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COMPUTING THE ASSOCIATED CYCLES OF CERTAIN HARISH-CHANDRA MODULES

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Abstract. Let $G_{\mathbb{R}}$ be a simple real linear Lie group with maximal compact subgroup $K_{\mathbb{R}}$ and assume that $\operatorname{rank}(G_{\mathbb{R}}) = \operatorname{rank}(K_{\mathbb{R}})$. In [MPVZ] we proved that for any representation X of Gelfand-Kirillov dimension $\frac{1}{2}\dim(G_{\mathbb{R}}/K_{\mathbb{R}})$, the polynomial on the dual of a compact Cartan subalgebra given by the dimension of the Dirac index of members of the coherent family containing X is a linear combination, with integer coefficients, of the multiplicities of the irreducible components occurring in the associated cycle. In this paper we compute these coefficients explicitly.

1. Introduction

Let $G_{\mathbb{R}}$ be a simple real linear Lie group with a Cartan involution θ and maximal compact subgroup $K_{\mathbb{R}} = G_{\mathbb{R}}^{\theta}$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of the complexified Lie algebra $\mathfrak{g}$ of $G_{\mathbb{R}}$; this decomposition is orthogonal with respect to the Killing form B. Let K be the complexification of $K_{\mathbb{R}}$ and G a complex Lie group (with Lie algebra $\mathfrak{g}$) containing K as the set of fixed points of the complex extension of θ. We assume throughout the paper that $\mathfrak{g}$ and $\mathfrak{k}$ have equal rank, i.e., there is a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ contained in $\mathfrak{k}$. We fix such $\mathfrak{h}$ and write W for the Weyl group of $(\mathfrak{g}, \mathfrak{h})$.

In this paper we are concerned with comparing two important invariants of $(\mathfrak{g}, K_{\mathbb{R}})$-modules. One is the Dirac index studied in [MPV]. It is defined

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using the Dirac operator $D \in U(\mathfrak{g}) \otimes C(\mathfrak{p})$, where $U(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$ and $C(\mathfrak{p})$ is the Clifford algebra of $\mathfrak{p}$ with respect to B. If M is a $(\mathfrak{g}, K_{\mathbb{R}})$-module, then D acts on $M \otimes S$ where S is a spin module for $C(\mathfrak{p})$. The Dirac cohomology of M is defined as

$$H_D(M) = \mathrm{Ker}(D)/(\mathrm{Im}(D) \cap \mathrm{Ker}(D);$$

it is a module for the spin double cover $\tilde{K}$ of $K_{\mathbb{R}}$ (finite-dimensional if M is admissible). This invariant was introduced in [V2], it turned out to be very interesting and also quite computable; see for example [HP1, HP2, HKP, HPR, HPP, HPZ, BP1, BP2, BPT, MP, MZ, DH].

Decomposing the $\tilde{K}$-module S as $S = S^+ \oplus S^-$ induces a decomposition of Dirac cohomology

$$H_D(M) = H_D(M)^+ \oplus H_D(M)^-.$$

The Dirac index of M is then defined as the virtual $\tilde{K}$-module

$$DI_v(M) = H_D(M)^+ - H_D(M)^-.$$

It is proved in [MPV] that Dirac index varies nicely over coherent families of $(\mathfrak{g}, K_{\mathbb{R}})$-modules. In particular, if $\{M_\lambda\}$ is such a coherent family, attached to a module M, then the function

$$\lambda \mapsto \dim DI_v(M_\lambda)$$

extends to a polynomial on $\mathfrak{h}^*$, which we denote by $DI_p(M)$.

Another very useful invariant of a Harish-Chandra module M is its associated cycle $AC(M)$, defined in [V1]. See [MPVZ] for a short review of the definition.

In concrete terms, for irreducible M, $AC(M)$ can be written as the formal sum

$$AC(M) = \sum_i m_i(M)\overline{\mathcal{O}_i},$$

where $\mathcal{O}_i \subset \mathfrak{p}$ are the real forms of a complex nilpotent G-orbit $\mathcal{O}^{\mathbb{C}} \subset \mathfrak{g}$, and the multiplicities $m_i(M)$ are nonnegative integers. The orbit $\mathcal{O}^{\mathbb{C}}$ is specified by the requirement that $\overline{\mathcal{O}^{\mathbb{C}}}$ is the associated variety of the annihilator of M.

If M is put into a coherent family $\{M_\lambda\}$, then the corresponding multiplicities extend to polynomials $m_i(M)$ on $\mathfrak{h}^*$. It was conjectured in [MPV], and proved in [MPVZ], that in certain special circumstances these multiplicity polynomials are related to the Dirac index polynomial by

$$DI_p(M) = \sum_i c_i m_i(M)$$

for some integers c_i. Such a relationship is true when the associated variety of the annihilator of M is contained in $\overline{\mathcal{O}^{\mathbb{C}}}$, with $\mathcal{O}^{\mathbb{C}}$ corresponding via Springer correspondence to the W-representation generated by the Weyl dimension polynomial P_K for K (P_K is defined by (3.1)).

The purpose of this paper is to complement [MPVZ] by explicitly computing the constants c_i in the classical cases other than $SU(p,q)$. The case $G_{\mathbb{R}} = SU(p,q)$ as well as the case of exceptional groups are done in [MPVZ].

We start by reviewing some facts about real forms of nilpotent orbits in Section 2, and assembling a few useful general facts about the computations in Section 3. Then we do the case-by-case computations in Sections 4 – 8.

2. Nilpotent orbits and their real forms

We recall that the list of the classical real groups for which the conjecture from [MPV] applies is given in [MPVZ], Section 6, Table 1, along with the relevant explanations. The groups on the list are the connected classical equal rank groups such that the W-representation σ_K generated by the Weyl dimension polynomial P_K for K is Springer. The list consists of

- $SU(p,q), \quad q \geq p \geq 1;$
- $SO_e(2p,2q+1), \quad q \geq p-1 \geq 0;$
- $\mathrm{Sp}(2n,\mathbb{R}), \quad n \geq 1;$
- $SO^*(2n), \quad n \geq 1;$
- $SO_e(2p,2q), \quad q \geq p \geq 1.$

The table in [MPVZ] also includes the nilpotent orbits $\mathcal{O}^{\mathbb{C}}$ corresponding to σ_K in each of the cases, as well as the number of real forms of these orbits. Here we explain how to get these real forms, and in particular how to write down the semisimple elements h of the corresponding $\mathfrak{sl}_2$-triples, which we need to begin our computations.

We start by recalling that complex nilpotent orbits in classical Lie algebras are in one-to-one correspondence with the set of partitions $[d_1, \cdots, d_k]$ with $d_1 \geq d_2 \geq \cdots \geq d_k \geq 1$ (if d_j occurs m times, we will simply write d_j^m) such that (see [CM, Chapter 5]):

- $d_1 + d_2 + \cdots + d_k = n$, when $\mathfrak{g} \simeq \mathfrak{sl}(n,\mathbb{C})$;
- $d_1+d_2+\cdots+d_k = 2n+1$ and the even d_j occur with even multiplicity, when $\mathfrak{g} \simeq \mathfrak{so}(2n+1,\mathbb{C})$;
- $d_1 + d_2 + \cdots + d_k = 2n$ and the odd d_j occur with even multiplicity, when $\mathfrak{g} \simeq \mathfrak{sp}(2n,\mathbb{C})$;
- $d_1 + d_2 + \cdots + d_k = 2n$ and the even d_j occur with even multiplicity, when $\mathfrak{g} \simeq \mathfrak{so}(2n,\mathbb{C})$; except that the partitions having all the d_j even and occurring with even multiplicity are each associated to *two* orbits.

We now recall the procedure which attaches $\mathfrak{sl}_2$-triples to complex nilpotent orbits (see [CM, Chapter 3]). For a positive integer i, define the Jordan block J_i to be the $i \times i$ matrix

$$J_i = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ . & . & . & . & \cdots & . \\ . & . & . & . & \cdots & . \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

For a positive integer n, write $[d_1, d_2, \cdots, d_k]$ for a partition of n. Define the $n \times n$ matrix

$$X_{[d_1,d_2,\cdots,d_k]} = \begin{pmatrix} J_{d_1} & 0 & 0 & 0 & \cdots & 0 \\ 0 & J_{d_2} & 0 & 0 & \cdots & 0 \\ . & . & . & . & \cdots & . \\ . & . & . & . & \cdots & . \\ 0 & 0 & 0 & 0 & \cdots & J_{d_k} \end{pmatrix}$$

Then $X_{[d_1,d_2,\cdots,d_k]}$ is a nilpotent element in the complex Lie algebra $\mathfrak{sl}_n$. Write $\mathcal{O}_{[d_1,d_2,\cdots,d_k]}$ for the complex nilpotent orbit under the adjoint group PSL_n of $\mathfrak{sl}_n$. It is convenient to attach to $\mathcal{O}_{[d_1,d_2,\cdots,d_k]}$ a Young tableau, i.e a left-justified arrangement of empty boxes of rows with size in the non-increasing order d_1, d_2, ..., d_k.

Let $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $X = J_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $Y = J_2^t = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Then $[H, X] = 2X$, $[H, Y] = -2Y$ and $[X, Y] = H$, so that H, X, Y span, over $\mathbb{C}$, the simple Lie algebra $\mathfrak{sl}_2$ of 2×2 complex matrices with zero trace. For a non-negative integer r, define the linear map $\rho_r : \mathfrak{sl}_2 \to \mathfrak{sl}_{r+1}$ by

$$\begin{aligned} \rho_r(H) &= \begin{pmatrix} r & 0 & 0 & 0 & \cdots & 0 \\ 0 & r-2 & 0 & 0 & \cdots & 0 \\ . & . & . & . & \cdots & . \\ . & . & . & . & \cdots & . \\ 0 & \cdot & \cdot & \cdot & -r+2 & 0 \\ 0 & \cdot & \cdot & \cdot & 0 & -r \end{pmatrix} \\ \rho_r(X) &= J_{r+1} \\ \rho_r(Y) &= \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ \mu_1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \mu_2 & 0 & 0 & \cdots & 0 \\ . & . & . & . & \cdots & . \\ . & . & . & . & \cdots & . \\ 0 & \cdot & \cdot & \cdot & \mu_r & 0 \end{pmatrix} \\ & \text{with } \mu_i = i(r+1-i) \text{ for } 1 \leq i \leq r \end{aligned}$$

ρ_r defines an irreducible representation of $\mathfrak{sl}_2$ of dimension $r+1$, and any finite dimensional irreducible representation of $\mathfrak{sl}_2$ is equivalent to ρ_r for some r. The map ρ_r induces the homomorphism $\Phi_{\mathcal{O}} : \mathfrak{sl}_2 \to \mathfrak{sl}_n$ defined by:

$$\Phi_{\mathcal{O}} = \bigoplus_{1 \le j \le k} \rho_{d_j - 1}$$

so that $\Phi_{\mathcal{O}}(X) = X_{[d_1, d_2, \cdots, d_k]}$. The standard $\mathfrak{sl}_2$-triple associated with the complex nilpotent orbit $\mathcal{O}_{[d_1,d_2,\cdots,d_k]}$ is

$$\{H_{[d_1,d_2,\cdots,d_k]}; X_{[d_1,d_2,\cdots,d_k]}; Y_{[d_1,d_2,\cdots,d_k]}\}$$

where $H_{[d_1,d_2,\cdots,d_k]} := \Phi_{\mathcal{O}}(H)$, $X_{[d_1,d_2,\cdots,d_k]} := \Phi_{\mathcal{O}}(X)$ and $Y_{[d_1,d_2,\cdots,d_k]} := \Phi_{\mathcal{O}}(Y)$. Choose the Cartan subalgebra consisting of $n \times n$ diagonal matrices $\operatorname{diag}(a_1, a_2, \cdots, a_n)$ with zero trace, and fix the positive system of roots $\{\epsilon_i - \epsilon_j \mid 1 \le i < j \le n\}$ whose corresponding Borel subalgebra consists of the upper tringular matrices of zero trace. Here ϵ_i is the complex linear form defined on the Cartan subalgebra such that $\epsilon_i(\operatorname{diag}(a_1, a_2, \cdots, a_n)) = a_i$. Then up to a Weyl group element of $\mathfrak{sl}_n$, the element $H_{[d_1,d_2,\cdots,d_k]}$ is conjugate to a dominant element

$$(h_1, h_2, \cdots, h_n) := \operatorname{diag}(h_1, h_2, \cdots, h_n)$$

with $h_1 \ge h_2 \ge \cdots \ge h_n$ and $h_1 + h_2 + \cdots + h_n = 0$. Associated with the orbit $\mathcal{O}_{[d_1,d_2,\cdots,d_k]}$ is the weighted Dynkin diagram

$$\overset{h_1-h_2}{\circ} \!-\!\!-\!\!-\! \overset{h_2-h_3}{\circ} \!-\!\!-\!\!-\! \cdots\cdots \!-\!\!-\!\!-\! \overset{h_{n-1}-h_n}{\circ}$$

Suppose now that $G_{\mathbb{R}} = SU(p,q)$, $K_{\mathbb{R}} = S(U(p) \times U(q))$ and $\mathfrak{g} = \mathfrak{sl}_{p+q}$, with $q \ge p \ge 1$ and $p + q = n$. The dominant h associated with the complex nilpotent orbit $\mathcal{O}^{\mathbb{C}} = \mathcal{O}_{[2^p;1^{q-p}]}$ is given by

$$h = (\underbrace{1, 1, \cdots, 1}_{p}, \underbrace{0, 0, \cdots, 0}_{q-p}, \underbrace{-1, -1, \cdots, -1}_{p})$$

along with the weighted Dynkin diagram

$$\underset{\epsilon_1-\epsilon_2}{\overset{0}{\circ}} \!-\!\!-\! \cdots\cdots \!-\!\!-\! \underset{\epsilon_p-\epsilon_{p+1}}{\overset{1}{\circ}} \!-\!\!-\! \underset{\epsilon_{p+1}-\epsilon_{p+2}}{\overset{0}{\circ}} \!-\!\!-\! \cdots\cdots \!-\!\!-\! \underset{\epsilon_q-\epsilon_{q+1}}{\overset{1}{\circ}} \!-\!\!-\! \underset{\epsilon_{q+1}-\epsilon_{q+2}}{\overset{0}{\circ}} \!-\!\!-\! \cdots\cdots \!-\!\!-\! \underset{\epsilon_{p+q}-\epsilon_{p+q-1}}{\overset{0}{\circ}}$$

Moreover, by Kostant-Sekiguchi, it is known that the nilpotent orbit $\mathcal{O}^{\mathbb{C}}$ has $p+1$ real forms which are in one to one correspondence with nilpotent K-orbits in $\mathfrak{p}$. More precisely, for $k = 0, 1, 2, \cdots, p$, let $I_0 = 0$ and

$$e_k = \left[\begin{array}{c|c} 0_p & \begin{matrix} I_k & 0 \\ 0 & 0 \end{matrix} \\ \hline \begin{matrix} 0 & 0 \\ 0 & I_{p-k} \end{matrix} & 0_q \end{array}\right]$$

For each k, the element e_k belongs to $\mathcal{O}^{\mathbb{C}}$. On the other hand, the K-orbit of e_k consists of matrices of the form

$$\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \quad \text{with } \operatorname{rank}(A) = k \text{ and } \operatorname{rank}(B) = p - k$$

In particular, if $k \neq k'$ then the K-orbits of e_k and $e_{k'}$ are disjoint. Choosing the positive system $\{\epsilon_i - \epsilon_j \mid 1 \leq i < j \leq p \text{ or } p+1 \leq i < j \leq p+q\}$, the (dominant) neutral element of the $\mathfrak{sl}_2$-triple corresponding to the real form $K \cdot e_k$ is

$$h_k = (\underbrace{1, 1, \cdots, 1}_{k}, \underbrace{-1, -1, \cdots, -1}_{p-k}, \underbrace{1, 1, \cdots, 1}_{p-k}, \underbrace{0, 0, \cdots, 0}_{q-p}, \underbrace{-1, -1, \cdots, -1}_{k})$$

