Classical Time Crystals

Alfred Shapere and Frank Wilczek

Classical Time Crystals

Alfred Shapere[1] and Frank Wilczek[2]

[1] *Department of Physics and Astronomy, University of Kentucky, Lexington, Kentucky 40502 USA*
[2] *Center for Theoretical Physics, Department of Physics,*
Massachusetts Institute of Technology, Cambridge, Massachusetts 02139 USA

We consider the possibility that classical dynamical systems display motion in their lowest energy state, forming a time analogue of crystalline spatial order. Challenges facing that idea are identified and overcome. We display arbitrary orbits of an angular variable as lowest-energy trajectories for nonsingular Lagrangian systems. Dynamics within orbits of broken symmetry provide a natural arena for formation of time crystals. We exhibit models of that kind, including a model with traveling density waves.

In this paper we will investigate a cluster of issues around the question of whether time-independent, conservative classical systems might exhibit motion in their lowest energy states. Fully quantum systems are the subject of a companion paper [1]. Related issues have been raised in a cosmological context [2][3], but those investigations consider quite different aspects, in which the time dependence introduced by the expansion of the universe plays a significant role. (The term "time crystal" has been used previously to describe periodic phenomena in other contexts [4, 5].)

General considerations. When a physical solution of a set of equations displays less symmetry than the equations themselves, we say the symmetry is spontaneously broken by that solution. Here the meaning of "physical solution" can be interpreted differently in different contexts, but one interesting case, that will concern us here, is of the lowest energy solutions of a time-independent, conservative, classical dynamical system. If such a solution exhibits motion, we will have broken time translation symmetry spontaneously. If the dynamical variable is an angular variable, then the motion will be periodic in time, so the time-translation symmetry is not entirely lost, but only reduced to a discrete subgroup. Spatial periodicity is, of course, associated with formation of ordinary crystals, so it is natural and suggestive to refer to the formation of time crystals.

It is very easy to construct simple Lagrangians or Hamiltonians whose lowest energy state is a spatial crystal. With $\phi(x)$ an angular variable, the potential energy functions

$$
\begin{aligned}
V_1(\phi) &= -\kappa_1 \frac{d\phi}{dx} + \frac{\lambda_1}{2}\left(\frac{d\phi}{dx}\right)^2 \\
V_2(\phi) &= -\frac{\kappa_2}{2}\left(\frac{d\phi}{dx}\right)^2 + \frac{\lambda_2}{4}\left(\frac{d\phi}{dx}\right)^4
\end{aligned} \quad (1)
$$

with all the Greek coefficients positive, are minimized for $\frac{d\phi_1}{dx} = \frac{\kappa_1}{\lambda_1}, \frac{d\phi_2}{dx} = \pm\sqrt{\frac{\kappa_2}{\lambda_2}}$ respectively. In both cases the spatial translation symmetry of the original potential is spontaneously broken; in the second case inversion symmetry is broken as well. The combined inversion $\phi(x) \to -\phi(-x)$ is preserved in both cases, as is a combined internal space-real space translation $\phi(x) \to \phi(x+\epsilon) - \frac{d\phi}{dx}\epsilon$.

From this one might surmise that time crystals are likewise easy to construct, at least mathematically. Moreover, higher powers of velocities appear quite naturally in models that portray the effects of finite response times, as we replace

$$
\left((\phi(t) - \phi(t-\delta))\right)^n \tilde{\to} \delta^n \dot{\phi}^n \quad (2)
$$

On second thought, however, reasons for doubt appear. Speaking broadly, what we're looking for seems perilously close to perpetual motion. Also, if the dynamical equations conserve energy, then the existence of a minimum-energy solution where the variables trace out an orbit implies that the energy function assumes its minimum value on a whole curve in $(\phi, \dot{\phi})$ space – not, as we expect generically, at an isolated point.

