Appendix A
Linear Programming and Duality

This appendix outlines linear programming and its duality relations. Readers are referred to text books such as Gass (1985)[1], Charnes and Cooper (1961)[2], Mangasarian (1969)[3] and Tone (1978)[4] for details. More advanced treatments may be found in Dantzig (1963)[5], Spivey and Thrall (1970)[6] and Nering and Tucker (1993).[7] Most of the discussions in this appendix are based on Tone (1978).

A.1 LINEAR PROGRAMMING AND OPTIMAL SOLUTIONS

The following problem, which minimizes a linear functional subject to a system of linear equations in nonnegative variables, is called a *linear programming problem*:

$$(P) \quad \min \; z = \boldsymbol{c}\boldsymbol{x} \tag{A.1}$$

$$\text{subject to} \quad A\boldsymbol{x} = \boldsymbol{b} \tag{A.2}$$

$$\boldsymbol{x} \geq \boldsymbol{0}, \tag{A.3}$$

where $A \in R^{m \times n}$, $\boldsymbol{b} \in R^m$ and $\boldsymbol{c} \in R^n$ are given, and $\boldsymbol{x} \in R^n$ is the vector of variables to be determined optimally to minimize the scalar z in the objective. $\boldsymbol{c}$ is a row vector and $\boldsymbol{b}$ a column vector. (A.3) is called a nonnegativity constraint. Also, we assume that $m < n$ and $\text{rank}(A) = m$.

A nonnegative vector of variables $\boldsymbol{x}$ that satisfies the constraints of (P) is called *a feasible solution* to the linear programming problem. A feasible solution that minimizes the objective function is called an *optimal solution.*

A.2 BASIS AND BASIC SOLUTIONS

We call a nonsingular submatrix $B \in R^{m \times m}$ of A a *basis* of A when it has the following properties: (1) it is of full rank and (2) it spans the space of solutions. We partition A into B and R and write symbolically:

$$A = [B \mid R], \tag{A.4}$$

where R is an $(m \times (n-m))$ matrix. The variable vector $\boldsymbol{x}$ is similarly divided into $\boldsymbol{x}^B$ and $\boldsymbol{x}^R$. $\boldsymbol{x}^B$ is called *basic* and $\boldsymbol{x}^R$ *nonbasic*. (A.2) can be expressed in terms of this partition as follows:

$$B\boldsymbol{x}^B + R\boldsymbol{x}^R = \boldsymbol{b}. \tag{A.5}$$

By multiplying the above equation by B^{-1}, we have:

$$\boldsymbol{x}^B = B^{-1}\boldsymbol{b} - B^{-1}R\boldsymbol{x}^R. \tag{A.6}$$

Thus, the basic vector $\boldsymbol{x}^B$ is expressed in terms of the nonbasic vector $\boldsymbol{x}^R$. By substituting this expression into the objective function in (A.1), we have:

$$z = \boldsymbol{c}^B B^{-1}\boldsymbol{b} - (\boldsymbol{c}^B B^{-1}R - \boldsymbol{c}^R)\boldsymbol{x}^R. \tag{A.7}$$

Now, we define a *simplex multiplier* $\boldsymbol{\pi} \in R^m$ and *simplex criterion* $\boldsymbol{p} \in R^{n-m}$ by

$$\boldsymbol{\pi} = \boldsymbol{c}^B B^{-1} \tag{A.8}$$
$$\boldsymbol{p} = \boldsymbol{\pi} R - \boldsymbol{c}^R, \tag{A.9}$$

where $\boldsymbol{\pi}$ and $\boldsymbol{p}$ are row vectors. The following vectors are called the *basic solution* corresponding to the basis B:

$$\bar{\boldsymbol{x}}^B = B^{-1}\boldsymbol{b} \tag{A.10}$$
$$\bar{\boldsymbol{x}}^R = \boldsymbol{0}. \tag{A.11}$$

Obviously the basic solution is feasible for (A.2) and (A.3).

A.3 OPTIMAL BASIC SOLUTIONS

We call a basis B *optimal* if it satisfies:

$$\bar{\boldsymbol{x}}^B = B^{-1}\boldsymbol{b} \geq \boldsymbol{0} \tag{A.12}$$
$$\boldsymbol{p} = \boldsymbol{\pi} R - \boldsymbol{c}^R \leq \boldsymbol{0}. \tag{A.13}$$

Theorem A.1 *The basic solution corresponding to an optimal basis is the optimal solution of linear programming* (P).

Proof. It is easy to see that $(\bar{\boldsymbol{x}}^B = B^{-1}\boldsymbol{b},\ \bar{\boldsymbol{x}}^R = \boldsymbol{0})$ is a feasible solution to (P). Furthermore,

$$z = \boldsymbol{c}^B \bar{\boldsymbol{x}}^B - \boldsymbol{p}\boldsymbol{x}^R. \tag{A.14}$$

Hence, by considering $\boldsymbol{p} \leq \boldsymbol{0}$, we find that z attains its minimum when $\boldsymbol{x}^R = \boldsymbol{0}$. □

The *simplex method* for linear programming starts from a basis, reduces the objective function monotonically by changing bases and finally attains an optimal basis.

A.4 DUAL PROBLEM

Given the linear programming (P) (called the *primal problem*), there corresponds the following *dual problem* with the row vector of variables $\boldsymbol{y} \in R^m$.

$$(D) \quad \max\ w = \boldsymbol{y}\boldsymbol{b} \tag{A.15}$$

$$\text{subject to} \quad \boldsymbol{y}A \leq \boldsymbol{c}, \tag{A.16}$$

and $\boldsymbol{y}$ not otherwise constrained.

Theorem A.2 *For each primal feasible solution $\boldsymbol{x}$ and each dual feasible solution $\boldsymbol{y}$,*

$$\boldsymbol{c}\boldsymbol{x} \geq \boldsymbol{y}\boldsymbol{b}. \tag{A.17}$$

That is, the objective function value of the dual maximizing problem never exceeds that of the primal minimizing problem.

Proof. By multiplying (A.2) from the left by $\boldsymbol{y}$, we have

$$\boldsymbol{y}A\boldsymbol{x} = \boldsymbol{y}\boldsymbol{b}. \tag{A.18}$$

By multiplying (A.16) from the right by $\boldsymbol{x}$ and noting $\boldsymbol{x} \geq \boldsymbol{0}$, we have:

$$\boldsymbol{y}A\boldsymbol{x} \leq \boldsymbol{c}\boldsymbol{x}. \tag{A.19}$$

Comparing (A.18) and (A.19),

$$\boldsymbol{c}\boldsymbol{x} \geq \boldsymbol{y}A\boldsymbol{x} = \boldsymbol{y}\boldsymbol{b}. \tag{A.20}$$

□

Corollary A.1 *If a primal feasible $\boldsymbol{x}^0$ and a dual feasible $\boldsymbol{y}^0$ satisfy*

$$\boldsymbol{c}\boldsymbol{x}^0 = \boldsymbol{y}^0\boldsymbol{b}, \tag{A.21}$$

then $\boldsymbol{x}^0$ is optimal for the primal and $\boldsymbol{y}^0$ is optimal for its dual.

Theorem A.3 (Duality Theorem) (*i*) *In a primal-dual pair of linear programs, if either the primal or the dual problem has an optimal solution, then the other does also, and the two optimal objective values are equal.*
(*ii*) *If either the primal or the dual problem has an unbounded solution, then the other has no feasible solution.* (*iii*) *If either problem has no solution then the other problem either has no solution or its solution is unbounded.*

Proof. (*i*) Suppose that the primal problem has an optimal solution. Then there exists an optimal basis B and $\boldsymbol{p} = \boldsymbol{\pi}R - \boldsymbol{c}^R \leq \boldsymbol{0}$ as in (A.13). Thus,

$$\boldsymbol{\pi}R \leq \boldsymbol{c}^R. \tag{A.22}$$

However, multiplying (A.8) on the right by B,

$$\boldsymbol{\pi}B = \boldsymbol{c}^B. \tag{A.23}$$

Hence,

$$\pi A = \pi\,[B|R] \leq \left[\mathbf{c}^B|\mathbf{c}^R\right] = \mathbf{c}. \tag{A.24}$$

Consequently,

$$\pi A \leq \mathbf{c}. \tag{A.25}$$

This shows that the simplex multiplier π for an optimal basis to the primal is feasible for the dual problem. Furthermore, it can be shown that π is optimal to the dual problem as follows: The basic solution ($\bar{\mathbf{x}}^B = B^{-1}\mathbf{b}$, $\bar{\mathbf{x}}^R = \mathbf{0}$) for the primal basis B has the objective value $z = \mathbf{c}^B B^{-1}\mathbf{b} = \pi\mathbf{b}$, while π has the dual objective value $w = \pi\mathbf{b}$. Hence, by Corollary A.1, π is optimal for the dual problem. Conversely, it can be demonstrated that if the dual problem has an optimal solution, then the primal problem does also and the two objective values are equal, by transforming the dual to the primal form and by observing its dual. (See Gass, *Linear Programming*, pp. 158-162, for details).

