Computer Construction of Quasi Optimal Portfolio for Stochastic Models with Jumps of Financial Markets

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Abstract. In the paper we propose a purely computational new method of construction of a quasi–optimal portfolio for stochastic models of a financial market with jumps. Here we present the method in the framework of a Black–Scholes–Merton model of an incomplete market (see, eg. [5], [7]), considering a well known optimal investment and consumption problem with the HARA type optimization functional. Our method is based on the idea to maximize this functional, taking into account only some subsets of possible portfolio and consumption processes. We show how to reduce the main problem to the construction of a portfolio maximizing a deterministic function of a few real valued parameters but under purely stochastic constraints. It is enough to solve several times an indicated system of stochastic differential equations (SDEs) with properly chosen parametrs. This is a generalization of an approach presented in [4] in connection with a well known classical Black–Scholes model.

Results of computer experiments presented here were obtained with the use of the *SDE–Solver* software package. This is our own professional C++ application to Windows system, designed as a scientific computing tool based on Monte Carlo simulations and serving for numerical and statistical construction of solutions to a wide class of systems of SDEs, including a broad class of diffusions with jumps driven by non-Gaussian random measures (consult [1], [4], [6], [9]).

The approach to construction of approximate optimal portfolio presented here should be useful in a stock market analysis, eg. for evolution based computer methods.

1 Optimal Investment and Consumption Problem for Stochastic Models with Jumps of Financial Markets

Let us recall that the Black–Scholes–Merton model of a financial market can be understood as a spacial case of the following system of $N+1$ SDEs

$$S_0(t) = S_0(0) + \int_0^t r(s)S_0(s)ds,$$

$$S_n(t) = S_n(0) + \int_0^t \mu_n(s)S_n(s)ds + \sum_{k=1}^{N} \int_0^t \sigma_{n,k}(s)S_n(s)dB_k(s)$$

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$$+ \int_0^t \rho_n(s) S_n(s) d\widetilde{N}^\lambda(s),$$

for $n = 1, \ldots, N$ and $t \in (0, T]$, and where we have the money market with a price $S_0(t)$ and N stocks with prices-per-share $S_1(t), \ldots, S_N(t)$, for $t \in [0, T]$.

We assume that processes $r = r(t)$, and $\mu_n = \mu_n(t)$, for $1 \le n \le N$, are $\mathbf{L}^1(\Omega \times [0, T])$–integrable, and processes $\sigma_{n,k} = \sigma_{n,k}(t)$ and $\rho_n = \rho_n(t)$ are $\mathbf{L}^2(\Omega \times [0, T])$–integrable on the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathrm{P})$ with a filtration $\{\mathcal{F}_t\}$, which is generated by N–dimensional Brownian motion process $(B_1(t), \ldots, B_N(t))$ and a compensated Poisson process $\widetilde{N}^\lambda(t)$.

Let the stochastic processes $\eta_0 = \eta_0(t)$ and $\eta_n = \eta_n(t)$, for $n = 1, 2, ..., N$, denote the number of shares of a bond and stocks, respectively. So, the value of the investor's holdings at time t is represented by the *wealth process*, which can be represented by

$$\mathbf{X}(t) \stackrel{\mathrm{df}}{=} \sum_{n=0}^{N} \eta_n(t) S_n(t) = \sum_{n=0}^{N} \pi_n(t), \tag{1}$$

where

$$\pi_n(t) \stackrel{\mathrm{df}}{=} \eta_n(t) S_n(t), \quad n = 0, 1, \ldots, N.$$

Let $\pi(t) = (\pi_1(t), \ldots, \pi_N(t))$. We say that the process

$$(\pi_0, \pi) = \{(\pi_0(t), \pi(t)) : t \in [0, T]\}$$

is the *portfolio process* or simply *portfolio* of an investor.

We define the *gains process* $\{G(t) : t \in [0, T]\}$ as a process which satisfies the following equation,

$$dG(t) = \sum_{n=0}^{N} \eta_n(t) dS_n(t) + \sum_{n=1}^{N} S_n(t) \delta_n(t) dt, \tag{2}$$

where $\delta_n = \delta_n(t)$ is the so called *divident rate payment process* of the nth stock, for all $0 < n \le N$. By the *consumption process* we understand here any non-negative, regular enough stochastic process $c = \{c(t) : t \in [0, T]\}$. Let $x > 0$ denote the *initial wealth* (or *endowment*, i.e. an amount of money an ivestor has to his disposal at time $t = 0$), what means that we have $X(0) = x$. Let $\Gamma(t) \stackrel{\mathrm{df}}{=} x - \int_0^t c(s) dt$. We say that the portfolio (π_0, π) is *Γ–financed*, when

$$\sum_{n=0}^{N} \pi_n(t) = \Gamma(t) + G(t), \tag{3}$$

with $G(0) = 0, \quad \Gamma(0) = x$.

