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# Some Low Distortion Metric Ramsey Problems*

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**Abstract.** In this note we consider the metric Ramsey problem for the normed spaces $\ell_p$. Namely, given some $1 \le p \le \infty$ and $\alpha \ge 1$, and an integer $n$, we ask for the largest $m$ such that every $n$-point metric space contains an $m$-point subspace which embeds into $\ell_p$ with distortion at most $\alpha$. In [1] it is shown that in the case of $\ell_2$, the dependence of $m$ on $\alpha$ undergoes a phase transition at $\alpha = 2$. Here we consider this problem for other $\ell_p$, and specifically the occurrence of a phase transition for $p \ne 2$. It is shown that a phase transition does occur at $\alpha = 2$ for every $p \in [1, 2]$. For $p > 2$ we are unable to determine the answer, but estimates are provided for the possible location of such a phase transition. We also study the analogous problem for isometric embedding and show that for every $1 < p < \infty$ there are arbitrarily large metric spaces, no four points of which embed isometrically in $\ell_p$.

## 1. Introduction

A Ramsey-type theorem states that large systems necessarily contain large, highly structured subsystems. Here we consider Ramsey-type problems for finite metric spaces, interpreting "highly structured" as having low distortion embedding in $\ell_p$.

A mapping between two metric spaces $f : M \to X$ is called an embedding of $M$ in $X$. The *distortion* of the embedding is defined as

$$\mathrm{dist}(f) = \sup_{\substack{x,y\in M\\ x\ne y}} \frac{d_X(f(x), f(y))}{d_M(x,y)} \cdot \sup_{\substack{x,y\in M\\ x\ne y}} \frac{d_M(x,y)}{d_X(f(x), f(y))}.$$

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The least distortion required to embed $M$ in $X$ is denoted by $c_X(M)$. When $c_X(M) \leq \alpha$ we say that $M$ $\alpha$-embeds in $X$. In this note we study the following notion.

**Definition 1** (Metric Ramsey Function). We denote by $R_X(\alpha, n)$ the largest integer $m$ such that every $n$-point metric space has a subspace of size $m$ that $\alpha$-embeds into $X$.

When $X = \ell_p$ we use the notations $c_p$ and $R_p$. Note that for $p \in [1, \infty]$, it is always true that $R_p(\alpha, n) \geq R_2(\alpha, n)$. When $\alpha = 1$ we drop it from the notation, i.e., $R_X(n) = R_X(1, n)$.

Bourgain et al. [4] study this function for $X = \ell_2$, as a metric analog of Dvoretzky's theorem [7]. They prove:

**Theorem 1** [4]. *For any* $\alpha > 1$ *there exists* $C(\alpha) > 0$ *such that* $R_2(\alpha, n) \geq C(\alpha) \log n$. *Furthermore, there exists* $\alpha_0 > 1$ *such that* $R_2(\alpha_0, n) = O(\log n)$.

In [1] the metric Ramsey problem is studied comprehensively. In particular, the following phase transition is established in the case of $X = \ell_2$.

**Theorem 2** [1]. *Let* $n \in \mathbb{N}$. *Then*:

1. *For every* $1 < \alpha < 2$: $c(\alpha) \log n \leq R_2(\alpha, n) \leq 2 \log n + C(\alpha)$, *where* $c(\alpha)$ *and* $C(\alpha)$ *may depend only on* $\alpha$.
2. *For every* $\alpha > 2$: $n^{c'(\alpha)} \leq R_2(\alpha, n) \leq n^{C'(\alpha)}$, *where* $c'(\alpha)$ *and* $C'(\alpha)$ *depend only on* $\alpha$ *and* $0 < c'(\alpha) \leq C'(\alpha) < 1$. *Moreover,* $c'(\alpha)$ *tends to* $1$ *as* $\alpha$ *tends to* $\infty$.

By Dvoretzky's theorem, the lower bound in part 2 of Theorem 2 implies in particular that if $\alpha > 2$, and $X$ is any infinite-dimensional normed space, then $R_X(\alpha, n) \geq n^{c'(\alpha)}$. Therefore, in our search for a possible phase transition for $R_p(\cdot, n)$, $p \neq 2$, it is natural to extend the upper bound in part 1 of Theorem 2 to this range. The main result proved in this note is the following:

**Theorem 3.** *There is an absolute constant* $c > 0$ *such that for every* $0 < \delta < 1$:

1. *For* $1 \leq p < 2$, $R_p(2 - \delta, n) \leq e^{c/\delta^2} \log n$.
2. *For* $2 < p < \infty$, $R_p(2^{2/p} - \delta, n) \leq e^{c/p^2\delta^2} \log n$.

Thus we extend the result of [1] to show that a phase transition occurs in the metric Ramsey problem for $\ell_p$, $p \in [1, 2)$, at $\alpha = 2$. The asymptotic behavior of $R_p(\alpha, n)$ for $p > 2$, and $\alpha \in [2^{2/p}, 2]$, is left as an open problem. In particular, we do not know whether or not this function undergoes a similar phase transition. We find this problem potentially significant: if there is a phase transition at 2 also in the range $2 < p < \infty$, then this result will certainly be of great interest. On the other hand, if it is possible to improve the lower bound in part 2 of Theorem 2 for $p > 2$ and certain distortions strictly less than 2, then this would involve an embedding technique that is different from the method used in [1], which does not distinguish between the various $\ell_p$ spaces.

The proof of the upper bound on $R_2(\alpha, n)$ for $\alpha < 2$ stated in Theorem 2 uses the Johnson-Lindenstrauss dimension reduction lemma for $\ell_2$ [10]. For $\ell_p$, $p \neq 2$, no such dimension reduction is known to hold. (Recent work [5], [11] shows that dimension reduction does not, in general, hold in $\ell_1$.) Our proof is based on a non-trivial modification of the random construction in [4], in the spirit of Erdös' upper bound on the Ramsey numbers [9], [3]. In the process we prove tight bounds on the embeddability of the metrics of complete bipartite graphs in $\ell_p$. Specifically we show that

$$c_p(K_{n,n}) = \begin{cases} 2 - \Theta(n^{-1}), & p \in [1,2], \\ 2^{2/p} - \Theta((pn)^{-1}), & p > 2. \end{cases}$$

The second part of this note addresses the isometric Ramsey problem for $p \in (1, \infty)$. It turns out that this problem is naturally tackled within the class of uniformly convex normed spaces (see Section 3 for the definition).

**Theorem 4** (Isometric Ramsey Problem). *Let $X$ be a uniformly convex normed space with* $\dim(X) \geq 2$. *Then* $R_X(1, n) = 3$ *for* $n \geq 3$.