The description in terms of Young tableaux of the complex orbit $\mathcal{O}^{\mathbb{C}}$ and of its real forms is as follows:

$$\mathcal{O}^{\mathbb{C}} \quad \begin{array}{l} \left.\begin{array}{|c|c|}\hline & \\ \hline : & : \\ \hline & \\ \hline \end{array}\right\} p \\ \left.\begin{array}{|c|}\hline \\ \hline : \\ \hline \\ \hline \end{array}\right\} q-p \end{array} \qquad\qquad \text{real forms for } \mathcal{O}^{\mathbb{C}} \quad \begin{array}{l} \left.\begin{array}{|c|c|}\hline + & - \\ \hline : & : \\ \hline + & - \\ \hline \end{array}\right\} k \in \{0, \dots, p\} \\ \left.\begin{array}{|c|c|}\hline - & + \\ \hline : & : \\ \hline - & + \\ \hline \end{array}\right\} p-k \\ \left.\begin{array}{|c|}\hline - \\ \hline : \\ \hline - \\ \hline \end{array}\right\} q-p \end{array}$$

Consider $G_{\mathbb{R}} = \mathrm{Sp}(2n, \mathbb{R})$ and $K_{\mathbb{R}} = U(n)$. The complexification $\mathfrak{g} = \mathfrak{sp}_{2n}$ of $G_{\mathbb{R}}$ is realized as the following set of matrices

$$\left\{ \begin{pmatrix} Z_1 & Z_2 \\ Z_3 & -Z_1^t \end{pmatrix} | Z_1 \; n \times n \text{ complex matrix, } Z_2, Z_3 \text{ symmetric complex matrices} \right\}$$

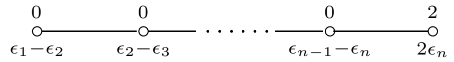
A Cartan subalgebra in $\mathfrak{g}$ consists of diagonal complex matrices of the form $\operatorname{diag}(a_1, a_2, \cdots, a_n, -a_1, -a_2, \cdots, -a_n)$. Fix the standard system of positive roots $\{\epsilon_i \pm \epsilon_j \mid 1 \leq i < j \leq n\} \cup \{2\epsilon_k \mid 1 \leq k \leq n\}$. As in type A, there is an explicit recipe which attaches an $\mathfrak{sl}_2$-triple to a complex nilpotent orbit (see [CM, 5.2.2]). We apply this recipe to the nilpotent orbit $\mathcal{O}^{\mathbb{C}} = \mathcal{O}_{[2^n]}$, using n chunks coinciding with $\{2\}$. We obtain (viewing $\mathfrak{sp}_{2n}$ as a subalgebra of $\mathfrak{sl}_{2n}$)

$$h := \operatorname{diag}(\underbrace{1, 1, \cdots, 1}_{n}, \underbrace{-1, -1, \cdots, -1}_{n})$$

which we will simply write

$$h = (\underbrace{1, 1, \cdots, 1}_{n})$$

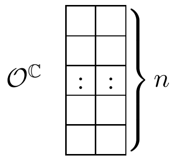
along with the weighted Dynkin diagram



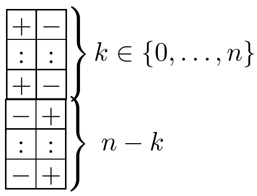
The same argument as in type A shows that $\mathcal{O}^{\mathbb{C}}$ possesses $n+1$ real forms with

$$h_k = (\underbrace{1,1,\cdots,1}_{k},\underbrace{-1,-1,\cdots,-1}_{n-k},\underbrace{-1,-1,\cdots,-1}_{k},\underbrace{1,1,\cdots,1}_{n-k})$$

The description in terms of Young tableaux of the complex orbit $\mathcal{O}^{\mathbb{C}}$ and of its real forms is as follows:



real forms



Consider $G_{\mathbb{R}} = SO_e(2p,2q+1)$ and $K_{\mathbb{R}} = S(O(2p)\times O(2q+1))$. The complexification $\mathfrak{g} = \mathfrak{so}_{2n+1}$ of $G_{\mathbb{R}}$, with $n = p+q$ and $q \geq p \geq 1$, is realized as the following set of matrices

$$\left\{ \begin{pmatrix} 0 & u & v \\ -v^t & Z_1 & Z_2 \\ -u^t & Z_3 & -Z_1^t \end{pmatrix} \mid u,v \in \mathbb{C}^n, Z_1\ n\times n \text{ complex}, Z_2, Z_3 \text{ skew-symmetric} \right\}$$

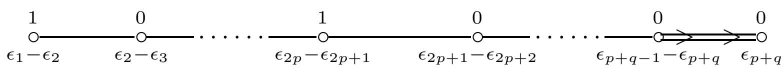
A Cartan subalgebra in $\mathfrak{g}$ consists of diagonal complex matrices of the form $\mathrm{diag}(0,a_1,a_2,\cdots,a_n,-a_1,-a_2,\cdots,-a_n)$ (first row and column of zeros). Fix the standard system of positive roots $\{\epsilon_i \pm \epsilon_j \mid 2 \leq i < j \leq n+1\} \cup \{\epsilon_k \mid 2 \leq k \leq n+1\}$. There is an explicit recipe which attaches an $\mathfrak{sl}_2$-triple to a complex nilpotent orbit (see [CM, 5.2.4]). We apply this recipe to the nilpotent orbit $\mathcal{O}^{\mathbb{C}} = \mathcal{O}_{[3,2^{2p-2},1^{2(q-p+1)}]}$, using the following chunks : $\{3\}$, $p-1$ $\{2;2\}$'s and $q-p+1$ $\{1;1\}$'s. We obtain (viewing $\mathfrak{so}_{2n+1}$ as a subalgebra of $\mathfrak{sl}_{2n+1}$)

$$h := \mathrm{diag}(0,2,\underbrace{1,1,\cdots,1}_{2p-2},\underbrace{0,0,\cdots,0}_{2(q-p+1)},\underbrace{-1,-1,\cdots,-1}_{2p-2},-2)$$

which we will simply write (dropping the first zero coordinate and shifting indices of ϵ_i's)

$$h = (2,\underbrace{1,1,\cdots,1}_{2p-2},\underbrace{0,0,\cdots,0}_{q-p+1})$$

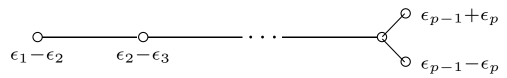
along with the weighted Dynkin diagram



The nilpotent orbit $\mathcal{O}^{\mathbb{C}}$ possesses 2 or 3 real forms depending wether $q > p-1$ or not. The description in terms of Young tableaux of the complex orbit $\mathcal{O}^{\mathbb{C}}$ and of its real forms is given below. The recipe to produce the real h's from the signed tableau can be stated as follows: the first row of length 3 gives a 2 in the first p coordinates if the row starts with a "+" and a 2 in the $p+1$ coordinate if it starts with a "-". For the rows of length two, a "+" (resp. "-") sign in the leftmost box provides $+1$ in the first p coordinates (resp. in the second group of coordinates $p+1, \cdots$); a "+" (resp. "-") sign in the rightmost box provides -1 in the first p coordinates (resp. in the second group of coordinates $p+1, \cdots$). In particular, we get

$$\begin{aligned} h_1^I &= (2, \underbrace{1, 1, \cdots, 1}_{p-1}, \underbrace{1, 1, \cdots, 1}_{p-1}, \underbrace{0, 0, \cdots, 0}_{q-p+1}) \\ h_1^{II} &= (2, \underbrace{1, 1, \cdots, -1}_{p-1}, \underbrace{1, 1, \cdots, 1}_{p-1}, \underbrace{0, 0, \cdots, 0}_{q-p+1}) \\ h_2 &= (\underbrace{1, 1, \cdots, 1}_{p-1}, 0, 2, \underbrace{1, 1, \cdots, 1}_{p-1}, \underbrace{0, 0, \cdots, 0}_{q-p}) \text{ only if } q > p-1 \end{aligned}$$

h_1^{II} is obtained from h_1^I by the outer automorphism $\epsilon_{p-1} + \epsilon_p \longleftrightarrow \epsilon_{p-1} - \epsilon_p$:



The description of the orbit $\mathcal{O}^{\mathbb{C}}$ and its real forms in terms of Young tableaux is as follows:

$$\mathcal{O}^{\mathbb{C}} \quad \begin{array}{l} \begin{array}{|c|c|c|}\hline \; & \; & \; \\ \hline \end{array} \\ \left.\begin{array}{|c|c|}\hline \; & \; \\ \hline : & : \\ \hline \; & \; \\ \hline \end{array}\right\} 2p-2 \\ \left.\begin{array}{|c|}\hline \; \\ \hline : \\ \hline \; \\ \hline \end{array}\right\} 2(q-p+1) \end{array} \qquad \text{real forms for } \mathcal{O}^{\mathbb{C}} \quad \begin{array}{l} \begin{array}{|c|c|c|}\hline + & - & + \\ \hline \end{array} \\ \left.\begin{array}{|c|c|}\hline + & - \\ \hline : & : \\ \hline + & - \\ \hline \end{array}\right\} 2p-2 \quad \text{I, II} \\ \left.\begin{array}{|c|}\hline - \\ \hline : \\ \hline - \\ \hline \end{array}\right\} 2(q-p+1) \end{array}$$

$$\begin{array}{l} \begin{array}{|c|c|c|}\hline - & + & - \\ \hline \end{array} \\ \left.\begin{array}{|c|c|}\hline + & - \\ \hline : & : \\ \hline + & - \\ \hline \end{array}\right\} 2p-2 \quad \text{only if } q > p-1 \\ \begin{array}{|c|}\hline + \\ \hline \end{array} \\ \left.\begin{array}{|c|}\hline - \\ \hline : \\ \hline - \\ \hline \end{array}\right\} 2(q-p)+1 \end{array}$$

Consider $G_{\mathbb{R}} = SO_e(2p, 2q)$ and $K_{\mathbb{R}} = SO(2p) \times SO(2q)$. The complexification $\mathfrak{g} = \mathfrak{so}_{2n}$ of $G_{\mathbb{R}}$, with $n = p + q$ and $q \geq p \geq 1$, is realized as the following set of matrices

$$\left\{ \begin{pmatrix} Z_1 & Z_2 \\ Z_3 & -Z_1^t \end{pmatrix} \mid Z_i \ n \times n \text{ complex matrices, } Z_2, Z_3 \text{ skew-symmetric} \right\}$$

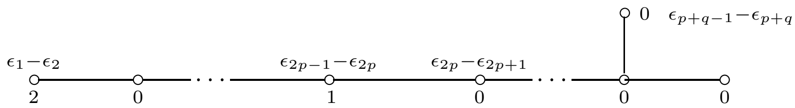
A Cartan subalgebra in $\mathfrak{g}$ consists of diagonal complex matrices of the form $\mathrm{diag}(a_1, a_2, \cdots, a_n, -a_1, -a_2, \cdots, -a_n)$. Fix the standard system of positive roots $\{\epsilon_i \pm \epsilon_j \mid 1 \leq i < j \leq n\}$. There is an explicit recipe which attaches an $\mathfrak{sl}_2$-triple to a complex nilpotent orbit (see [CM, 5.2.6]). We apply this recipe to the nilpotent orbit $\mathcal{O}^{\mathbb{C}} = \mathcal{O}_{[3,2^{2p-2},1^{2(q-p)+1}]}$, using the following chunks : $\{3;1\}$, $p-1$ $\{2;2\}$'s and $q-p$ $\{1;1\}$'s. We obtain (viewing $\mathfrak{so}_{2n}$ as a subalgebra of $\mathfrak{sl}_{2n}$)

$$h := \mathrm{diag}(2, \underbrace{1, 1, \cdots, 1}_{2p-2}, \underbrace{0, 0, \cdots, 0}_{2(q-p+1)}, \underbrace{-1, -1, \cdots, -1}_{2p-2}, -2)$$

which we will simply write

$$h = (2, \underbrace{1, 1, \cdots, 1}_{2p-2}, \underbrace{0, 0, \cdots, 0}_{q-p+1})$$

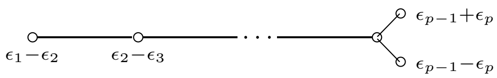
along with the weighted Dynkin diagram



The nilpotent orbit $\mathcal{O}^{\mathbb{C}}$ possesses 3 or 4 real forms depending whether $q > p$ or not. Using a recipe analogous to the one used for type B, we get

$$\begin{aligned}
h_1^I &= (2, \underbrace{1, 1, \cdots, 1}_{p-1}, \underbrace{1, 1, \cdots, 1}_{p-1}, \underbrace{0, 0, \cdots, 0}_{q-p+1}) \\
h_1^{II} &= (2, \underbrace{1, 1, \cdots, -1}_{p-1}, \underbrace{1, 1, \cdots, 1}_{p-1}, \underbrace{0, 0, \cdots, 0}_{q-p+1}) \\
h_2^I &= (\underbrace{1, 1, \cdots, 1}_{p-1}, 0, 2, \underbrace{1, 1, \cdots, 1}_{p-1}, \underbrace{0, 0, \cdots, 0}_{q-p}) \\
h_2^{II} &= (\underbrace{1, 1, \cdots, 1}_{p-1}, 0, 2, \underbrace{1, 1, \cdots, -1}_{p-1}, \underbrace{0, 0, \cdots, 0}_{q-p}) \text{ only if } q = p
\end{aligned}$$

As before, h_i^{II} is obtained from h_i^I by the outer automorphism:



The description of the complex orbit $\mathcal{O}^{\mathbb{C}}$ and its real forms in terms of Young tableaux is as follows:

$$\mathcal{O}^{\mathbb{C}} \quad \begin{array}{ll} \begin{array}{|c|c|c|}\hline \ & \ & \ \\ \hline \end{array} & \\ \left.\begin{array}{|c|c|}\hline \ & \ \\ \hline \vdots & \vdots \\ \hline \ & \ \\ \hline \end{array}\right\} & 2p-2 \\ \left.\begin{array}{|c|}\hline \ \\ \hline \vdots \\ \hline \ \\ \hline \end{array}\right\} & 2(q-p)+1 \end{array} \qquad \text{real forms for } \mathcal{O}^{\mathbb{C}} \quad \begin{array}{ll} \begin{array}{|c|c|c|}\hline + & - & + \\ \hline \end{array} & \\ \left.\begin{array}{|c|c|}\hline + & - \\ \hline \vdots & \vdots \\ \hline + & - \\ \hline \end{array}\right\} & 2p-2 \quad \text{I, II} \\ \left.\begin{array}{|c|}\hline - \\ \hline \vdots \\ \hline - \\ \hline \end{array}\right\} & 2(q-p)+1 \end{array}$$

$$\begin{array}{ll} \begin{array}{|c|c|c|}\hline - & + & - \\ \hline \end{array} & \\ \left.\begin{array}{|c|c|}\hline + & - \\ \hline \vdots & \vdots \\ \hline + & - \\ \hline \end{array}\right\} & 2p-2 \quad (\text{I, II if } q=p) \\ \left.\begin{array}{|c|}\hline - \\ \hline \vdots \\ \hline - \\ \hline \end{array}\right\} & q-p \end{array}$$

Consider

$$G_{\mathbb{R}} = SO^*(2n) = SO(2n,\mathbb{C}) \cap \mathfrak{gl}(n,\mathbb{H}),$$