Applying the general semimartingale version of Itô formula one can check that if conditions (2) and (3) are satisfied, then the wealth process

$$X \equiv X^{x,c,\pi} = \{X^{x,c,\pi}(t) : t \in [0, T]\},$$

defined by (1), can be obtained as a solution to the following SDE

$$dX(t) = (rX(t) - c(t))dt + \sum_{n=1}^{N}(\mu_n(t) + \delta_n(t) - r)\pi_n(t)dt + \\ + \sum_{n=1}^{N}\pi_n(t)\left(\sum_{m=1}^{N}\sigma_{n,m}(t)dB_m(t) + \rho_n(t)d\widetilde{N}^{\lambda}(t)\right), \quad (4)$$

with an initial condition of the form $\mathbf{X}(0) = x = \sum_{n=0}^{N}\eta_n(0)S_n(0)$.

From (2) and (3), after application of the classical Itô formula, it follows that the following – very important in our approach – equation must be satisfied

$$\sum_{n=0}^{N} S_n(t)d\eta_n(t) = \sum_{n=1}^{N} S_n(t)\delta_n(t)dt - c(t)dt. \quad (5)$$

In optimization problems utility functions can be chosen in many different ways, however the typical choice for scientific investigations is the HARA model, what means that we chose *utility function* given by

$$U^{(p)}(x) \overset{\mathrm{df}}{=} x^p/p, \quad x > 0, \quad (6)$$

for $p \in (-\infty, 0)$ or $p \in (0,1)$.

The *risk aversion coefficient* is then defined by the formula

$$R \overset{\mathrm{df}}{=} -x\frac{d^2}{dx^2}U^{(p)}(x)\Big/\frac{d}{dx}U^{(p)}(x) = 1 - p. \quad (7)$$

Here we are interested in the following optimization problem.

For a given utility function $U^{(p)} = U^{(p)}(c)$ and initial wealth $x > 0$, we look for an *optimal portfolio* $\widehat{\pi}(t) = \{\widehat{\pi}_1(t), \ldots, \widehat{\pi}_N(t)\}$ and an *optimal consumption process* $\widehat{c} = \widehat{c}(t)$, such that for the *value function* of the form

$$V_{c,\pi}(x) \overset{\mathrm{df}}{=} \mathrm{E}\left[\int_0^T U^{(p)}(c(t))\mathrm{e}^{-\int_0^t \beta(s)ds}dt\right] \quad (8)$$

the following condition is satisfied

$$V_{\hat{c},\hat{\pi}}(x) = \sup_{(c,\pi)\in\mathcal{A}(x)} V_{c,\pi}(x). \quad (9)$$

Here the condition $(c,\pi) \in \mathcal{A}(x)$ means, that the processes $c = c(t)$ and $\pi = \pi(t)$ are subject to the stochastic constraints, what means that $c = c(t)$ is positive on $[0,T]$ and the corresponding wealth process satisfing SDE (4) is such that

$$X^{x,c,\pi}(t) \geq 0 \ \text{ a.s.} \qquad \text{for} \ \ t \in [0,T]. \quad (10)$$

2 An Example of a Quasi–optimal Portfolio

An attempt to answer the question how to apply computational methods to solve directly and effectively optimizations problem (9) through analytical methods, e.g. making use of the Malliavin calculus (see [3], [8]) or trying to get a hedging strategy by constructing a relevant replicating portfolio (see eg. [5], etc.) is not an obvious task in our framework (see eg. [7]).

So, our aim is to describe a method of computer construction of a quasi–optimal portfolio solving approximate problem related to (9).

The method is based on the idea to maximize functional (8), taking into account only some subsets of possible portfolio processes derived from equations (4) and (5), and chosing the class of admissible positive consumption processes arbitrally, in a convenient reasonable way.

We show how to reduce the main problem to the construction of a portfolio maximizing a deterministic function of a few real valued parameters but under purely stochastic constraints.

In order to make the further exposition easier, we restrict ourselves to the one dimensional ($N = 1$) Black–Scholes–Merton model, which can be described in the following form

$$S_0(t) = S_0(0) + r \int_0^t S_0(s)ds \tag{11}$$

$$S_1(t) = S_1(0) + \mu \int_0^t S_1(s)ds + \sigma \int_0^t S_1(s)dB(s) + \rho \int_0^t S_n(s-)d\widetilde{N}^\lambda(s), \tag{12}$$

for $t \in [0, T]$, and initial conditions such that $S_0(0) > 0$, $S_1(0) > 0$.