(*ii*) (a) If the objective function value of the primal problem is unbounded below and the dual problem has a feasible solution, then by Theorem A.2,

$$w = \mathbf{y}\mathbf{b} \leq -\infty. \tag{A.26}$$

Thus, we have a contradiction. Hence, the dual has no feasible solution.

(b) On the other hand, if the objective function value of the dual problem is unbounded upward, it can be shown by similar reasoning that the primal problem is not feasible.

(*iii*) To demonstrate (*iii*), it is sufficient to show the following as an example in which both primal and dual problems have no solution.

<Primal>	min	$-x$	<Dual>	max	y
	subject to	$0 \times x = 1$		subject to	$y \times 0 \leq -1$
		$x \geq 0$			

where x and y are scalar variables. □

A.5 SYMMETRIC DUAL PROBLEMS

The following two LPs, ($P1$) and ($D1$), are mutually dual.

$$\begin{aligned} (P1) \quad \min\ z &= \mathbf{c}\mathbf{x} \\ \text{subject to}\quad A\mathbf{x} &\geq \mathbf{b} \\ \mathbf{x} &\geq \mathbf{0}. \end{aligned} \tag{A.27}$$

$$\begin{aligned} (D1) \quad \max\ w &= \mathbf{y}\mathbf{b} \\ \text{subject to}\quad \mathbf{y}A &\leq \mathbf{c} \qquad (A.28) \\ \mathbf{y} &\geq \mathbf{0}. \qquad (A.29) \end{aligned}$$

The reason is that, by introducing a nonnegative slack $\boldsymbol{\lambda} \in R^m$, $(P1)$ can be rewritten as $(P1')$ below and its dual turns out to be equivalent to $(D1)$.

$$
\begin{array}{rrcl}
(P1') & \min\ z & = & \boldsymbol{cx} \\
\text{subject to} & A\boldsymbol{x} - \boldsymbol{\lambda} & = & \boldsymbol{b} \\
 & \boldsymbol{x} & \geq & \mathbf{0}, \quad \boldsymbol{\lambda} \ \geq \ \mathbf{0}.
\end{array}
\tag{A.30}
$$

This form of mutually dual problems can be depicted as Table A.1, which is expressed verbally as follows:

For the inequality $\geq$ ($\leq$) constraints of the primal (dual) problem, the corresponding dual (primal) variables must be nonnegative. The constraints of the dual (primal) problem are bound to inequality $\leq$ ($\geq$). The objective function is to be maximized (minimized).

This pair of LPs are called symmetric primal-dual problems. The duality theorem above holds for this pair, too.

Table A.1. Symmetric Primal-Dual Problem

		x_1	...	...	x_n	≥ 0	
$0 \leq$	y_1	a_{11}	...	...	a_{1n}		b_1
	.	...	...	...	...		.
	.	...	...	...	...	$\geq$	.
	.	...	...	...	...		.
	y_m	a_{m1}	...	...	a_{mn}		b_m
			$\leq$				
		c_1	...	...	c_n		

A.6 COMPLEMENTARITY THEOREM

Let us transform the symmetric primal-dual problems into equality constraints by introducing nonnegative slack variables $\boldsymbol{\lambda} \in R^m$ and $\boldsymbol{\mu} \in R^n$, respectively.

$$
\begin{array}{rrcl}
(P1') & \min\ z & = & \boldsymbol{cx} \\
\text{subject to} & A\boldsymbol{x} - \boldsymbol{\lambda} & = & \boldsymbol{b} \\
 & \boldsymbol{x} & \geq & \mathbf{0}, \quad \boldsymbol{\lambda} \ \geq \ \mathbf{0}.
\end{array}
$$

$$
\begin{array}{rrcl}
(D1') & \max\ w & = & \boldsymbol{yb} \\
\text{subject to} & \boldsymbol{y}A + \boldsymbol{\mu} & = & \boldsymbol{c} \\
 & \boldsymbol{y} & \geq & \mathbf{0}, \quad \boldsymbol{\mu} \ \geq \ \mathbf{0}.
\end{array}
\begin{array}{r} \\ \text{(A.31)} \\ \text{(A.32)} \end{array}
$$

Then, the optimality condition in Duality Theorem A.3 can be stated as follows:

Theorem A.4 (Complementarity Theorem) *Let $(\boldsymbol{x}, \boldsymbol{\lambda})$ and $(\boldsymbol{y}, \boldsymbol{\mu})$ be feasible to $(P1')$ and $(D1')$, respectively. Then, $(\boldsymbol{x}, \boldsymbol{\lambda})$ and $(\boldsymbol{y}, \boldsymbol{\mu})$ are optimal* *to* $(P1')$ *and* $(D1')$ *if and only if it holds:*

$$\boldsymbol{\mu x} = \boldsymbol{y\lambda} = 0. \tag{A.33}$$

Proof.

$$\text{From } \boldsymbol{\mu x} = 0, \text{ we have } (\boldsymbol{c} - \boldsymbol{y}A)\boldsymbol{x} = 0 \Rightarrow \boldsymbol{cx} = \boldsymbol{y}A\boldsymbol{x}. \tag{A.34}$$

$$\text{From } \boldsymbol{y\lambda} = 0, \text{ we have } \boldsymbol{y}(A\boldsymbol{x} - \boldsymbol{b}) = 0 \Rightarrow \boldsymbol{y}A\boldsymbol{x} = \boldsymbol{yb}. \tag{A.35}$$

Thus, $\boldsymbol{cx} = \boldsymbol{yb}$. By the duality theorem, $\boldsymbol{x}$ and $\boldsymbol{y}$ are optimal for the primal and the dual, respectively. □

By (A.33), we have

$$\sum_{j=1}^{n} \mu_j x_j = 0, \quad \sum_{i=1}^{m} y_i \lambda_i = 0. \tag{A.36}$$

By nonnegativity of each term in these two expressions,

$$\mu_j x_j = 0 \quad (j = 1, \ldots, n) \tag{A.37}$$

$$y_i \lambda_i = 0. \quad (i = 1, \ldots, m) \tag{A.38}$$

Thus, either the variable μ_j or the variable x_j must be zero for each j and either y_i or λ_i must be zero for each i. We called this property *complementarity.*

A.7 FARKAS' LEMMA AND THEOREM OF THE ALTERNATIVE

Theorem A.5 (Farkas' Lemma, Theorem of the Alternative) *For each* $(m \times n)$ *matrix* A *and each vector* $\boldsymbol{b} \in R^m$, *either*

$$(I) \quad A\boldsymbol{x} = \boldsymbol{b} \quad \boldsymbol{x} \geq \boldsymbol{0}$$

has a solution $\boldsymbol{x} \in R^n$ *or*

$$(II) \quad \boldsymbol{y}A \leq \boldsymbol{0} \quad \boldsymbol{yb} > 0$$

has a solution $\boldsymbol{y} \in R^m$ *but never both.*

Proof. For (I), we consider the following the primal-dual pair of LPs:

$$(P2) \qquad \min \ z = \boldsymbol{0x}$$
$$A\boldsymbol{x} = \boldsymbol{b}$$
$$\boldsymbol{x} \geq \boldsymbol{0}$$

$$(D2) \qquad \max \ w = \boldsymbol{yb}$$
$$\boldsymbol{y}A \leq \boldsymbol{0}.$$

If (I) has a feasible solution, then it is optimal for $(P2)$ and hence, by the duality theorem, the optimal objective value of $(D2)$ is 0. Therefore, (II) has no solution.