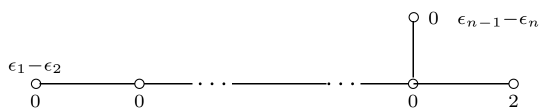
$K_{\mathbb{R}} = U(n)$ and $\mathfrak{g} = \mathfrak{so}_{2n}^*$. For $\mathfrak{so}_{2n}$, a Cartan subalgebra in $\mathfrak{g}$ consists of diagonal complex matrices of the form $\mathrm{diag}(a_1, a_2, \cdots, a_n, -a_1, -a_2, \cdots, -a_n)$. Fix the standard system of positive roots $\{\epsilon_i \pm \epsilon_j \mid 1 \le i < j \le n\}$. Using a recipe analogous to that of type D, one can attach an $\mathfrak{sl}_2$-triple to a complex nilpotent orbit. We apply this recipe to the nilpotent orbit $\mathcal{O}^{\mathbb{C}} = \mathcal{O}_{[2^n]}$ to obtain (viewing $\mathfrak{so}_{2n}^*$ as a subalgebra of $\mathfrak{sl}_{2n}$)

$$h := \mathrm{diag}(\underbrace{1, 1, \cdots, 1}_{n}, \underbrace{-1, -1, \cdots, -1}_{n})$$

which we will simply write

$$h = (\underbrace{1, 1, \cdots, 1}_{n})$$

along with the weighted Dynkin diagram



The nilpotent orbit $\mathcal{O}^{\mathbb{C}}$ possesses $\frac{n}{2}+1$ real forms if n is even, and $\frac{n+1}{2}$ real forms otherwise. Using a recipe analogous to the one used for type D, we get

$$h_k = (\underbrace{1, 1, \cdots, 1}_{2k}, \underbrace{-1, -1, \cdots, -1}_{n-2k}) \text{ for } n \text{ even, and } k = 0, \cdots, \frac{n}{2},$$

$$h_k = (\underbrace{1, 1, \cdots, 1}_{2k}, 0, \underbrace{-1, -1, \cdots, -1}_{n-2k-1}) \text{ for } n \text{ odd, and } k = 0, \cdots, \frac{n-1}{2}.$$

Finally, the description of the complex orbit $\mathcal{O}^{\mathbb{C}}$ and its real forms in terms of Young tableaux is as follows:

<u>n even:</u>

$$\mathcal{O}^{\mathbb{C}} \quad \left.\begin{array}{|c|c|}\hline & \\ \hline & \\ \hline : & : \\ \hline & \\ \hline & \\ \hline\end{array}\right\} n \qquad\qquad \text{real forms} \quad \begin{array}{|c|c|}\hline + & - \\ \hline : & : \\ \hline + & - \\ \hline - & + \\ \hline : & : \\ \hline - & + \\ \hline\end{array} \begin{array}{l} \left.\vphantom{\begin{array}{c}+\\:\\+\end{array}}\right\} k \in \{0, \dots, \tfrac{n}{2}\} \\ \left.\vphantom{\begin{array}{c}-\\:\\-\end{array}}\right\} \tfrac{n}{2} - k \end{array}$$

<u>n odd:</u>

$$\mathcal{O}^{\mathbb{C}} \quad \begin{array}{l} \left.\begin{array}{|c|c|}\hline & \\ \hline : & : \\ \hline & \\ \hline\end{array}\right\} n-1 \\ \left.\begin{array}{|c|}\hline \\ \hline \\ \hline\end{array}\right\} 2 \end{array} \qquad\qquad \text{real forms} \quad \begin{array}{l} \begin{array}{|c|c|}\hline + & - \\ \hline : & : \\ \hline + & - \\ \hline - & + \\ \hline : & : \\ \hline - & + \\ \hline\end{array} \begin{array}{l} \left.\vphantom{\begin{array}{c}+\\:\\+\end{array}}\right\} k \in \{0, \dots, \tfrac{n-1}{2}\} \\ \left.\vphantom{\begin{array}{c}-\\:\\-\end{array}}\right\} \tfrac{n-1}{2} - k \end{array} \\ \begin{array}{|c|}\hline + \\ \hline\end{array} \end{array}$$

3. Some general facts

Let $\mathcal{O}_i$ be a real form of the orbit $\mathcal{O}^{\mathbb{C}}$, and denote the corresponding semisimple element of the (normal) $\mathfrak{sl}(2)$-triple by $h \in \mathfrak{h}$. As in [MPVZ], we attach to h the θ-stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ such that $\mathfrak{l}$ is the centralizer of h in $\mathfrak{g}$, and $\mathfrak{u}$ is the sum of negative eigenspaces for $\operatorname{ad} h$ on $\mathfrak{g}$. We fix a choice of $\Delta^+ = \Delta^+(\mathfrak{g}, \mathfrak{h})$; in examples, this will always be the standard positive root system. This defines a choice $\Delta_c^+ = \Delta^+(\mathfrak{k}, \mathfrak{h}) = \Delta(\mathfrak{k}, \mathfrak{h}) \cap \Delta^+$ of positive compact roots. Let $\Delta_n^+ := \Delta^+ \setminus \Delta_c^+$ be the set of positive noncompact roots. Denote by ρ_c (resp. ρ_n) half the sum of positive compact (resp. noncompact) roots. The Weyl dimension polynomial

$$P_K(\lambda) = \prod_{\alpha \in \Delta_c^+} \frac{\langle \lambda, \alpha \rangle}{\langle \rho_c, \alpha \rangle} \tag{3.1}$$

will always be defined with respect to this fixed positive root system Δ_c^+. If A is a subset of $\Delta(\mathfrak{g},\mathfrak{h})$, write $\rho(A)$ for half the sum of the roots in A.

We choose $\Delta^+(\mathfrak{l})$ compatibly with Δ^+, i.e., $\Delta^+(\mathfrak{l}) = \Delta(\mathfrak{l}) \cap \Delta^+$. This also gives a choice for positive roots of $\mathfrak{l} \cap \mathfrak{k}$, and fixes the Weyl dimension polynomial $P_{L\cap K}$. Denote by $\Delta_n^+(\mathfrak{l})$ the set of noncompact roots in $\Delta^+(\mathfrak{l})$, and by $\Delta(\mathfrak{p}_1)$ the set of noncompact roots that are 1 on h.

The constant $c = c_i$ we are going to compute is attached to $\mathcal{O}_i$ as in [MPVZ]. It is defined by equation [MPVZ, (6.4)]; this is up to sign the same equation as [MPVZ, (5.9)], but the sign is made precise using [MPVZ, Remark 3.8, equation (6.1)], and the discussion around (6.1). The equation is

$$(-1)^N \sum_{\substack{A\subseteq\Delta_n^+(\mathfrak{l})\\ C\subseteq\Delta(\mathfrak{p}_1)}} (-1)^{\#A+\#C} P_K(\lambda-\rho_n(\mathfrak{l})+2\rho(A)-2\rho(C)) = cP_{L\cap K}(\lambda), \tag{3.2}$$

where

$$N = \#\{\alpha\in\Delta^+ \,\big|\, \alpha(h)>0\}. \tag{3.3}$$

The computations we are going to make will be easier if equation (3.2) is turned into an analogue of equation [MPVZ, (6.5)]:

Proposition 3.1. *Assume that $\rho_n(\mathfrak{l})$ is orthogonal to all roots of $\mathfrak{l}\cap\mathfrak{k}$. Then*

$$(-1)^{N+\#\Delta_n^+(\mathfrak{l})} \sum_{\substack{A\subseteq\Delta_n^+(\mathfrak{l})\\ C\subseteq\Delta(\mathfrak{p}_1)}} (-1)^{\#A+\#C} P_K(\lambda-2\rho(A)-2\rho(C)) = cP_{L\cap K}(\lambda). \tag{3.4}$$

Proof. This follows by passing from summation over A to summation over the complement of A in $\Delta_n^+(\mathfrak{l})$. For any $A\subseteq\Delta_n^+(\mathfrak{l})$,

$$-\rho_n(\mathfrak{l})+2\rho(A) = \rho_n(\mathfrak{l}) - 2\rho(\Delta_n^+(\mathfrak{l})\setminus A)$$

and

$$(-1)^{\#A} = (-1)^{\#\Delta_n^+(\mathfrak{l})}(-1)^{\#(\Delta_n^+(\mathfrak{l})\setminus A)},$$

so (3.2) can be rewritten as

$$(-1)^{N+\#\Delta_n^+(\mathfrak{l})} \sum_{\substack{A\subseteq\Delta_n^+(\mathfrak{l})\\ C\subseteq\Delta(\mathfrak{p}_1)}} (-1)^{\#A+\#C} P_K(\lambda+\rho_n(\mathfrak{l})-2\rho(A)-2\rho(C)) = cP_{L\cap K}(\lambda).$$

We now replace λ by $\lambda-\rho_n(\mathfrak{l})$; since $\rho_n(\mathfrak{l})$ is orthogonal to the roots of $\mathfrak{l}\cap\mathfrak{k}$, $P_{L\cap K}(\lambda-\rho_n(\mathfrak{l})) = P_{L\cap K}(\lambda)$, and the statement follows. ∎

In each of the examples we will consider, one can check directly that indeed $\rho_n(\mathfrak{l})$ is orthogonal to all roots of $\mathfrak{l}\cap\mathfrak{k}$, and hence we can compute the constant c using (3.4). A little more systematic way of checking this assumption, which will be easy to apply in all cases we consider, is given by the following lemma.

LEMMA 3.2. *Suppose that all simple factors of $\mathfrak{l}_0$ are either compact or noncompact Hermitian. Assume also that $\Delta^+(\mathfrak{l})$ (induced by Δ^+) is Borel - de Siebenthal for each noncompact factor of $\mathfrak{l}$. Then $\rho_n(\mathfrak{l})$ is orthogonal to all roots of $\mathfrak{l} \cap \mathfrak{k}$.*

PROOF. Since ρ_n of any compact factor is 0, it is enough to prove the statement for each of the noncompact factors. Denote by $\mathfrak{d}$ one of these factors, and let $\mathfrak{d} = \mathfrak{c} \oplus \mathfrak{s}$ be its Cartan decomposition (so $\mathfrak{c} = \mathfrak{d} \cap \mathfrak{k}$ and $\mathfrak{s} = \mathfrak{d} \cap \mathfrak{p}$). Since $\mathfrak{d}$ is Hermitian,

$$\mathfrak{s} = \mathfrak{s}^+ \oplus \mathfrak{s}^-$$

as a $\mathfrak{c}$-module. If $\Delta^+(\mathfrak{d})$ is a Borel-de Siebenthal positive root system with respect to a compact Cartan subalgebra of $\mathfrak{d}$, then $\Delta_n^+(\mathfrak{d})$ must be equal to $\Delta(\mathfrak{s}^+)$ or $\Delta(\mathfrak{s}^-)$, and we can assume $\Delta_n^+(\mathfrak{d}) = \Delta(\mathfrak{s}^+)$. It follows that $2\rho_n(\mathfrak{d})$ is the weight of the one-dimensional $\mathfrak{c}$-module $\bigwedge^{\mathrm{top}} \mathfrak{s}^+$, and so it must be orthogonal to the roots of $\mathfrak{c}$. $\square$

REMARK 3.3. If $\mathfrak{l}$ has a simple noncompact factor that is not Hermitian, and if $\Delta^+(\mathfrak{l})$ is any positive root system for $\mathfrak{l}$, then $\rho_n(\mathfrak{l})$ is not orthogonal to all roots of $\mathfrak{l} \cap \mathfrak{k}$. Indeed, if $\mathfrak{d} = \mathfrak{c} \oplus \mathfrak{s}$ is the Cartan decomposition of one such factor, then $\mathfrak{c}$ is semisimple and hence has no nontrivial one-dimensional modules. So if $\rho_n(\mathfrak{d})$ were orthogonal to all roots of $\mathfrak{c}$, it would have to be 0, but that is not possible since $\mathfrak{d}$ is noncompact.

The following proposition will enable us to get our constants for some of the real forms of $\mathcal{O}^{\mathbb{C}}$ without having to do computations.

PROPOSITION 3.4. *Let h_1 and h_2 correspond to two real forms of $\mathcal{O}^{\mathbb{C}}$. Assume that there is an automorphism σ of $\mathfrak{g}$ such that*

1. *σ preserves the compact Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$;*
2. *σ commutes with the Cartan involution, so it preserves $\mathfrak{k}$ and $\mathfrak{p}$;*
3. *$\sigma(\Delta_c^+) = \Delta_c^+$;*
4. *$\sigma(h_1) = h_2$.*

Then the constants c_1, c_2 corresponding to h_1, h_2 are related by

$$c_2 = (-1)^{n+N_1+N_2}\, c_1,$$

where

$$n = \#[\Delta_n^+(\mathfrak{l}_2) \cap (-\sigma(\Delta_n^+(\mathfrak{l}_1)))] = \#\{\alpha \in \Delta_n^+(\mathfrak{l}_1) \mid \sigma\alpha \in (-\Delta^+)\},$$

and N_1, N_2 are defined as in (3.3), i.e.,

$$N_i = \#\{\alpha \in \Delta^+ \mid \alpha(h_i) > 0\}, \qquad i = 1, 2.$$

PROOF. If $\mathfrak{l}_i$ denotes the centralizer of h_i in $\mathfrak{g}$, then it is clear that $\sigma(\mathfrak{l}_1) = \mathfrak{l}_2$. Moreover, the conditions we put on σ ensure that $P_K \circ \sigma = P_K$, and also $P_{L_2 \cap K} \circ \sigma = P_{L_1 \cap K}$.

We let σ act on roots by

$$\sigma(\alpha) = \alpha \circ \sigma^{-1}.$$

Then it is clear that σ takes $\Delta(\mathfrak{p}_1)_1$ (the set of noncompact roots that are 1 on h_1) to $\Delta(\mathfrak{p}_1)_2$ (the set of noncompact roots that are 1 on h_2). Furthermore, σ takes $\Delta_n^+(\mathfrak{l}_1)$ to $\widetilde{\Delta_n^+}(\mathfrak{l}_2)$, where $\widetilde{\Delta_n^+}(\mathfrak{l}_2)$ is a positive root system for $\mathfrak{l}_2$, possibly different from $\Delta_n^+(\mathfrak{l}_2)$ which is defined using Δ^+. It follows that $\sigma(\rho_n(\mathfrak{l}_1)) = \tilde{\rho}_n(\mathfrak{l}_2)$, where $\tilde{\rho}_n(\mathfrak{l}_2)$ is the half sum of roots in $\widetilde{\Delta_n^+}(\mathfrak{l}_2)$.

Also, for any $A \subseteq \Delta_n^+(\mathfrak{l}_1)$, $C \subseteq \Delta(\mathfrak{p}_1)_1$, we clearly have

$$2\rho(\sigma(A)) = \sigma(2\rho(A)), \qquad 2\rho(\sigma(C)) = \sigma(2\rho(C)).$$

Writing the equation (3.2) for c_1, we get

$$(-1)^{N_1} \sum_{\substack{A \subseteq \Delta_n^+(\mathfrak{l}_1) \\ C \subseteq \Delta(\mathfrak{p}_1)_1}} (-1)^{\#A + \#C} P_K(\lambda - \rho_n(\mathfrak{l}_1) + 2\rho(A) - 2\rho(C)) = c_1 P_{L_1 \cap K}(\lambda).$$

We now replace λ by $\sigma^{-1}(\lambda)$ and use the equalities $P_{L_1 \cap K} \circ \sigma^{-1} = P_{L_2 \cap K}$, $P_K \circ \sigma^{-1} = P_K$. We also replace summing over A and C by summing over $\sigma(A)$ and $\sigma(C)$. We obtain

(3.5)
$$\begin{aligned} &(-1)^{N_1} \sum_{\substack{\sigma(A) \subseteq \widetilde{\Delta_n^+}(\mathfrak{l}_2) \\ \sigma(C) \subseteq \Delta(\mathfrak{p}_1)_2}} (-1)^{\#\sigma(A) + \#\sigma(C)} P_K(\lambda - \tilde{\rho}_n(\mathfrak{l}_2) + 2\rho(\sigma(A)) - 2\rho(\sigma(C))) \\ &= c_1 P_{L_2 \cap K}(\lambda). \end{aligned}$$

We now want to pass from summing over $\sigma(A) \subseteq \widetilde{\Delta_n^+}(\mathfrak{l}_2)$ to summing over $A' \subseteq \Delta_n^+(\mathfrak{l}_2)$. To do this, we define

$$\Delta_1 = \Delta_n^+(\mathfrak{l}_2) \cap \widetilde{\Delta_n^+}(\mathfrak{l}_2); \qquad \Delta_2 = \Delta_n^+(\mathfrak{l}_2) \setminus \Delta_1 = \Delta_n^+(\mathfrak{l}_2) \cap (-\widetilde{\Delta_n^+}(\mathfrak{l}_2));$$

so

$$\Delta_n^+(\mathfrak{l}_2) = \Delta_1 \cup \Delta_2; \qquad \widetilde{\Delta_n^+}(\mathfrak{l}_2) = \Delta_1 \cup (-\Delta_2).$$