In the model (11)–(12) all paremeters, i.e. $S_0(0)$, $S_1(0)$, r, μ, σ, and ρ are given positive real numbers. So, the processes S_0, S_1 can be described in the explicit closed form:

$$S_0(t) = S_0(0)\ \mathrm{e}^{rt}, \tag{13}$$

$$S_1(t) = S_1(0)\ \mathrm{e}^{\{(\mu-\sigma^2/2)t+\sigma B(t)+log(1+\rho)N^\lambda(t)\}}. \tag{14}$$

Our quasi–optimal portfolio is now given by $\pi = (\pi_0, \pi_1)$, where

$$\pi_0(t) = \eta_0(t)S_0(t), \quad \pi_1(t) = \eta_1(t)S_1(t), \quad t \in [0, T]. \tag{15}$$

In the example we have chosen for computer experiments presented here we reduced the class of admissible portolio process to those which are of the following form

$$\eta_1(t) = p_1 S_1(t), \quad t \in [0, T]. \tag{16}$$

We also restric ourselves to the class of consumption processes defined by

$$c(t) = c_0 S_0(t) + c_1 S_1(t), \quad t \in [0, T], \tag{17}$$

In (16) and (17) parameters p_1, c_0, c_1 are deterministic (real) variables, subject to some stochastic constraints, and which should be determined in an optimal way.

It is not difficult to notice that in such circumstances the wealth process $X(t) = X^{c_0,c_1,p_1}(t)$, defined by (4), solves the following Itô SDE

$$\begin{aligned} dX(t) = &\Big(rX(t) + (\mu + \delta - r)\eta(t)S_1(t) - \big(c_0 S_0(t) + c_1 S_1(t)\big)\Big)dt \\ &+\sigma\eta_1(t)S_1(t)dB(t) + \rho * \eta_1(t)S_1(t-)d\widetilde{N}^\lambda(t), \quad t \in (0, T], \quad X(0) = x. \end{aligned} \tag{18}$$

Making use of the equation (5), it is also possible to check that the first component of the portfolio, i.e. the proces $\eta_0 = \eta_0(t)$ solves the following SDE

$$d\eta_0(t) = \Big((-\mu+\delta)p_1 S_1^2(t) - \big(c_0 S_0(t) + c_1 S_1(t)\big)\Big)/S_0(t)dt - \\ -p_1\sigma S_1^2(t)/S_0(t)dB(t) - p_1\sigma S_1^2(t)/S_0(t)d\widetilde{N}^{\lambda}(t), \quad t \in (0,T], \quad (19)$$
$$\eta(0) = (x - p_1 S_1^2(0))/S_0(0).$$

In this way we arrive at the following problem.

For a given utility function $U^{(p)} = U^{(p)}(c)$ and initial wealth $x > 0$, we look for optimal values of parameters $\widehat{c_0}$, $\widehat{c_1}$, $\widehat{p_1}$, such that for the *value function* of the form

$$V_{c_0,c_1,p_1}(x) \stackrel{\text{df}}{=} \mathrm{E}\left[\int_0^T U^{(p)}(c_0 S_0(t) + c_1 S_1(t))\mathrm{e}^{-\beta t}dt\right] \quad (20)$$

the following condition is satisfied

$$V_{\widehat{c_0},\widehat{c_1}\widehat{p_1}}(x) = \sup_{(c_0,c_1,p_1)\in\mathcal{A}(x)} V_{c_0,c_1,p_1}(x). \quad (21)$$

Now the condition $(c_0, c_1, p_1) \in \mathcal{A}(x)$ means, that the consumption and wealth processes, defined by (17) and (18), are such that we have

$$X^{c_0,c_1,p_1}(t) \geq 0 \text{ a.s.}, \qquad c_0 S_0(t) + c_1 S_1(t) \geq 0 \text{ a.s.} \quad \text{for} \quad t \in [0,T]. \quad (22)$$

We see that, having to our disposal stochastic processes solving SDEs (11), (12), and (18), we are able to solve the problem (20)–(22). In order to get values of the value function (20) using the *SDE–Solver* software it is enough to solve the system of two equations

$$dY(t) = \big(c_0 r S_0(t) + c_1 \mu S_1(t)\big)\ dt + c_1\sigma_1 S_1(t)\ dB(t), \quad (23)$$
$$dZ(t) = U^{(p)}(Y(t))\mathrm{e}^{\beta t}dt, \quad (24)$$

for $t \in (0,T]$, and with initial conditions $Y(0) = c_0 S_0(0) + c_1 S_1(0)$, $Y(0) = 0$, and finally to compute

$$V_{c_0,c_1,p_1}(x) = \mathrm{E}\ Z(T). \quad (25)$$

Then, making use of formulae (19), (16), (15), and (17), one can easily construct quasi optimal portfolio and quasi optimal consumption processes.

