}}$ EuroCG, 2013; for omitted proofs, see [2].

2 The Rectilinear Steiner Arborescence Problem

Let S be a set of N points in the plane whose coordinates are nonnegative integers. The points in S are called *sinks*. A *rectilinear tree* T is a connected acyclic collection of horizontal and vertical line segments that intersect only