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A201198: Counting Walks on Jacobi Graphs: An Application of Orthogonal Polynomials

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If the tridiagonal symmetric *Jacobi* matrix $\mathbf{J}_N$ associated with the three term recurrence of an orthogonal polynomial system (*OPS*) in one variable has nonnegative integer entries one can interpret $\mathbf{J}_N$ as the vertex-vertex (or adjacency) matrix of a graph which will be denoted also by $\mathbf{J}_N$. In this case we will call the graph a *Jacobi* graph associated with the corresponding *OPS*. We consider two types, the open graphs and the closed ones. In the closed case the vertex no. N is connected with the vertex no. 1.

a) Open case: $\mathbf{J}_N \equiv \mathbf{J}_N\left(\{c_k\}_0^{N-1}, \{b_k\}_1^{N-1}\right)$ $c_k \in \mathbb{N}_0,\ b_k \in \mathbb{N}$

$$\mathbf{J}_N = \begin{pmatrix} c_0 & b_1 & & & \\ b_1 & c_1 & b_2 & & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & & b_{N-2} & c_{N-2} & b_{N-1} \\ & & & b_{N-1} & c_{N-1} \end{pmatrix}. \tag{1}$$

b) Closed case: $J_N \equiv J_N\left(\{c_k\}_0^{N-1}, \{b_k\}_0^{N-1}\right)$ $c_k \in \mathbb{N}_0,\ b_k \in \mathbb{N}$

$$J_N = \begin{pmatrix} c_0 & b_1 & & & b_0 \\ b_1 & c_1 & b_2 & & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & & b_{N-2} & c_{N-2} & b_{N-1} \\ b_0 & & & b_{N-1} & c_{N-1} \end{pmatrix}. \tag{2}$$

See the *Figures 1* and *2* for the skeleton of these graphs where multilines have been denoted by a single line with multiplicity.

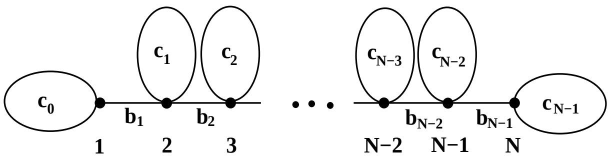

FIG. 1 : Open Chain Graph Skeleton $\mathbf{J}_N$

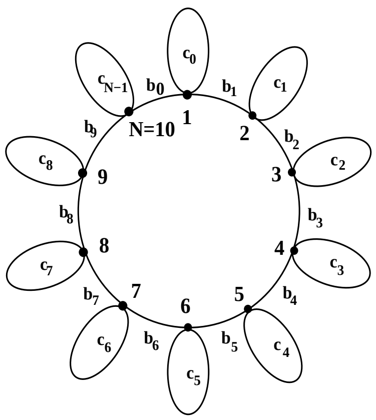

FIG. 2 : Closed Chain Graph Skeleton J_N , N = 10

For example, if $c_k = 0$ and $b_k = 1$ for $k \in \{0, 1, ..., N-1\}$, one has the graphs $\mathbf{J}_N = P_N$, the simple path with N vertices and $N-1$ lines, and $\mathbf{J}_N = C_N$, the cycle graph with N vertices and N lines.

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Obviously these graphs belong to the *Chebyshev* polynomials $S_N(x)$, members of the classical *Jacobi* *OPS*, whose coefficient array is shown in [4] A049310. Their recurrence is $S_n(x) - x\,S_{n-1}(x) + S_{n-2}(x) = 0$, $S_{-1}(x) = 0$ and $S_0(x) = 1$. In general the three term recurrence for any monic *OPS* with positive moment functional is, due to *Favard*'s theorem, *e.g.*, [1], p. 21-22, Theorem 4.4. (we need $b_n^2 > 0$ for $n \in \mathbb{N}$ and $c_n \in \mathbb{R}$, for $n \in \mathbb{N}_0$)

$$\tilde{P}_n(x) \;=\; (x \,-\, c_{n-1})\,\tilde{P}_{n-1}(x) \,-\, b_{n-1}^2\,\tilde{P}_{n-2}(x)\ , \tag{3}$$

with standard input $\tilde{P}_{-1}(x) = 0$ and $\tilde{P}_0(x) = 1$. Note that $b_0^2 < \infty$ multiplies 0 and is therefore free, sometimes it is put to 1, but one can keep it arbitrary. The b_0 used for closed graphs should not be confused with this arbitrary number. One also needs the first and sometimes higher associated polynomials, obtained by shifting the recurrence coefficients by 1 or more. In cases like the *Chebyshev-S* *OPS*, where these coefficients are $n-$independent, these associated polynomial systems coincide with the original ones, the zeroth order *OPS*.

$$\tilde{P}_n^{[m]}(x) \;=\; (x \,-\, c_{n-1+m})\,\tilde{P}_{n-1}^{[m]}(x) \,-\, b_{n-1+m}^2\,\tilde{P}_{n-2}^{[m]}(x)\ ,\ \tilde{P}_{-1}^{[m]}(x) \,=\, 0,\ \tilde{P}_{0}^{[m]}(x) \,=\, 1\ ,\ m \,\in\, N_0\ . \tag{4}$$

These *OPS* with nonnegative coefficients play a rôle in walk counting on *Jacobi* graphs which is now explained in detail.