Since $\ell_p$ is uniformly convex for $p \in (1, \infty)$, the conclusion of Theorem 4 holds in these cases. Note that the theorem does not apply for $\ell_1$ and $\ell_\infty$ which are not uniformly convex. Specifically, it is known that $\ell_\infty$ is universal in that it contains an isometric copy of every finite metric space, whence $R_\infty(n) = n$. It is known [6] that any four-point metric space is isometrically embeddable in $\ell_1$, and therefore $R_1(n) \geq 4$ for $n \geq 4$. The determination of $R_1(n)$ is left as an open problem.

## 2. An Upper Bound for $\alpha < 2$

In this section we prove that for any $\alpha < \min\{2, 2^{2/p}\}$, $R_p(\alpha, n) = O(\log n)$. Our technique both improves and simplifies the technique of [4], which is itself in the spirit of Erdös' original upper bound for the Ramsey coloring numbers. The basic idea is to exploit a universality property of random graphs $G \in G(n, \frac{1}{2})$. Namely, that any fixed graph of constant size appears as an induced subgraph of every induced subgraph of $G$ of size $\Omega(\log n)$. More precisely, we define the following notion of universality.

**Definition 2.** Let $H$ be a graph. A graph $G$ is called $(H, s)$-universal if every set of $s$ vertices in $G$ contains an induced subgraph isomorphic to $H$.

**Proposition 1.** *For every $k$-vertex graph $H$ there exists a constant $C > 0$ and an integer $n_0$ such that for any $n > n_0$ there exists an $(H, C \log n)$-universal graph on $n$ vertices. Furthermore*,

$$C \leq O(k^2 2^{\binom{k}{2}}) \quad \text{and} \quad n_0 \leq O(k^3 2^{\binom{k}{2}}).$$

Such facts are well known in random graph theory, and similar arguments can be found for example in [13]. We sketch the standard details for the sake of completeness.

Recall that a family of sets $\mathcal{F}$ is called *almost disjoint* if $|A \cap B| \leq 1$ for every $A, B \in \mathcal{F}$. In what follows, given a set $S$ and an integer $k$, we denote by $\binom{S}{k}$ the set of all $k$-point subsets of $S$.

**Lemma 2.** *For every integer $k$ and a finite set $S$ of cardinality $s = |S| > 2k^2$, there exists an almost disjoint family $K \subset \binom{S}{k}$, such that $|K| \geq \lfloor s/2k \rfloor^2$.*

*Proof.* Let $p$ be a prime satisfying $s/2k \leq p \leq s/k$, and assume that

$$L = \{(i, j); i, j \in \mathbb{Z}_p,\ i \in \{0, \ldots, k-1\}\} \subseteq S.$$

For each $a, b \in \mathbb{Z}_p$ (the field of residues modulo $p$), define

$$A_{a,b} = \{(i, j);\ j \equiv ai + b \pmod{p},\ i \in \{0, \ldots, k-1\}\},$$

and take $K = \{A_{a,b} | a, b \in \mathbb{Z}_p\}$. The set $K$ is easily checked to satisfy the requirements. □

As usual $G(n, \frac{1}{2})$ denotes the model of random graphs in which each edge on $n$ vertices is chosen independently with probability $\frac{1}{2}$.

**Lemma 3.** *Let $H$ be a $k$-vertex graph and let $s > 2k^2$. The probability that a random graph $G \in G(s, \frac{1}{2})$ does not contain an induced subgraph isomorphic to $H$, is at most $(1 - 2^{-\binom{k}{2}})^{\lfloor s/2k \rfloor^2}$.*

*Proof.* Construct, as in Lemma 2, an almost disjoint family $\mathcal{F}$ of $\lfloor s/2k \rfloor^2$ subsets of $\{1, \ldots, s\}$, the vertex set of $G$. If $F_1 \neq F_2 \in \mathcal{F}$, then the event that the restriction of $G$ to $F_1$ (resp. $F_2$) is isomorphic to $H$ is independent. Hence, the probability that none of the sets $F \in \mathcal{F}$ spans a subgraph isomorphic to $H$ is at most $(1 - 2^{-\binom{k}{2}})^{\lfloor s/2k \rfloor^2}$. □

*Proof of Proposition* 1. Let $G$ be a random graph in $G(n, \frac{1}{2})$. By the previous lemma, the expected number of sets of $s$ vertices which contain no induced isomorphic copy of $H$ is at most $\binom{n}{s}(1 - 2^{-\binom{k}{2}})^{\lfloor s/2k \rfloor^2}$. If this number is $< 1$, then there is an $(H, s)$-universal graph, as claimed. It is an easy matter to check that this holds with the parameters as stated. □

A class $\mathcal{C}$ of finite metric spaces is called a *metric class* if it is closed under isometries. $\mathcal{C}$ is said to be *hereditary* if $M \in \mathcal{C}$ and $N \subset M$ imply $N \in \mathcal{C}$. We call a metric space $(X, d)$ a $\{0, 1, 2\}$ metric space if for all $x, y \in X$, $d(x, y) \in \{0, 1, 2\}$. There is a simple 1:1 correspondence between graphs and $\{0, 1, 2\}$ metrics. Namely, associated with a $\{0, 1, 2\}$ metric space $M = (X, d)$ is the graph $G = (X, E)$ where $\{x, y\} \in E$ iff $d_M(x, y) = 1$.

**Lemma 4.** *Let $\mathcal{C}$ be a hereditary metric class of finite metric spaces, and suppose that there exists some finite $\{0, 1, 2\}$ metric space $M_0$ which is not in $P$. Then there exist metric spaces $M = M_n$ of arbitrarily large size $n$ such that every subspace $S \subset M_n$ with at least $C \log n$ points is not in $\mathcal{C}$. The constant $C$ depends only on the cardinality of $M_0$.*

*Proof.* Let $H_0$ be the graph corresponding to the metric space $M_0$. We apply Proposition 1, to construct arbitrarily large graphs $G_n = (V_n, E_n)$ with $|V_n| = n$, in which every set of $\geq C \log n$ vertices contains an induced subgraph isomorphic to $H_0$. Let $M_n$ be the $n$-point metric space corresponding to $G_n$. It follows that every subspace of $M_n$ of size $\geq C \log n$ contains a metric subspace that is isometric to $M_0$. Since $\mathcal{C}$ is hereditary, $S \notin \mathcal{C}$. $\square$

Note that $\{M; M \text{ is a metric space}, c_p(M) \leq \alpha\}$ is a hereditary metric class. Therefore, in order to show that for $\alpha < 2$, $R_p(\alpha, n) = O(\log n)$, it is enough to find a $\{0, 1, 2\}$ metric space whose $\ell_p$ distortion is greater than $\alpha$. We use the complete bipartite graphs $K_{n,n}$. The $\ell_p$-distortion of $K_{n,n}$, $1 \leq p < \infty$, is estimated in the following proposition.

