Amm 69 (1962) A110 A11965 - A11971 A5493

THEOREM. $R(\rho)<1$, *i.e. for all one-to-one harmonic mappings of the given annulus* $\rho<|\zeta|<1$ *onto an annulus* $r<|z|<1$ *the inequality* $r\leqq R(\rho)<1$ *holds.*

Proof. Choose a closed subannulus A of $\rho<|\zeta|<1$, for instance $(1+3\rho)/4 \leqq|\zeta| \leqq(3+\rho)/4$. By Harnack's inequality for positive harmonic functions $u(\xi,\eta)$ in $\rho<|\zeta|<1$ there exists a constant $k>1$, depending only on ρ, such that the inequalities

$$k^{-1}u(\xi_2, \eta_2) \leqq u(\xi_1, \eta_1) \leqq ku(\xi_2, \eta_2)$$

are valid for any pair of points ζ_1 and ζ_2 in A. The function $u(\xi, \eta)=1+x(\xi, \eta)$ is harmonic and positive in $\rho<|\zeta|<1$. Furthermore there must be a point ζ_1 in A where $x(\xi_1, \eta_1)>r$ and a point ζ_2 where $x(\xi_2, \eta_2)<-r$. Then

$$1 - r > 1 + x(\xi_2, \eta_2) = u(\xi_2, \eta_2) \geqq k^{-1}u(\xi_1, \eta_1) = k^{-1}(1 + x(\xi_1, \eta_1)) > k^{-1}(1 + r).$$

From these inequalities we conclude $(1+r)/(1-r)<k$ or $r<(k-1)/(k+1)$. This implies $R(\rho)\leqq(k-1)/(k+1)<1$, *q.e.d.*

Clearly the proof applies to the higher dimensional case as well. It would be of interest to determine the exact value of $R(\rho)$ and to see whether it is identical with the number $2\rho/(1+\rho^2)$.

ON THE NUMBER OF PARTITIONINGS OF A SET OF n DISTINCT OBJECTS

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Introduction. By partitioning a set S we mean dividing the set into mutually exclusive subsets. Two partitionings are considered the same if and only if every subset in one of them is also a subset in the other. In this paper we shall investigate the number of different partitionings of a set of n distinct objects.

The Difference Equation. Let us denote by $P(n)$ the number of partitionings of a set S of n distinct objects. Let us choose one object of S and name it a. Consider the case that a and i other objects form one subset, and the remaining $n-i-1$ objects are partitioned in one way or another. The result is a partitioning of S. After a has been chosen, the additional i objects may be chosen in

$$\binom{n-1}{i}$$

ways. Since there are $P(n-i-1)$ ways to partition the remaining $n-i-1$ objects, there are

$$\binom{n-1}{i}\cdot P(n-i-1)$$

P(n) are Bell nos

ways to partition S in such a way that a is in a subset of order $i+1$. Therefore,

$$P(n) = \sum_{i=0}^{i=n-1} \binom{n-1}{i} \cdot P(n-i-1). \tag{1}$$

Calculation of $P(n)$. We next describe what we believe is an efficient way of calculating $P(j)$ for $j=1, 2, \cdots, n$. For simplicity, let us define $P(0)=1$, and then,

$$P(1) = \sum_{i=0}^{0} \binom{0}{i} \cdot P(0-i) = 1$$

as required.

The value of $P(n)$ can be computed by means of a matrix array $((A_{ij}))$ constructed according to the following rules:

1. $A_{1j} = A_{(j-1)1}$
2. $A_{ij} = A_{(i-1)j} + A_{(i-1)(j+1)}$.

As is well known from the study of difference tables ([1]), the entries in the first column of $((A_{ij}))$ are given by

$$A_{n1} = \sum_{i=0}^{n-1} \binom{n-1}{i} A_{1(i+1)}. \tag{2}$$

Comparing Equations 1 and 2, we see that if

$$A_{11} = P(0), \qquad A_{12} = P(1), \cdots, A_{1n} = P(n-1),$$

then $A_{n1} = P(n)$. The first seven values of $P(n)$ are computed in Table 1 below:

$P(1) =$	1	1	2	5	15	52	203	A110
$P(2) =$	2	3	7	20	67	255		← A11968
$P(3) =$	5	10	27	87	322			← A11969
$P(4) =$	15	37	114	409				← A11970
$P(5) =$	52	151	523					A11967
$P(6) =$	203	674						A11966 Δ = A11971
$P(7) =$	877							↓ 5493 A11965

(110)

TABLE 1—Computation of $P(1)$ through $P(7)$ by means of the array $((A_{ij}))$.