Definition 1: number of walks

$w_{N,L}(p_n \,\to\, p_m)$ denotes the number of walks (paths) of length L on an open $\mathbf{J}_N$ graph from vertex p_n to vertex p_m. Similarly, we use $\mathrm{w}_{N,L}(p_n \,\to\, p_m)$ for a closed J_N graph.

Of course, one has to have symmetry: $w_{N,L}(p_n \,\to\, p_m) \,=\, w_{N,L}(p_m \,\to\, p_n)$ for all $n, m \,\in\, \{1, 2, ..., N\}$. One defines $w_{N,0}(p_n \,\to\, p_m) \,:=\, \delta_{n,m}$, with *Kronecker* symbol δ. Similarly for $\mathrm{w}_{N,L}$. The following proposition, which holds in fact for all symmetric adjacency matrices, not just tridiagonal ones, is obvious. See also [2], part III, D. The Matrix Method.

Proposition 1: Matrix power elements

$$w_{N,L}(p_n \,\to\, p_m) \;=\; ((\mathbf{J}_N)^L)_{n,m}\ , \tag{5}$$

and analogously for $\mathrm{w}_{N,L}(p_n \,\to\, p_m) \,=\, ((J_N)^L)_{n,m}$. The walk counting symmetry is reflected in the symmetry of the adjacency matrices.

Definition 2: Average number of round trips

$$w_{N,L} \;:=\; \frac{1}{N}\sum_{n=1}^{N} w_{N,L}(p_n \,\to\, p_n). \tag{6}$$

$\mathrm{w}_{N,L}$ is defined analogously. Because $w_{N,L}(p_n \,\to\, p_n)$ is the number of walks starting and ending with vertex p_n (also for $\mathrm{w}_{N,L}(p_n \,\to\, p_n)$), this is the number of round trips related to vertex p_n. If a graph is vertex-transitive (meaning that the graph without labeling looks the same from each vertex) $w_{N,L}$, or $\mathrm{w}_{N,L}$, is the number of round trips from any of the vertices. This happens, *e.g.*, for each $\mathbf{C}_N$, $N \,\geq\, 2$, graph or for the $\mathbf{P}_1$ and $\mathbf{P}_2$ graphs. In general $w_{N,L}$ or $\mathrm{w}_{N,L}$, is then the average number of round trips pro vertex which may be a fraction.

Proposition 2: Normalized sums of powers of eigenvalues of adjacency matrix

$$w_{N,L} \;=\; \frac{1}{N}\,Tr(\mathbf{J}_N{}^L) \;=\; \frac{1}{N}\sum_{n=1}^{N}(x_n^{(N)})^L, \tag{7}$$

with the eigenvalues of the adjacency matrix $\mathbf{J}_N$, and a similar proposition holds for for $\mathrm{w}_{N,L}$. This proposition is true for every symmetrc adjacency matrix. These eigenvalues are the zeros of the characteristic (monic) polynomials $\tilde{P}_N(x) \,=\, Det(x\,\mathbf{1}_N \,-\, \mathbf{J}_N)$. This leads to a simple formula for the *o.g.f.*

of $w_{N,L}$, called $G_N(z) \,:=\, \sum_{L=0}^{\infty} w_{N,L}\, z^L$ (with a similiar definition for $\mathcal{G}_N(z)$ in the closed graph case), namely

Proposition 3: O.g.f. $\mathbf{G_N(z)}$ or $\mathcal{G}_\mathbf{N}(\mathbf{z})$ ([3] Theorem 5)

$$G_N(z) \;=\; \frac{1}{N}\,\frac{x\,\tilde{P}'_N(x)}{\tilde{P}_N(x)}\bigg|_{x=1/z}\;. \tag{8}$$

Proof:

This proposition holds also for every (symmetric) adjacency matrix.

Here for $G_N(z)$ with the characteristic polynomial $\tilde{P}_N(x)$ of $\mathbf{J_N}$.

$E_N(z) \,:=\, \prod_{n=1}^{N} (1 \,-\, x_n^{(N)}\, z) \;=\; z^N\, \tilde{P}_N\left(\frac{1}{z}\right)\;.$

$(log\, E_N(z))' \;=\; -\frac{1}{z} \sum_{n=1}^{N} (x_n^{(N)}\, z)\, \frac{1}{1 - x_n^{(N)}\, z} \;=\; -\frac{1}{z} \sum_{L=0}^{\infty} \sum_{n=1}^{N} (x_n^{(N)})^{L+1}\, z^{L+1} \;=$

$-\frac{N}{z} \sum_{L=1}^{\infty} \left(\frac{1}{N} \sum_{n=1}^{N} (x_n^{(N)})^L\right) z^L \;=\; -\frac{N}{z}\, (G_N(z) \,-\, 1).$

Hence $G_N(z) \;=\; 1 \,-\, \frac{z}{N}\, log\left(z^N\, \tilde{P}_N\left(\frac{1}{z}\right)\right)' \;=\; \frac{1}{N}\,\frac{x\,\tilde{P}'_N(x)}{\tilde{P}_N(x)}\bigg|_{x=1/z}$ □

For closed graphs the same formula holds but with $\tilde{P}_N(x)$ replaced by the characteristic (monic) polynomial $\widetilde{Pc}_N(x)$ for the adjacency matrix $J_\mathbf{N}$ from eq. (2). Now the the tridiagonality of the adjacency matrix of the corresponding open graph will enter. By expanding $Det(x\, \mathbf{1_N} \,-\, J_\mathbf{N})$ repeatedly one finds the following proposition expressing $\widetilde{Pc}_N(x)$ in terms of the open graph's characteristic polynomial and its first associate one.