It follows that for the half sums of roots $\rho_n(\mathfrak{l}_2), \tilde{\rho}_n(\mathfrak{l}_2)$ we have

$$\tilde{\rho}_n(\mathfrak{l}_2) = \rho_n(\mathfrak{l}_2) - 2\rho(\Delta_2). \tag{3.6}$$

For any $A' \subseteq \Delta_n^+(\mathfrak{l}_2)$, let

$$A'_1 = A' \cap \Delta_1; \qquad A'_2 = A' \cap \Delta_2;$$

so $A' = A'_1 \cup A'_2$. To each such A' we attach

$$\tilde{A} = A'_1 \cup (-(\Delta_2 \setminus A'_2)) \subseteq \widetilde{\Delta_n^+}(\mathfrak{l}_2).$$

Then the correspondence $A' \leftrightarrow \tilde{A}$ defines a bijection between the subsets of $\Delta_n^+(\mathfrak{l}_2)$ and the subsets of $\widetilde{\Delta_n^+}(\mathfrak{l}_2)$. Using (3.6), we see that

$$(3.7)\quad 2\rho(\tilde{A}) - \tilde{\rho}_n(\mathfrak{l}_2) = 2\rho(A_1') - (2\rho(\Delta_2) - 2\rho(A_2')) - (\rho_n(\mathfrak{l}_2) - 2\rho(\Delta_2)) = \\ 2\rho(A_1') + 2\rho(A_2') - \rho_n(\mathfrak{l}_2) = 2\rho(A') - \rho_n(\mathfrak{l}_2).$$

It follows that we can rewrite (3.5) into a sum over A' instead of a sum over $\tilde{A} = \sigma(A)$, taking into account that

$$(-1)^{\#\tilde{A}} = (-1)^n(-1)^{\#A'},$$

with n as in the statement of the proposition. If we also rename A' by A and $\sigma(C)$ by C, we get

$$(-1)^{N_1+n} \sum_{\substack{A\subseteq\Delta_n^+(\mathfrak{l}_2)\\ C\subseteq\Delta(\mathfrak{p}_1)_2}} (-1)^{\#A+\#C} P_K(\lambda - \rho_n(\mathfrak{l}_2) + 2\rho(A) - 2\rho(C)) = c_1 P_{L\cap K}(\lambda).$$

If we compare this with the equation (3.2) written for c_2, we immediately get the statement of the proposition. ∎

4. The case $G_{\mathbb{R}} = SU(p,q)$, $p \le q$

This case was treated in [MPVZ] and we just record the results here. The real forms of $\mathcal{O}^{\mathbb{C}}$ correspond to

$$(4.1)\qquad h_k = (\underbrace{1,\dots,1}_{k}, \underbrace{-1,\dots,-1}_{p-k} \,|\, \underbrace{1,\dots,1}_{p-k}, \underbrace{0,\dots,0}_{q-p}, \underbrace{-1,\dots,-1}_{k}),$$

with $k = 0, 1, \dots, p$. The corresponding constants $c = c_k^{p,q}$ can be computed from the formula (3.4).

The set $\Delta_n^+(\mathfrak{l})$ is

$$\Delta_n^+(\mathfrak{l}) = \{\varepsilon_i - \varepsilon_j \,\big|\, 1 \le i \le k,\ p+1 \le j \le 2p-k\} \cup \\ \cup\ \{\varepsilon_i - \varepsilon_j \,\big|\, k+1 \le i \le p,\ p+q-k+1 \le j \le p+q\}.$$

The set $\Delta(\mathfrak{p}_1)$ is empty if $q = p$, and if $q > p$, then

$$\Delta(\mathfrak{p}_1) = \{\varepsilon_i - \varepsilon_j \,\big|\, 1 \le i \le k,\ 2p-k+1 \le j \le p+q-k\} \cup \\ \cup\ \{\varepsilon_i - \varepsilon_j \,\big|\, 2p-k+1 \le i \le p+q-k,\ k+1 \le j \le p\}.$$

We evaluate (3.4) at $\lambda = \lambda_0$, where

$$\lambda_0 = (q, q-1, \dots, q-k+1, p, p-1, \dots, k+1 \\ \,|\, p-k, \dots, 1, q-k, \dots, p-k+1, k, \dots, 1).$$

For each choice of $A \subseteq \Delta_n^+(\mathfrak{l})$ and $C \subseteq \Delta(\mathfrak{p}_1)$, we set

$$\Lambda = \lambda_0 - 2\rho(A) - 2\rho(C).$$

For $q > p$, we show that there is exactly one $C \subseteq \Delta(\mathfrak{p}_1)$ for which $P_K(\Lambda)$ can be nonzero:

$$C = \{\varepsilon_i - \varepsilon_j \,\big|\, 1 \le i \le k; 2p-k+1 \le j \le p+q-k\},$$

with $\#C = k(q-p)$.

Then we show that all Λ as above, with $P_K(\Lambda) \neq 0$, are of the form

$$\Lambda = (i_1, \dots, i_k; j_1, \dots, j_{p-k} \,|\, j_1, \dots, j_{p-k}; q, \dots, p+1; i_1, \dots, i_k),$$

with $i_1, \dots, i_k; j_1, \dots, j_{p-k}$ a shuffle of $p, \dots, 1$, i.e.,

$$i_1 > \dots > i_k; \quad j_1 > \dots > j_{p-k}; \quad \{i_1, \dots, i_k; j_1, \dots, j_{p-k}\} = \{p, \dots, 1\}.$$

For each such Λ there is a unique corresponding A, consisting of roots $\alpha_{a,b}$, $1 \le a \le k$, $1 \le b \le p-k$, where

$$\alpha_{a,b} = \begin{cases} \varepsilon_a - \varepsilon_{p+b}, & \text{if } i_a < j_b; \\ \varepsilon_{k+a} - \varepsilon_{p+q-k+b}, & \text{if } i_a > j_b. \end{cases}$$

In particular, for each A involved, $\#A = k(p-k)$.

This leads to the following result.

THEOREM 4.1. *Let $G_{\mathbb{R}} = SU(p,q)$, and let $k \in \{0, 1, \dots, p\}$ correspond to the real form of $\mathcal{O}^{\mathbb{C}}$ given by (4.1). Then $c_k^{p,q} = (-1)^{k(p+q-k)}\binom{p}{k}$.*

5. The case $G_{\mathbb{R}} = SO_e(2p, 2q+1)$, $q \ge p-1 \ge 0$

There are three real forms of $\mathcal{O}^{\mathbb{C}}$ if $q \ge p \ge 1$, and two real forms if $q = p-1$.

5.1. *The first real form.* This real form exists for all $q \ge p-1 \ge 0$. The corresponding h is

$$h_1 = (2, \underbrace{1, \dots, 1}_{p-1} \,|\, \underbrace{1, \dots, 1}_{p-1}, \underbrace{0, \dots, 0}_{q-p+1}).$$

Since $\mathfrak{l} = \mathfrak{l}_1$ is built from roots that vanish on h_1, we see that

$$\Delta_n^+(\mathfrak{l}) = \{\varepsilon_i - \varepsilon_j \,\big|\, 2 \le i \le p,\, p+1 \le j \le 2p-1\}.$$

It follows that for any $A \subseteq \Delta_n^+(\mathfrak{l})$,

$$2\rho(A) = (0; a_1, \dots, a_{p-1} \,|\, -b_1, \dots, -b_{p-1}; 0, \dots, 0), \tag{5.1}$$

with

$$0 \le a_i, b_j \le p-1; \qquad \textstyle\sum_i a_i = \sum_j b_j. \tag{5.2}$$

Furthermore, recall that $\Delta(\mathfrak{p}_1)$ consists of noncompact roots that are 1 on h_1. So

$$\Delta(\mathfrak{p}_1) = \{\varepsilon_1 - \varepsilon_j \,\big|\, p+1 \le j \le 2p-1\} \cup \\ \cup\ \{\varepsilon_i \pm \varepsilon_j \,\big|\, 2 \le i \le p,\, 2p \le j \le p+q\} \cup \{\varepsilon_i \,\big|\, 2 \le i \le p\}.$$

It follows that for any $C \subseteq \Delta(\mathfrak{p}_1)$,

$$2\rho(C) = (c; d_1, \dots, d_{p-1} \,|\, -c_1, \dots, -c_{p-1}; e_1, \dots, e_{q-p+1}), \tag{5.3}$$

with

$$\begin{aligned} &0 \le c_j \le 1; \qquad 0 \le c \le p-1; \qquad c = \textstyle\sum_j c_j; \\ &0 \le d_i \le 2(q-p+1)+1; \qquad -(p-1) \le e_j \le p-1. \end{aligned} \tag{5.4}$$

Note that for $q = p-1$, there are no coordinates after $2p-1$, so there are no zeros at the end of $2\rho(A)$, and there are no e_j. Otherwise, all of the above holds in this special case.

By (5.1),

$$\rho_n(\mathfrak{l}) = (0, p-1, \dots, p-1 \,|\, -p+1, \dots, -p+1, 0, \dots, 0).$$

This is clearly orthogonal to all roots of $\mathfrak{l} \cap \mathfrak{k}$, which are equal to

$$\begin{aligned} \Delta(\mathfrak{l} \cap \mathfrak{k}) = \{\varepsilon_i - \varepsilon_j \,\big|\, 2 \le i, j \le p\} \cup \{\varepsilon_i - \varepsilon_j \,\big|\, p+1 \le i, j \le 2p-1\} \cup \\ \cup \{\varepsilon_i \pm \varepsilon_j \,\big|\, 2p \le i, j \le p+q\}. \end{aligned} \tag{5.5}$$

By Proposition 3.1, this means that we can determine the constant $c = c_1^{p,q}$ from the equation (3.4). To do this, we take $\lambda = \lambda_0$, where

(5.6)

$$\begin{aligned}
\lambda_0 = (\tfrac{1}{2}; q+\tfrac{1}{2}, q-\tfrac{1}{2} \dots, q-p+\tfrac{5}{2} \,|\, \\
|\, -1, -2, \dots, -(p-1); q-p+1, q-p, \dots, 1) \quad &\text{if } \ q \ge p \ge 2; \\
\lambda_0 = (\tfrac{1}{2} \,|\, q, q-1, \dots, 1) \qquad \text{if } \ p = 1, q \ge 1;& \\
\lambda_0 = (\tfrac{1}{2}; p-\tfrac{1}{2}, p-\tfrac{3}{2} \dots, \tfrac{3}{2} \,|\, -1, -2, \dots, -(p-1)) \quad &\text{if } \ p \ge 2, q = p-1; \\
\lambda_0 = (\tfrac{1}{2} \,|\,) \qquad \text{if } \ p = 1, q = 0.&
\end{aligned}$$

Proposition 5.1. *Let $\Lambda = \lambda_0 - 2\rho(A) - 2\rho(C)$, with λ_0 given by* (5.6), *and with $A \subseteq \Delta_n^+(\mathfrak{l})$ and $C \subseteq \Delta(\mathfrak{p}_1)$. If $P_K(\Lambda) \neq 0$, then:*

1. *If $p = 1$, then $A = C = \emptyset$ and $\Lambda = \lambda_0$.*
2. *If $p \ge 2$ and $q = p-1$, then $A = C = \emptyset$ and $\Lambda = \lambda_0$.*
3. *If $q \ge p \ge 2$, then*

$$\begin{aligned} &A = \emptyset; \\ &C = \{\varepsilon_i - \varepsilon_j \,\big|\, 2 \le i \le p,\, 2p \le j \le p+q\}; \\ &\Lambda = (\tfrac{1}{2}; p-\tfrac{1}{2}, p-\tfrac{3}{2}, \dots, \tfrac{3}{2} \,|\, -1, -2, \dots, -(p-1); q, q-1, \dots, p). \end{aligned}$$

Proof. We first note that if $p = 1$, then

$$h_1 = (2 \,|\, 0, \dots, 0) \ \text{ or } \ h_1 = (2),$$

so $\Delta_n^+(\mathfrak{l}) = \Delta(\mathfrak{p}_1) = \emptyset$ and both A and C are automatically empty. It follows that the only possible Λ is $\Lambda = \lambda_0$, and this proves the proposition for $p = 1$. We continue by induction on p. Let us assume that $p \geq 2$, that $q \geq p-1$ is arbitrary, and that the statement of the proposition is true for $G_{\mathbb{R}} = SO_e(2p-2, 2p-3)$, i.e., when p, q are replaced by $p' = p-1$, and $q' = p-2$.

By (5.6), (5.1) and (5.3), we have

$$\begin{aligned} \Lambda = (\tfrac{1}{2} - c; q + \tfrac{1}{2} - a_1 - d_1, q - \tfrac{1}{2} - a_2 - d_2, \dots, q - p + \tfrac{5}{2} - a_{p-1} - d_{p-1} \,| \\ | -1 + b_1 + c_1, -2 + b_2 + c_2, \dots, -(p-1) + b_{p-1} + c_{p-1}; \\ q - p + 1 - e_1, q - p - e_2, \dots, 1 - e_{q-p+1}) \end{aligned} \tag{5.7}$$

(the third row of the above equation is not there if $q = p-1$).

Using (5.2) and (5.4), we see that the coordinates $\Lambda_{p+1}, \dots, \Lambda_{p+q}$ are in the following intervals:

$$\begin{aligned} \Lambda = (\dots | \underbrace{-1 + b_1 + c_1}_{[-1, p-1]}, \underbrace{-2 + b_2 + c_2}_{[-2, p-2]}, \dots, \underbrace{-(p-1) + b_{p-1} + c_{p-1}}_{[-(p-1), 1]}; \\ \underbrace{q - p + 1 - e_1}_{[q-2p+2, q]}, \underbrace{q - p - e_2}_{[q-2p+1, q-1]}, \dots, \underbrace{1 - e_{q-p+1}}_{[-(p-2), p]}) \end{aligned}$$

(the second row of the above equation is not there if $q = p-1$).

So $\Lambda_{p+1}, \dots, \Lambda_{p+q}$ are q integers between $-(p-1)$ and q. Moreover, $P_K(\Lambda) \neq 0$ implies that these integers are nonzero, different from each other, and no two of them are opposite integers. If $q \geq p$, it follows that $q, q-1, \dots, p$ must each be equal to some Λ_i, and the only possibility for that is

$$\Lambda_{2p} = q, \ \Lambda_{2p+1} = q-1, \ \dots, \ \Lambda_{p+q} = p.$$

So $e_1, \dots, e_{q-p+1}$ are all equal to $-(p-1)$, and hence

$$\varepsilon_i - \varepsilon_j \in C, \ \varepsilon_i + \varepsilon_j \notin C, \qquad 2 \leq i \leq p, \ 2p \leq j \leq p+q$$

(if $q = p-1$, the above says nothing and should be skipped).

This implies

$$q - p + 1 \leq d_i \leq q - p + 2, \qquad 1 \leq i \leq p-1, \tag{5.8}$$

with $d_i = q - p + 1$ if $\varepsilon_{i+1} \notin C$, and $d_i = q - p + 2$ if $\varepsilon_{i+1} \in C$. (If $q = p-1$, this gives no new information about the d_i. The following arguments all work also in case $q = p-1$ if we delete the last group of coordinates, $q, q-1, \dots, p$.)