Dynamical equations. That easy/impossible dichotomy carries over into the dynamical equations. If one simply turns the space derivatives in Eqn. (1) into time derivatives, then the resulting Lagrangians

$$
\begin{aligned}
L_1(\phi, \dot{\phi}) &= -\kappa_1 \dot{\phi} + \frac{\lambda_1}{2}\dot{\phi}^2 \\
L_2(\phi, \dot{\phi}) &= -\frac{\kappa_2}{2}\dot{\phi}^2 + \frac{\lambda_2}{4}\dot{\phi}^4
\end{aligned} \quad (3)
$$

are associated with the energy functions

$$
\begin{aligned}
E_1(\phi, \dot{\phi}) &= \frac{\lambda_1}{2}\dot{\phi}^2 \\
E_2(\phi, \dot{\phi}) &= -\frac{\kappa_2}{2}\dot{\phi}^2 + \frac{3\lambda_2}{4}\dot{\phi}^4 .
\end{aligned} \quad (4)
$$

The first of these is minimized at $\dot{\phi}_1 = 0$, the second at $\dot{\phi}_2 = \pm\sqrt{\frac{\kappa_2}{3\lambda_2}}$. So the analogue of our first symmetry-breaking example in Eqn.(1) has collapsed, but the second survives, with a quantitative change.

On the other hand if we convert the space derivatives in Eqn. (1) into momenta, the resulting Hamiltonians are

$$H_1(p, \phi) = -\kappa_1 p + \frac{\lambda_1}{2}p^2$$

$$H_2(p, \phi) = -\frac{\kappa_2}{2}p^2 + \frac{\lambda_2}{4}p^4. \tag{5}$$

We find precisely the original algebraic structure for the minimum-energy solutions, *viz.* $p_1 = \frac{\kappa_1}{\lambda_1}, p_2 = \pm\sqrt{\frac{\kappa_2}{\lambda_2}}$ respectively. Their physical implications are entirely different, though. Indeed, they correspond to $\dot\phi_1 = \dot\phi_2 = 0$: thus no symmetry breaking occurs, in either case.

This disappointing consequence of the Hamiltonian formalism is quite general. Hamilton's equations of motion $\dot p_j = -\frac{\partial H}{\partial q^j}$, $\dot q^j = \frac{\partial H}{\partial p_j}$ indicate that the energy function $E(p_j(0), q^j(0)) = H(p_j(0), q^j(0))$, regarded as a function of the dynamical variables at a chosen initial time, is minimized for a trajectory with $\dot p_j = \dot q^j = 0$, since the gradients on the right-hand side of Hamilton's equations vanish.

How do we reconcile this very general null result in the Hamiltonian approach, with our positive result in the Lagrangian approach? The point is that the Lagrangian L_2, which gave symmetry breaking, cannot be converted into a Hamiltonian smoothly. Indeed, expressing the algebraic recipe for the Hamiltonian

$$H(p, \phi) = p\dot\phi - L = p\dot\phi + \frac{\kappa}{2}\dot\phi^2 - \frac{1}{4}\dot\phi^4 \tag{6}$$

(in which we have set $\lambda_2 = 1$ for simplicity and dropped all '2' subscripts) as a function of

$$p = \frac{\partial L}{\partial\dot\phi} = \dot\phi^3 - \kappa\dot\phi \tag{7}$$

leads to a multi-valued function [6], with cusps where $\frac{\partial p}{\partial\dot\phi} = 0$, *i.e.* $p = \mp\frac{2\kappa^{3/2}}{3^{3/2}}$, corresponding precisely to the energy minima $\dot\phi = \pm\sqrt{\kappa/3}$. (See Figure 1.) For $\kappa \leq 0$, $H(p)$ is regular, but as κ passes through zero there is a swallowtail catastrophe.

At the cusps the usual condition that the gradient should vanish at a minimum does not apply, and so our null result for smooth Hamiltonian systems is avoided.

For classical physics the Lagrangian formalism is adequate, so let us follow that direction out further. A logical next step would be to add a potential $V(\phi)$ to L. Doing that, however, leads us directly into the problem with energy conservation that we anticipated earlier. Minimizing V, we will find a preferred value for $\phi = \phi_0$, but minimizing the kinetic part will favor motion in ϕ, and there is a conflict.