On the other hand, $(D2)$ has a feasible solution $\boldsymbol{y} = \boldsymbol{0}$ and is not infeasible. Hence, if $(P2)$ is infeasible, $(D2)$ is unbounded upward. Thus, (II) has a solution.

□

A.8 STRONG THEOREM OF COMPLEMENTARITY

Theorem A.6 *For each skew-symmetric matrix K $(= -K^T) \in R^{n\times n}$, the inequality*

$$K\boldsymbol{x} \geq \boldsymbol{0}, \quad \boldsymbol{x} \geq \boldsymbol{0} \tag{A.39}$$

has a solution $\bar{\boldsymbol{x}}$ such that

$$K\bar{\boldsymbol{x}} + \bar{\boldsymbol{x}} > \boldsymbol{0}. \tag{A.40}$$

Proof. Let $\boldsymbol{e}_j$ $(j = 1, \ldots, n)$ be the j-th unit vector and the system (P_j) be,

$$(P_j) \qquad \begin{array}{l} K\boldsymbol{x} \geq \boldsymbol{0} \\ \boldsymbol{x} \geq \boldsymbol{0}, \quad \boldsymbol{e}_j\boldsymbol{x} > 0. \end{array}$$

If (P_j) has a solution $\bar{\boldsymbol{x}}^j \in R^n$, then we have:

$$(K\bar{\boldsymbol{x}}^j)_j \geq \boldsymbol{0}, \quad \bar{\boldsymbol{x}}^j \geq \boldsymbol{0}, \quad \boldsymbol{e}_j\bar{\boldsymbol{x}}^j = (\bar{\boldsymbol{x}}^j)_j > 0$$

and hence

$$(K\bar{\boldsymbol{x}}^j)_j + (\bar{\boldsymbol{x}}^j)_j > 0.$$

If (P_j) has no solution, then by Farkas' lemma the following system has a solution $\bar{\boldsymbol{v}}^j \in R^n$, $\bar{\boldsymbol{w}}^j \in R^n$.

$$(D_j) \qquad \begin{array}{l} K\boldsymbol{v} = \boldsymbol{e}_j + \boldsymbol{w} \\ \boldsymbol{v} \geq \boldsymbol{0}, \quad \boldsymbol{w} \geq \boldsymbol{0}. \end{array}$$

This solution satisfies:

$$(K\bar{\boldsymbol{v}}^j)_j = 1 + (\bar{\boldsymbol{w}}^j)_j > 0$$

and hence

$$(K\bar{\boldsymbol{v}}^j)_j + (\bar{\boldsymbol{v}}^j)_j > 0.$$

Since, for each $j = 1, \ldots, n$, either $\bar{\boldsymbol{x}}^j$ or $\bar{\boldsymbol{v}}^j$ exists, we can define a vector $\bar{\boldsymbol{x}}$ by summing over j. Then $\bar{\boldsymbol{x}}$ satisfies:

$$K\bar{\boldsymbol{x}} + \bar{\boldsymbol{x}} > \boldsymbol{0}.$$

□

Let a primal-dual pair of LPs with the coefficient $A \in R^{m\times n}$, $\boldsymbol{b} \in R^m$ and $\boldsymbol{c} \in R^n$ be $(P1')$ and $(D1')$ in Section A.6. Suppose they have optimal solutions $(\bar{\boldsymbol{x}},\ \bar{\boldsymbol{\lambda}})$ for the primal and $(\bar{\boldsymbol{y}},\ \bar{\boldsymbol{\mu}})$ for the dual, respectively. Then, by the complementarity condition in Theorem A.4, we have:

$$\bar{\mu}_j \bar{x}_j = 0 \quad (j = 1, \ldots, n) \tag{A.41}$$

$$\bar{y}_i \bar{\lambda}_i = 0 \quad (i = 1, \ldots, m) \tag{A.42}$$

However, a stronger theorem holds:

Theorem A.7 (Strong Theorem of Complementarity) *The primal-dual pair of LPs $(P1')$ and $(D1')$ have optimal solutions such that, in the complementarity condition* (A.41) *and* (A.42), *if one member of the pair is* 0, *then the other is positive.*

Proof. Observe the system:

$$\begin{aligned} A\boldsymbol{x} - r\boldsymbol{b} &\geq \boldsymbol{0} \\ -\boldsymbol{y}A + r\boldsymbol{c} &\geq \boldsymbol{0} \\ \boldsymbol{y}\boldsymbol{b} - \boldsymbol{c}\boldsymbol{x} &\geq 0 \\ \boldsymbol{x} \geq \boldsymbol{0},\ \ \boldsymbol{y} &\geq \boldsymbol{0},\ \ r \geq 0. \end{aligned}$$

We define a matrix K and a vector $\boldsymbol{w}$ by:

$$K = \begin{pmatrix} O & A & -\boldsymbol{b} \\ -A^T & O & \boldsymbol{c}^T \\ \boldsymbol{b}^T & -\boldsymbol{c} & 0 \end{pmatrix}$$

$$\boldsymbol{w}^T = \left(\boldsymbol{y}^T,\ \boldsymbol{x},\ r\right)^T.$$

Then, by Theorem A.6, the system

$$K\boldsymbol{w} \geq \boldsymbol{0}, \quad \boldsymbol{w} \geq \boldsymbol{0}$$

has a solution $\widetilde{\boldsymbol{w}}^T = \left(\widetilde{\boldsymbol{y}}^T,\ \widetilde{\boldsymbol{x}},\ \widetilde{r}\right)^T$ such that

$$K\widetilde{\boldsymbol{w}} + \widetilde{\boldsymbol{w}} > \boldsymbol{0}.$$

This results in the following inequalities:

$$A\widetilde{\boldsymbol{x}} - \widetilde{r}\boldsymbol{b} + \widetilde{\boldsymbol{y}}^T > \boldsymbol{0} \tag{A.43}$$

$$-\widetilde{\boldsymbol{y}}A + \widetilde{r}\boldsymbol{c} + \widetilde{\boldsymbol{x}}^T > \boldsymbol{0} \tag{A.44}$$

$$\widetilde{\boldsymbol{y}}\boldsymbol{b} - \boldsymbol{c}\widetilde{\boldsymbol{x}} + \widetilde{r} > 0 \tag{A.45}$$

We have two cases for $\widetilde{r}$.
(i) If $\widetilde{r} > 0$, we define $\bar{\boldsymbol{x}}$ and $\bar{\boldsymbol{\lambda}}$ by

$$\bar{\boldsymbol{x}} = \widetilde{\boldsymbol{x}}/\widetilde{r}, \quad \bar{\boldsymbol{y}} = \widetilde{\boldsymbol{y}}/\widetilde{r} \tag{A.46}$$

$$\bar{\boldsymbol{\lambda}} = A\bar{\boldsymbol{x}} - \boldsymbol{b}, \quad \bar{\boldsymbol{\mu}} = \boldsymbol{c} - \bar{\boldsymbol{y}}A. \tag{A.47}$$

Then, $(\bar{\boldsymbol{x}}, \ \bar{\boldsymbol{\lambda}})$ is a feasible solution of $(P1')$ and $(\bar{\boldsymbol{y}}, \ \bar{\boldsymbol{\mu}})$ is a feasible solution of $(D1')$. Furthermore, $\bar{\boldsymbol{y}}\boldsymbol{b} \geq \boldsymbol{c}\bar{\boldsymbol{x}}$. Hence, these solutions are optimal for the primal-dual pair LPs. In this case, (A.43) and (A.44) result in

$$\bar{\boldsymbol{\lambda}} + \bar{\boldsymbol{y}} \quad > \quad \mathbf{0} \tag{A.48}$$

$$\bar{\boldsymbol{\mu}} + \bar{\boldsymbol{x}} \quad > \quad \mathbf{0}. \tag{A.49}$$

Thus, strong complementarity holds as asserted in the theorem.
(ii) If $\tilde{r} = 0$, it cannot occur that both $(P1')$ and $(D1')$ have feasible solutions. The reason is: if they have feasible solutions $\boldsymbol{x}^*$ and $\boldsymbol{y}^*$, then

$$A\boldsymbol{x}^* \geq \boldsymbol{b}, \quad \boldsymbol{x}^* \geq \mathbf{0}, \quad \boldsymbol{y}^* A \leq \boldsymbol{c}, \quad \boldsymbol{y}^* \geq \mathbf{0}. \tag{A.50}$$

Hence, we have:

$$\boldsymbol{c}\tilde{\boldsymbol{x}} \geq \boldsymbol{y}^* A\tilde{\boldsymbol{x}} \geq 0 \geq \tilde{\boldsymbol{y}} A\boldsymbol{x}^* \geq \tilde{\boldsymbol{y}}\boldsymbol{b}. \tag{A.51}$$

This contradicts (A.45) in the case $\tilde{r} = 0$. Thus, the case $\tilde{r} = 0$ cannot occur.