**Proposition 5.** *For every* $1 \leq p \leq 2$,

$$2\left(\frac{n-1}{n}\right)^{1/p} \leq c_p(K_{n,n}) \leq 2\sqrt{\frac{n-1}{n}}.$$

*For every* $2 \leq p < \infty$,

$$2^{2/p}\left(\frac{n-1}{n}\right)^{1/p} \leq c_p(K_{n,n}) \leq 2^{2/p}\left(1-\frac{1}{2n}\right)^{1/p}.$$

Before proving Proposition 5, we deduce the main result of this section:

**Theorem 5.** *There is an absolute constant* $c > 0$ *such that for every* $0 < \delta < 1$, *if* $1 \leq p \leq 2$, *then*

$$R_p(2-\delta, n) \leq e^{c/\delta^2} \log n,$$

*and if* $2 < p < \infty$, *then*

$$R_p(2^{2/p}-\delta, n) \leq e^{c/p^2\delta^2} \log n.$$

*Proof.* Proposition 1 implies that there is an absolute constant $C$ such that for every $n \geq 2^{Ck^3}$ there exists a $\{0, 1, 2\}$ metric space $M_n$ such that any subset $S \subset M_n$ of cardinality at least $2^{Ck^2} \log n$ contains an isometric copy of $K_{k,k}$.

We start with $1 \leq p \leq 2$. Let $k = \lfloor 2/\delta \rfloor + 1$. By Proposition 5,

$$c_p(K_{k,k}) \geq 2\left(1-\frac{1}{k}\right)^{1/p} > 2\left(1-\frac{\delta}{2}\right) = 2-\delta,$$

so that for $n$ large enough ($\geq e^{C'/\delta^3}$), and hence for all $n$ (by proper choice of constants),

$$R_p(2-\delta, n) \leq e^{C'/\delta^2} \log n.$$

When $p > 2$ take $k = 2\lfloor 4/p\delta \rfloor$. In this case one easily verifies that

$$c_p(K_{k,k}) \geq 2^{2/p}\left(1-\frac{1}{k}\right)^{1/p} \geq 2^{2/p}-\delta,$$

from which the required result follows as above. $\square$

In order to prove Proposition 5, we need some preparation.

**Lemma 6.** *Let $A = (a_{ij})$ be an $n \times n$ matrix and $2 \le p < \infty$. Then*

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\left|\sum_{k=1}^{n} a_{ik} - \sum_{k=1}^{n} a_{jk}\right|^{p} + \left|\sum_{k=1}^{n} a_{ki} - \sum_{k=1}^{n} a_{kj}\right|^{p}\right) \le \frac{(2n)^p}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}|^p.$$

*Proof.* We identify $\ell_p^{n^2}$ with the space of all $n \times n$ matrices $A = (a_{ij})$, equipped with the $\ell_p$ norm:

$$\|A\|_p = \left(\sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}|^p\right)^{1/p}.$$

Define a linear operator $T : \mathbb{R}^{n^2} \to \mathbb{R}^{n^2} \oplus \mathbb{R}^{n^2}$ by

$$T(a_{ij}) = \left(\sum_{k=1}^{n} a_{ik} - \sum_{k=1}^{n} a_{jk}\right)_{ij} \oplus \left(\sum_{k=1}^{n} a_{ki} - \sum_{k=1}^{n} a_{kj}\right)_{ij}.$$

For $q \ge 1$ denote $\|T\|_{q\to q} = \max_{A\neq 0} \|T(A)\|_q/\|A\|_q$. Our goal is to show that $\|T\|_{p\to p} \le 2^{1-1/p}n$. By a result from the complex interpolation theory for linear operators (see [2]), for $2 \le p \le \infty$, $\|T\|_{p\to p} \le \|T\|_{2\to 2}^{2/p} \cdot \|T\|_{\infty\to\infty}^{1-2/p}$. It is therefore enough to prove the required estimate for $p = 2$ and $p = \infty$. The case $p = \infty$ is simple:

$$\|T(A)\|_\infty = \max_{1\le i,j\le n} \max\left\{\left|\sum_{k=1}^{n} a_{ik} - \sum_{k=1}^{n} a_{jk}\right|, \left|\sum_{k=1}^{n} a_{ki} - \sum_{k=1}^{n} a_{kj}\right|\right\} \le 2n\|A\|_\infty.$$

For $p = 2$ we have to show that

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\left|\sum_{k=1}^{n} a_{ik} - \sum_{k=1}^{n} a_{jk}\right|^{2} + \left|\sum_{k=1}^{n} a_{ki} - \sum_{k=1}^{n} a_{kj}\right|^{2}\right) \le 2n^2\sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}|^2.$$

This inequality follows from the following elementary identity:

$$\begin{aligned}2n^2\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^2 \;=\; &\sum_{i=1}^{n}\sum_{j=1}^{n}\left[\left(\sum_{k=1}^{n} a_{ik} - \sum_{k=1}^{n} a_{jk}\right)^2 + \left(\sum_{k=1}^{n} a_{ki} - \sum_{k=1}^{n} a_{kj}\right)^2\right]\\ &+ 2\sum_{i=1}^{n}\sum_{j=1}^{n}\left(na_{ij} - \sum_{k=1}^{n} a_{ik} - \sum_{k=1}^{n} a_{kj}\right)^2.\end{aligned}$$

□

**Corollary 7.** *Let $1 \le p < \infty$ and $x_1, \dots, x_n, y_1, \dots, y_n \in \ell_p$. Then if $2 \le p < \infty$,*

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\|x_i - x_j\|_p^p + \|y_i - y_j\|_p^p\right) \le 2^{p-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\|x_i - y_j\|_p^p.$$

*If* $1 \le p \le 2$, *then*

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\|x_i - x_j\|_p^p + \|y_i - y_j\|_p^p\right) \le 2\sum_{i=1}^{n}\sum_{j=1}^{n}\|x_i - y_j\|_p^p.$$