Using (5.8) together with the inequalities (5.2), (5.4) for a_i and c, we go back to (5.7) and conclude that $\Lambda_1, \dots, \Lambda_p$ are in the following intervals:

$$\Lambda = (\underbrace{\tfrac{1}{2} - c}_{[-p+\frac{3}{2}, \frac{1}{2}]}; \underbrace{q + \tfrac{1}{2} - a_1 - d_1}_{[-\frac{1}{2}, p-\frac{1}{2}]}, \underbrace{q - \tfrac{1}{2} - a_2 - d_2}_{[-\frac{3}{2}, p-\frac{3}{2}]}, \dots, \underbrace{q - p + \tfrac{5}{2} - a_{p-1} - d_{p-1}}_{[-p+\frac{3}{2}, \frac{3}{2}]} | \dots)$$

So $\Lambda_1, \dots, \Lambda_p$ are p half-integers between $-p+\frac{3}{2}$ and $p-\frac{1}{2}$. Moreover, $P_K(\Lambda) \neq 0$ implies that these half-integers are different from each other, and no two of them are opposite. It follows that one of them must be equal to $p - \frac{1}{2}$, and the only possibility is

$$\Lambda_2 = p - \frac{1}{2}.$$

So $a_1 = 0$ and $d_1 = q - p + 1$. It follows that

$$\begin{aligned} &\varepsilon_2 - \varepsilon_j \notin A, \qquad p+1 \le j \le 2p-1; \\ &\varepsilon_2 \notin C, \end{aligned}$$

and hence

$$0 \le b_j \le p-2, \qquad 1 \le j \le p-1.$$

These improved inequalities for the b_j together with inequalities (5.4) for the c_j imply

$$\begin{aligned} \Lambda =(\dots \,|\, &\underbrace{-1+b_1+c_1}_{[-1,p-2]}, \underbrace{-2+b_2+c_2}_{[-2,p-3]}, \dots, \underbrace{-(p-1)+b_{p-1}+c_{p-1}}_{[-(p-1),0]}; \\ &q, q-1, \dots, p). \end{aligned}$$

Since $\Lambda_{p+1}, \dots, \Lambda_{2p-1}$ are $p-1$ nonzero integers between $-(p-1)$ and $p-2$, with no two of them equal or opposite to each other, we conclude that

$$\Lambda_{2p-1} = -(p-1).$$

This implies

$$b_{p-1} = c_{p-1} = 0,$$

and hence

$$\begin{aligned} &\varepsilon_i - \varepsilon_{2p-1} \notin A, \quad 2 \le i \le p; \\ &\varepsilon_1 - \varepsilon_{2p-1} \notin C, \end{aligned}$$

and

$$\begin{aligned} &0 \le a_i \le p-2, \quad 2 \le i \le p-1; \\ &0 \le c \le p-2. \end{aligned}$$

We see that

$$\begin{aligned} \Lambda = (\tfrac{1}{2} - c; p - \tfrac{1}{2}, q - \tfrac{1}{2} - a_2 - d_2, \dots, q - p + \tfrac{5}{2} - a_{p-1} - d_{p-1} \,|\, \\ |\, -1 + b_1 + c_1, \dots, -(p-2) + b_{p-2} + c_{p-2}, -(p-1); \\ q, q-1, \dots, p) \end{aligned}$$

(the third row is not there if $q = p-1$).

We now consider the subalgebra $\mathfrak{g}' \cong \mathfrak{so}(2p-2, 2p-3)$ of $\mathfrak{g}$ built on coordinates

$$\varepsilon_1, \varepsilon_3, \dots, \varepsilon_p; \varepsilon_{p+1}, \dots, \varepsilon_{2p-2},$$

so the coordinates 2 and $2p-1, 2p, \dots, p+q$ are deleted. We also consider the real form of $\mathcal{O}_{K'}$ given by

$$h_1' = (2, \underbrace{1, \dots, 1}_{p-2} \,|\, \underbrace{1, \dots, 1}_{p-2}),$$

with centralizer $\mathfrak{l}' = \mathfrak{l} \cap \mathfrak{g}'$. Then

$$\begin{aligned}
\Delta_n^+(\mathfrak{l}') &= \Delta_n^+(\mathfrak{l}) \setminus \{\varepsilon_i - \varepsilon_j \,\big|\, i = 2 \text{ or } j = 2p-1\};\\
\Delta(\mathfrak{p}_1') &= \{\varepsilon_1 - \varepsilon_{p+1}, \dots, \varepsilon_1 - \varepsilon_{2p-2}; \varepsilon_3, \dots, \varepsilon_p\}.
\end{aligned}$$

We set

$$\begin{aligned}
A' &= A \cap \Delta_n^+(\mathfrak{l}') = A;\\
C' &= C \cap \Delta(\mathfrak{p}_1') = C \setminus \{\varepsilon_i - \varepsilon_j \,\big|\, 2 \le i \le p,\ 2p \le j \le p+q\}.
\end{aligned}$$

Then

$$\begin{aligned}
2\rho(A') &= (0; a_2, \dots, a_{p-1} \,|\, -b_1, \dots, -b_{p-2})\\
&= (0; a_1', \dots, a_{p-2}' \,|\, -b_1', \dots, -b_{p-2}');\\
2\rho(C') &= (c; d_2 - (q-p+1), \dots, d_{p-1} - (q-p+1) \,|\, -c_1, \dots, -c_{p-2})\\
&= (c'; d_1', \dots, d_{p-1}' \,|\, -c_1', \dots, -c_{p-2}'),
\end{aligned}$$

where we define

$$a_i' = a_{i+1}; \quad b_i' = b_i; \quad c_i' = c_i; \quad c' = c; \quad d_i' = d_{i+1} - (q-p+1).$$

The numbers $a_i', b_i', c_i', c', d_i'$ satisfy analogues of (5.2) and (5.4). We define λ_0' by (5.6), but for $G_{\mathbb{R}} = SO_e(2p-2, 2p-3)$, i.e.,

$$\lambda_0' = (\frac{1}{2}; p - \frac{3}{2}, \dots, \frac{3}{2} \,|\, -1, -2, \dots, -(p-2)).$$

Then A', C' and

$$\Lambda' = \lambda_0' - 2\rho(A') - 2\rho(C')$$

satisfy all conditions of the proposition, but p, q are reduced to $p' = p-1$, $q' = p-2$. Moreover, $P_K(\Lambda) \neq 0$ is equivalent to $P_{K'}(\Lambda') \neq 0$. Therefore the inductive assumption implies that $A' = C' = \emptyset$, and that $\Lambda' = \lambda_0'$. This implies the statement of the proposition for A, C and Λ. ∎

To compute the constant $c_1^{p,q}$, we have to compute $P_{L\cap K}(\lambda_0)$ where λ_0 is given by (5.6), and $P_K(\Lambda)$ for Λ determined in Proposition 5.1. The main ingredients for this computation are given in the following lemma.

Lemma 5.2. *(i) Let P_p^1 be the Weyl dimension formula polynomial for $\mathfrak{so}(2p)$, $p \ge 1$, and let $\lambda_p = (p - \frac{1}{2}, p - \frac{3}{2}, \dots, \frac{1}{2})$. Then*

$$P_p^1(\lambda_p) = 2^{p-1}.$$

(ii) Let P^2_q be the Weyl dimension formula polynomial for $\mathfrak{so}(2q+1)$, $q \geq 1$, and let P^2_0 be the constant polynomial 1*. Furthermore, let $\mu_q = (q, q-1, \dots, 1)$ if $q \geq 1$, and $\mu_0 = 0$. Then*

$$P^2_q(\mu_q) = 2^q.$$

PROOF. (i) Let n^1_p be the numerator of P^1_p; the denominator is then

$$d^1_p = n^1_p(\rho_{\mathfrak{so}(2p)}) = n^1_p(p-1, p-2, \dots, 1, 0).$$

The factors of $n^1_p(\lambda_p)$ that correspond to the roots $\varepsilon_i - \varepsilon_j$ clearly cancel with the corresponding factors of d^1_p. Denoting by $m^1_p(\lambda_p)$ respectively e^1_p the product of factors of $n^1_p(\lambda_p)$ respectively d^1_p corresponding to the roots $\varepsilon_i + \varepsilon_j$, we have

$$\begin{aligned} m^1_p(\lambda_p) &= (2p-2)(2p-3)\dots(p+1)p\, m^1_{p-1}(\lambda_{p-1}); \\ e^1_p &= (2p-3)(2p-4)\dots p(p-1)\, e^1_{p-1}. \end{aligned}$$

It follows that

$$P^1_p(\lambda_p) = \frac{m^1_p(\lambda_p)}{e^1_p} = \frac{(2p-2)m^1_{p-1}(\lambda_{p-1})}{(p-1)e^1_{p-1}} = 2P^1_{p-1}(\lambda_{p-1}).$$

Since P^1_1 is the constant polynomial 1, this proves (i).

(ii) There is nothing to prove for $q = 0$, and it is obvious that

$$P^2_1(\mu_1) = \frac{1}{1/2} = 2.$$

For $q \geq 2$, let n^2_q be the numerator of P^2_q; the denominator is then

$$d^2_q = n^2_q(\rho_{\mathfrak{so}(2q+1)}) = n^2_q(q - \frac{1}{2}, q - \frac{3}{2}, \dots, \frac{3}{2}, \frac{1}{2}).$$

The factors of $n^2_q(\mu_q)$ that correspond to the roots $\varepsilon_i - \varepsilon_j$ clearly cancel with the corresponding factors of d^2_q. Denoting by $m^2_q(\mu_q)$ respectively e^2_q the product of factors of $n^2_q(\mu_q)$ respectively d^2_q corresponding to the roots $\varepsilon_i + \varepsilon_j$ and ε_i, we have

$$\begin{aligned} m^2_q(\mu_q) &= (2q-1)(2q-2)\dots(q+2)(q+1)q\, m^2_{q-1}(\mu_{q-1}); \\ e^2_q &= (2q-2)(2q-3)\dots(q+1)q(q-\frac{1}{2})\, e^2_{q-1}. \end{aligned}$$

It follows that

$$P^2_q(\mu_q) = \frac{m^2_q(\mu_q)}{e^2_q} = \frac{(2q-1)m^2_{q-1}(\mu_{q-1})}{(q-\frac{1}{2})e^2_{q-1}} = 2P^2_{q-1}(\mu_{q-1}).$$

The statement now follows by induction. $\square$

To compute $P_{L\cap K}(\lambda_0)$, we recall (5.5), which shows that $\mathfrak{l}\cap\mathfrak{k}$ is up to center typically a product of three factors: the $\mathfrak{u}(p-1)$ on coordinates $2,\dots,p$, the $\mathfrak{u}(p-1)$ on coordinates $p+1,\dots,2p-1$, and the $\mathfrak{so}(2(q-p+1)+1)$ on coordinates $2p,\dots,p+q$. (If $p=1$, then the first two factors are missing, if $q=p-1$ then the third factor is missing, and if $p=1$ and $q=0$, then $\mathfrak{l}\cap\mathfrak{k}$ is one-dimensional. What we say below applies also to these cases with obvious modifications.)

It is clear from the definition (5.6) of λ_0 that for each of the first two factors, the corresponding coordinates of λ_0 differ from the ρ of the factor by a weight orthogonal to the roots of the factor, so in the notation of Lemma 5.2,

$$P_{L\cap K}(\lambda_0) = P^2_{q-p+1}(\mu_{q-p+1}) = 2^{q-p+1}. \tag{5.9}$$

To compute $P_K(\Lambda)$, we first write $\Lambda = (\Lambda_L \mid \Lambda_R)$ and note that

$$P_K(\Lambda) = P^1_p(\Lambda_L)P^2_q(\Lambda_R).$$

To use Lemma 5.2, we have to rearrange coordinates of Λ_L and Λ_R, and use the fact that P^1_p is skew for the Weyl group of $\mathfrak{so}(2p)$, while P^2_q is skew for the Weyl group of $\mathfrak{so}(2q+1)$. To rearrange Λ_L to λ_p, we only need to bring the $\frac{1}{2}$ from the first coordinate to the p-th coordinate, and hence

$$P^1_p(\Lambda_L) = (-1)^{p-1}2^{p-1}. \tag{5.10}$$

To bring Λ_R to $\mu_q = (q,\dots,1)$, we need to change $p-1$ signs, and then bring coordinates $p-1, p-2, \dots, 1$, in that order, all the way to the right. The sign produced in this way is

$$(-1)^{(p-1)+(q-p+1)+(q-p+2)+\cdots+(q-1)} = (-1)^{(p-1)(q-p+1)+\frac{(p-1)p}{2}}.$$

Since

$$\frac{(p-1)p}{2} \equiv [\frac{p}{2}] \mod 2, \tag{5.11}$$

Lemma 5.2 implies that

$$P^2_q(\Lambda_R) = (-1)^{(p-1)(q-p+1)+[\frac{p}{2}]}2^q. \tag{5.12}$$

Now we substitute (5.9), (5.10) and (5.12) into (3.4). Since

$$\#A + \#C = \#C = (p-1)(q-p+1),$$

and since N from (3.3) is easily checked to satisfy

$$N \equiv p \mod 2, \tag{5.13}$$

we see that the total sign is $(-1)^{[\frac{p}{2}]+1}$, and we conclude the following result.

THEOREM 5.3. *Let $G_{\mathbb{R}} = SO_e(2p, 2q+1)$, $q \geq p-1 \geq 0$, and let $c_1^{p,q}$ be the constant corresponding to the first real form of $\mathcal{O}^{\mathbb{C}}$. Then*

$$c_1^{p,q} = (-1)^{[\frac{p}{2}]+1}2^{2p-2}.$$

5.2. *The second real form.* This real form exists for all $q \geq p-1 \geq 0$. The corresponding h is

$$h_2 = (2, \underbrace{1, \dots, 1}_{p-2}, -1 \,|\, \underbrace{1, \dots, 1}_{p-1}, \underbrace{0, \dots, 0}_{q-p+1}).$$

This real form is conjugate to the first real form by the automorphism $\sigma = s_{\varepsilon_p}$, the reflection with respect to the short noncompact root ε_p. The automorphism σ of $\mathfrak{g}$ clearly satisfies the conditions of Proposition 3.4. Moreover, the number n from Proposition 3.4 is $p-1$; the roots from $\Delta_n^+(\mathfrak{l}_1)$ that σ sends to $-\Delta^+$ are

$$\varepsilon_p - \varepsilon_j, \qquad p+1 \leq j \leq 2p-1.$$

Moreover, by (5.13), $N_1 \equiv p \mod 2$, and another short computation shows that N_2 is always even. The total sign in Proposition 3.4 is thus

$$(-1)^{n+N_1+N_2} = -1,$$

so

$$c_2^{p,q} = -c_1^{p,q} = (-1)^{[\frac{p}{2}]} 2^{2p-2}. \tag{5.14}$$

5.3. *The third real form.* This real form exists for $q \geq p \geq 1$, so we assume this condition in the following. The corresponding h is

$$h_3 = (\underbrace{1, \dots, 1}_{p-1}, 0 \,|\, 2, \underbrace{1, \dots, 1}_{p-1}, \underbrace{0, \dots, 0}_{q-p}).$$

Since $\mathfrak{l} = \mathfrak{l}_3$ is built from roots that vanish on h_3, we see that

$$\begin{aligned} \Delta_n^+(\mathfrak{l}) = \{\varepsilon_i - \varepsilon_j \,\big|\, 1 \leq i \leq p-1,\, p+2 \leq j \leq 2p\} \\ \cup \{\varepsilon_p \pm \varepsilon_j \,\big|\, 2p+1 \leq j \leq p+q\} \cup \{\varepsilon_p\}. \end{aligned}$$

It follows that for any $A \subseteq \Delta_n^+(\mathfrak{l})$,

$$2\rho(A) = (a_1, \dots, a_{p-1}; x \,|\, 0; -b_1, \dots, -b_{p-1}; y_1, \dots, y_{q-p}), \tag{5.15}$$

with

$$\begin{aligned} &0 \leq a_i, b_j \leq p-1; \qquad \textstyle\sum_i a_i = \sum_j b_j; \\ &0 \leq x \leq 2(q-p)+1; \qquad -1 \leq y_j \leq 1. \end{aligned} \tag{5.16}$$

Furthermore, recall that $\Delta(\mathfrak{p}_1)$ consists of noncompact roots that are 1 on h_3. So

$$\begin{aligned} \Delta(\mathfrak{p}_1) = \{\varepsilon_i \pm \varepsilon_j \,\big|\, 1 \leq i \leq p-1,\, 2p+1 \leq j \leq p+q\} \cup \{\varepsilon_i \,\big|\, 1 \leq i \leq p-1\} \\ \cup\ \{\varepsilon_j \pm \varepsilon_p \,\big|\, p+2 \leq j \leq 2p\} \cup \{\varepsilon_{p+1} - \varepsilon_i \,\big|\, 1 \leq i \leq p-1\}. \end{aligned}$$

It follows that for any $C \subseteq \Delta(\mathfrak{p}_1)$,

$$2\rho(C) = (c_1, \dots, c_{p-1}; u \,|\, v; d_1, \dots, d_{p-1}; e_1, \dots, e_{q-p}), \tag{5.17}$$

with

$$\begin{aligned} -1 \le c_i \le 2(q-p)+1; \qquad -(p-1) \le u \le p-1; \\ 0 \le v \le p-1; \qquad 0 \le d_j \le 2; \qquad -(p-1) \le e_j \le p-1. \end{aligned} \tag{5.18}$$

If we write (5.15) for $A = \Delta_n^+(\mathfrak{l})$, we get

$$\rho_n(\mathfrak{l}) = (p-1,\dots,p-1; 2(q-p)+1 \,|\, 0; -p+1,\dots,-p+1; 0,\dots,0).$$

This is clearly orthogonal to all roots of $\mathfrak{l} \cap \mathfrak{k}$, which are equal to

$$\begin{aligned} \Delta(\mathfrak{l} \cap \mathfrak{k}) = \{\varepsilon_i - \varepsilon_j \,\big|\, 1 \le i,j \le p-1\} \cup \{\varepsilon_i - \varepsilon_j \,\big|\, p+2 \le i,j \le 2p\} \\ \cup \{\varepsilon_i \pm \varepsilon_j \,\big|\, 2p+1 \le i,j \le p+q\} \cup \{\varepsilon_{2p+1},\dots,\varepsilon_{p+q}\}. \end{aligned} \tag{5.19}$$

By Proposition 3.1, this means that the constant $c = c_3^{p,q}$ satisfies (3.4) for any λ, and we will compute $c_3^{p,q}$ by using this for $\lambda = \lambda_0$, where

$$\begin{aligned} \lambda_0 =& (q - \frac{3}{2}, q - \frac{5}{2} \dots, q-p+\frac{1}{2}; q-p+\frac{1}{2} \\ & \,|\, p-1; 0, -1, \dots, -(p-2); q-p, q-p-1, \dots, 1) \end{aligned} \tag{5.20}$$

(out of the 5 groups of coordinates separated by semicolons and the bar, the first and the fourth group are missing if $p = 1$, and the fifth group is missing if $q = p$).