□

A.9 LINEAR PROGRAMMING AND DUALITY IN GENERAL FORM

As a more general LP, we consider the case when there are both nonnegative variables $\boldsymbol{x}^1 \in R^k$ and sign-free variables $\boldsymbol{x}^2 \in R^{n-k}$ and both inequality and equality constraints are to be satisfied as follows:

$$\begin{array}{lll} (LP) & \min & z = \boldsymbol{c}^1\boldsymbol{x}^1 + \boldsymbol{c}^2\boldsymbol{x}^2 \\ & \text{subject to} & A_{11}\boldsymbol{x}^1 + A_{12}\boldsymbol{x}^2 \geq \boldsymbol{b}^1 \\ & & A_{21}\boldsymbol{x}^1 + A_{22}\boldsymbol{x}^2 = \boldsymbol{b}^2 \\ & & \boldsymbol{x}^1 \geq \mathbf{0} \\ & & \boldsymbol{x}^2 \ \text{ free,} \end{array} \tag{A.52}$$

where $A_{11} \in R^{\,l \times k}$, $A_{12} \in R^{\,l \times (n-k)}$, $A_{21} \in R^{(m-l) \times k}$ and $A_{22} \in R^{(m-l) \times (n-k)}$. The corresponding dual problem is expressed as follows, with variables $\boldsymbol{y}^1 \in R^l$ and $\boldsymbol{y}^2 \in R^{m-l}$.

$$\begin{array}{lll} (DP) & \max & w = \boldsymbol{y}^1\boldsymbol{b}^1 + \boldsymbol{y}^2\boldsymbol{b}^2 \\ & \text{subject to} & \boldsymbol{y}^1 A_{11} + \boldsymbol{y}^2 A_{21} \leq \boldsymbol{c}^1 \\ & & \boldsymbol{y}^1 A_{12} + \boldsymbol{y}^2 A_{22} = \boldsymbol{c}^2 \\ & & \boldsymbol{y}^1 \geq \mathbf{0} \\ & & \boldsymbol{y}^2 \ \text{ free.} \end{array} \tag{A.53}$$

It can be easily demonstrated that the two problems are mutually primal-dual and the duality theorem holds between them. Table A.2 depicts the general form of the duality relation of Linear Progrmming.

Table A.2. General Form of Duality Relation

		$\geq \mathbf{0}$	free		
		$\boldsymbol{x}^1$	$\boldsymbol{x}^2$		
$\mathbf{0} \leq$	$\boldsymbol{y}^1$	A_{11}	A_{12}	$\geq$	$\boldsymbol{b}^1$
free	$\boldsymbol{y}^2$	A_{21}	A_{22}	$=$	$\boldsymbol{b}^2$
		$\leq$	$=$		
		$\boldsymbol{c}^1$	$\boldsymbol{c}^2$		

Now, we introduce slack variables $\boldsymbol{\lambda}^1 \in R^{\,l}$ and $\boldsymbol{\mu}^1 \in R^k$ to (LP) and (DP) and rewrite them as (LP') and (DP') below:

$$
\begin{aligned}
(LP') \quad \min \quad & z = \boldsymbol{c}^1\boldsymbol{x}^1 + \boldsymbol{c}^2\boldsymbol{x}^2 \\
\text{subject to} \quad & A_{11}\boldsymbol{x}^1 + A_{12}\boldsymbol{x}^2 - \boldsymbol{\lambda}^1 = \boldsymbol{b}^1 \\
& A_{21}\boldsymbol{x}^1 + A_{22}\boldsymbol{x}^2 = \boldsymbol{b}^2 \\
& \boldsymbol{x}^1 \geq \mathbf{0} \\
& \boldsymbol{x}^2 \ \text{free} \\
& \boldsymbol{\lambda}^1 \geq \mathbf{0}.
\end{aligned}
\tag{A.54}
$$

$$
\begin{aligned}
(DP') \quad \max \quad & w = \boldsymbol{y}^1\boldsymbol{b}^1 + \boldsymbol{y}^2\boldsymbol{b}^2 \\
\text{subject to} \quad & \boldsymbol{y}^1 A_{11} + \boldsymbol{y}^2 A_{21} + \boldsymbol{\mu}^1 = \boldsymbol{c}^1 \\
& \boldsymbol{y}^1 A_{12} + \boldsymbol{y}^2 A_{22} = \boldsymbol{c}^2 \\
& \boldsymbol{y}^1 \geq \mathbf{0} \\
& \boldsymbol{y}^2 \ \text{free} \\
& \boldsymbol{\mu}^1 \geq \mathbf{0}.
\end{aligned}
\tag{A.55}
$$

We then have the following complementarity theorem:

Corollary A.2 (Complementarity Theorem in General Form)
Let $(\boldsymbol{x}^1,\ \boldsymbol{x}^2,\ \boldsymbol{\lambda}^1)$ and $(\boldsymbol{y}^1,\ \boldsymbol{y}^2,\ \boldsymbol{\mu}^1)$ be feasible to (LP') and (DP'), respectively. Then they are optimal to (LP') and (DP') if and only if the relation below holds.

$$\boldsymbol{\mu}^1\boldsymbol{x}^1 = \boldsymbol{y}^1\boldsymbol{\lambda}^1 = 0. \tag{A.56}$$

Also, there exist optimal solutions that satisfy the following strong complementarity.

Corollary A.3 (Strong Theorem of Complementarity) *In the optimal solutions to the primal-dual pair LPs,* (LP') *and* (DP')*, there exist ones such that, in the complementarity condition* (A.56)*, if one of the pair is* 0*, then the other is positive.*

Notes

1. S.I. Gass (1985), *Linear Programming,* 5th ed., McGraw-Hill.
2. A. Charnes and W.W. Cooper (1961), *Management Models and Industrial Applications of Linear Programming,* (Volume 1 & 2), John Wiley & Sons.
3. O.L. Mangasarian (1969), *Nonlinear Programming,* McGraw-Hill.
4. K. Tone (1978), *Mathematical Programming,* (in Japanese) Asakura, Tokyo.
5. G.B. Dantzig (1963), *Linear Programming and Extensions* (Princeton: Princeton University Press).
6. W.A. Spivey and R.M. Thrall (1970), *Linear Optimization* (New York: Holt, Rinehart and Winston).
7. E.D. Nering and A.W. Tucker (1993), *Linear Programming and Related Problems* (New York: Academic Press).

Appendix B
Introduction to DEA-Solver

This is an introduction and manual for the attached DEA-Solver. There are two versions of DEA-Solver, the "Learning Version" (called **DEA-Solver-LV**, in the attached CD) and the "Professional Version" (called **DEA-Solver-PRO**: visit the DEA-Solver website at: **http://www.saitech-inc.com/** for further information). This manual serves both versions. DEA-Solver was developed by Kaoru Tone. All responsibility is attributed to Tone, but not to Cooper and Seiford in any dimension.

B.1 PLATFORM

The platform for this software is Microsoft Excel 97/2000 or later (a trademark of Microsoft Corporation).