*Proof.* By summation it is clearly enough to prove these inequalities for $x_1, \dots, x_n$, $y_1, \dots, y_n \in \mathbb{R}$. If $2 \le p < \infty$, then the required result follows from an application of Lemma 6 to the matrix $a_{ij} = x_i - y_j$. If $1 \le p \le 2$, then consider $\ell_p$ equipped with the metric $d(x, y) = \|x - y\|_p^{p/2}$. It is well known (see [14]) that $(\ell_p, d)$ embeds isometrically in $\ell_2$, so that the case $1 \le p \le 2$ follows from the case $p = 2$. □

**Remark.** In [8] Enflo defined the notion on generalized roundness of a metric space. A metric space $(M, d)$ is said to have generalized roundness $q \ge 0$ if for every $x_1, \dots, x_n, y_1, \dots, y_n \in M$,

$$\sum_{i=1}^{n}\sum_{j=1}^{n}(d(x_i, x_j)^q + d(y_i, y_j)^q) \le 2\sum_{i=1}^{n}\sum_{j=1}^{n} d(x_i, y_j)^q.$$

Enflo proved that Hilbert space has generalized roundness 2 and in [12] the concept of generalized roundness was investigated and was shown to be equivalent to the notion of negative type (see [6] and [14] for the definition). Particularly, it was proved in [12] that for $1 \le p < 2$, $\ell_p$ has generalized roundness $p$, which is precisely the second statement in Corollary 7. For the case $p = 1$, simpler more direct proofs can be given which do not use reduction to the case $p = 2$, see, e.g., [6]. Observe that Lemma 6 would follow simply by convexity had it not been for the additional factor $\frac{1}{2}$ on the right-hand side. This factor is crucial for our purposes, and this is why the interpolation argument was needed.

*Proof of Proposition* 5. We identify $K_{n,n}$ with the metric on $\{u_1, \dots, u_n, v_1, \dots, v_n\}$ where $d(u_i, u_j) = d(v_i, v_j) = 2$ for all $i \ne j$, and $d(u_i, v_j) = 1$ for every $1 \le i, j \le n$. Fix some $1 \le p < \infty$ and let $f : \{u_1, \dots, u_n, v_1, \dots, v_n\} \to \ell_p$ be an embedding such that for every $x, y \in K_{n,n}$, $d(x, y) \le \|f(x) - f(y)\|_p \le Ld(x, y)$. Then

$$\sum_{i=1}^{n}\sum_{j=1}^{n}(\|f(u_i) - f(u_j)\|_p^p + \|f(v_i) - f(v_j)\|_p^p) \ge 2n(n-1)2^p$$

and

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\|f(u_i) - f(v_j)\|_p^p \le n^2 L^p.$$

For $1 \le p \le 2$ Corollary 7 gives

$$2n(n-1)^p 2^p \le 2n^2 L^p \quad\Longrightarrow\quad L \ge 2\left(\frac{n-1}{n}\right)^{1/p}.$$

For $2 \leq p < \infty$ we get that

$$2n(n-1)2^p \leq 2^{p-1}n^2L^p \quad \Longrightarrow \quad L \geq 2^{2/p}\left(\frac{n-1}{n}\right)^{1/p}.$$

This proves the required lower bounds on $c_p(K_{n,n})$.

To prove the upper bound assume first that $p = 2$ and denote by $\{e_i\}_{i=1}^{\infty}$ the standard unit vectors in $\ell_2$. Define $f : K_{n,n} \to \ell_2^{2n}$ by

$$f(u_i) = \sqrt{2}\left(e_i - \frac{1}{n}\sum_{j=1}^{n} e_j\right),$$
$$f(v_i) = \sqrt{2}\left(e_{n+i} - \frac{1}{n}\sum_{j=1}^{n} e_{n+j}\right).$$

Then for $i \neq j$, $\|f(u_i) - f(u_j)\|_2 = \|f(v_i) - f(v_j)\|_2 = 2 = d(u_i, u_j) = d(v_i, v_j)$. On the other hand,

$$\begin{aligned}\|f(u_i) - f(v_j)\|_2 &= \sqrt{\|f(u_i)\|_2^2 + \|f(v_j)\|_2^2}\\ &= \sqrt{4\left(1-\frac{1}{n}\right)^2 + 4(n-1)\cdot\frac{1}{n^2}} = 2\sqrt{\frac{n-1}{n}}.\end{aligned}$$

This finishes the calculation of $c_2(K_{n,n})$. For $1 \leq p < 2$, since for every $\varepsilon > 0$ and for every $k$, $\ell_p$ contains a $(1+\varepsilon)$ distorted copy of $\ell_2^k$, we get the estimate $c_p(K_{n,n}) \leq 2\sqrt{(n-1)/n}$.

The case $2 < p < \infty$ requires a different embedding. We begin by describing an embedding with distortion $2^{2/p}$ and then explain how to modify it so as to reduce the distortion by a factor of $(1 - 1/2n)^{1/p}$. Let $z_1, \dots, z_n$ be a collection of $n$ mutually orthogonal $\pm 1$ vectors of dimension $m = O(n)$. (For example, the first $n$ rows in an $m \times m$ Hadamard matrix.) In our first embedding we define $f(u_i)$ as the $(2m)$-dimensional vector $(z_i, 0)$, namely, $z_i$ concatenated with $m$ zeros. Likewise, $f(v_i) = (0, z_i)$ for all $i$. Now $\|f(u_i) - f(u_j)\|_p = 2(m/2)^{1/p}$ and $\|f(u_i) - f(v_j)\|_p = (2m)^{1/p}$, and so $f$ has distortion $2^{2/p}$. To get the $(1 - 1/2n)^{1/p}$ improvement, note that for some $m \leq 4n$ it is possible to select the $z_i$ so that the $m$th coordinate in all of them is $+1$. Modify the previous construction to an embedding into $2m - 1$ dimensions as follows: now $g(u_i)$ is $z_i$ concatenated with $m - 1$ zeros, whereas $g(v_i)$ has zeros in the first $m - 1$ coordinates, 1 in the $m$th and this is followed by the first $m - 1$ coordinates of the vector $z_i$. The easy details are omitted. □

**Remark.** The upper bounds in Proposition 5 were not used in the proof of Theorem 5. Apart from their intrinsic interest, these upper estimates show that the above technique cannot prove an upper bound of $O(\log n)$ on $R_2(2 - \varepsilon, n)$ which is independent of $\varepsilon$. In fact, this can never be achieved using $\{0, 1, 2\}$ metric spaces due to the following proposition.