We can elucidate this issue as it arises for a general Lagrangian system. Suppose that the energy function of a system with many degrees of freedom is minimized by nonzero velocities $\dot\phi_0^k \neq 0$, so that

$$0 = \left.\frac{\partial E}{\partial\dot\phi^k}\right|_{\dot\phi_0^k} = \left.\left(\frac{\partial^2 L}{\partial\dot\phi^k\partial\dot\phi^j}\right)\right|_{\dot\phi_0^k}\dot\phi_0^j. \tag{8}$$
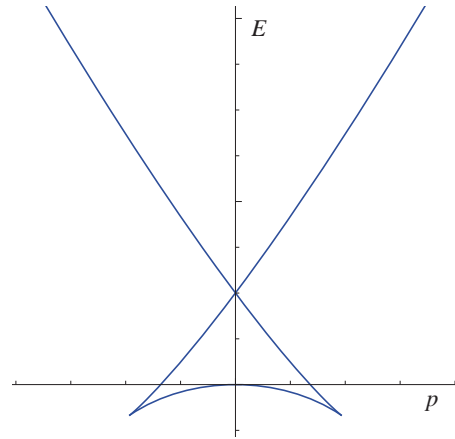

FIG. 1: Energy is a multivalued function of momentum.

Then in the equations of motion

$$0 = \frac{d}{dt}\left(\frac{\partial L}{\partial\dot\phi^k}\right) - \frac{\partial L}{\partial\phi^k} = \left(\frac{\partial^2 L}{\partial\dot\phi^j\partial\dot\phi^k}\right)\ddot\phi^j + \dots \tag{9}$$

the coefficient of the acceleration in the direction $\ddot\phi^j \propto \dot\phi_0^j$ vanishes at $\dot\phi_0^k$. In that case the equations of motion, which generally serve to determine the accelerations, require supplementation. (As we shall discuss below, there are physically interesting models that avoid any singularities of this type.)

Brick Wall Solutions: Upon integrating

$$E = \frac{3}{4}\dot\phi^4 - \frac{\kappa}{2}\dot\phi^2 + V(\phi) \tag{10}$$

directly we obtain

$$t(\phi) = \int^\phi \frac{d\phi}{\pm\sqrt{\frac{\kappa}{3} \pm \sqrt{\left(\frac{\kappa}{3}\right)^2 + \frac{4}{3}(E - V(\phi))}}} \tag{11}$$

where the $\pm$ signs are independent.

The argument of the inner square root is non-negative if and only if $V(\phi) \leq E + \kappa^2/12 \equiv \Delta$, where $\Delta \equiv E - E_0 \geq 0$ is the energy above the minimum kinetic energy $E_0 = -\frac{\kappa^2}{12}$. The inequality is saturated when $\dot\phi = \pm\sqrt{\frac{\kappa}{3}}$, i.e., when the kinetic energy is minimized. Close to a point ϕ_t where this happens,

$$\dot\phi \approx \pm\sqrt{\frac{\kappa}{3}} \pm \sqrt{\frac{1}{\kappa}V'(\phi_t)(\phi_t - \phi)}. \tag{12}$$

Since ϕ cannot continue past ϕ_t without violating the bound $V(\phi) \leq \Delta$, it suddenly reverses direction, $\dot\phi = \pm\sqrt{\frac{\kappa}{3}} \to \mp\sqrt{\frac{\kappa}{3}}$. Such a reversal conserves energy, but requires a sudden jump in momentum. This is analogous to the turning point of a "brick-wall" potential enforced by an infinitely massive source. Unless ϕ_t is an extremum of $V(\phi)$, the acceleration diverges at ϕ_t, as required by the equations of motion (9).