B.2 INSTALLATION OF DEA-SOLVER

The accompanying installer will install DEA-Solver and sample problems in the attached CD-ROM to the hard disk (C:) of your PC. Click Setup.EXE in the folder "DEA-Solver" in the CD-ROM. Just follow the instruction on the screen. The folder in the hard disk is "C:\DEA-Solver" which includes the code DEA-Solver-LV(V3).xls and another folder "Samples(LV3)." A shortcut to DEA-Solver.xls will be automatically put on the Desktop. If you want to install "DEA-Solver" to another drive or to other folder (not to "C:\DEA-Solver"), just copy it to the disk or to the folder you designate. For the "Professional Version" an installer will automatically install "DEA-Solver."

B.3 NOTATION OF DEA MODELS

DEA-Solver applies the following notation for describing DEA models.

<Model Name> - <I or O> - <C or V or GRS>

where I or O corresponds to "Input"- or "Output"-orientation and C, V or GRS to "Constant", "Variable" or "General" returns to scale. For example, "AR-I-C" means the Input oriented Assurance Region model under Constant returns-to-scale assumption. In some cases, "I or O" and/or "C or V" are omitted. For example, "CCR-I" indicates the Input oriented CCR model which is naturally under constant returns-to-scale. "FDH" (= Free Disposal Hull) has no extensions. The abbreviated model names correspond to the following models,

1. CCR = Charnes-Cooper-Rhodes model (Chapters 2, 3)
2. BCC = Banker-Charnes-Cooper model (Chapters 4, 5)
3. IRS = Increasing Returns-to-Scale model (Chapter 5)
4. DRS = Decreasing Returns-to-Scale model (Chapter 5)

5. GRS = Generalized Returns-to-Scale model (Chapter 5)
6. AR = Assurance Region model (Chapter 6)
7. ARG = Assurance Region Global model (Chapter 6)
8. NCN = Non-controllable variable model (Chapter 7)
9. NDSC = Non-discretionary variable model (Chapter 7)
10. BND = Bounded variable model (Chapter 7)
11. CAT = Categorical variable model (Chapter 7)
12. SYS = Different Systems model (Chapter 7)
13. SBM = Slacks-Based Measure model (Chapter 4)
14. Weighted SBM = Weighted Slacks-Based Measure model (Chapter 4)
15. Cost = Cost efficiency model (Chapter 8)
16. New-Cost = New-Cost efficiency model (Chapter 8)
17. Revenue = Revenue efficiency model (Chapter 8)
18. New-Revenue = New-Revenue efficiency model (Chapter 8)
19. Profit = Profit efficiency model (Chapter 8)
20. New-Profit = New-Profit efficiency model (Chapter 8)
21. Ratio = Ratio efficiency model (Chapter 8)
22. Bilateral = Bilateral comparison model (Chapter 7)
23. FDH = Free Disposal Hull model (Chapter 4)
24. Window = Window Analysis (Chapter 9)
25. Super-efficiency = Super-efficiency model (Chapter 10)

B.4 INCLUDED DEA MODELS

The "Learning Version" includes all models and can solve problems with up to 50 DMUs; The "Professional Version" includes *Malmquist, Scale elasticity, Congestion* and *Undesirable output* models in addition to the above models and can deal with large-scale problems within the capacity of Excel worksheet.

B.5 PREPARATION OF THE DATA FILE

The data file should be prepared in an Excel Workbook prior to execution of DEA-Solver. The formats are as follows:

B.5.1 The CCR, BCC, IRS, DRS, GRS, SBM, Super-Efficiency and FDH Models

Figure B.1 shows an example of data file for these models.

1. **The first row (Row 1)**
   The first row (Row 1) contains Names of Problem and Input/Output Items, i.e.,
   Cell A1 = Problem Name
   Cell B1, C1, ...= Names of I/O items.
   The heading (I) or (O), showing them as being input or output should head the names of I/O items. The items without an (I) or (O) heading will not be considered as inputs and outputs. The ordering of (I) and (O) items is arbitrary.

2. **The second row and after**
   The second row contains the name of the first DMU and I/O values for the corresponding I/O items. This continues up to the last DMU.

3. **The scope of data domain**
   A data set should be bordered by at least one blank column at right and at least one blank row at bottom. This is a necessity for knowing the scope of the data domain. The data set should start from the top-left cell (A1).

4. **Data sheet name**
   A preferable sheet name is "DAT" (not "Sheet 1"). Never use names "Score", "Rank", "Projection", "Weight", "WeightedData", "Slack", "RTS", "Window", "Graph1" and "Graph2" for data sheet. These are reserved for this software.

The sample problem "Hospital(CCR)" in Figure B.1 has 12 DMUs with two inputs "(I)Doctor" and "(I)Nurse" and two outputs "(O)Outpatient" and "(O)Inpatient". The data set is bordered by one blank column (F) and by one blank row (14). The GRS model has the constraint $L \leq \sum_{j=1}^{n} \lambda_j \leq U$. The values of $L(\leq 1)$ and $U(\geq 1)$ must be supplied through the Message-Box on the display by request. Defaults are $L = 0.8$ and $U = 1.2$.

As noted in **1.** above, items without an (I) or (O) heading will not be considered as inputs or outputs. So, if you delete "(I)" from "(I)Nurse" to "Nurse," then "Nurse" will not be accounted for in this efficiency evaluation. Thus you can add (delete) items freely to (from) inputs and outputs without changing your data set.

	A	B	C	D	E	F
1	Hospital	(I)Doctor	(I)Nurse	(O)Outpatient	(O)Inpatient	
2	A	20	151	100	90	
3	B	19	131	150	50	
4	C	25	160	160	55	
5	D	27	168	180	72	
6	E	22	158	94	66	
7	F	55	255	230	90	
8	G	33	235	220	88	
9	H	31	206	152	80	
10	I	30	244	190	100	
11	J	50	268	250	100	
12	K	53	306	260	147	
13	L	38	284	250	120	
14						

Figure B.1. Sample.xls in Excel Sheet

B.5.2 The AR Model

Figure B.2 exhibits an example of data for the AR (Assurance Region) model. This problem has the same inputs and outputs as in Figure B.1. The constraints for the assurance region are described in rows 15 and 16 after "one blank row" at 14. This blank row is necessary for separating the data set and the assurance region constraints. These rows read as follows: the ratio of weights "(I)Doctor" vs. "(I)Nurse" is not less than 1 and not greater than 5 and that for "(O)Outpatient" vs. "(O)Inpatient" is not greater than 0.2 and not less than 0.5. Let the weights for Doctor and Nurse be $v(1)$ and $v(2)$, respectively. Then the first constraint implies

$$1 \leq v(1)/v(2) \leq 5.$$

Similarly, the second constraint means that the weights $u(1)$ (for Outpatient) and $u(2)$ (for Inpatient) satisfies the relationship

$$0.2 \leq u(1)/u(2) \leq 0.5.$$

Notice that the weights constraint can be applied between inputs and outputs as well.

	A	B	C	D	E	F
1	Hospital	(I)Doctor	(I)Nurse	(O)Outpatient	(O)Inpatient	
2	A	20	151	100	90	
3	B	19	131	150	50	
4	C	25	160	160	55	
5	D	27	168	180	72	
6	E	22	158	94	66	
7	F	55	255	230	90	
8	G	33	235	220	88	
9	H	31	206	152	80	
10	I	30	244	190	100	
11	J	50	268	250	100	
12	K	53	306	260	147	
13	L	38	284	250	120	
14						
15	1	(I)Doctor	(I)Nurse	5		
16	0.2	(O)Outpatient	(O)Inpatient	0.5		
17						

Figure B.2. Sample-AR.xls in Excel Sheet

B.5.3 The ARG Model

Instead of restricting ratios of virtual multipliers, this model imposes bounds on the virtual input (output) relative to the total virtual input (output). For example, in the above hospital case, the virtual input of Doctor is expressed by v(1) × (Number of) Doctor and the total virtual input is denoted by v(1) × (Number of) Doctor + v(2) × (Number of) Nurse, where v(1) and v(2) are weights to Doctor and Nurse, respectively. We impose lower and upper bounds, L and U, to the ratio of these two factors. Thus, we have constraints as expressed below.

$$L \leq \frac{v(1) \times \text{Doctor}}{v(1) \times \text{Doctor} + v(2) \times \text{Nurse}} \leq U.$$

In the Excel worksheet, we designate L and U along with the input (output) name as exhibited in Figure B.3. This means that $L = 0.5$ and $U = 0.8$ for Doctor in the above expression. See Section 6.3 in Chapter 6.