3 Results of Computer Experiments

We solved the optimization problem desribed by formulae (11)–(22), with the following fixed values of constant parameters: $T = 1$, $r = 0.2$, $\mu = 0.15$, $\delta = 0.05$, $\sigma = 0.35$, $\rho = 0.35$, $\beta = 0$, and also with $\beta \in \{0, 0.1, 0.2, 0.4, 0.8\}$, $x = 50$, $S_{00} = 50$, $S_{10} = 50$.

The optimal solution for $\beta = 0.0$ is of the following form:

$$\widehat{c}_0 = 1.0, \quad \widehat{c}_1 = 0.00, \quad \widehat{p}_1 = 0.0, \qquad V_{\widehat{c}_0,\widehat{c}_1,\widehat{c}p_1} = 16.3.$$

From a large amount of data obtained in our experiments we present here only the optimal solution for $\beta = 0$.

In Table 1 below some values of function V_{c_0,c_1,p_1} are presented with corresponding values of parameters c_0, c_1, p_1.

Table 1. Values of V_{c_0,c_1,p_1} from (20)

$\mathbf{V}_{c_0,c_1,p_1}(x)$	$\mathbf{c_0}$	$\mathbf{c_1}$	$\mathbf{p_1}$	β
16.3	**1.00**	**0.00**	**0.00**	0.00
12.6	0.30	0.30	0.00	0.00
13.8	0.80	−0.10	0.01	0.00
12.6	0.30	0.30	0.01	0.00
12.4	0.30	0.30	0.02	0.00
14.6	1.00	0.00	0.00	0.20
12.9	0.40	0.40	0.01	0.20
12.4	0.80	−0.10	0.01	0.20
11.3	0.30	0.30	0.02	0.20
13.3	1.00	0.00	0.00	0.40
11.8	0.80	−0.05	0.01	0.40
10.3	0.30	0.30	0.02	0.40
11.1	1.00	0.00	0.00	0.80
8.6	0.30	0.30	0.02	0.80

In all runs of the system of equations (11), (12), (18), (23), (24) leading to computation of values of the expression in (25) with the *SDE–Solver*, we had 1000 trajectories of the solution, which were constructed on the grid given by 1000 subintervals of lenght 0.001 of the interval $[0, 1]$. Numerical and statistical approximation methods involved are described in [1], [2], [4], [6].

Another example (simplier one, with value function depending only on two parameters) of a quasi optimal portfolio and quasi optimal consumption processes that can be generalized in the same way as in our example presented here is discussed in [1]. Instead of (16), (17), and (20), the following conditions describe the optimization problem:

$$\eta_0(t) = p_0 S_0(t), \quad c(t) = c_2 X(t),$$

$$V_{c_2,p_0}(x) \stackrel{\text{df}}{=} \mathrm{E}\left[\int_0^T U^{(p)}(c_2 X(t)) \mathrm{e}^{-\beta t} dt\right].$$

Graphical representations, visualizing trajectories and some statistical properties of quasi optimal processes $\eta_0 = \eta_0(t)$, $\eta_1 = \eta_1(t)$, $X = X(t)$, and $c = c(t)$, are included there.

4 Conclusions

We strongly insist that even such rough approximations of the optimal investment and consumption problem as presented here are of important practical interest. One can get quite useful ideas about properties of stochastic processes solving the problem, how they depend on parameters of the stochastic model of financial market, investor preferences, etc. Of course, much more work on improvement of suggested method of construction of quasi optimal portfolio has to be done. It is also quite obvious that the method can be easily extended onto much more sofisticated stochastic models of financial market. There are various questions of mathematical nature, that should be answered in the future, e.g. on the correctness and convergence of the proposed approximate method, when more and more parameters enlargig properly the sets of admissible portfolio and consumption processes, are included.

Our computer experiments indicate to some extent the direction of further development of computational methods and computer software useful in practical solution of such complicated problems as construction of optimal strategies for investors, when stochastic model of the financial market is investigated in the framework of a system of SDEs of Itô or another, more general, type. For example, our approach can be in a very simple and natural way implemented for parallel computing systems.

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