Proposition 5.4. *Let $\Lambda = \lambda_0 - 2\rho(A) - 2\rho(C)$, with λ_0 given by* (5.20), *and with $A \subseteq \Delta_n^+(\mathfrak{l})$ and $C \subseteq \Delta(\mathfrak{p}_1)$. Then $\Lambda_{p+1} = 0$. In particular, $P_K(\Lambda) = 0$.*

Proof. By (5.20), (5.15) and (5.17), we have

$$\begin{aligned} \Lambda = (q - \tfrac{3}{2} - a_1 - c_1, \dots, q-p+\tfrac{1}{2} - a_{p-1} - c_{p-1}; q-p+\tfrac{1}{2} - x - u \,| \\ | \, p-1-v; b_1 - d_1, -1 + b_2 - d_2, \dots, -(p-2) + b_{p-1} - d_{p-1}; \\ q-p-y_1-e_1, q-p-1-y_2-e_2, \dots, 1 - y_{q-p} - e_{q-p}). \end{aligned}$$

If $p = 1$, then the situation is much simpler; in particular, since the coordinates of h_3 are 0 or 2, $\Delta(\mathfrak{p}_1)$ is empty, so $C = \emptyset$ and $2\rho(C) = 0$. It follows that $\Lambda_{p+1} = \Lambda_2 = p - 1 = 0$, and so $P_K(\Lambda) = 0$ as claimed.

So the proposition is true for $p = 1$. We continue by induction on p. Let us assume that $p \ge 2$, that $q \ge p$ is arbitrary, and that the statement of the proposition is true for $G_{\mathbb{R}} = SO_e(2p-2, 2p-1)$, i.e., when p, q are replaced by $p' = p-1$, $q' = p-1$.

By (5.20), (5.16) and (5.18), the coordinates $\Lambda_{p+1},\dots,\Lambda_{p+q}$ are in the following intervals:

(5.21)

$$\Lambda=(\dots|\underbrace{p-1-v}_{[0,p-1]};\underbrace{b_1-d_1}_{[-2,p-1]},\underbrace{-1+b_2-d_2}_{[-3,p-2]},\dots,\underbrace{-(p-2)+b_{p-1}-d_{p-1}}_{[-p,1]};$$
$$\underbrace{q-p-y_1-e_1}_{[q-2p,q]},\underbrace{q-p-1-y_2-e_2}_{[q-2p-1,q-1]},\dots,\underbrace{1-y_{q-p}-e_{q-p}}_{[-(p-1),p+1]}).$$

So $\Lambda_{p+1},\dots,\Lambda_{p+q}$ are q integers between $-p$ and q. Moreover, $P_K(\Lambda)\neq 0$ implies that these integers are nonzero, and no two of them are equal or opposite to each other. It follows that $q,q-1,\dots,p+1$ must each be equal to some Λ_i, and the only possibility for that is

$$\Lambda_{2p+1}=q,\ \Lambda_{2p+2}=q-1,\ \dots,\ \Lambda_{p+q}=p+1.$$

It follows that $y_1,\dots,y_{q-p}$ are all equal to -1, and that $e_1,\dots,e_{q-p}$ are all equal to $-(p-1)$. So

$$\begin{aligned}&\varepsilon_p-\varepsilon_j\in A,\ \varepsilon_p+\varepsilon_j\notin A, && 2p+1\le j\le p+q;\\ &\varepsilon_i-\varepsilon_j\in C,\ \varepsilon_i+\varepsilon_j\notin C, && 1\le i\le p-1,\ 2p+1\le j\le p+q.\end{aligned}$$

This implies

$$\begin{aligned}&q-p\le x\le q-p+1;\\ &q-p-1\le c_i\le q-p+1, \qquad 1\le i\le p-1.\end{aligned}\tag{5.22}$$

Note that $x=q-p$ if $\varepsilon_p\notin A$ and $x=q-p+1$ if $\varepsilon_p\in A$. Similarly, $c_i=q-p-1$ if $\varepsilon_{p+1}-\varepsilon_i\in C$, $\varepsilon_i\notin C$; $c_i=q-p$ if $\varepsilon_{p+1}-\varepsilon_i\in C$, $\varepsilon_i\in C$ or $\varepsilon_{p+1}-\varepsilon_i\notin C$, $\varepsilon_i\notin C$; and $c_i=q-p+1$ if $\varepsilon_{p+1}-\varepsilon_i\notin C$, $\varepsilon_i\in C$.

Using the same arguments as above, we can also conclude from (5.21) that

$$\Lambda_{2p}=-p.$$

This implies that $b_{p-1}=0$ and $d_{p-1}=2$. It follows that

$$\begin{aligned}&\varepsilon_i-\varepsilon_{2p}\notin A, \quad 1\le i\le p-1;\\ &\varepsilon_{2p}\pm\varepsilon_p\in C,\end{aligned}$$

so

$$\begin{aligned}&0\le a_i\le p-2, \quad 1\le i\le p-1;\\ &-(p-2)\le u\le p-2.\end{aligned}\tag{5.23}$$

Using the improved inequalities (5.22) and (5.23), we see that $\Lambda_1,\dots,\Lambda_p$ are in the following intervals:

$$\Lambda=(\underbrace{q-\tfrac{3}{2}-a_1-c_1}_{[-\frac12,p-\frac12]},\underbrace{q-\tfrac{5}{2}-a_2-c_2}_{[-\frac32,p-\frac32]},\dots,\underbrace{q-p+\tfrac12-a_{p-1}-c_{p-1}}_{[-(p-\frac32),\frac32]};$$
$$\underbrace{q-p+\tfrac12-x-u}_{[-(p-\frac32),p-\frac32]}\,|\,\dots$$

So $\Lambda_1,\dots,\Lambda_p$ are p half-integers between $-(p-\frac{3}{2})$ and $p-\frac{1}{2}$, such that no two of them are equal or opposite to each other. It follows that

$$\Lambda_1=p-\frac{1}{2},$$

and consequently $a_1=0$, $c_1=q-p+1$. Therefore,

$$\varepsilon_1-\varepsilon_j\notin A,\quad p+2\le j\le 2p;$$
$$\varepsilon_{p+1}-\varepsilon_1\in C,\ \varepsilon_1\notin C,$$

and we conclude that

$$0\le b_j\le p-2,\quad 1\le j\le p-1;$$
$$1\le v\le p-1.$$

We see that

$$\Lambda=(p-\tfrac12,q-\tfrac52-a_2-c_2,\dots,q-p+\tfrac12-a_{p-1}-c_{p-1};q-p+\tfrac12-x-u\,|$$
$$|\,p-1-v;b_1-d_1,\dots,-(p-3)+b_{p-2}-d_{p-2},-p;$$
$$q,q-1,\dots,p+1)$$

(if $q=p$, the coordinates $q,\dots,p+1$ are not there; if $p=2$ there are no coordinates involving a_i,c_i,b_i or d_i).

We now consider the subalgebra $\mathfrak{g}'\cong\mathfrak{so}(2p-2,2p-1)$ of $\mathfrak{g}$ built on coordinates

$$\varepsilon_2,\varepsilon_3,\dots,\varepsilon_p;\varepsilon_{p+1},\dots,\varepsilon_{2p-1},$$

so the coordinates 1 and $2p,2p+1,\dots,p+q$ are deleted. We also consider the real form of $\mathcal{O}_{K'}$ given by

$$h_3'=(\underbrace{1,\dots,1}_{p-2},0\,|\,2,\underbrace{1,\dots,1}_{p-2}),$$

with centralizer $\mathfrak{l}'=\mathfrak{l}\cap\mathfrak{g}'$. Then

$$\Delta_n^+(\mathfrak{l}')=\{\varepsilon_i-\varepsilon_j\,\big|\,2\le i\le p-1,\ p+2\le j\le 2p-1\}\cup\{\varepsilon_p\};$$
$$\Delta(\mathfrak{p}_1')=\{\varepsilon_2,\dots,\varepsilon_{p-1}\}\cup\{\varepsilon_j\pm\varepsilon_p\,\big|\,p+2\le j\le 2p-1\}$$
$$\cup\{\varepsilon_{p+1}-\varepsilon_i\,\big|\,2\le i\le p-1\}.$$

We set

$$\begin{aligned}A' &= A \cap \Delta_n^+(\mathfrak{l}') = A \setminus \{\varepsilon_p - \varepsilon_j \,\big|\, 2p+1 \leq j \leq p+q\};\\ C' &= C \cap \Delta(\mathfrak{p}_1')\\ &= C \setminus \{\varepsilon_{2p} \pm \varepsilon_p;\, \varepsilon_{p+1} - \varepsilon_1;\, \varepsilon_i - \varepsilon_j \,\big|\, 1 \leq i \leq p-1,\ 2p+1 \leq j \leq p+q\}.\end{aligned}$$

Then

$$\begin{aligned}2\rho(A') &= (a_2, \dots, a_{p-1}; x - (q-p) \,|\, 0; -b_1, \dots, -b_{p-2})\\ &= (a_1', \dots, a_{p-2}'; x' \,|\, 0; -b_1', \dots, -b_{p-2}');\\ 2\rho(C') &= (c_2 - (q-p), \dots, c_{p-1} - (q-p); u \,|\, v-1; d_1, \dots, d_{p-2}) =\\ &= (c_1', \dots, c_{p-2}'; u' \,|\, v'; d_1', \dots, d_{p-2}'),\end{aligned}$$

where we define

$$\begin{aligned}&a_i' = a_{i+1}; \quad x' = x - (q-p); \quad b_i' = b_i;\\ &c_i' = c_{i+1} - (q-p); \quad u' = u; \quad v' = v - 1; \quad d_i' = d_i.\end{aligned}$$

The numbers $a_i', x', b_i', c_i', u', v', d_i'$ satisfy analogues of (5.16) and (5.18).

We define λ_0' by (5.20), but for $G_{\mathbb{R}} = SO_e(2p-2, 2p-1)$, i.e.,

$$\lambda_0' = (p - \frac{5}{2}, p - \frac{7}{2}, \dots, \frac{1}{2}; \frac{1}{2}; \,|\, p-2; 0, -1, \dots, -(p-3)).$$

Then A', C' and

$$\Lambda' = \lambda_0' - 2\rho(A') - 2\rho(C')$$

satisfy all conditions of the proposition, but p, q are reduced to $p' = p-1$, $q' = p-1$. Therefore the inductive assumption implies that $\Lambda_p' = 0$. So $v' = p-2$, and therefore $v = p-1$ and $\Lambda_{p+1} = 0$. It follows that $P_K(\Lambda) = 0$, since Λ is orthogonal to the compact root ε_{p+1}. ∎

Proposition 5.4 implies that the left hand side of (3.4) is 0 in this case. On the other hand,

$$P_{L \cap K}(\lambda_0) \neq 0$$

by (5.20) and (5.19). We conclude

Theorem 5.5. *For $G_{\mathbb{R}} = SO_e(2p, 2q+1)$, $q \geq p \geq 1$, the constant corresponding to the third real form is*

$$c_3^{p,q} = 0.$$

6. The case $G_{\mathbb{R}} = \mathrm{Sp}(2n, \mathbb{R})$, $n \geq 1$

The real forms of $\mathcal{O}^{\mathbb{C}}$ correspond to integers p such that $0 \leq p \leq n$. We denote $n - p$ by q. The h corresponding to p is

$$h_p = (\underbrace{1, \ldots, 1}_{p}, \underbrace{-1, \ldots, -1}_{q}), \qquad p = 0, 1, \ldots, n.$$

Since $\mathfrak{l} = \mathfrak{l}_p$ is built from roots that vanish on h_p, we see that

$$\Delta_n^+(\mathfrak{l}) = \{\varepsilon_i + \varepsilon_{p+j} \,\big|\, 1 \leq i \leq p,\, 1 \leq j \leq q\}.$$

It follows that for any $A \subseteq \Delta_n^+(\mathfrak{l})$,

$$2\rho(A) = (a_1, \ldots, a_p \,|\, b_1, \ldots, b_q), \tag{6.1}$$

with

$$0 \leq a_i \leq q, \qquad 0 \leq b_j \leq p, \qquad \textstyle\sum_i a_i = \sum_j b_j. \tag{6.2}$$

In particular,

$$\rho_n(\mathfrak{l}) = (q, \ldots, q \,|\, p, \ldots, p),$$

and this is clearly orthogonal to the roots of $\mathfrak{l} \cap \mathfrak{k}$, which are given by

$$\Delta^+(\mathfrak{l} \cap \mathfrak{k}) = \{\varepsilon_i - \varepsilon_j \,\big|\, 1 \leq i < j \leq p\} \cup \{\varepsilon_{p+i} - \varepsilon_{p+j} \,\big|\, 1 \leq i < j \leq q\}.$$

So the constants $c = c_p^n$ can be calculated from (3.4). Since it is clear that in our present case

$$\Delta(\mathfrak{p}_1) = \emptyset,$$

(3.4) becomes

$$\sum_{A \subseteq \Delta_n^+(\mathfrak{l})} (-1)^{\#A} P_K(\lambda - 2\rho(A)) = c P_{L \cap K}(\lambda). \tag{6.3}$$

We take $\lambda = \lambda_0$, where

$$\lambda_0 = (n, n-1, \ldots, q+1 \,|\, n, n-1, \ldots, p+1), \tag{6.4}$$

(If p is 0 or n, then there is only one group of coordinates in the above expression, and $\lambda_0 = (n, n-1, \ldots, 1)$.)

Since λ_0 differs from $\rho_{\mathfrak{l} \cap \mathfrak{k}}$ by a weight orthogonal to all roots of $\mathfrak{l} \cap \mathfrak{k}$,

$$P_{L \cap K}(\lambda_0) = 1.$$

So to compute c_p^n we have to compute the left side of (6.3). The following proposition describes the relevant A and the corresponding Λ.