	A	B	C	D	E	F
1	Sample-ARG	(I)Doctor	(I)Nurse	(O)Outpatien	(O)Inpatient	
2	A	20	151	100	90	
3	B	19	131	150	50	
4	C	25	160	160	55	
5	D	27	168	180	72	
6	E	22	158	94	66	
7	F	55	255	230	90	
8	G	33	235	220	88	
9	H	31	206	152	80	
10	I	30	244	190	100	
11	J	50	268	250	100	
12	K	53	306	260	147	
13	L	38	284	250	120	
14						
15	0.5	(I)Doctor	0.8			
16	0.2	(I)Nurse	0.3			
17	0.2	(O)Outpatien	0.5			
18	0.4	(O)Inpatient	0.8			
19						

Figure B.3. Sample-ARG.xls in Excel Sheet

B.5.4 The NCN and NDSC Models

The non-controllable and non-discretionary models have basically the same data format as the CCR model. However, the uncontrollable inputs or outputs must have the headings (IN) or (ON), respectively. Figure B.4 exhibits the case where 'Doctor' is an uncontrollable (i.e., "non-discretionary" or "exogenously fixed") input and 'Inpatient' is an uncontrollable output.

	A	B	C	D	E	F
1	Hospital	(IN)Doctor	(I)Nurse	(O)Outpatient	(ON)Inpatient	
2	A	20	151	100	90	
3	B	19	131	150	50	
4	C	25	160	160	55	
5	D	27	168	180	72	
6	E	22	158	94	66	
7	F	55	255	230	90	
8	G	33	235	220	88	
9	H	31	206	152	80	
10	I	30	244	190	100	
11	J	50	268	250	100	
12	K	53	306	260	147	
13	L	38	284	250	120	
14						

Figure B.4. Sample-NCN (NDSC).xls in Excel Sheet

B.5.5 The BND Model

The bounded inputs or outputs must have the headings (IB) or (OB). Their lower and upper bounds should be designated by the columns headed by (LB) and (UB), respectively. These (LB) and (UB) columns must be inserted immediately after the corresponding (IB) or (OB) column. Figure B.5 implies that 'Doctor' and 'Inpatient' are bounded variables and their lower and upper bounds are given by the columns (LB)Doc., (UB)Doc., (LB)Inpat., and (UB)Inpat, respectively.

	A	B	C	D	E	F	G	H	I
1	Hospital	(IB)Doc.	(LB)Doc.	(UB)Doc.	(I)Nurse	(O)Outpat	(OB)Inpat.	(LB)Inpat.	(UB)Inpat.
2	A	20	15	22	151	100	90	80	100
3	B	19	15	23	131	150	50	45	55
4	C	25	20	25	160	160	55	50	60
5	D	27	21	27	168	180	72	70	76
6	E	22	20	25	158	94	66	60	80
7	F	55	45	56	255	230	90	80	100
8	G	33	31	36	235	220	88	80	95
9	H	31	29	33	206	152	80	70	90
10	I	30	28	31	244	190	100	90	110
11	J	50	45	50	268	250	100	90	120
12	K	53	45	54	306	260	147	130	160
13	L	38	30	40	284	250	120	110	130
14									

Figure B.5. Sample-BND.xls in Excel Sheet

B.5.6 The CAT, SYS and Bilateral Models

These models have basically the same data format as the CCR model. However, in the last column they must have an integer showing their category, system or bilateral group, as follows.

For the CAT model, the number starts from 1 (DMUs under the most difficult environment or with the most severe competition), 2 (in the second group of difficulty) and so on. It is recommended that the numbers be continuously assigned starting from 1.

For the SYS model, DMUs in the same system should have the same integer starting from 1.

For the Bilateral model, DMUs must be divided into two groups, denoted by 1 or 2.

Figure B.6 exhibits a sample data format for the CAT model.

	A	B	C	D	E	F
1	Hospital	(I)Doctor	(I)Nurse	(O)Outpatient	(O)Inpatient	Cat.
2	A	20	151	100	90	1
3	B	19	131	150	50	2
4	C	25	160	160	55	2
5	D	27	168	180	72	2
6	E	22	158	94	66	1
7	F	55	255	230	90	1
8	G	33	235	220	88	2
9	H	31	206	152	80	1
10	I	30	244	190	100	1
11	J	50	268	250	100	2
12	K	53	306	260	147	2
13	L	38	284	250	120	2
14						

Figure B.6. Sample-CAT.xls in Excel Sheet

B.5.7 The Cost and New-Cost Models

The unit cost columns must have the heading (C) followed by the *input* name. The ordering of columns is arbitrary. If an input has no cost column, its cost is regarded as zero. Figure B.7 is a sample.

	A	B	C	D	E	F	G	H
1	Hospital	(I)Doctor	(C)Doctor	(I)Nurse	(C)Nurse	(O)Outpat.	(O)Inpat.	
2	A	20	500	151	100	100	90	
3	B	19	350	131	80	150	50	
4	C	25	450	160	90	160	55	
5	D	27	600	168	120	180	72	
6	E	22	300	158	70	94	66	
7	F	55	450	255	80	230	90	
8	G	33	500	235	100	220	88	
9	H	31	450	206	85	152	80	
10	I	30	380	244	76	190	100	
11	J	50	410	268	75	250	100	
12	K	53	440	306	80	260	147	
13	L	38	400	284	70	250	120	
14								

Figure B.7. Sample-Cost(New-Cost).xls in Excel Sheet

B.5.8 The Revenue and New-Revenue Models

The unit price columns must have the heading (P) followed by the *output* name. The ordering of columns is arbitrary. If an output has no price column, its price is regarded as zero. See Figure B.8 for an example.

	A	B	C	D	E	F	G	H
1	Hospital	(I)Doctor	(I)Nurse	(O)Outpat.	(P)Outpat.	(O)Inpat.	(P)Inpat.	
2	A	20	151	100	550	90	2010	
3	B	19	131	150	400	50	1800	
4	C	25	160	160	480	55	2200	
5	D	27	168	180	600	72	3500	
6	E	22	158	94	400	66	3050	
7	F	55	255	230	430	90	3900	
8	G	33	235	220	540	88	3300	
9	H	31	206	152	420	80	3500	
10	I	30	244	190	350	100	2900	
11	J	50	268	250	410	100	2600	
12	K	53	306	260	540	147	2450	
13	L	38	284	250	295	120	3000	
14								

Figure B.8. Sample-Revenue(New-Revenue).xls in Excel Sheet

B.5.9 The Profit, New-Profit and Ratio Models

As a combination of *Cost* and *Revenue* models, these models have cost columns headed by (C) for inputs and price columns headed by (P) for outputs.

B.5.10 The Window Models

Figure B.9 exhibits an example of data format for a Window Analysis model. Top-left corner (A1) contains the problem name, e.g., "Car" in this example. The next right cell (B1) must include the first time period, e.g., "89." The second row beginning from the B column exhibits "I/O items", e.g., "(I)Sales" and "(O)Profit." The name of DMUs appears from the third row in the column A. The contents (observed data) follow in the third row and after. This style is repeated until the last time period. Notice that each time period is placed at the top-left corner of the corresponding frame and (I)/(O) items have the same names throughout the time period. It is not necessary to insert headings (I)/(O) to the I/O names of the second time period and after. I/O items are determined as designated in the first time period. Figure B.9 demonstrates performance of four car-manufacturers, i.e., Toyota, Nissan, Honda and Mitsubishi, during five time periods, i.e., from (19)89 to (19)93, in terms of the input "Sales" and the output "Profit."