**Proposition 8.** *Let X be an n-point* $\{0, 1, 2\}$ *metric space. Then* $c_2(X) \leq 2\sqrt{(n-1)/n}$.

*Proof.* We think of $X$ as a metric on $\{1, \ldots, n\}$ and denote $d(i, j) = d_{ij}$. Define an $n \times n$ matrix $A = (a_{ij})$ as follows:

$$a_{ij} = \begin{cases} 2 & \text{if} \quad i = j, \\ 0 & \text{if} \quad d_{ij} = 2, \\ \dfrac{2}{n} & \text{if} \quad d_{ij} = 1. \end{cases}$$

We claim that $A$ is positive semidefinite. Indeed, for any $z \in \mathbb{R}^n$,

$$\begin{aligned} \langle Az, z \rangle &= \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} z_i z_j \\ &\geq \sum_{i=1}^{n} 2z_i^2 - \sum_{i \neq j} \frac{2}{n} |z_i| \cdot |z_j| \\ &\geq \sum_{i=1}^{n} 2z_i^2 - \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{2}{n} |z_i| \cdot |z_j| \\ &= 2\|z\|_2^2 - \frac{2}{n} \|z\|_1^2 \geq 2\|z\|_2^2 - \frac{2}{n} n \|z\|_2^2 = 0. \end{aligned}$$

In particular, it follows that $A$ has a square root, denoted $A^{1/2}$. Let $e_1, \ldots, e_n$ be the standard unit vectors in $\mathbb{R}^n$. Define $f : X \to \mathbb{R}^n$ by $f(i) = A^{1/2} e_i$. Now,

$$\|f(i) - f(j)\|_2^2 = \langle Ae_i, e_i \rangle + \langle Ae_j, e_j \rangle - 2\langle Ae_i, e_j \rangle = a_{ii} + a_{jj} - 2a_{ij},$$

so that if $d_{ij} = 1$, then $\|f(i) - f(j)\|_2 = \sqrt{4 - 4/n}$ and if $d_{ij} = 2$, then $\|f(i) - f(j)\|_2 = 2$. It follows that

$$\operatorname{dist}(f) = 2\sqrt{\frac{n-1}{n}}. \qquad \square$$

## 3. The Isometric Ramsey Problem

In this section we prove that for $n \geq 3$, $1 < p < \infty$, $R_p(n) = R_p(1, n) = 3$. In fact, we show that this is true for any uniformly convex normed space. We begin by sketching an argument that is specific to $\ell_2$:

**Proposition 9.** *$R_2(n) = 3$ for $n \geq 3$.*

*Proof.* That $R_2(n) \geq 3$ follows since any metric space on three points embeds isometrically in $\ell_2^2$. To show that $R_2(n) \leq 3$, we construct a metric space on $n > 3$ points, no four-point subspace of which embeds isometrically in $\ell_2$. Fix an integer $n > 3$ and let $\{a_i\}_{i=0}^n$ be an increasing sequence such that $a_0 = 0, a_1 = 1$, and for $1 \leq i < n, a_{i+1} \geq 2(n+1)a_i$. Fix some $0 < \varepsilon < 1/(2a_n)$. It is easily verified that $d(i, j) = |i - j| - \varepsilon a_{|i-j|}$ is a metric on $\{1, 2, \ldots, n\}$. We show that for $\varepsilon$ small enough no four points in $(\{1, \ldots, n\}, d)$ embed isometrically in $\ell_2$. Fix four integers $1 \leq i_1 < i_2 < i_3 < i_4 \leq n$ and set $j = i_2 - i_1$,

$k = i_3 - i_2$, $l = i_4 - i_3$. Suppose that for every $\varepsilon > 0$ there exists an isometric embedding $f : (\{i_1, i_2, i_3, i_4\}, d) \to \ell_2^3$. Without loss of generality we may assume that $f(i_1) = (\alpha, \beta, \gamma)$, $f(i_2) = (0, 0, 0)$, $f(i_3) = (k - \varepsilon a_k, 0, 0)$, and $f(i_4) = (p, q, 0)$. Then

$$\begin{aligned} 2\alpha(k - \varepsilon a_k) &= 2\langle f(i_1), f(i_3)\rangle \\ &= \|f(i_1) - f(i_2)\|_2^2 + \|f(i_3) - f(i_2)\|_2^2 - \|f(i_3) - f(i_1)\|_2^2 \\ &= (j - \varepsilon a_j)^2 + (k - \varepsilon a_k)^2 - (j + k - \varepsilon a_{j+k})^2. \end{aligned}$$

Hence,

$$\alpha \le -j + \frac{\varepsilon}{k}[(k + j)a_{k+j} - ja_j - ka_k - ja_k] + O(\varepsilon^2).$$

Similarly,

$$p \ge (k + l) + \frac{\varepsilon}{k}[(k + l)a_k - (k + l)a_{k+l} - ka_k + la_l] + O(\varepsilon^2).$$

Now

$$\begin{aligned} &j + k + l - \varepsilon a_{j+k+l} \\ &\qquad = \|f(i_4) - f(i_1)\|_2 \\ &\qquad \ge p - \alpha \\ &\qquad \ge j + k + l \\ &\qquad\quad + \frac{\varepsilon}{k}[(k + l)a_k - (k + l)a_{k+l} + la_l - (k + j)a_{k+j} + ja_j + ja_k] \\ &\qquad\quad + O(\varepsilon^2). \end{aligned}$$

Letting $\varepsilon$ tend to zero we deduce that

$$\begin{aligned} a_{j+k+l} &\le \left(1 + \frac{j}{k}\right)a_{k+j} + \left(1 + \frac{l}{k}\right)a_{k+l} - \frac{l}{k}a_l - \frac{j}{k}a_j - \frac{j + k + l}{k}a_k \\ &< 2(n + 1)a_{j+k+l-1}, \end{aligned}$$

which is a contradiction. □

The argument above is quite specific to $\ell_2$, and so we now consider any uniformly convex normed space. The modulus of uniform convexity of a normed space $X$ is defined by

$$\delta_X(\varepsilon) = \inf\left\{1 - \frac{\|a + b\|}{2};\ \|a\|, \|b\| \le 1 \text{ and } \|a - b\| \ge \varepsilon\right\}.$$

$X$ is said to be uniformly convex if $\delta_X(\varepsilon) > 0$ for every $0 < \varepsilon \le 2$. The $L_p$ spaces $1 < p < \infty$ are known to be uniformly convex. For a uniformly convex space $X$, $\delta_X$ is known to be continuous and strictly increasing on $(0, 2]$.

