Phase Transition of Active Rotators in Complex Networks

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Abstract. We study the nonequilibrium phenomena of a coupled active rotator model in complex networks. From a numerical Langevin simulation, we find the peculiar phase transition not only on globally connected network but also on other complex networks and reveal the corresponding phase diagram. In this model, two phases — stationary and quasi-periodic moving phases — are observed, in which microscopic dynamics are thoroughly investigated. We extend our study to the nonidentical oscillators and the more heterogeneous degree distribution of complex networks.

Keywords: active rotator model, phase transitions, complex networks.

1 Introduction

Various coupled oscillatory systems in nature have been known to exhibit many interesting behaviors including synchronization. Collective synchronization has attracted much interest due to the beauty of simultaneousness and the spontaneous emergence in such phenomena as the synchronous flashing of fireflies, the chorusing of crickets, and the clapping of hands after an astonishing orchestral performance [1]. In order to understand such synchronized behaviors, nonlinear coupled oscillators have been studied extensively with various models. Among them, the Kuramoto model is one of the most studied models due to its simplicity and analytical tractability [2,3]. The Kuramoto model has been extended with many variations for applications in diverse systems [3]. One natural extension is to add external fields, which implies the external current applied to a neuron to describe an excitable systems. This is also known to be an *active rotator* model when each oscillator has the constant natural frequency [3,4].

Most studies of the active rotator model have assumed that all oscillators are connected to each other, i.e., globally connected network, or sometimes 2 and 3-dimensional regular lattice is used [4,5,6]. However, such a type of interaction has a limitation when applied to most real systems. Therefore we need to consider such nontrivial connectivity and extend the study of synchronization to complex networks. Thus, in the present paper, we report our study of active rotator model in complex networks.

J. Zhou (Ed.): Complex 2009, Part I, LNICST 4, pp. 242–246, 2009.

2 Model System

The dynamics of N coupled limit-cycle oscillators having the phase $\{\phi_i(t)|i = 1, 2, \ldots, N\}$ is described by the set of equations

$$\frac{d\phi_i}{dt} = \omega_i - b\sin\phi_i - \frac{K}{\langle k\rangle}\sum_{j=1}^{N} a_{ij}\sin(\phi_i - \phi_j) + \eta_i(t). \tag{1}$$

The first term ω_i represents the natural frequency of the ith oscillator, which is assumed the random normal distribution having the correlation $\langle\omega_i\omega_j\rangle = \sigma^2\delta_{ij}$ with the variance σ^2 and the mean $\langle\omega_i\rangle = \omega_0$. The second and third terms indicate the pinning force and the coupling between the oscillators respectively; the coupling strength K is set to be a positive one ($K > 0$), so the interacting oscillators favor their phase difference minimized. The adjacency matrix element $a_{ij} = 1(0)$ if oscillators i and j are connected (disconnected), and $\langle k\rangle$ denotes the mean degree given by $\sum_i k_i/N$, where the degree $k_i = \sum_j a_{ij}$. In the last term of Eq. (1), $\eta_i(t)$ is the Gaussian white noise with properties $\langle\eta_i(t)\rangle = 0, \langle\eta_i(t)\eta_j(t')\rangle = 2D\delta(t - t')\delta_{ij}$.

When all oscillators are connected to each other, i.e., $a_{ij} = 1$ for all $i \neq j$, and $b = 0$, $D = 0$, the model corresponds to the original Kuramoto model [2]. If all oscillators are identical and $b = 0$, it describes the thermodynamic system of classical XY spins, where D plays role of the temperature of the spin systems [6]. When all oscillators have the same frequency, we call the system as active rotators.

Collective phase synchronization is conveniently described by the order parameter defined by

$$r(t)e^{i\theta(t)} \equiv \frac{1}{N}\sum_{j=1}^{N} e^{i\phi_j(t)}, \tag{2}$$

where $r > 0$ implies emergence of the phase synchronization. Then we take the time average of $r(t)$ such as $r \equiv \overline{r(t)} = (2/T)\sum_{t=T/2+1}^{T} r(t)$, where the over line represents the time averaging and we set T to enough large number after confirming the state passes over the transient period. In the case of the original Kuramoto model, the time averaged r delivers most information since $r(t)$ saturates to a value r. However, active rotators do not always go to the stationary phase but show periodic behavior. Therefore, Shinomoto *et al.* [4] introduced another order parameter σ and a kind of fluctuation measure $\widetilde{\chi}$ defined by

$$\sigma e^{i\varphi} \equiv \overline{r(t)e^{i\theta(t)}} = \frac{2}{T}\sum_{t=T/2+1}^{T} r(t)e^{i\theta(t)}, \tag{3}$$

$$\widetilde{\chi} \equiv N \cdot \overline{|r(t)e^{i\theta(t)} - \sigma e^{i\varphi}|^2}. \tag{4}$$

One can easily show that $\widetilde{\chi}$ is equivalent to $N \cdot [\overline{r^2(t)} - \sigma^2]$, which measures the difference between r and σ.