2

Small oscillations about the minimum of a generic potential $V(\phi) \approx \frac{1}{2}\mu(\phi - \phi_0)^2$ exhibit turning points of this type, with bounded orbits that oscillate between $\phi_t = \phi_0 - \sqrt{2\Delta/\mu}$ and $\phi_0 + \sqrt{2\Delta/\mu}$. In the limit of small Δ, the orbits ricochet about the minimum, with nearly constant speed $|\dot\phi| = \sqrt{\frac{\kappa}{3}}$, reconciling the apparently contradictory conditions $\dot\phi = \pm\sqrt{\frac{\kappa}{3}}$ and $\phi = \phi_0$.

Generalization: A natural generalization of the model considered above is obtained by allowing κ to be a function of ϕ. Then the energy function (10) may be written

$$E(\phi, \dot\phi) = \frac{3}{4}(\dot\phi^2 - \frac{1}{3}\kappa(\phi))^2 + \widetilde{V}(\phi) \qquad (13)$$

with $\widetilde{V}(\phi) \equiv V(\phi) - \frac{1}{12}\kappa^2$. The solution is again given by Eqn. (11), and we expect a similar phenomenology of low-energy orbits. Indeed, the energy is minimized with respect to $\dot\phi$, as above, by $\dot\phi = \pm\sqrt{\kappa(\phi)/3}$; expanding $\widetilde{V}(\phi)$ about a minimum we generically find $\widetilde{V}(\phi) \approx V_0 + \frac{1}{2}\mu(\phi - \phi_0)$, where V_0 and μ are constants. Thus, as before, low-energy orbits oscillate about ϕ_0 with nearly constant speed $\sqrt{\kappa(\phi_0)/3}$.

In the special case $\widetilde{V}(\phi) = V_0$, the minimum-energy orbits are well-behaved solutions of the first-order equation $\dot\phi = \pm\sqrt{\kappa(\phi)/3}$. By choosing $\kappa(\phi)$ appropriately, any orbit $\phi(t)$ can be realized, in many ways, as the stable minimum energy solution to a Lagrangian of this type.

For constant $\widetilde{V}$ (which we now set to zero without loss of generality), conservation of energy $E \geq 0$ leads to

$$\dot\phi^2 - \frac{1}{3}\kappa(\phi) = \pm\sqrt{\frac{4}{3}E}. \qquad (14)$$

This equation is of a familiar form; it expresses the conservation of a pseudo-energy $\mathcal{E} = \pm\sqrt{\frac{4}{3}E}$ for a particle with mass $m = 2$ and potential $\mathcal{V}(\phi) = -\kappa(\phi)/3$. This result allows us to infer the qualitative dynamics, based on familiar mechanical concepts. The only turning points are of the usual variety: putting $\dot\phi = 0$ into Eqn. (14) we find turning points where $\mathcal{E} = \mathcal{V}(\phi_t) = -\kappa(\phi_t)/3$. The motion is confined to a region where $\mathcal{V}(\phi) \leq \mathcal{E}$. Thus the model can support motions in which the velocity changes sign smoothly, but these motions require *higher* energy than the minimal orbit, which is unidirectional.

Avoiding Singularities: For general potentials $\widetilde{V}(\phi)$, we have noted that low-energy solutions of Eqn. (13) typically display singular behavior – infinite acceleration – at turning points. However, if $\widetilde{V}(\phi)$ is bounded above, solutions of sufficiently high energy will be smooth provided that $E \geq \widetilde{V}_{\max}$.

Quantum mechanics can ameliorate the singularities. In interesting cases the Hamiltonian is a multivalued function of the momentum. This implies that the momentum does not provide a complete set of commuting observables. Nor, therefore, does the position. Wave functions must be defined over expanded spaces [6, 7].

Naturally Flat Directions; Double Sombrero: It can be natural to have energy constant along an orbit, if the points of the orbit are related by symmetry. If we want this situation to occur along a trajectory for the minimum-energy state, then the symmetry must must be spontaneously broken.