Assume that $X$ is a uniformly convex normed space and $a, b \in X\setminus\{0\}$. Then

$$\begin{aligned}\left\|\frac{a}{\|a\|}+\frac{b}{\|b\|}\right\| &= \left\|\left(\frac{1}{\|a\|}+\frac{1}{\|b\|}\right)(a+b)-\frac{a}{\|b\|}-\frac{b}{\|a\|}\right\| \\ &\geq \left(\frac{1}{\|a\|}+\frac{1}{\|b\|}\right)\|a+b\|-\frac{\|a\|}{\|b\|}-\frac{\|b\|}{\|a\|} \\ &= 2-\left(\frac{1}{\|a\|}+\frac{1}{\|b\|}\right)(\|a\|+\|b\|-\|a+b\|).\end{aligned}$$

Now,

$$\begin{aligned}\delta_X\left(\left\|\frac{a}{\|a\|}-\frac{b}{\|b\|}\right\|\right) &\leq 1-\frac{1}{2}\cdot\left\|\frac{a}{\|a\|}+\frac{b}{\|b\|}\right\| \\ &\leq \frac{1}{2}\cdot\left(\frac{1}{\|a\|}+\frac{1}{\|b\|}\right)(\|a\|+\|b\|-\|a+b\|).\end{aligned}$$

Hence

$$\left\|\frac{a}{\|a\|}-\frac{b}{\|b\|}\right\| \leq \delta_X^{-1}\left(\frac{1}{2}\cdot\left(\frac{1}{\|a\|}+\frac{1}{\|b\|}\right)(\|a\|+\|b\|-\|a+b\|)\right).$$

Take $x, y, z \in X$ and apply this inequality for $a = x-y$, $b = y-z$. It follows that

$$\begin{aligned}&\left\|y-\left(\frac{\|y-z\|}{\|x-y\|+\|y-z\|}\cdot x+\frac{\|x-y\|}{\|x-y\|+\|y-z\|}\cdot z\right)\right\| \\ &\qquad \leq \frac{\|x-y\|\cdot\|y-z\|}{\|x-y\|+\|y-z\|}\cdot\delta_X^{-1}\left(\frac{\|x-y\|+\|y-z\|-\|x-z\|}{\min\{\|x-y\|,\|y-z\|\}}\right). \qquad (1)\end{aligned}$$

This inequality is the way uniform convexity is going to be applied in the sequel. Indeed, we have the following "metric" consequence of it:

**Lemma 10.** *Let $X$ be a uniformly convex normed space and let $x_1, x_2, x_3, x_4 \in X$ be distinct. Then*

$$\begin{aligned}&\frac{\|x_1-x_2\|+\|x_2-x_3\|-\|x_1-x_3\|}{2\|x_2-x_3\|} \\ &\qquad \leq \delta_X^{-1}\left(\frac{\|x_1-x_3\|+\|x_3-x_4\|-\|x_1-x_4\|}{\min\{\|x_1-x_3\|,\|x_3-x_4\|\}}\right) \\ &\qquad\quad +\delta_X^{-1}\left(\frac{\|x_2-x_3\|+\|x_3-x_4\|-\|x_2-x_4\|}{\min\{\|x_2-x_3\|,\|x_3-x_4\|\}}\right).\end{aligned}$$

Lemma 10 is a quantitative version of the fact that in a uniformly convex space, if $\|x_1-x_4\|$ is approximately $\|x_1-x_3\|+\|x_3-x_4\|$ and $\|x_2-x_4\|$ is approximately $\|x_2-x_3\|+\|x_3-x_4\|$, then $\|x_1-x_3\|$ is approximately $\|x_1-x_2\|+\|x_2-x_3\|$. This fact is geometrically evident since the first assumption implies that $x_3$ is almost on the line segment connecting $x_1$ and $x_4$ and $x_2$ is almost on the line segment connecting $x_1$ and $x_3$. It follows that $x_2$ is almost on the line segment connecting $x_1$ and $x_3$, as required. Since we are dealing with bi-Lipschitz embeddings, we must formulate this phenomenon without referring to "line segments."

*Proof of Lemma* 10. Define

$$\lambda = \frac{\|x_3 - x_4\|}{\|x_1 - x_3\| + \|x_3 - x_4\|} \quad \text{and} \quad \mu = \frac{\|x_3 - x_4\|}{\|x_2 - x_3\| + \|x_3 - x_4\|}.$$

An application of (1) twice gives

$$\begin{aligned} \|x_3 - (\lambda x_1 + (1-\lambda)x_4)\| \;\le\; & \frac{\|x_1 - x_3\| \cdot \|x_3 - x_4\|}{\|x_1 - x_3\| + \|x_3 - x_4\|} \\ & \cdot \delta_X^{-1}\left(\frac{\|x_1 - x_3\| + \|x_3 - x_4\| - \|x_1 - x_4\|}{\min\{\|x_1 - x_3\|, \|x_3 - x_4\|\}}\right) \end{aligned}$$

and

$$\begin{aligned} \|x_3 - (\mu x_2 + (1-\mu)x_4)\| \;\le\; & \frac{\|x - 2 - x_3\| \cdot \|x_3 - x_4\|}{\|x_2 - x_3\| + \|x_3 - x_4\|} \\ & \cdot \delta_X^{-1}\left(\frac{\|x_2 - x_3\| + \|x_3 - x_4\| - \|x_2 - x_4\|}{\min\{\|x_2 - x_3\|, \|x_3 - x_4\|\}}\right). \end{aligned}$$

By symmetry, we may assume without loss of generality that $\lambda \le \mu$. Now,