A	B	C	D	E	F	G	H	I	J	K
Car	89		90		91		92		93	
DMU	(I)Sales	(O)Profit	Sales	Profit	Sales	Profit	Sales	Profit	Sales	Profit
Toyota	719	400	800	539	850	339	894	125	903	103
Nissan	358	92	401	139	418	120	427	34	390	0
Honda	264	74	275	100	280	65	291	54	269	33
Mitsubishi	190	44	203	49	231	66	255	56	262	57

Figure B.9. Sample-Window.xls in Excel Sheet

B.5.11 Weighted SBM Model

This model requires weights to inputs/outputs as data. They should be given at the rows below the main body of data set with one inserted blank row. See Figure B.10. The first column (A) has **WeightI** or **WeightO** designating input or output, respectively, and the weights to inputs or outputs follow consecutively in the order of input (output) items recorded at the top row. The values are relative, since the software normalizes them properly. Refer to (4.81)-(4.83) in Chapter 4. If they are vacant, weights are regarded as even. Figure B.10 designates that weights to Doctor and Nurse are 10:1 and those to Outpatient and Inpatient are 1:5.

	A	B	C	D	E	F
1	WSBM	(I)Doctor	(I)Nurs	(O)Outpatie	(O)Inpatient	
2	A	20	151	100	90	
3	B	19	131	150	50	
4	C	25	160	160	55	
5	D	27	168	180	72	
6	E	22	158	94	66	
7	F	55	255	230	90	
8	G	33	235	220	88	
9						
10	WeightI	10	1			
11	WeightC	1	5			
12						

Figure B.10. Sample-Weighted SBM.xls in Excel Sheet

B.6 STARTING DEA-SOLVER

After completion of the data file in an Excel sheet on an Excel book as mentioned above, close the data file and click either the icon or the file "DEA-Solver" in Explorer. This starts DEA-Solver. First, click "Enable Macros" and then follow the instructions on the display.

Otherwise if the file "DEA-Solver" is already open (loaded), click "Tools" on the *Menu Bar*, then select "Macro" and click "Macros." Finally, click "Run" on the Macro.

This Solver proceeds as follows,

1. Selection of a DEA model

2. Selection of a data set in Excel Worksheet

3. Selection of a Workbook for saving the results of computation and

4. DEA computation

B.7 RESULTS

The results of computation are stored in the selected Excel workbook. The following worksheets contain the results, although some models lack some of them.

1. **Worksheet "Summary"**
   This worksheet shows statistics on data and a summary report of results obtained.

2. **Worksheet "Score"**
This worksheet contains the DEA-score, reference set, λ-value for each DMU in the reference set and ranking in input and in the descending order of efficiency scores. A part of a sample Worksheet "Score" is displayed in Figure B.11, where it is shown that DMUs A, B and D are efficient (Score=1) and DMU C is inefficient (Score=0.882708) with the reference set composed of B ($\lambda_B = 0.9$) and D ($\lambda_D = 0.13889$) and so on.

No.	DMU	Score	Rank	Reference set (lambda)					
1	A	1	1	A	1				
2	B	1	1	B	1				
3	C	0.8827083	8	B	0.9	D	0.13888889		
4	D	1	1	D	1				
5	E	0.7634995	12	A	0.5794409	B	5.72E-02	D	0.1526401
6	F	0.8347712	10	B	0.2	D	1.11111111		
7	G	0.9019608	7	A	0.2588235	B	1.29411765		
8	H	0.7963338	11	A	0.3866921	B	1.35E-02	D	0.6183983
9	I	0.9603922	4	A	0.6470588	B	0.83529412		
10	J	0.8706468	9	D	1.3888889				
11	K	0.955098	6	A	0.86	D	0.96666667		
12	L	0.9582043	5	A	0.6470588	B	1.23529412		

Figure B.11. A Sample Score Sheet

3. **Worksheet "Rank"**
This worksheet contains the ranking of DMUs in the descending order of efficiency scores.

4. **Worksheet "Projection"**
This worksheet contains projections of each DMU onto the efficient frontier by the chosen model.

5. **Worksheet "Weight"**
Optimal weights $v(i)$ and $u(i)$ for inputs and outputs are exhibited in this worksheet. $v(0)$ corresponds to the constraints $\sum_j^n \lambda_j \geq l$ and $u(0)$ to $\sum_j^n \lambda_j \leq u$. In the BCC model where $l = u = 1$ holds, $u(0)$ stands for the value of the dual variable for this constraint.

6. **Worksheet "WeightedData"**
This worksheet shows the optimal weighted I/O values, $x_{ij}v(i)$ and $y_{rj}u(r)$ for each DMU$_j$ (for $j = 1, \ldots, n$).

7. **Worksheet "Slack"**
This worksheet contains the input excesses s^- and output shortfalls s^+

for each DMU. In the radial models, e.g., CCR and BCC, s^- and s^+ are calculated by using the formula (3.10) for the input-oriented case. Hence, notice that the (total) input-slacks are obtained as $s^- + (1-\theta)x_o$. In the non-radial models, e.g., SBM (Slacks-based measure), s^- and s^+ are defined via (4.48), and they indicate the total slacks of the concerned DMU.

8. **Worksheet "RTS"**
In case of the BCC, AR-I-V and AR-O-V models, the returns-to-scale characteristics are recorded in this worksheet. For inefficient DMUs, returns-to-scale characteristics are those of the (input- or output-oriented) projected DMUs on the frontier.

9. **Graphsheet "Graph1"**
The bar chart of the DEA scores is exhibited in this graphsheet. This graph can be redesigned using the Graph functions of Excel.

10. **Graphsheet "Graph2"**
The bar chart of the DEA scores in the ascending order is exhibited in this graphsheet. A sample of Graph2 is exhibited in Figure B.12.

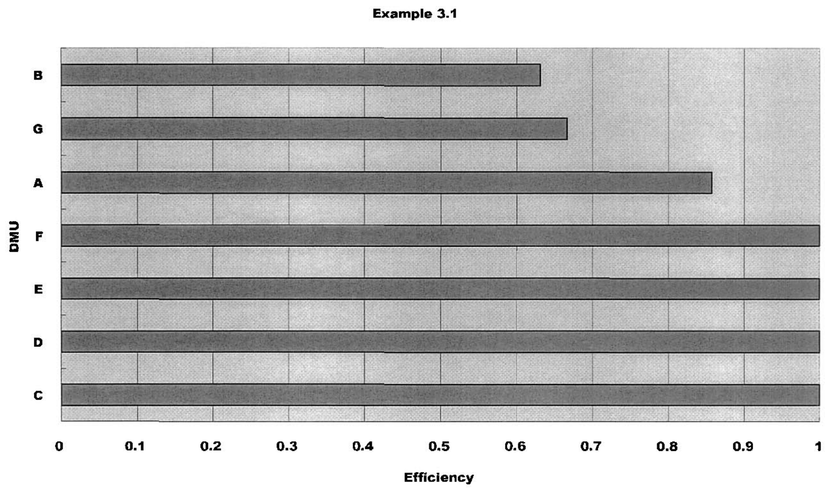

Figure B.12. A Sample Graph2

11. **Worksheets "Window$k$"**
These sheets are only for Window models and k ranges from 1 to L (the length of time periods in the data). The contents are similar to Table 11.1 in Chapter 11. They also include two graphs, 'Variations through Window' and 'Variations by Term'. We will illustrate them in the case of Sample-Window.xsl in Figure B.9. Let $k = 3$ (so we deal with three adjacent years, for example). The results of computation in the case of "Window-I-C" are summarized in Table B.1.

Table B.1. Window Analysis by Three Adjacent Years

Maker	89	90	91	92	93	Average	C Average
Toyota	0.826	1	0.592			0.806	
		1	0.592	0.208		0.600	
			1	0.351	0.286	0.546	0.651
Nissan	0.381	0.515	0.426			0.441	
		0.515	0.426	0.118		0.353	
			0.720	0.200	0	0.307	0.367
Honda	0.416	0.540	0.345			0.434	
		0.540	0.345	0.275		0.387	
			0.582	0.465	0.308	0.452	0.424
Mitsubishi	0.344	0.358	0.424			0.375	
		0.358	0.424	0.326		0.369	
			0.716	0.551	0.545	0.604	0.449

From this table we can see row-wise averages of scores for each maker, which we call "Average through Window." The graph "Variations through Window" exhibits these averages. See Figure B.13.