Proposition 4: Characteristic polynomials for closed Jacobi graphs

$$\widetilde{Pc}_N(x) \;=\; \tilde{P}_N(x) \,-\, b_0^2\, \tilde{P}^{[1]}_{N-2}(x) \,-\, 2 \prod_{k=0}^{N-1} b_k\;. \tag{9}$$

If the classical *OPS* recurrences are scanned for possible graph candidates one has to keep in mind that the three term recurrence for the monic polynomials has a freedom (a γ-transformation) explained in the following *lemma*.

Lemma 1: Freedom in the monic three term recurrence

With $\{\tilde{P}_n(x)\}$ satisfying the recurrence eq. (3), together with the given two inputs, also the monic polynomial system $\{\hat{P}_n(x)\}$, with $\hat{P}_n(x) \,:=\, \gamma^n\, \tilde{P}_n\left(\frac{x}{\gamma}\right)$, with any $\gamma \,\in\, \mathbb{R} \setminus 0$, satisfies the same type of recurrence and inputs, but with $\hat{c}_{n-1} \,=\, \gamma\, c_{n-1}$ and $\hat{b}^2_{n-1} \,=\, \gamma^2\, b^2_{n-1}$.

Proof: Elementary. □

Example 1: The adjacency matrix for the graph $\mathbf{C}_2$ has $c_{n-1} \,=\, 0$ and $b_{n-1} \,=\, 2$, therefore the characteristic polynomial is $\tilde{C}_2(x) \,=\, 2^2\, S_2\left(\frac{x}{2}\right) \,=\, x^2 \,-\, 4$.

Now we scan the three families of classical *OPS* for a possible Jacobi graph interpretation.

a) Jacobi polynomials: $\{\tilde{P}_n^{(\alpha,\beta)}(x)\}$, $\alpha \,>\, -1, \beta \,>\, -1$ (for orthogonality on the interval $[-1, +1]$ with a certain known weight function $w^{(\alpha,\beta)}(x)$):

The recurrence for the monic polynomials is for $n \geq 3$

$$\begin{aligned}
\tilde{P}_n^{(\alpha,\beta)}(x) &= \left(x - \frac{\beta^2 - \alpha^2}{4\,(n + \frac{\alpha+\beta}{2})\,(n + \frac{\alpha+\beta}{2} - 1)}\right) \tilde{P}_{n-1}^{(\alpha,\beta)}(x) \\
&\quad - \frac{1}{4}\,\frac{(n-1)\,(n+\alpha-1)\,(n+\beta-1)\,(n+\alpha+\beta-1)}{(n + \frac{\alpha+\beta}{2} - 1)^2\,(n + \frac{\alpha+\beta-1}{2})\,(n + \frac{\alpha+\beta-3}{2})}\,\tilde{P}_{n-2}^{(\alpha,\beta)}(x)\;, \\
\tilde{P}_{-1}^{(\alpha,\beta)}(x) &= 0,\ \tilde{P}_0^{(\alpha,\beta)}(x) = 1\;. \qquad (10)
\end{aligned}$$