The computation of $P(n)$ from $P(n-1)$, using the technique described above, required $n-1$ additions; thus the computation of $P(n)$ from $P(0)=1$ required $\{n(n-1)\}/2$ additions and no multiplications. This is far more efficient than the direct use of Eq. 1 which requires as many additions, not to mention multiplication and computation of binomial coefficients.

Klein-Barmen [2] gives the following closed form for the number $S(n, m)$ of partitionings of a set of n distinct objects into m nonempty subsets (also known as Stirling numbers of the second kind [3]);

$$S(n, m) = \frac{1}{(m-1)!} \sum_{j=0}^{m-1} (-1)^j \binom{m-1}{j} (m-j)^{n-1}. \tag{3}$$

Using this expression we can write

$$P(n) = \sum_{m=1}^{n} \frac{1}{(m-1)!} \sum_{j=0}^{m-1} (-1)^j \binom{m-1}{j} (m-j)^{n-1}. \tag{4}$$

The computation of $P(n)$ by means of Eq. 4 is clearly far more tedious than by the method of the present paper.

A Generating Function for $P(n)$. The purpose of this section is to prove that $E(x) = \exp(e^x + x - 1)$ is an exponential generating function for the sequence $P(1), P(2), \cdots$; that is,

$$P(n) = \left. \frac{d^{n-1}E(x)}{dx^{n-1}} \right|_{x=0}. \tag{5}$$

It is convenient first to prove the following

LEMMA.

$$\frac{d^{n-1}E(x)}{dx^{n-1}} = e^{-1}e^{e^x} \cdot \sum_{k=1}^{n} a(n, k)e^{kx},$$

where the $a(n, k)$ are Stirling numbers of the second kind.

Proof. By induction on n. The lemma is obvious for $n=1$; assume that

$$\frac{d^{n-2}E(x)}{dx^{n-2}} = e^{-1}e^{e^x} \cdot \sum_{k=1}^{n-1} a(n-1, k)e^{kx}.$$

Then

$$\frac{d^{n-1}E(x)}{dx^{n-1}} = \frac{d}{dx} \cdot \frac{d^{n-2}E(x)}{dx^{n-2}}$$

$$= e^{-1}e^{e^x} \left\{ \sum_{k=1}^{n-1} a(n-1, k)ke^{kx} + e^x \cdot \sum_{k=1}^{n-1} a(n-1, k)e^{kx} \right\}.$$

Thus

$$\frac{d^{n-1}E(x)}{dx^{n-1}} = e^{-1}e^{e^x} \cdot \sum_{k=1}^{n} a(n, k)e^{kx},$$

with

$$a(n, k) = ka(n-1, k) + a(n-1, k-1), \qquad (6)$$
$$a(1, 1) = 1.$$

But Eq. 6 is precisely the defining relation for Stirling numbers of the second kind, completing the proof of the lemma.

The proof of Eq. 5 now follows directly, for

$$\left.\frac{d^{n-1}E(x)}{dx^{n-1}}\right|_{x=0} = \sum_{k=1}^{n} a(n, k) = P(n).$$

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3. John Riordan, An Introduction to Combinatorial Analysis, Wiley, New York, 1958, pp. 32–34, 48.

CLASSROOM NOTES

EDITED BY JOHN M. H. OLMSTED, Southern Illinois University

This department welcomes brief expository articles on problems and topics closely related to classroom experience in courses that are normally available to undergraduate students, from the freshman year through early graduate work. Items of interest to teachers, such as pedagogical tactics, course improvement, new proofs and counterexamples, and fresh viewpoints in general, are invited. All material should be sent to John M. H. Olmsted, Department of Mathematics, Southern Illinois University, Carbondale, Illinois.

AN ALGEBRAIC ALGORITHM FOR THE REPRESENTATION PROBLEMS OF THE AHMES PAPYRUS

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The *Ahmes* (or *Rhind*) *Papyrus*, a famous Egyptian mathematical document described in [1], was largely concerned with the following problem: Given a rational number, e.g., 5/7, express it as a sum of reciprocals of distinct integers (thus, $5/7=1/2+1/6+1/21$). Apparently, the Egyptians had a convenient notation for reciprocals, but not for fractions in general. The picture was somewhat complicated by the fact that they also possessed a special symbol for 2/3; but in our treatment, we will not allow 2/3 to be used as an "admissible component." We prove the following.