$$\begin{aligned} & \left\| x_2 - \frac{\lambda(1-\mu)}{\mu(1-\lambda)}x_1 - \frac{\mu-\lambda}{\mu(1-\lambda)}x_3 \right\| \\ & \quad = \frac{1}{\mu}\left\| \mu x_2 + (1-\mu)x_4 - x_3 + \frac{1-\mu}{1-\lambda}(x_3 - \lambda x_1 - (1-\lambda)x_4) \right\| \\ & \quad \le \frac{1}{\mu}\|x_3 - \mu x_2 - (1-\mu)x_4\| + \frac{1-\mu}{\mu(1-\lambda)} \cdot \|x_3 - \lambda x_1 - (1-\lambda)x_4\| \\ & \quad \le \frac{\|x_2 - x_3\| + \|x_3 - x_4\|}{\|x_3 - x_4\|} \cdot \frac{\|x_2 - x_3\| \cdot \|x_3 - x_4\|}{\|x_2 - x_3\| + \|x_3 - x_4\|} \\ & \qquad \cdot \delta_X^{-1}\left(\frac{\|x_2 - x_3\| + \|x_3 - x_4\| - \|x_2 - x_4\|}{\min\{\|x_2 - x_3\|, \|x_3 - x_4\|\}}\right) \\ & \qquad + \frac{\|x_2 - x_3\|}{\|x_3 - x_4\|} \frac{\|x_1 - x_3\| + \|x_3 - x_4\|}{\|x_1 - x_3\|} \frac{\|x_1 - \|x_3\| \cdot \|x_3 - x_4\|}{\|x_1 - x_3\| + \|x_3 - x_4\|} \\ & \qquad \cdot \delta_X^{-1}\left(\frac{\|x_1 - x_3\| + \|x_3 - x_4\| - \|x_1 - x_4\|}{\min\{\|x_1 - x_3\|, \|x_3 - x_4\|\}}\right) \\ & \quad = \|x_2 - x_3\|\delta_X^{-1}\left(\frac{\|x_1 - x_3\| + \|x_3 - x_4\| - \|x_1 - x_4\|}{\min\{\|x_1 - x_3\|, \|x_3 - x_4\|\}}\right) \\ & \qquad + \|x_2 - x_3\|\delta_X^{-1}\left(\frac{\|x_2 - x_3\| + \|x_3 - x_4\| - \|x_2 - x_4\|}{\min\{\|x_2 - x_3\|, \|x_3 - x_4\|\}}\right). \end{aligned}$$

Additionally,

$$\begin{aligned} \|x_2 - x_1\| \;\le\; & \left\| x_2 - \frac{\lambda(1-\mu)}{\mu(1-\lambda)}x_1 - \frac{\mu-\lambda}{\mu(1-\lambda)}x_3 \right\| \\ & + \left\| x_1 - \frac{\lambda(1-\mu)}{\mu(1-\lambda)}x_1 - \frac{\mu-\lambda}{\mu(1-\lambda)}x_3 \right\| \\ = \; & \left\| x_2 - \frac{\lambda(1-\mu)}{\mu(1-\lambda)}x_1 - \frac{\mu-\lambda}{\mu(1-\lambda)}x_3 \right\| + \frac{\mu-\lambda}{\mu(1-\lambda)}\|x_1 - x_3\| \end{aligned}$$

and

$$\begin{aligned}\|x_2 - x_3\| &\le \left\| x_2 - \frac{\lambda(1-\mu)}{\mu(1-\lambda)}x_1 - \frac{\mu-\lambda}{\mu(1-\lambda)}x_3 \right\| \\ &\quad + \left\| x_3 - \frac{\lambda(1-\mu)}{\mu(1-\lambda)}x_1 - \frac{\mu-\lambda}{\mu(1-\lambda)}x_3 \right\| \\ &= \left\| x_2 - \frac{\lambda(1-\mu)}{\mu(1-\lambda)}x_1 - \frac{\mu-\lambda}{\mu(1-\lambda)}x_3 \right\| + \frac{\lambda(1-\mu)}{\mu(1-\lambda)}\|x_1 - x_3\|.\end{aligned}$$

Summing up these estimates gives the required result. □

We can now prove the main result of this section:

**Theorem 6.** *Let $X$ be a uniformly convex normed space with* $\dim(X) \ge 2$. *Then for every $n \ge 3$, $R_X(n) = 3$. Moreover, for every $\delta : (0,2] \to (0,\infty)$ which is continuous, increasing, and $\delta \le \delta_{\ell_2}$, let $UC_\delta$ be the class of all normed spaces $X$ with $\delta_X \ge \delta$. Then for each $n \ge 3$ there is a constant $\varepsilon_n(\delta) > 0$ such that $R_{UC_\delta}(1+\varepsilon_n(\delta), n) = 3$.*

The proof of Theorem 6 proceeds by constructing a space in which each quadruple violates the conclusion of Lemma 10. The construction is done iteratively, by adding one point at a time.

*Proof of Theorem* 6. That $R_X(n) \ge 3$ follows since any three-point metric embeds isometrically into any two-dimensional normed space, by a standard continuity argument.

Fix some $\delta : (0,2] \to (0,\infty)$ which is continuous, increasing, and $\delta \le \delta_{\ell_2}$. We shall construct inductively a sequence $\{M_n\}_{n=3}^\infty$ of metric spaces and numbers $\{\eta_n\}_{n=3}^\infty$ such that:

(a) For every $n \ge 3$, $\eta_n > 0$. Each $M_n$ is a metric on $\{1,\ldots,n\}$, and we denote $d_{ij}^n = d_{M_n}(i,j)$.
(b) For every $1 \le i < j < k \le n$,

$$\begin{aligned}&d_{i,j}^n + d_{jk}^n - d_{i,k}^n - \eta_n \\ &\qquad \ge 2d_{j,k}^n\left[\delta^{-1}\left(\frac{d_{i,k}^n + d_{k,n}^n - d_{i,n}^n}{\min\{d_{i,k}^n, d_{k,n}^n\}}\right) + \delta^{-1}\left(\frac{d_{j,k}^n + d_{k,n}^n - d_{j,n}^n}{\min\{d_{j,k}^n, d_{k,n}^n\}}\right)\right].\end{aligned}$$

Lemma 10 immediately implies that there is a constant $\varepsilon_n(\delta) > 0$ such that for every $1 \le i < j < k < l \le n$ and for every normed space $X$ with $\delta_X \ge \delta$,

$$c_X(\{i,j,k,l\}, d_{M_n}) \ge 1 + \varepsilon_n(\delta),$$

as required.

$M_3$ is the equilateral metric on $\{1,2,3\}$, in which case $\eta_3 = 1$. We construct $M_{n+1} = (\{1,\ldots,n+1\}, d^{n+1})$ as an extension of $M_n$, by setting

$$d_{n,n+1}^{n+1} = 1 - s/2 \quad \text{and} \quad \forall 1 \le i < n, \qquad d_{i,n+1}^{n+1} = d_{in}^n + 1 - s.$$

This is indeed a definition of a metric as long as $0 < s \le \min\{1, 2\min_{1\le i<n} d^n_{i,n}\}$ (this fact follows from a simple case analysis).