Because of the denominators, given the above mentioned range of α and β which implies $-\dfrac{\alpha+\beta}{2} < 1$, one has to take care of the $n = 1$ and $n = 2$ cases. One uses $\tilde{P}_1^{(\alpha,\beta)}(x) = x + \frac{\alpha-\beta}{2+\alpha+\beta}$, as deduced from the non-monic case. This formula is also taken, by definition, for the cases $\alpha + \beta = 0$ and $\alpha + \beta + 1 = 0$ (even though in the derivation of the non-monic $n = 1$ formula these cases are first excluded). For the case $\alpha + \beta + 1 = 0$ the $n = 2$ formula is $\tilde{P}_2^{(\alpha,-(\alpha+1))}(x) = x^2 + \frac{4}{3}\,(\alpha + \frac{1}{2})\,x - \frac{1}{3} + \frac{2}{3}\,\alpha^2 + \frac{2}{3}\,\alpha$.
The search for characteristic polynomials of graphs proceeds in two steps. First one considers the recurrence of type eq. (3), written for $\hat{P}_n(x)$ according to *lemma 1*, for $n \geq 3$, then the $n = 2$ and $n = 1$ cases are studied separately. For $n = 0$ the input is always 1. $\hat{c}_{n-1} = \gamma\,c_{n-1}$ in eq. (3) should become a nonnegative integer for $n \geq 3$. Because the n-dependence of c_{n-1} does not allow a cancellation of the denominators in the first term of the recurrence (10) , and γ is n-independent, this can only happen if $\beta^2 = \alpha^2$, and then c_{n-1} vanishes identically for $n \geq 3$. In the second term of the recurrence there is no $0/0$ problem for $n \geq 3$. Because the rational number multiplying $-\dfrac{1}{4}$ has monic polynomials in n (of degree 4) in the numerator as well as in the denominator, it cannot become a square other than possibly 1. If $\alpha = \beta$ this is indeed possible precisely for $\alpha = \pm\dfrac{1}{2}$. Also for $\alpha = -\beta$ this is true for $\alpha = \pm\dfrac{1}{2}$. Therefore one has in both cases $\hat{b}_{n-1}^2 = \gamma^2\,\dfrac{1}{4}$, and this determines $\gamma = \pm 2\,k$, with $k \in \mathbb{N}$, and then $\hat{b}_{n-1} = k$ (it has to be positive) for $n \geq 3$.
Now the low n values have to be considered. For $n = 1$ one has for $\alpha = \beta$ also $c_0 = 0$, hence $\hat{c}_0 = 0$. For $\alpha = -\beta$ one has $c_0 = -\alpha$, hence $\hat{c}_0 = -\gamma\alpha = \mp 2\,k\,\alpha$. This picks from the two α solutions from the $n \geq 3$ case $\alpha = +\dfrac{1}{2}$ for negative γ and $\alpha = -\dfrac{1}{2}$ for positive γ. Thus $\hat{c}_0 = k$ in both cases. $\mathbf{J}_N$ from eq. (1) shows that for the graph there are k self-loops for vertex no. 1 in these two cases $(\alpha,\beta) = (+\dfrac{1}{2}, -\dfrac{1}{2})$ and $(-\dfrac{1}{2}, +\dfrac{1}{2})$. Remember that no $\hat{b}_0$ term appears.
As for $n = 2$, only the case $\alpha + \beta + 1$, *i.e.*, $\alpha = \beta = -\dfrac{1}{2}$ is special (for the other cases the recurrence is valid for $n = 2$ with the above determined $\hat{P}_1(x)$). $\tilde{P}_2^{(-\frac{1}{2},-\frac{1}{2})}(x) = x^2 - \dfrac{1}{2}$ (not $x^2 - \dfrac{1}{4}$ as one might incorrectly guess from the recurrence with the rational in the second term taken as 1). Thus $\hat{P}_2^{(-\frac{1}{2},-\frac{1}{2})}(x) = x^2 - 2\,k^2$. However, from the recurrence (3), written for the monic $\hat{P}$-polynomials, this corresponds to $\hat{c}_1 = 0$ (because $\hat{c}_0 = 0$ in this case) and $\hat{b}_1^2 = 2\,k^2$, and therefore this case has to be discarded because $\hat{b}_1$ is not integer. This concludes the analysis, and the result is stated in the following *proposition*.

Proposition 5: Jacobi graphs for the Jacobi OPS

i) $\alpha = \beta = +\dfrac{1}{2}$: $\hat{c}_{n-1} = 0$, for $n \geq 1$ and $\hat{b}_{n-1} = k$, $k \in \mathbb{N}$, for $n \geq 2$. These graphs look like $\mathbf{P}_N$ graphs but with k-multi-lines but no self-loops. The characteristic polynomials are $\hat{P}_n^{(\frac{1}{2},\frac{1}{2})}(k;x) = k^n\,S_n\left(\dfrac{x}{k}\right)$ with *Chebyshev* $S-$polynomials. Therefore the fundamental graph for $k = 1$ is $\mathbf{P}_N$ with characteristic polynomial $S_N(x)$, and the more general case is covered by a $\gamma-$transformation according to *lemma 1*.

ii) $\alpha = -\beta = \pm\frac{1}{2}$: $\hat{c}_{n-1} = 0$ for $n \geq 2$, $\hat{b}_{n-1} = k$, $k \in \mathbb{N}$, for $n \geq 2$. $\hat{c}_0 = k$. Both cases lead to identical graphs which look like the ones from case **i**) but now the vertex no. 1 has a $k-$self-loop. The characteristic polynomial is in both cases $\hat{P}_n^{(\pm\frac{1}{2},\mp\frac{1}{2})}(k;x) = k^n\left(S_n\left(x/k\right) - S_{n-1}\left(x/k\right)\right)$, with correlated signs. Therefore, the fundamental graph with $k = 1$ looks like P_N but with a self-loop at vertex no. 1.

The characteristic polynomial for the fundamental $k = 1$ graph in case **ii**) has a recurrence like the S-polynomials, but with the inputs 1 and $x-1$ for $n = 0$ and $n = 1$, respectively. Therefore these are the polynomials $S_n(x) - S_{n-1}(x)$. Their coeffcient table is found under A130777.

b) Laguerre-Sonin(e) polynomials: $\{\tilde{L}_n^{(\alpha}(x)\}$, $\alpha > -1$ (for orthogonality on $[0, +\infty)$ with a certain known weight function $w^{(\alpha)}(x)$):

The recurrence for the monic polynomials is, for $n \geq 1$,

$$\begin{aligned}
\tilde{L}_n^{(\alpha)}(x) &= (x - (2n-1+\alpha))\,\tilde{L}_{n-1}^{(\alpha)}(x) + (n-1+\alpha)\,(n-1)\,\tilde{L}_{n-2}^{(\alpha)}(x)\ , \\
\tilde{L}_{-1}^{(\alpha}(x) &= 0, \quad \tilde{L}_0^{(\alpha)}(x) = 1\ .
\end{aligned} \tag{11}$$

Here the analysis is much simpler than above. $\hat{c}_{n-1} = \gamma\,(2n-1+\alpha)$, $n \geq 1$, has to become a positive integer, and $\hat{b}_{n-1}^2 = \gamma^2\,(n-1)\,(n-1-\alpha)$ has to become a squared integer. This only works for $\alpha = 0$, the usual *Laguerre* polynomials, and $\gamma = k \in \mathbb{N}$. Therefore the following proposition holds.