Proposition 6.1. *Let $\Lambda = \lambda_0 - 2\rho(A)$, with λ_0 given by* (6.4)*, and with $A \subseteq \Delta_n^+(\mathfrak{l})$.*

(i) *If p and q are both odd, then $P_K(\Lambda) = 0$ for all Λ as above.*

(ii) *Suppose that at least one of p, q is even, and suppose that for some A the corresponding Λ satisfies $P_K(\Lambda) \neq 0$. Then:*

1. *If $p = 0$ or $q = 0$, then $A = \emptyset$ and $\Lambda = \lambda_0 = (n, n-1, \ldots, 1)$.*

2. *If* $0 < p < n$, *let* $r = [\frac{p}{2}]$ *and* $s = [\frac{q}{2}]$. *Then there is a shuffle*

$$1 \le i_1 < \cdots < i_r \le r+s; \qquad 1 \le j_1 < \cdots < j_s \le r+s$$

of $1, 2, \ldots, r+s$ *such that*

$$A = \{\alpha_{u,v}, \beta_{u,v} \,\big|\, 1 \le u \le r,\, 1 \le v \le s\} \cup B,$$

where

$$\alpha_{u,v} = \varepsilon_{p+1-u} + \varepsilon_{n+1-v}; \qquad \beta_{u,v} = \begin{cases} \varepsilon_{p+1-u} + \varepsilon_{p+v}, & i_u < j_v; \\ \varepsilon_u + \varepsilon_{n+1-v}, & i_u > j_v. \end{cases}$$

and

$$B = \begin{cases} \emptyset, & p, q \text{ even}; \\ \{\varepsilon_{r+1} + \varepsilon_{p+j} \,|\, s+1 \le j \le q\}, & p \text{ odd}; \\ \{\varepsilon_i + \varepsilon_{p+s+1} \,|\, r+1 \le i \le p\}, & q \text{ odd}. \end{cases}$$

The corresponding Λ *has coordinates*

$$\Lambda_1 = n+1-i_1, \ldots, \Lambda_r = n+1-i_r; \qquad \Lambda_{p-r+1} = i_r, \ldots, \Lambda_p = i_1;$$
$$\Lambda_{p+1} = n+1-j_1, \ldots, \Lambda_{p+s} = n+1-j_s; \qquad \Lambda_{n-s+1} = j_s, \ldots, \Lambda_n = j_1,$$

and possibly in addition

$$\Lambda_{p-r} = n-r-s, \quad \text{if } p \text{ is odd}; \qquad \Lambda_{n-s} = n-r-s, \quad \text{if } q \text{ is odd}.$$

PROOF. The statement is obviously true for any n if $p = 0$ or $q = 0$. Hence it is true for $n = 1$. If $n = 2$ and $p = q = 1$, there are two cases:

$$A = \emptyset, \qquad \text{or} \qquad A = \{\varepsilon_1 + \varepsilon_2\}.$$

If $A = \emptyset$, then $\Lambda = \Lambda_0 = (n \,|\, n)$, so $P_K(\Lambda) = 0$. If $A = \{\varepsilon_1 + \varepsilon_2\}$, then $\Lambda = (n-1 \,|\, n-1)$, and again $P_K(\Lambda) = 0$. So the proposition is true for $n = 2$.

We proceed by induction on n. Assume that $n > 2$ and $p, q \ge 1$, and assume that the proposition is true for $n-2$.

Using the definitions and the inequalities (6.2), we see that

$$\Lambda = (\underbrace{n-a_1}_{[p,n]}, \underbrace{n-1-a_2}_{[p-1,n-1]}, \ldots, \underbrace{q+1-a_p}_{[1,q+1]} \,|\, \underbrace{n-b_1}_{[q,n]}, \underbrace{n-1-b_2}_{[q-1,n-1]}, \ldots, \underbrace{p+1-b_q}_{[1,p+1]}).$$

So the coordinates of Λ are n integers between 1 and n, and assuming that $P_K(\Lambda) \ne 0$, they have to be different from each other, i.e., Λ has to be a permutation of $(n, \ldots, 1)$. In particular, some Λ_i must be equal to n and there are two possibilities:

$$\Lambda_1 = n \qquad \text{or} \qquad \Lambda_{p+1} = n. \tag{6.5}$$

Assume first that $\Lambda_1 = n$. Then

$$a_1 = 0,$$

and it follows that

$$\varepsilon_1 + \varepsilon_{p+j} \notin A, \qquad 1 \le j \le q.$$

This implies that

$$0 \le b_j \le p-1, \qquad 1 \le j \le q,$$

and so

$$\Lambda = (n, \underbrace{n-1-a_2}_{[p-1,n-1]}, \dots, \underbrace{q+1-a_p}_{[1,q+1]} \,|\, \underbrace{n-b_1}_{[q+1,n]}, \underbrace{n-1-b_2}_{[q,n-1]}, \dots, \underbrace{p+1-b_q}_{[2,p+1]}).$$

If $p=1$, then there is only $\Lambda_1 = n$ in the left group of coordinates, and we see there is no place to put the coordinate 1. Therefore, if $p=1$ then Λ_1 can not be n, hence $\Lambda_{p+1} = n$, so we are in the second case which we treat below. If $p>1$, then there is exactly one place where 1 can be, i.e.,

$$\Lambda_p = 1.$$

This implies

$$a_p = q,$$

and therefore

$$\varepsilon_p + \varepsilon_{p+j} \in A, \qquad 1 \le j \le q.$$

It follows that

$$1 \le b_j \le p-1, \qquad 1 \le j \le q,$$

and so

$$\Lambda = (n, \underbrace{n-1-a_2}_{[p-1,n-1]}, \dots, \underbrace{q+2-a_{p-1}}_{[2,q+2]}, 1 \,|\, \underbrace{n-b_1}_{[q+1,n-1]}, \underbrace{n-1-b_2}_{[q,n-2]}, \dots, \underbrace{p+1-b_q}_{[2,p]}).$$

Let now $\mathfrak{g}' \cong \mathfrak{sp}(2(n-2), \mathbb{R})$ be the subalgebra of $\mathfrak{g}$ built on coordinates $2, \dots, p-1, p+1, \dots, n$, and let $\mathfrak{l}' = \mathfrak{l} \cap \mathfrak{g}'$. Then

$$\Delta_n^+(\mathfrak{l}') = \Delta_n^+(\mathfrak{l}) \setminus \{\varepsilon_1 + \varepsilon_{p+j}, \varepsilon_p + \varepsilon_{p+j} \,\big|\, 1 \le j \le q\},$$

and we set

$$A' = A \setminus \{\varepsilon_p + \varepsilon_{p+j} \,\big|\, 1 \le j \le q\}.$$

We define λ_0 as in (6.4), but with n replaced by $n-2$ and p replaced by $p-2$. Then Λ' corresponding to A' can be obtained from Λ by deleting coordinates Λ_1 and Λ_p, and decreasing all the other coordinates by 1. We now see that Λ is a permutation of $(n, \dots, 1)$ if and only if Λ' is a permutation of $(n-2, \dots, 1)$. By inductive assumption, this is equivalent to A' and Λ' being defined by a shuffle as in the statement of the proposition, and this clearly implies the same statement for A and Λ.

The other possibility in (6.5) is handled analogously: $\Lambda_{p+1} = n$ implies

$$b_1 = 0,$$

and it follows that

$$\varepsilon_i + \varepsilon_{p+1} \notin A, \qquad 1 \le i \le p.$$

This implies that

$$0 \leq a_i \leq q-1, \qquad 1 \leq i \leq p,$$

and so

$$\Lambda = (\underbrace{n-a_1}_{[p+1,n]}, \underbrace{n-1-a_2}_{[p,n-2]}, \dots, \underbrace{q+1-a_p}_{[2,q]} \,|\, n, \underbrace{n-1-b_2}_{[q-1,n-1]}, \dots, \underbrace{p+1-b_q}_{[1,p+1]}).$$

If $q=1$, then there is only $\Lambda_{p+1}=n$ in the right group of coordinates, and we see there is no place to put the coordinate 1. Therefore, if $q=1$, Λ_{p+1} can not be n and we are back to the first case that we already handled. If $q>1$, then there is exactly one place where 1 can be, i.e.,

$$\Lambda_n = 1.$$

This implies

$$b_q = p,$$

and therefore

$$\varepsilon_i + \varepsilon_n \in A, \qquad 1 \leq i \leq p.$$

It follows that

$$1 \leq a_i \leq q-1, \qquad 1 \leq i \leq p,$$

and so

$$\Lambda = (\underbrace{n-a_1}_{[p+1,n-1]} \underbrace{n-1-a_2}_{[p,n-2]}, \dots, \underbrace{q+1-a_p}_{[2,q]} \,|\, n, \underbrace{n-1-b_2}_{[q-1,n-1]}, \underbrace{n-2-b_3}_{[q-2,n-2]}, \dots, \underbrace{p+2-b_{q-1}}_{[2,p+1]}, 1).$$

We now reason in the same way as in the first case, and conclude that the proposition follows from the inductive assumption for $n-2$ with p staying the same and q being replaced by $q-2$. ∎

To finish the computation of the constant c_p^n, we first note that for every A described in Proposition 6.1(ii)

$$\#A = \frac{pq}{2}. \tag{6.6}$$

Namely, the $\alpha_{u,v}$ and $\beta_{u,v}$ make for $2rs$ elements of A. In addition, the set B has 0 elements if p and q are even, s elements if p is odd, and r elements if q is odd. So the total number of elements is

$$2rs = \frac{pq}{2}, \qquad p,q \text{ even};$$
$$2rs + s = (2r+1)s = p\frac{q}{2}, \qquad p \text{ odd};$$
$$2rs + r = r(2s+1) = \frac{p}{2}q, \qquad q \text{ odd}.$$

On the other hand, since Λ is a permutation of $(n,\dots,1)$, $P_K(\Lambda)$ is equal to ± 1. To compute the sign, we need to find the parity of the permutation bringing Λ to $(n,\dots,1)$. This parity can be found by counting the number of inversions in Λ when compared with $(n,\dots,1)$, i.e., counting the number of pairs (i,j), $1\le i<j\le n$, such that $\Lambda_i<\Lambda_j$. We know from Proposition 6.1 that

$$\Lambda=(n{+}1{-}i_1,\dots,n{+}1{-}i_r,i_r,\dots,i_1\,|\,n{+}1{-}j_1,\dots,n{+}1{-}j_s,j_s,\dots,j_1) \tag{6.7}$$

if p and q are both even. It is clear that $i_r,\dots,i_1$ are in inversion with $n+1-j_1,\dots,n+1-j_s$; that is rs inversions. The further inversions are possible only between groups

$$n+1-i_1,\dots,n+1-i_r \qquad \text{and} \qquad n+1-j_1,\dots,n+1-j_s, \tag{6.8}$$

and

$$i_r,\dots,i_1 \qquad \text{and} \qquad j_s,\dots,j_1. \tag{6.9}$$

If i_u is in inversion with j_v, i.e., $i_u<j_v$, then $n+1-i_u>n+1-j_v$, i.e., $n+1-i_u$ is not in inversion with $n+1-j_v$. The converse also holds, and it follows that the total number of inversions in groups (6.8) and (6.9) is again rs. So the total number of inversions in case p and q are even is

$$2rs=\frac{pq}{2}.$$

If p is odd, then Λ is again given by (6.7), except that there is in addition $r+s+1$ between $n-i_r$ and i_r. This coordinate is in inversion with the coordinates $n+1-j_1,\dots,n+1-j_s$, and with no others, so the total number of inversions in this case is

$$2rs+s=\frac{pq}{2}.$$

Similarly, if q is odd, then Λ is given by (6.7), with the addition of $r+s+1$ between $n+1-j_s$ and j_s. This coordinate is in inversion with the coordinates $i_r,\dots,i_1$, and with no others, so the total number of inversions in this case is

$$2rs+r=\frac{pq}{2}.$$

So we have proved that for each Λ from Proposition 6.1,

$$P_K(\Lambda)=(-1)^{\frac{pq}{2}}.$$

Combined with (6.6), and with the fact that N from (3.3) is in this case

$$N=\binom{p}{2}+pq+p\equiv[\frac{p+1}{2}] \mod 2,$$

this tells us that the nonzero contributions to the sum in (6.3) are all equal to $(-1)^{[\frac{p+1}{2}]}$. Since the number of nonzero summands is by Proposition 6.1 equal to the number of (r,s)-shuffles of $r+s$, i.e., to $\binom{r+s}{r}$, we have proved:

THEOREM 6.2. *Let $G_{\mathbb{R}} = \mathrm{Sp}(2n,\mathbb{R})$, $n \geq 1$, and let p, $0 \leq p \leq n$, be an integer. Let $r = [\frac{p}{2}]$ and let $s = [\frac{n-p}{2}]$. Then the constant c_p^n for the real form of $\mathcal{O}^{\mathbb{C}}$ corresponding to p is*

$$c_p^n = \begin{cases} 0, & \text{if } n \text{ is even and } p \text{ is odd;} \\ (-1)^{[\frac{p+1}{2}]}\binom{r+s}{r}, & \text{if } n \text{ is odd, or if } n \text{ is even and } p \text{ is even.} \end{cases}$$

7. THE CASE $G_{\mathbb{R}} = SO_e(2p,2q)$, $q \geq p \geq 1$

There are three real forms of $\mathcal{O}^{\mathbb{C}}$ if $q > p > 1$, four if $q = p > 1$, and two if p=1.

7.1. *The first real form.* This real form is defined in all cases; it corresponds to

$$h_1 = (2, \underbrace{1,\dots,1}_{p-1} \,|\, \underbrace{1,\dots,1}_{p-1}, \underbrace{0,\dots,0}_{q-p+1}).$$

Since $\mathfrak{l} = \mathfrak{l}_1$ is built from roots that vanish on h_1, we see that

$$\Delta_n^+(\mathfrak{l}) = \{\varepsilon_i - \varepsilon_j \,|\, 2 \leq i \leq p,\, p+1 \leq j \leq 2p-1\}.$$

It follows that for any $A \subseteq \Delta_n^+(\mathfrak{l})$,

$$2\rho(A) = (0; a_1, \dots, a_{p-1} \,|\, -b_1, \dots, -b_{p-1}; 0, \dots, 0), \tag{7.1}$$

with

$$0 \leq a_i, b_j \leq p-1; \qquad \textstyle\sum_i a_i = \sum_j b_j. \tag{7.2}$$

Furthermore, recall that $\Delta(\mathfrak{p}_1)$ consists of noncompact roots that are 1 on h_1. So

$$\Delta(\mathfrak{p}_1) = \{\varepsilon_1 - \varepsilon_j \,|\, p+1 \leq j \leq 2p-1\} \cup \{\varepsilon_i \pm \varepsilon_j \,|\, 2 \leq i \leq p,\, 2p \leq j \leq p+q\}.$$

It follows that for any $C \subseteq \Delta(\mathfrak{p}_1)$,

$$2\rho(C) = (c; d_1, \dots, d_{p-1} \,|\, -c_1, \dots, -c_{p-1}; e_1, \dots, e_{q-p+1}), \tag{7.3}$$

with

$$\begin{aligned} &0 \leq c_j \leq 1; \qquad 0 \leq c \leq p-1; \qquad c = \textstyle\sum_j c_j; \\ &0 \leq d_i \leq 2(q-p+1); \qquad -(p-1) \leq e_j \leq p-1 \end{aligned} \tag{7.4}$$

(if $p = 1$, then

$$h_1 = (2 \,|\, 0, \dots, 0),$$

so $\Delta_n^+(\mathfrak{l}) = \Delta(\mathfrak{p}_1) = \emptyset$).