We are left to check condition (b). Fix $1 \le i < j < k \le n$. If $k \ne n$, then

$$\begin{aligned}
&d^{n+1}_{i,j} + d^{n+1}_{j,k} - d^{n+1}_{i,k} - \eta_n \\
&\quad = d^n_{i,j} + d^n_{j,k} - d^n_{i,k} - \eta_n \\
&\quad \ge 2d^n_{j,k}\left[\delta^{-1}\left(\frac{d^n_{i,k}+d^n_{k,n}-d^n_{i,n}}{\min\{d^n_{i,k}, d^n_{k,n}\}}\right) + \delta^{-1}\left(\frac{d^n_{j,k}+d^n_{k,n}-d^n_{j,n}}{\min\{d^n_{j,k}, d^n_{k,n}\}}\right)\right] \\
&\quad \ge 2d^n_{j,k}\left[\delta^{-1}\left(\frac{d^n_{i,k}+(d^n_{k,n}+1-s)-(d^n_{i,n}+1-s)}{\min\{d^n_{i,k}, d^n_{k,n}+1-s\}}\right)\right. \\
&\qquad \left. + \delta^{-1}\left(\frac{d^n_{j,k}+(d^n_{k,n}+1-s)-(d^n_{j,n}+1-s)}{\min\{d^n_{j,k}, d^n_{k,n}+1-s\}}\right)\right] \\
&\quad = 2d^{n+1}_{j,k}\left[\delta^{-1}\left(\frac{d^{n+1}_{i,k}+d^{n+1}_{k,n+1}-d^{n+1}_{i,n+1}}{\min\{d^{n+1}_{i,k}, d^{n+1}_{k,n+1}\}}\right)\right. \\
&\qquad \left. + \delta^{-1}\left(\frac{d^{n+1}_{j,k}+d^{n+1}_{k,n+1}-d^{n+1}_{j,n+1}}{\min\{d^{n+1}_{j,k}, d^{n+1}_{k,n+1}\}}\right)\right].
\end{aligned}$$

It remains to check (b) for the quadruple $\{i, j, n, n+1\}$. Condition (b) for $M_n$ implies that

$$d^{n+1}_{ij} + d^{n+1}_{jn} - d^{n+1}_{in} \ge \eta_n.$$

On the other hand,

$$\begin{aligned}
&2d^{n+1}_{j,n}\left[\delta^{-1}\left(\frac{d^{n+1}_{i,n}+d^{n+1}_{n,n+1}-d^{n+1}_{i,n+1}}{\min\{d^{n+1}_{i,n}, d^{n+1}_{n,n+1}\}}\right) + \delta^{-1}\left(\frac{d^{n+1}_{j,n}+d^{n+1}_{n,n+1}-d^{n+1}_{j,n+1}}{\min\{d^{n+1}_{j,n}, d^{n+1}_{n,n+1}\}}\right)\right] \\
&\quad = 2d^n_{j,n}\left[\delta^{-1}\left(\frac{s/2}{\min\{d^n_{i,n}, 1-s/2\}}\right) + \delta^{-1}\left(\frac{s/2}{\min\{d^n_{j,n}, 1-s/2\}}\right)\right],
\end{aligned}$$

so that condition (b) will hold when $s$ is small enough such that the quantity above is at most $\eta_n/2$ and with $\eta_{n+1} = \eta_n/2$. □

**Corollary 11.** *For all* $1 < p < \infty$, $R_p(n) = 3$ *for* $n \ge 3$.

We end this section with a simple lower bound for the isometric Ramsey problem for graphs. We do not know the asymptotically tight bound in this setting.

**Proposition 12.** *Let $G$ be an unweighted graph of order $n$. Then there is a set of* $\Omega\left(\sqrt{\log n/\log\log n}\right)$ *vertices in $G$ whose metric embeds isometrically into* $\ell_2$.

*Proof.* Let $\Delta$ be the diameter of $G$. The shortest path between two diameterically far vertices is isometrically embeddable in $\ell_2$. On the other hand, the Bourgain et al. theorem

[4] yields, for every $0 < \varepsilon < 1$, a subset $N \subset V$ which is $(1+\varepsilon)$ embeddable in Hilbert space and $|N| = \Omega((\varepsilon/\log(2/\varepsilon))\log n)$. When $\varepsilon = 1/2\Delta$, such an embedding is an isometry. Hence we can always extract a subset of $V$ which is isometrically embeddable in $\ell_2$ with cardinality

$$\Omega\left(\max\left\{\Delta, \frac{\log n}{\Delta\log\Delta}\right\}\right) = \Omega\left(\sqrt{\frac{\log n}{\log\log n}}\right),$$

as claimed. □


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## References


1. Y. Bartal, N. Linial, M. Mendel, and A. Naor. On metric Ramsey-type phenomena. *Ann. of Math.*, to appear.
2. J. Bergh and J. Löfström. *Interpolation Spaces. An Introduction*. Grundlehren der Mathematischen Wissenschaften, No. 223. Springer-Verlag, Berlin, 1976.
3. B. Bollobás. *Random Graphs*, second edition. Cambridge University Press, Cambridge, 2001.
4. J. Bourgain, T. Figiel, and V. Milman. On Hilbertian subsets of finite metric spaces. *Israel J. Math.*, 55(2):147–152, 1986.
5. W. Brinkman and M. Charikar. On the impossibility of dimension reduction in $\ell_1$. In *Proceedings of the* 44*th Annual Symposium on Foundations of Computer Science*, pages 514–523, 2003.
6. M. M. Deza and M. Laurent. *Geometry of Cuts and Metrics*. Springer-Verlag, Berlin, 1997.
7. A. Dvoretzky. Some results on convex bodies and Banach spaces. In *Proceedings of the International Symposium on Linear Spaces* (*Jerusalem*, 1960), pages 123–160. Jerusalem Academic Press, Jerusalem, 1961.
8. P. Enflo. On a problem of Smirnov. *Ark. Mat.*, 8:107–109, 1969.
9. P. Erdös. Some remarks on the theory of graphs. *Bull. Amer. Math. Soc.*, 53:292–294, 1947.
10. W. B. Johnson and J. Lindenstrauss. Extensions of Lipschitz mappings into a Hilbert space. In *Conference in Modern Analysis and Probability* (*New Haven*, *Conn.*, 1982), pages 189–206. American Mathematical Society, Providence, RI, 1984.
11. J. R. Lee and A. Naor. Embedding the diamond graph in $L_p$ and dimension reduction in $L_1$. *Geom. Functional Anal.*, to appear.
12. C. J. Lennard, A. M. Tonge, and A. Weston. Generalized roundness and negative type. *Michigan Math. J.*, 44(1):37–45, 1997.
13. C. Lund and M. Yannakakis. The approximation of maximum subgraph problems. In *Proceedings of ICALP*, volume 700 of Lecture Notes in Computer Science, pages 40–51. Springer-Verlag, Berlin, 1993.
14. J. H. Wells and L. R. Williams. *Embeddings and Extensions in Analysis*. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 84. Springer-Verlag, New York, 1975.


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