Planar Capacitated Dominating Set Is $W[1]$-Hard

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Abstract. Given a graph G together with a capacity function $c: V(G) \rightarrow \mathbb{N}$, we call $S \subseteq V(G)$ a capacitated dominating set if there exists a mapping $f : (V(G) \setminus S) \rightarrow S$ which maps every vertex in $(V(G) \setminus S)$ to one of its neighbors such that the total number of vertices mapped by f to any vertex $v \in S$ does not exceed $c(v)$. In the Planar Capacitated Dominating Set problem we are given a planar graph G, a capacity function c and a positive integer k and asked whether G has a capacitated dominating set of size at most k. In this paper we show that Planar Capacitated Dominating Set is $W[1]$-hard, resolving an open problem of Dom et al. [*IWPEC, 2008*]. This is the first bidimensional problem to be shown $W[1]$-hard. Thus Planar Capacitated Dominating Set can become a useful starting point for reductions showing parameterized intractablility of planar graph problems.

1 Introduction

In the Dominating Set problem we are given a graph G and asked for the smallest set of vertices such that every vertex in the graph either belongs to this set or has a neighbor which does. This basic problem in algorithms and complexity has been studied extensively, and finds applications in various domains. Dominating Set has a special place in parameterized complexity [5,8,13]. It is the most well-known $W[2]$-complete problem and is a standard starting point for reductions that show intractability of parameterized problems [5]. Even though the Dominating Set problem is a fundamentally hard problem in the parameterized W-hierarchy, it has been used as a benchmark problem for developing *sub-exponential time* FPT algorithms [1,3,11], and also for obtaining *linear kernels* on planar graphs [2,8,12,13], and more generally, graphs that exclude a fixed graph H as a minor.

Different applications of Dominating Set have initiated studies of different generalizations and variations of the problem. These include Connected Dominating Set, Partial Dominating Set, and Capacitated Dominating Set to name a few. In this paper we focus on one such generalization, namely Capacitated Dominating Set. Given a graph G together with a capacity function

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$c : V(G) \to \mathbb{N}$, we call $S \subseteq V(G)$ a capacitated dominating set if there exists a mapping $f : (V(G) \setminus S) \to S$ which maps every vertex in $(V(G) \setminus S)$ to one of its neighbors such that the total number of vertices mapped by f to any vertex $v \in S$ does not exceed $c(v)$. The Capacitated Dominating Set problem is defined as follows.

> Capacitated Dominating Set (CDS): Given a graph G, a capacity function c and a positive integer k, determine whether there exists a capacitated dominating set S of G containing at most k vertices.

Dom et al. initiated the study of CDS from the perspective of Parameterized Complexity, and showed that CDS is $W[1]$-hard parameterized by solution size and the treewidth of the input graph [4]. Like Dominating Set, CDS has become a useful source for showing W-hardness, especially when the parameter is the structure of the input graph [7,10]. It has been recently used to show the first W-hardness results for problems parameterized by the cliquewidth of the input graph [10].

Many graph problems that are W-hard in general turn out to be FPT when restricted to planar graphs. This is true for Dominating Set and many of its variants, and hence it is very natural to consider the parameterized complexity of Planar Capacitated Dominating Set, the restriction of CDS to planar graphs. For most planar graph problems, an FPT algorithm can be obtained by combining a combinatorial bound on the treewidth of non-trivial instances with a dynamic programming algorithm for graphs of bounded treewidth. In fact for most problems restricted to planar graphs we have subexponential time parameterized algorithms using *bidimensionality* theory [3]. PCDS, however, is an exception to this rule. In particular, it can easily be shown by using bidimensionality that any planar graph that has a capacitated dominating set of size at most k has treewidth $O(\sqrt{k})$. On the other hand, Dom et al. showed that CDS is $W[1]$-hard when parameterized by solution size and the treewidth of the input graph [4]. Thus, bidimensionality alone was not enough to tackle this problem and it was an intriguing question whether PCDS could still turn out to be FPT by a non-trivial use of planarity. We show that these hopes were futile by giving a $W[1]$-hardness reduction for PCDS. Planar Capacitated Dominating Set is the first bidimensional problem to be shown $W[1]$-hard. We believe that Planar Capacitated Dominating Set can become a useful starting point for reductions showing parameterized intractablility of planar graph problems.

2 Preliminaries

We will work with both undirected and directed graphs. Given a graph G, the vertex set of G is $V(G)$ and the edge set of G is $E(G)$. . For a graph G, $n = |V(G)|$ and $m = |E(G)|$. With $N_G(u)$ we denote all vertices that are adjacent to u and the *degree* of u is $d_G(u) = |N_G(u)|$. Let f be the function associated with a capacitated dominating set S. Given $u \in S$ and $v \in V \setminus S$, we say that u *dominates* v if $f(v) = u$; moreover, every vertex $u \in S$ dominates itself. Note