Proposition 6: Jacobi graphs for the Laguerre-Sonin(e) OPS

Graphs are only possible for $\alpha = 0$, $\hat{c}_{n-1} = k\,(2n-1)$, and $\hat{b}_{n-1} = k\,(n-1)$, for $n \geq 1$. The characteristic polynomials are $\hat{P}(k;x) = k^n\,\tilde{L}_n^{(0)}\left(\frac{x}{k}\right)$. The fundamental graph is $\mathbf{L}_N$ with $k = 1$, and characteristic polynomial $\tilde{L}_N(x)$, the usual $(\alpha = 0)$ monic *Laguerre* polynomial.

The example L_4 is shown in *Figure 3*.

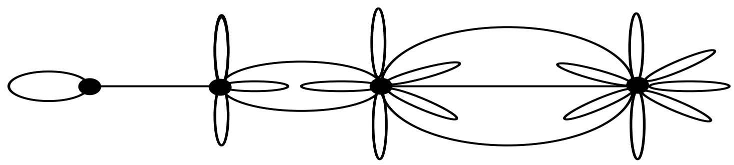

Fig. 3: L a g u e r r e Graph L_4

In the remaining **c)** *Hermite* class of classical orthogonal polynomials no non-trivial graph candidates $\mathbf{J}_N$ can be found because the recurrence relation for the monic polynomials is

$$\begin{aligned}
\tilde{H}_n(x) &= x\,\tilde{H}_{n-1}(x) - \frac{n-1}{2}\,\tilde{H}_{n-2}(x)\ , \\
\tilde{H}_{-1}(x) &= 0,\ \tilde{H}_0(x) = 1\ .
\end{aligned} \tag{12}$$

Therefore $\hat{b}_{n-1}^2 = \gamma^2\,\frac{n-1}{2}$ can become a squared positive integer for every $n \geq 2$ but only if γ is taken as $n-$dependent. For example, for $n = 2$ it would be $\sqrt{2}\,k$, with $k \in \mathbb{N}$, and for $n = 3$ it would have to be chosen as k, etc. This n dependence of γ is not allowed in lemma 1. The only trivial possibilities are the graphs for $N = 1$ with $\hat{c}_0 = 0$ and characteristic polynomial x, and $N = 2$ with $\hat{c}_0 = 0$ and $\hat{b}_1 = k \in \mathbb{N}$ and characteristic polynomial $\hat{H}_2(x) = (\sqrt{2}\,k)^2\,\tilde{H}_2\left(\frac{x}{\sqrt{2}\,k}\right)$. This is a graph like $\mathbf{P}_2$ with $k-$multi-lines. The fundamental graph with $k = 1$ is $\mathbf{P}_2$. For each $N \geq 3$ one cannot find only one γ.

This ends the discussion of classical *OPS* which can function as characteristic polynomials for adjacency matrices of graphs. Of course, non-classical *OPS* can also be good candidates. For example, take the $\mathbf{P}_N$ graph with a self-loop for each of its vertices. This means that all three diagonals of the *Jacobi* matrix are composed of 1s. This leads to the characteristic polynomials whose coefficient table is given in A104562 for $N \geq 1$.

The array for the total number of return trips $w(N,L)$ for the (open) $\mathbf{P}_N$ graphs, with characteristic *Chebyshev* S-polynomials, can be fond under A198632 which is the corresponding number triangle $a(K,N) = w(N, 2\,(K-N+2))$, $K+1 \geq n \geq 1$. The *o.g.f.* $GS_N(z)$ for $w(N,L)$ is

$$GS_N(z) \;=\; y\;\frac{S_N'(y)}{S_N(y)}\bigg|_{y=\frac{1}{z}}\;, \tag{13}$$

and it can be rewritten, with the help of the identity

$$2\,S_N'(y) \;=\; \frac{1}{(1-(\frac{y}{2})^2)}\left(\frac{y}{2}\,S_N(y)\;-\;(N+1)\,T_{N+1}\left(\frac{y}{2}\right)\right)\,, \tag{14}$$

and the *Binet-de Moivre* identities for the *Chebyshev* S- and T-polynomials as (see [3], p. 245, eqs. (3.8a) to (3.8d), where in eq. (3.8d) *tanh* should be replaced by *coth*)

$$GS_N(z) \;=\; \frac{1}{\sqrt{1\,-\,(2\,z)^2)}}\left\{\,(N\,+\,1)\,coth\left((N+1)\,log\frac{2\,z}{1\,-\,\sqrt{1\,-\,(2\,z)^2}}\right)\;-\;\frac{1}{\sqrt{1\,-\,(2\,z)^2}}\right\}\,. \tag{15}$$

It satisfies the *Riccati* equation (compare [3], p. 247, *Theorem 11*, eq. (3.10))

$$(1-(2\,z)^2)\,z\,GS_N'(z)\;-\;(2\,+\,(2\,z)^2)\,GS_N(z)\;+\;(1-(2\,z)^2)\,(GS_N(z))^2\;+\;N\,(N+2)\;=\;0. \tag{16}$$

For the cycle graphs $\mathbf{C}_N$ the number of return trips for any of the N vertices $\mathrm{w}(N,L)$ is found under A199571, which is the number triangle version $a(K,L) = \mathrm{w}(N, K-N+1)$, $K \geq 0$, $n = 1, ..., K+1$.