Consider first a Lagrangian with a "sombrero" kinetic term, together with the classic sombrero potential:

$$
\begin{aligned}
L &= \frac{1}{4}(\dot\psi_1^2 + \dot\psi_2^2 - \kappa)^2 - V(\psi_1, \psi_2) \\
V &= -\frac{\mu}{2}(\psi_1^2 + \psi_2^2) + \frac{\lambda}{4}(\psi_1^2 + \psi_2^2)^2 \qquad (15)
\end{aligned}
$$

This defines a "double sombrero" model, exhibiting circular motion at constant speed in the lowest-energy state. We may rewrite this model and its generalizations in terms of polar fields ρ and ϕ, where $\psi_1 + i\psi_2 = \rho e^{i\phi} \equiv \varphi$. Then the double sombrero Lagrangian takes the form

$$L = \frac{1}{4}(\dot\rho^2 + \rho^2\dot\phi^2 - \kappa)^2 + \frac{\mu}{2}\rho^2 - \frac{\lambda}{4}\rho^4. \qquad (16)$$

If ρ is set equal to its value $\sqrt{2\mu/\lambda}$ at the minimum of $V(\rho)$, this reduces to our original Lagrangian (3). Generalizing, any Lagrangian with a kinetic term that is a polynomial in $\dot\phi$, $\dot\rho$, and ρ, and a potential energy depending only on ρ, will preserve the symmetry $\phi \to \phi + \eta$.

Charge and Locking: The charge operator associated with the original (broken) symmetry is $Q = -\int i(\varphi^*\pi_{\varphi^*} - \varphi\pi_\varphi)$ where $\pi_\varphi = \frac{\partial L}{\partial\dot\varphi}$ depends only on $\dot\phi$ and ρ. Thus in states with constant, non-vanishing values of ρ and $\dot\phi$ we have a non-zero, uniform density of Q. This is significant in two ways:

First: If we suppose that our system is embedded in a larger symmetry-conserving bath and undergoes a transition to the symmetry-breaking state, *e.g.* that it is a material body cooled through a phase transition, then the transition will necessarily be accompanied by radiation of an appropriate balancing charge.

Second: Although invariance under both infinitesimal time-translation $\phi(t) \to \phi(t + \epsilon)$ and infinitesimal phase (charge) translation $\phi \to \phi + \eta$ are broken by constant-$\dot\phi$ solutions $\phi(t) = \omega t + \beta$, the combined transformation with $\omega\epsilon + \eta = 0$ leaves the solution invariant. Thus there is a residual "locked" symmetry. To exploit it, we can go to a sort of rotating frame, by using the shifted Hamiltonian $\widetilde{H} = H - \omega Q$ to compute the evolution [3, 8]. (Here we normalize Q so that φ has unit charge.) In the rotating frame, the equations of motion will not contain any explicit time dependence, but there will be a sort of effective chemical potential (associated however with a *broken* symmetry). The most interesting effects will arise at interfaces between the locked phase and the normal phase, or between different locked phases, as exemplified in the preceding paragraph.

Space-Time Structure; More Complex States: We can also contemplate slightly more complex examples, that support qualitatively different, richer physical effects. If there is a potential for $\nabla\varphi$, or ultimately for $\nabla\rho$, that favors gradients, then we can have a competition between the energetic desirability of putting ρ at the energetic minimum and accommodating non-zero gradients. Unlike the case of time derivatives, there is no general barrier to reaching a stable compromise. To keep things

simple, let us suppress the underlying φ structure and consider the potential

$$V(\rho) = \frac{\kappa_1}{2} \left(1 - a\rho^2 - b\left(\frac{d\rho}{dx}\right)^2 \right)^2 \qquad (17)$$

with $a, b > 0$. This potential is minimized by

$$\rho_0(x) = \sqrt{\frac{1}{a}} \sin(\sqrt{\frac{a}{b}}x + \alpha),$$

which reduces the translation symmetry to a discrete subgroup. Constant $\dot{\phi}$ produces a charge density wave.

If we also have a term of the form

$$V_{\text{gradient}} = \frac{\kappa_2}{2} \left(\frac{d\phi}{dx} - \mu\frac{d\rho}{dx} \right)^2 \qquad (18)$$

then at the minimum $\phi_0(x)$ will develop spatial structure as well, according to $\phi_0(x) = \mu\rho_0(x) + \beta$, breaking the phase (charge) symmetry completely. (Note that V_{gradient} respects the symmetry $\phi \to \phi + \eta$.)