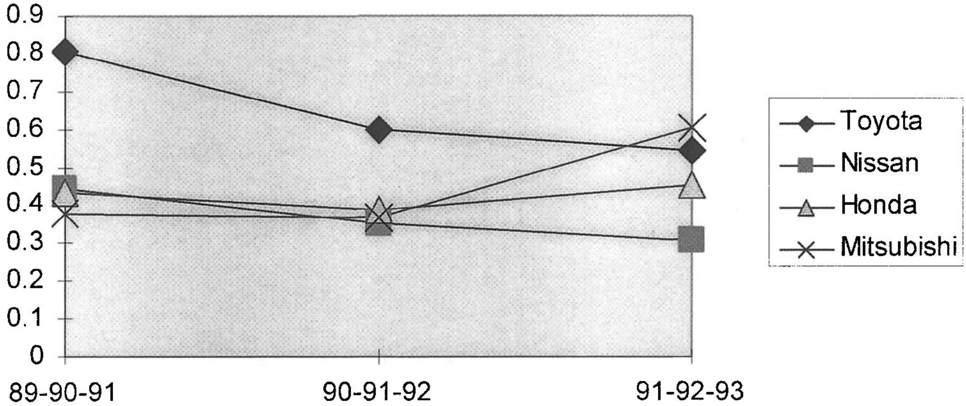

Figure B.13. Variations through Window

We can also evaluate column-wise averages of scores for each maker, which we call "Average by Term." The graph "Variations by Term" exhibits these averages. See Figure B.14.

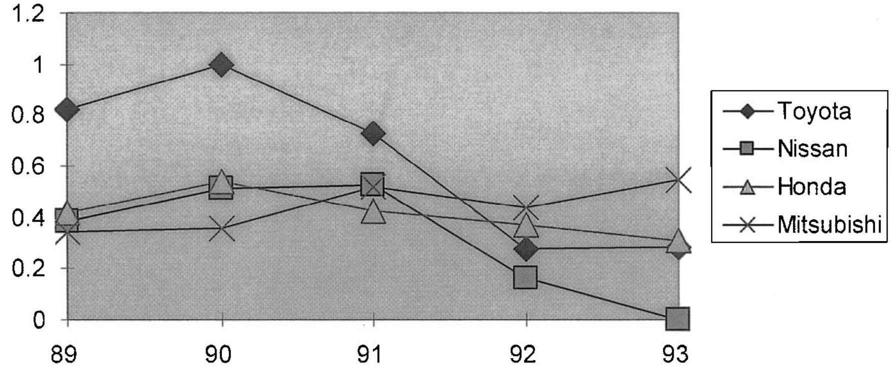

Figure B.14. Variations by Term

Note. The BCC, AR-I-V and AR-O-V models contain all the worksheets except "Window." The CCR, IRS, DRS, GRS, AR-I-C, AR-O-C and SBM models contain all sheets except "RTS" and "Window." The NCN, BND, CAT, SYS, Cost, Revenue, Profit, Ratio and FDH models produce "Summary," "Rank," "Score," "Projection," "Graph1" and "Graph2." The Bilateral model shows "Summary," "Score" and "Rank" sheets. The Window models return only "Window" and "Summary" sheets.

B.8 DATA LIMITATIONS

B.8.1 Problem Size

The "Learning Version" in the attached CD can solve problems with up to 50 DMUs. For the "Professional Version," the problem size is unlimited in terms of the number of DMUs and I/O items within the capacity of an Excel worksheet and the main memory of PC. More concretely, the data limitations for the "Professional Version" are as follows;

1. No. of DMUs must be less than 60000.

2. If No. of DMUs $\times$ (No. of Inputs + No. of Outputs + 2) $\geq$ 60000, then the "Projection" sheet will not be provided.

B.8.2 Inappropriate Data for Each Model

DMUs with the following irregular data are excluded from the comparison group as "inappropriate" DMUs. They are listed in the Worksheet "Summary." We will adopt the following notations for this purpose.

$xmax$ ($xmin$) = the max (min) input value of the DMU concerned

ymax (*ymin*) = the max (min) output value of the DMU concerned
costmax (*costmin*) = the max (min) unit cost of the DMU concerned
pricemax (*pricemin*) = the max (min) unit price of the DMU concerned

1. For the *CCR, BCC-I, IRS, DRS, GRS, CAT* and *SYS* models, a DMU with no positive value in inputs, i.e., $xmax \leq 0$, will be excluded from computation. Zero or minus values are permitted if there is at least one positive value in the inputs of the DMU concerned.

   For the *BCC-O* model, DMUs with no positive value in outputs, i.e., $ymax \leq 0$, will be excluded from computation.

2. For the *AR* model, i.e., *AR-I-C, AR-I-V, AR-O-C* and *AR-O-V*, DMUs with $xmin < 0$, $xmax \leq 0$ or $ymax \leq 0$ will be excluded from the comparison group.

3. For the *FDH* model, DMUs with no positive input value, i.e., $xmax \leq 0$, or a negative input value, i.e., $xmin < 0$, will be excluded from computation.

4. For the *Cost* model, DMUs with $xmax \leq 0$, $xmin < 0$, $costmax \leq 0$, or $costmin < 0$ are excluded. DMUs with the current input cost ≤ 0 will also be excluded.

5. For the *Revenue*, *Profit* and *Ratio* models, DMUs with no positive input value, i.e., $xmax \leq 0$, no positive output value, i.e., $ymax \leq 0$, or with a negative output value, i.e., $ymin < 0$, will be excluded from computation. Furthermore, in the *Revenue* model, DMUs with $pricemax \leq 0$, or $pricemin < 0$ will be excluded from the comparison group. In the *Profit* model DMUs with $costmax \leq 0$ or $costmin < 0$ will be excluded. Finally, in the *Ratio* model, DMUs with $pricemax \leq 0$, $pricemin < 0$, $costmax \leq 0$ or $costmin < 0$ will be excluded.

6. For the *NCN* and *BND* models, negative input and output values are automatically set to zero by the program. DMUs with $xmax \leq 0$ in the controllable (discretionary) input variables will be excluded from the comparison group as "inappropriate" DMUs. In the *BND* model, the lower bound and the upper bound must enclose the given (observed) value, otherwise these values will be adjusted to the given value.

7. For the *Window-I-C* and *Window-I-V* models, no restriction exists for output data, i.e., positive, zero or negative values for outputs are permitted. However, DMUs with $xmax \leq 0$ will be characterized as being zero efficiency. This is for purpose of completing the score matrix. So, care is needed for interpreting the results in this case. If the number of DMUs per one period (term) exceeds 255, no graph will be produced.

8. For the *SBM* model, nonpositive inputs or outputs are replaced by a small positive value.

9. For the *Bilateral* model, we cannot compare two groups if some inputs are zero for one group while the other group has all positive values for the corresponding input item.

B.9 SAMPLE PROBLEMS AND RESULTS

The attached "DEA-Solver LV (learning version)" includes the sample problems and results for all models in the folder "Samples".

The "Professional Version" is available via http://www.saitech-inc.com.

B.10 SUMMARY OF HEADINGS TO INPUTS/OUTPUTS

Table B.2 exhibits headings to input/output and samples.

Table B.2. Headings to Inputs/Outputs

Heading	Description	Example	Models employed
(I)	Input	(I)Employee	All models
(O)	Output	(O)Sales	All models
(IN)	Non-controllable or Non-discretionary input	(IN)Population	NCN (Non-controllable) NDSC (Non-discretionary)
(ON)	Non-controllable or Non-discretionary output	(ON)Area	As above
(IB)	Bounded input	(IB)Doctor	BND (Bounded variable)
(OB)	Bounded output	(OB)Attendance	As above
(LB)	Lower bound of bounded variable	(LB)Doctor	As above
(UB)	Upper bound of bounded variable	(UB)Doctor	As above
(C)	Unit cost of input	(C)Manager	Cost, New-Cost, Profit, New-Profit, Ratio
(P)	Unit price of output	(P)Laptop	Revenue, New-Revenue, Profit, New-Profit, Ratio

Appendix C
Bibliography

Comprehensive bibliography of 2800 DEA references is available in the attached CD-ROM.

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