For the **open Laguerre graphs** L_N, which are not vertex-transitive, one computes the average number of round trips from the ordinary *Laguerre* polynomials, using the well known differential-difference identity $x\,L_N'(x) \,=\, N\,(L_N(x) \,-\, L_{N-1}(x))$, written for the monic ploynomials $\tilde{L}_N(x) \,=\, N!\,(-1)^N\,L_N(x)$,

$$x\,\tilde{L}_N'(x) \;=\; N\,(\tilde{L}_N\;+\;N\,\tilde{L}_{N-1})\;. \tag{17}$$

This immediately leads, from *proposition 3*, to the following proposition.

Proposition 7: O.g.f. for average round trips for open Laguerre graphs $\mathbf{L_N}$

$$GL_N(z) \;=\; 1\;+\;N\,\frac{\tilde{L}_{N-1}\left(\frac{1}{z}\right)}{\tilde{L}_N\left(\frac{1}{z}\right)} \tag{18}$$

$$=\;\frac{\sum_{k=0}^{N-1}(-1)^k\,(1-\frac{k}{N})\,\binom{N}{N-k}\,\frac{N!}{(N-k)!}\,z^k}{\sum_{k=0}^{N}(-1)^k\,\binom{N}{N-k}\,\frac{N!}{(N-k)!}\,z^k}\;. \tag{19}$$

The last equation used the explicit form of the *Laguerre* polynomials. See also [3], Note 6, p. 244, and *Corollary 19* with $\alpha \,=\, 0$.

This *o.g.f.* is not an elementary function, it is

$$GL_N(z) \;=\; 1\;-\;\frac{{}_1F_1(-N+1,1;1/z)}{{}_1F_1(-N,1;1/z)}\;. \tag{20}$$

It satisfies (see [3], *Theorem 21*, p. 251) the following *Riccati* equation.

$$z^2\,\frac{d}{dz}\,GL_N(z) \;=\; N\,z\,(GL_N(z))^2\;-\;GL_N(z)\;+\;1. \tag{21}$$

Example 2: O.g.f. for average number of round trips on $\mathbf{L_4}$

$$GL_4(z) = \frac{1 - 12\,x + 36\,x^2 - 24\,x^3}{1 - 16\,x + 72\,x^2 - 96\,x^3 + 24\,x^4}\,. \tag{22}$$

This generates the sequence [1, 4, 28, 232, 2056, 18784, 174112, 1625152, 15220288, 142777600, 1340416768, 12588825088, 118252556800,...] found under A199579.

In *Table 1* the array of the average round trip numbers $w(N,L)$ for the *Laguerre* graphs L_N computed from these *o.g.f.* s is given for $N = 1, ..., 10$ and $L = 0, ..., 8$. This appears as triangle $a(K,N) = w(N, K-N+1)$ under A201198.

In *Table 3* the *o.g.f.* s $GL_N(z)$ are shown for $N = 1, ..., 9$.

We are here not going into a detailed computation of the general number of walks $w_{N,L}(p_n \to p_m)$. We only mention its *o.g.f.* $G_N(p_n \to p_m; z) = \sum_{L=0}^{\infty} w_{N,L}(p_n \to p_m)\, z^L = \sum_{L=0}^{\infty} (\mathbf{J}^L)_{n,m}\, z^L = [(\mathbf{1}_N - z\,\mathbf{J}_N)^{-1}]_{n,m}$. This leads with the resolvent (*Green*'s function) for $\mathbf{J}_N$ given by $\mathbf{G}_N(x) = (\mathbf{A}_N(x))^{-1}$, with $\mathbf{A}_N(x) := x\,\mathbf{1}_N - \mathbf{J}_N$ to

$$G_N(p_n \to p_m; z) = \frac{1}{z}\left[\mathbf{G}_N\left(\frac{1}{z}\right)\right] = x\frac{C_{m,n}(\mathbf{A}_N(x))}{Det\,\mathbf{A}_N(x)}\bigg|_{x=\frac{1}{z}}\,. \tag{23}$$

Here $C_{n,m}(\mathbf{M})$ is the (n,m)-cofactor of the matrix $\mathbf{M}$.