We can engineer similar phenomena involving $\dot{\phi}$ most easily if we work at the level of the energy function. One can derive general energy functions involving powers of $\dot{\phi}$ from Lagrangians of the same kind, so long as there are no terms linear in $\dot{\phi}$. Thus if we have additional term

$$E_{\text{kinetic}}(\phi) = \frac{\kappa_3}{2} \left(\left(\frac{d\phi}{dx}\right)^2 - \frac{1}{v^2}\dot{\phi}^2 \right)^2 \qquad (19)$$

then at the minimum we have

$$\phi_0(x, t) = \mu\rho_0(x, t) + \beta \qquad (20)$$
$$\rho_0(x, t) = \sqrt{\frac{1}{a}} \sin\left(\sqrt{\frac{a}{b}}(x \pm vt) + \tilde{\alpha} \right). \qquad (21)$$

Here in Eqn. (21) we have adapted our solution $\rho_0(x)$ for the potential (17) by taking $\alpha = \pm vt + \tilde{\alpha}$. In doing this

we assume that the energy intrinsically associated with time derivatives of ρ vanishes (or that it is dominated by the locking effects of Eqns. (18, 19)). Both spatial and time translation are spontaneously broken, as is reflected in the disposable constants $\tilde{\alpha}, \beta$, and so is time-reversal T, as reflected in the disposable sign.

Combining Eqns. (20, 21), we now have a traveling charge density wave. Thus this example exhibits its time-dependence in a physically tangible form. The residual continuous symmetry is reduced to a combined discrete time-space-charge transformation. Although our construction has been specific and opportunistic, it serves to establish the existence of a universality class that, since it is characterized by symmetry, should be robust. It is noteworthy that cyclic motion of ϕ in internal space has given rise to linear motion in physical space.

Relativistic Lagrangians: All of our constructions above have been nonrelativistic. In a relativistic theory there are relations among the coefficients of time and space gradient terms. The relativistic quartic term $L \propto ((\partial_0\phi)^2 - (\nabla\phi)^2)^2$ leads to an energy that is unbounded below, for large gradients of one kind or another. But use of a sextic enables positive energy. Indeed, the energy function for $((\partial_0\phi)^2 - (\nabla\phi)^2)^n$ is

$$((2n-1)(\partial_0\phi)^2 + (\nabla\phi)^2)((\partial_0\phi)^2 - (\nabla\phi)^2)^{n-1}. \qquad (22)$$

For n odd this is semi-positive definite, with a zero at $(\partial_0\phi)^2 = (\nabla\phi)^2$ unless $n = 1$. For n even it has no definite sign. Bounded energy requires only that the leading term have odd n and a positive coefficient and that the coefficient of the $n = 1$ term be non-negative. This consideration seems to have been overlooked and might help to constrain the models of [2] [3].

[1] F. Wilczek, arXiv:1202.2359 [quant-ph].

[2] C. Armendariz-Picon, T. Damour and V. F. Mukhanov, Phys. Lett. B **458**, 209 (1999).

[3] A. Arkani-Hamed, H. Cheng, M. Luty, S. Mukohyama, JHEP **0405** (05): 074 (2004).

[4] A. Winfree, Biological Rhythm Research **8**, 1 (1977); A. Winfree, *The Geometry of Biological Time, 2nd ed.* (Springer, 2001).

[5] Y. N. Srivastava, A. Widom and E. Sassaroli, in *Proceedings of the International Conference on Macroscopic Quantum Coherence*, eds. E. Sassaroli, Y. Srivastava, J. Swain and A. Widom (World Scientific, 1998) p. 1.

[6] M. Henneaux, C. Teitelboim, and J. Zanelli, Phys. Rev. **A36**, 4417 (1987).

[7] A. Shapere and F. Wilczek, "Branched Quantization", to appear.

[8] A. Nicolis and F. Piazza, arXiv:1112.5174 [hep-th].