Finally, the **closed Laguerre graphs** $\mathbf{Lc}_N$, with $\mathbf{Lc}_1 = \mathbf{L}_1$ and $\mathbf{Lc}_2$ with adjacency matrix $[[1,2],[2,3]]$ (not the one for $\mathbf{L}_2$ which has adjacency matrix $[[1,1,],[1,3]]$) and $b_0 = 2$ for symmetry reason, have for $N \geq 3$ the characteristic polynomials of the adjacency matrices

$$\widetilde{Lc}_N(x) = \tilde{L}_N(x) - 4\,\tilde{L}^{[1]}_{N-2}(x) - 4\,(N-1)!, \tag{24}$$

with the classical monic *Laguerre* polynomials (parameter $\alpha = 0$) and their monic first associated ones $\tilde{L}^{[1]}_n(x)$, with coefficient triangle given under A199577. For $N = 1$ and $N = 2$ $\widetilde{Lc}_N(x)$ is $z - 1$ and $z^2 - 4\,z - 1$, respectively. This leads to the following *o.g.f.* for the total number of round trips on $\mathbf{Lc}_N$ (note that the average number of round trips will in general be a fraction).

Proposition 8: O.g.f. for total number of round trips for closed Laguerre graphs

$$GLc_1(z) = GL_1(z) = \frac{1}{1-z}\,, \tag{25}$$

$$GLc_2(z) = GL_2(z) = \frac{1-2\,z}{1 - 4\,z - z^2}\,, \tag{26}$$

$$GLc_N(z) = x\,\frac{\widetilde{Lc}'_N(x)}{\widetilde{Lc}_N(x)}\bigg|_{x=\frac{1}{z}}\,, \quad N \geq 3\,. \tag{27}$$

This could be rewritten by insering the formula for $\widetilde{Lc}'_N(x)$.

A sketch of the graph $\mathbf{Lc_4}$ is given in *Figure 4*.

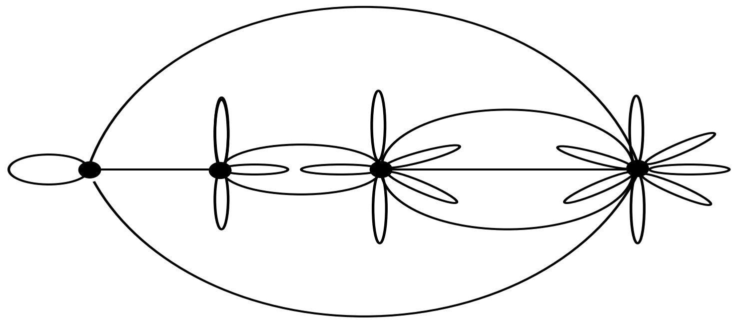

Fig. 4: Closed L a g u e r r e Graph $\boldsymbol{Lc}_{\,4}$

Example 3: O.g.f. for total number of round trips on $\mathbf{Lc_4}$

$$GLc_4(z) \,=\, 4\,\frac{1\,-\,12\,x\,+\,34\,x^2\,-\,16\,x^3}{1\,-\,16\,x\,+\,68\,x^2\,-\,64\,x^3\,-\,44\,x^4}\;. \tag{28}$$

This generates the sequence $4*[1, 4, 30, 256, 2356, 22384, 215640, 2090176, 20315536, 197702464, ...]$ found as 4∗A201200. In this case the average number is integer (this happens, *e.g.*, also in the case $N = 8$).

In *Table 2* the array of the total round trip numbers $wc(N, L)$ for these closed *Laguerre* graphs $\mathbf{Lc}_N$, computed from these *o.g.f.* s, is given for $N = 1, ..., 10$ and $L = 0, ..., 8$. This appears as triangle $a(K, N) = w(N, K - N + 1)$ under A201199.

In *Table 4* the *o.g.f.* s $GLc_N(z) = \mathcal{G}_N(z)$ are shown for $N = 1, ..., 9$.

In the limit $N \to \infty$ one finds for the open $\mathbf{P}_N$ graphs the *o.g.f.* of the average number of round trips $GS_N(z)/N$ from eq. (15) to become *o.g.f.* $\dfrac{1}{\sqrt{1-(2\,z)^2}}$ which generates the sequence of central binomial numbers interspersed with 0s, $[1, 0, 2, 0, 6, 0, 20, 0, 70, 0, 252, 0, 924, ...,]$ which is A126869 . For the cyclic graphs $\mathbf{C}_N$ the number of round trips for each vertex approximates for large N also these central binomial numbers numbers, as one derives from the *o.g.f.* given in A199571.

For the open *Laguerre* graphs $\mathbf{L}_N$ the average number of round trips diverges for $N \to \infty$. One finds for the scaled *o.g.f.* $GL_N(\frac{z}{N})$ from the *Riccati* equation (21) the *o.g.f.* for the *Catalan* numbers $c(z) = \dfrac{1}{2\,z}\,(1\,-\,\sqrt{1-4\,z}\,)$. Therefore, $\lim\limits_{N\to\infty}\dfrac{w(N, L)}{N^L} = C_L$, with the *Catalan* numbers C_L given in A000108.

This is an enlarged version of a talk given by the author in 1997 at the “Kombinatorik” meeting at TU Braunschweig, Germany.

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Table 1: **A201198: Array of average number of round trips $w_{N,L}$ for open Laguerre graphs $\mathbf{L_N}$**

N \ L	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8** ...
1	1	1	1	1	1	1	1	1	1
2	1	2	6	20	68	232	792	2704	9232
3	1	3	15	87	531	3303	20691	129951	816939
4	1	4	28	232	2056	18784	174112	1625152	15220288
5	1	5	45	485	5645	68245	841725	10495525	131661325
6	1	6	66	876	12636	190296	2935656	45927216	724547376
7	1	7	91	1435	24703	445627	8259727	155635459	2962913527
8	1	8	120	2192	43856	922048	19964736	440311936	9826743424
9	1	9	153	3177	72441	1739529	43098777	1089331497	27897922233
10	1	10	190	4420	113140	3055240	85252600	2429880400	70250453200
⋮									

Table 2: A201199: Array of total number of round trips $w_{N,L}$ for closed Laguerre graphs Lc_N

N \ L	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8** ...
1	1	1	1	1	1	1	1	1	1
2	2	4	12	40	136	464	1584	5408	18464
3	3	9	53	357	2489	17509	123449	870893	6144769
4	4	16	120	1024	9424	89536	862560	8360704	81262144
5	5	25	233	2545	29985	367505	4599521	58216113	741355649
6	6	36	404	5400	78392	1188336	18460016	290899680	4623415648
7	7	49	645	10213	176473	3195829	59473593	1125306973	21514466689
8	8	64	968	17728	355536	7493504	162671840	3597143040	80497036736
9	9	81	1385	28809	657953	15826041	392792273	9945708777	255176534209
10	10	100	1908	44440	1138840	30790000	860218416	24549157600	710660174944
⋮									

Table 3: O.g.f $\mathbf{GL_N(z)}$ for average number of round trips on open Laguerre graphs $\mathbf{L_N}$.

N	$\mathbf{GL_N(z)}$
1	$\frac{1}{1-z}$
2	$\frac{1-2\,z}{1-4\,z+2\,z^2}$
3	$\frac{1-6\,z+6\,z^2}{1-9\,z+18\,z^2-6\,z^3}$
4	$\frac{1-12\,z+36\,z^2-24\,z^3}{1-16\,z+72\,z^2-96\,z^3+24\,z^4}$
5	$\frac{1-20\,z+120\,z^2-240\,z^3+120\,z^4}{1-25\,z+200\,z^2-600\,z^3+600\,z^4-120\,z^5}$
6	$\frac{1-30\,z+300\,z^2-1200\,z^3+1800\,z^4-720\,z^5}{1-36\,z+450\,z^2-2400\,z^3+5400\,z^4-4320\,z^5+720\,z^6}$
7	$\frac{1-42\,z+630\,z^2-4200\,z^3+12600\,z^4-15120\,z^5+5040\,z^6}{1-49\,z+882\,z^2-7350\,z^3+29400\,z^4-52920\,z^5+35280\,z^6-5040\,z^7}$
8	$\frac{1-56\,z+1176\,z^2-11760\,z^3+58800\,z^4-141120\,z^5+141120\,z^6-40320\,z^7}{1-64\,z+1568\,z^2-18816\,z^3+117600\,z^4-376320\,z^5+564480\,z^6-322560\,z^7+40320\,z^8}$
9	$\frac{1-72\,z+2016\,z^2-28224\,z^3+211680\,z^4-846720\,z^5+1693440\,z^6-1451520\,z^7+362880\,z^8}{1-81\,z+2592\,z^2-42336\,z^3+381024\,z^4-1905120\,z^5+5080320\,z^6-6531840\,z^7+3265920\,z^8-362880\,z^9}$
$\vdots$	

Table 4: O.g.f $\mathcal{G}_{\mathbf{N}}(\mathbf{z})$ for total number of round trips on closed Laguerre graphs $\mathbf{Lc_N}$.

N	$\mathcal{G}_{\mathbf{N}}(\mathbf{z})$
1	$\frac{1}{1-z}$
2	$2\,\frac{1-2\,z}{1-4\,z-z^2}$
3	$\frac{3-18\,z+14\,z^2}{1-9\,z+14\,z^2-2\,z^3}$
4	$4\,\frac{1-12\,z+34\,z^2-16\,z^3}{1-16\,z+68\,z^2-64\,z^3-44\,z^4}$
5	$\frac{5-100\,z+588\,z^2-1080\,z^3+368\,z^4}{1-25\,z+196\,z^2-540\,z^3+368\,z^4-16\,z^5}$
6	$2\,\frac{3-90\,z+892\,z^2-3456\,z^3+4692\,z^4-1272\,z^5}{1-36\,z+446\,z^2-2304\,z^3+4692\,z^4-2544\,z^5-856\,z^6}$
7	$\frac{7-294\,z+4390\,z^2-28840\,z^3+83208\,z^4-89712\,z^5+20448\,z^6}{1-49\,z+878\,z^2-7210\,z^3+27736\,z^4-44856\,z^5+20448\,z^6-864\,z^7}$
8	$8\,\frac{1-56\,z+1173\,z^2-11640\,z^3+57130\,z^4-131280\,z^5+117576\,z^6-23328\,z^7}{1-64\,z+1564\,z^2-18624\,z^3+114260\,z^4-350080\,z^5+470304\,z^6-186624\,z^7-32112\,z^8}$
9	$\frac{9-648\,z+18116\,z^2-252504\,z^3+1875000\,z^4-7342560\,z^5+14022240\,z^6-10756800\,z^7+1901376\,z^8}{1-81\,z+2588\,z^2-42084\,z^3+375000\,z^4-1835640\,z^5+4674080\,z^6-5378400\,z^7+1901376\,z^8-85824\,z^9}$
$\vdots$	