By (7.1),

$$\rho_n(\mathfrak{l}) = (0, p-1, \dots, p-1 \,|\, -p+1, \dots, -p+1, 0, \dots, 0).$$

This is clearly orthogonal to all roots of $\mathfrak{l} \cap \mathfrak{k}$, which are equal to

$$\begin{aligned} \Delta(\mathfrak{l} \cap \mathfrak{k}) = {} & \{\varepsilon_i - \varepsilon_j \,|\, 2 \leq i, j \leq p\} \cup \{\varepsilon_i - \varepsilon_j \,|\, p+1 \leq i, j \leq 2p-1\} \\ & \cup \{\varepsilon_i \pm \varepsilon_j \,|\, 2p \leq i, j \leq p+q\}. \end{aligned} \tag{7.5}$$

By Proposition 3.1, this means that we can determine the constant $c = c_1^{p,q}$ from the equation (3.4). We apply (3.4) for $\lambda = \lambda_0$, where

$$\begin{aligned}
&\lambda_0 = (\frac{1}{2}; q-\frac{1}{2}, q-\frac{3}{2}\dots, q-p+\frac{3}{2}\,| \\
&\qquad | -\frac{3}{2}, -\frac{5}{2}, \dots, -(p-\frac{1}{2}); q-p+\frac{1}{2}, \dots, \frac{3}{2}, \frac{1}{2}) \qquad \text{if } p \geq 2; \\
&\lambda_0 = (\frac{1}{2}\,|\, q-\frac{1}{2}, q-\frac{3}{2}, \dots, \frac{1}{2}) \qquad \text{if } p = 1.
\end{aligned} \tag{7.6}$$

PROPOSITION 7.1. *Let $\Lambda = \lambda_0 - 2\rho(A) - 2\rho(C)$, with λ_0 given by* (7.6), *and with $A \subseteq \Delta_n^+(\mathfrak{l})$ and $C \subseteq \Delta(\mathfrak{p}_1)$. If $P_K(\Lambda) \neq 0$, then:*

1. *If $p = 1$, then $A = C = \emptyset$ and $\Lambda = \lambda_0$.*
2. *If $p \geq 2$, then*

$$\begin{aligned}
&A = \emptyset; \\
&C = \{\varepsilon_i - \varepsilon_j \,\big|\, 2 \leq i \leq p,\ 2p \leq j \leq p+q-1\}; \\
&\Lambda = (\frac{1}{2}; p-\frac{1}{2}, p-\frac{3}{2}, \dots, \frac{3}{2}\,| -\frac{3}{2}, -\frac{5}{2}, \dots, -(p-\frac{1}{2}); q-\frac{1}{2}, q-\frac{3}{2}, \dots, p+\frac{1}{2}, \frac{1}{2}).
\end{aligned}$$

PROOF. The proposition is clear if $p = 1$, since in that case $\Delta_n^+(\mathfrak{l}) = \Delta(\mathfrak{p}_1) = \emptyset$. We continue by induction on p. Let $p \geq 2$ and let $q \geq p$ be arbitrary. We assume that the proposition is true for $p' = p-1$ and $q' = p-1$, and we show it is then also true for p and q.

By (7.6), (7.1) and (7.3), we have

$$\begin{aligned}
\Lambda = (\tfrac{1}{2} - c; q - \tfrac{1}{2} - a_1 - d_1, q - \tfrac{3}{2} - a_2 - d_2, \dots, q - p + \tfrac{3}{2} - a_{p-1} - d_{p-1}\,| \\
| -\tfrac{3}{2} + b_1 + c_1, -\tfrac{5}{2} + b_2 + c_2, \dots, -(p-\tfrac{1}{2}) + b_{p-1} + c_{p-1}; \\
q - p + \tfrac{1}{2} - e_1, q - p - \tfrac{1}{2} - e_2, \dots, \tfrac{3}{2} - e_{q-p}, \tfrac{1}{2} - e_{q-p+1}).
\end{aligned}$$

Using (7.2) and (7.4), we see that the coordinates $\Lambda_{p+1}, \dots, \Lambda_{p+q}$ are in the following intervals:

$$\begin{aligned}
\Lambda = (\dots | \underbrace{-\tfrac{3}{2} + b_1 + c_1}_{[-\frac{3}{2}, p-\frac{3}{2}]}, \underbrace{-\tfrac{5}{2} + b_2 + c_2}_{[-\frac{5}{2}, p-\frac{5}{2}]}, \dots, \underbrace{-(p-\tfrac{1}{2}) + b_{p-1} + c_{p-1}}_{[-(p-\frac{1}{2}), \frac{1}{2}]}; \\
\underbrace{q - p + \tfrac{1}{2} - e_1}_{[q-2p+\frac{3}{2}, q-\frac{1}{2}]}, \underbrace{q - p - \tfrac{1}{2} - e_2}_{[q-2p+\frac{1}{2}, q-\frac{3}{2}]}, \dots, \underbrace{\tfrac{3}{2} - e_{q-p}}_{[-(p-\frac{5}{2}), p+\frac{1}{2}]} \underbrace{\tfrac{1}{2} - e_{q-p+1}}_{[-(p-\frac{3}{2}), p-\frac{1}{2}]}).
\end{aligned}$$

So $\Lambda_{p+1}, \dots, \Lambda_{p+q}$ are q half-integers between $-(p-\frac{1}{2})$ and $q-\frac{1}{2}$. Moreover, $P_K(\Lambda) \neq 0$ implies that no two of these half-integers are equal or opposite to each other. If $q > p$, it follows that $q-\frac{1}{2}, q-\frac{3}{2}, \dots, p+\frac{1}{2}$ must each be equal to some Λ_i, and the only possibility for that is

$$\Lambda_{2p} = q - \frac{1}{2},\ \Lambda_{2p+1} = q - \frac{3}{2},\ \dots,\ \Lambda_{p+q-1} = p + \frac{1}{2}.$$

So $e_1, \dots, e_{q-p}$ are all equal to $-(p-1)$, and hence

$$\varepsilon_i - \varepsilon_j \in C,\ \varepsilon_i + \varepsilon_j \notin C, \qquad 2 \le i \le p,\ 2p \le j \le p+q-1.$$

(If $q = p$, the above says nothing and should be skipped.)

This implies

$$q - p \le d_i \le q - p + 2, \qquad 1 \le i \le p-1, \tag{7.7}$$

with d_i being $q-p$ if $\varepsilon_{i+1} \pm \varepsilon_{p+q} \notin C$, $d_i = q-p+2$ if $\varepsilon_{i+1} \pm \varepsilon_{p+q} \in C$, and $d_i = q-p+1$ if one of the roots $\varepsilon_{i+1} \pm \varepsilon_{p+q}$ is in C while the other is not in C. (If $q = p$, this gives no new information about the d_i. The following arguments all work also in case $q = p$ if we delete the group of coordinates from place $2p$ to place $p+q-1$.)

Looking at the bounds for coordinates $\Lambda_{p+1}, \dots, \Lambda_{2p-1}$ and Λ_{p+q}, we see that they are p half-integers between $-(p-\frac{1}{2})$ and $p-\frac{1}{2}$, such that no two of them are equal or opposite to each other. It follows that some of these Λ_j must be equal to $\pm(p-\frac{1}{2})$. There are two possibilities:

$$\Lambda_{2p-1} = -(p-\frac{1}{2}) \qquad \text{or} \qquad \Lambda_{p+q} = p - \frac{1}{2}.$$

Let us first examine the possibility that $\Lambda_{p+q} = p - \frac{1}{2}$. If this is true, then $e_{p+q} = -(p-1)$, so

$$\varepsilon_{i+1} - \varepsilon_{p+q} \in C, \quad \varepsilon_{i+1} + \varepsilon_{p+q} \notin C, \qquad 1 \le i \le p-1,$$

and it follows that

$$d_i = q - p + 1, \qquad 1 \le i \le p-1.$$

Using this together with the inequalities (7.2), (7.4) for a_i and c, we see

$$\Lambda = (\underbrace{\tfrac{1}{2} - c}_{[-(p-\frac{3}{2}), \frac{1}{2}]} ; \underbrace{p - \tfrac{3}{2} - a_1}_{[-\frac{1}{2}, p-\frac{3}{2}]}, \underbrace{p - \tfrac{5}{2} - a_2}_{[-\frac{3}{2}, p-\frac{5}{2}]}, \dots, \underbrace{\tfrac{1}{2} - a_{p-1}}_{[-(p-\frac{3}{2}), \frac{1}{2}]} \,|\, \dots).$$

So $\Lambda_1, \dots, \Lambda_p$ are p half-integers between $-(p-\frac{3}{2})$ and $p-\frac{3}{2}$. Moreover, $P_K(\Lambda) \ne 0$ implies that no two of these half-integers are equal or opposite to each other. This is impossible, and so Λ_{p+q} can not be $p - \frac{1}{2}$.

It follows that

$$\Lambda_{2p-1} = -(p - \frac{1}{2}),$$

and hence

$$b_{p-1} = 0; \qquad c_{p-1} = 0.$$

This implies that

$$\begin{aligned} &\varepsilon_i - \varepsilon_{2p-1} \notin A, \qquad 2 \le i \le p; \\ &\varepsilon_1 - \varepsilon_{2p-1} \notin C, \end{aligned}$$

and therefore

$$\begin{aligned} &0 \le a_i \le p-2, \quad 1 \le i \le p-1; \\ &0 \le c \le p-2. \end{aligned} \tag{7.8}$$

Using (7.8) and (7.7), we see that $\Lambda_1, \dots, \Lambda_p$ are in the following intervals:

$$\begin{aligned} \Lambda = (&\underbrace{\tfrac{1}{2} - c}_{[-(p-\frac{5}{2}),\frac{1}{2}]} ; \underbrace{q - \tfrac{1}{2} - a_1 - d_1}_{[-\frac{1}{2},p-\frac{1}{2}]}, \underbrace{q - \tfrac{3}{2} - a_2 - d_2}_{[-\frac{3}{2},p-\frac{3}{2}]}, \dots, \\ &\underbrace{q - p + \tfrac{3}{2} - a_{p-1} - d_{p-1}}_{[-(p-\frac{3}{2}),\frac{3}{2}]} | \dots). \end{aligned}$$

So $\Lambda_1, \dots, \Lambda_p$ are p half-integers between $-(p - \frac{3}{2})$ and $p - \frac{1}{2}$. As before, these half-integers must be different from each other, and no two of them are opposite, so one of them must be equal to $p - \frac{1}{2}$, and the only possibility is

$$\Lambda_2 = p - \frac{1}{2}.$$

So

$$a_1 = 0; \qquad d_1 = q - p.$$

It follows that

$$\begin{aligned} &\varepsilon_2 - \varepsilon_j \notin A, \qquad p+1 \le j \le 2p-1; \\ &\varepsilon_2 \pm \varepsilon_{p+q} \notin C, \end{aligned}$$

and hence

$$\begin{aligned} &0 \le b_j \le p-2, \qquad 1 \le j \le p-1; \\ &-(p-2) \le e_{q-p+1} \le p-2. \end{aligned}$$

So we see that

$$\begin{aligned} \Lambda = (\tfrac{1}{2} - c; p - \tfrac{1}{2}, q - \tfrac{3}{2} - a_2 - d_2, \dots, q - p + \tfrac{3}{2} - a_{p-1} - d_{p-1} \,| \\ | -\tfrac{3}{2} + b_1 + c_1, \dots, -(p - \tfrac{3}{2}) + b_{p-2} + c_{p-2}, -(p - \tfrac{1}{2}); \\ q - \tfrac{1}{2}, q - \tfrac{3}{2}, \dots, p + \tfrac{1}{2}, \tfrac{1}{2} - e_{q-p+1}) \end{aligned}$$

(the coordinates $q - \frac{1}{2}, q - \frac{3}{2}, \dots, p + \frac{1}{2}$ are not there if $q = p$).

We now consider the subalgebra $\mathfrak{g}' \cong \mathfrak{so}(2p-2, 2p-3)$ of $\mathfrak{g}$ built on coordinates

$$\varepsilon_1; \varepsilon_3, \dots, \varepsilon_p; \varepsilon_{p+1}, \dots, \varepsilon_{2p-2}; \varepsilon_{p+q},$$

so the coordinates 2 and $2p-1, 2p, \dots, p+q-1$ are deleted. We also consider the real form of $\mathcal{O}_{K'}$ given by

$$h'_1 = (2, \underbrace{1, \dots, 1}_{p-2} \,|\, \underbrace{1, \dots, 1}_{p-2}, 0),$$

with centralizer $\mathfrak{l}' = \mathfrak{l} \cap \mathfrak{g}'$. Then

$$\Delta_n^+(\mathfrak{l}') = \{\varepsilon_i - \varepsilon_j \,|\, 3 \le i \le p,\, p+1 \le j \le 2p-2\};$$
$$\Delta(\mathfrak{p}_1') = \{\varepsilon_1 - \varepsilon_j \,|\, p+1 \le j \le 2p-2\} \cup \{\varepsilon_i \pm \varepsilon_{p+q} \,|\, 3 \le i \le p\}.$$

We set

$$A' = A \cap \Delta_n^+(\mathfrak{l}') = A;$$
$$C' = C \cap \Delta(\mathfrak{p}_1') = C \setminus \{\varepsilon_i - \varepsilon_j \,|\, 2 \le i \le p,\ 2p \le j \le p+q-1\}.$$

Then

$$\begin{aligned} 2\rho(A') &= (0; a_2, \dots, a_{p-1} \,|\, -b_1, \dots, -b_{p-2}; 0) \\ &= (0; a_1', \dots, a_{p-2}' \,|\, -b_1', \dots, -b_{p-2}'; 0); \\ 2\rho(C') &= (c; d_2 - (q-p), \dots, d_{p-1} - (q-p) \,|\, -c_1, \dots, -c_{p-2}; e_{q-p+1}) \\ &= (c'; d_1', \dots, d_{p-1}' \,|\, -c_1', \dots, -c_{p-2}'; e_{q-p+1}'), \end{aligned}$$

where we define

$$a_i' = a_{i+1}; \quad b_i' = b_i; \quad c_i' = c_i; \quad c' = c;$$
$$d_i' = d_{i+1} - (q-p); \quad e_{q-p+1}' = e_{q-p+1}.$$

The numbers $a_i', b_i', c_i', c', d_i'$ satisfy analogues of (7.2) and (7.4). We define λ_0' by (7.6), but for $G_{\mathbb{R}} = SO_e(2p-2, 2p-1)$, i.e.,

$$\lambda_0' = (\frac{1}{2}; p - \frac{3}{2}, \dots, \frac{3}{2} \,|\, -\frac{3}{2}, -\frac{5}{2}, \dots, -(p-\frac{3}{2}); \frac{1}{2}).$$

Then A', C', and

$$\Lambda' = \lambda_0' - 2\rho(A') - 2\rho(C')$$

satisfy all conditions of the proposition, but p, q are reduced to $p' = p-1$, $q' = p-1$. Moreover, $P_K(\Lambda) \neq 0$ is equivalent to $P_{K'}(\Lambda') \neq 0$. Therefore the inductive assumption implies that $A' = C' = \emptyset$, and that $\Lambda' = \lambda_0'$. This implies the statement of the proposition for A, C and Λ. ∎

In view of (3.4), to compute the constant $c = c_1^{p,q}$ we need to compute $P_{L\cap K}(\lambda_0)$ and $P_K(\Lambda)$, where λ_0 is given by (7.6), and Λ is given by Proposition 7.1.

To compute $P_{L\cap K}(\lambda_0)$, we note that we described $\mathfrak{l} \cap \mathfrak{k}$ in (7.5); it has up to three factors, two of which are $\mathfrak{u}(p-1)$, and the third is $\mathfrak{so}(2(q-p+1))$. From the shape of λ_0 it now follows that, in the notation of Lemma 5.2,

$$P_{L\cap K}(\lambda_0) = P_{q-p+1}^1(\lambda_{q-p+1}),$$

and we see that Lemma 5.2(i) implies that

$$P_{L\cap K}(\lambda_0) = 2^{q-p}.$$

To compute $P_K(\Lambda)$, with Λ as in Proposition 7.1, we first write $\Lambda = (\Lambda_L \,|\, \Lambda_R)$ and note that

$$P_K(\Lambda) = P_p^1(\Lambda_L) P_q^1(\Lambda_R).$$

To use Lemma 5.2, we have to rearrange coordinates of Λ_L and Λ_R, using the fact that P_p^1 is skew for the Weyl group of $\mathfrak{so}(2p)$, while P_q^1 is skew for the Weyl group of $\mathfrak{so}(2q)$. Moreover, both polynomials are invariant under sign changes of the variables; this follows since the sign change of the j-th coordinate switches roots $\varepsilon_i - \varepsilon_j$ and $\varepsilon_i + \varepsilon_j$.

To rearrange Λ_L to λ_p, we only need to bring the $\frac{1}{2}$ from the first coordinate to the p-th coordinate, and hence

$$P_p^1(\Lambda_L) = (-1)^{p-1} 2^{p-1}.$$

To bring Λ_R to $\mu_q = (q, \dots, 1)$, after removing the signs which does not change the expression, we need to bring coordinates $p - \frac{1}{2}, p - \frac{3}{2}, \dots, \frac{3}{2}$, in that order, to the right of $p + \frac{1}{2}$, leaving $\frac{1}{2}$ at the end. The sign produced in this way is

$$(-1)^{(q-p)+(q-p+1)+\dots+(q-2)} = (-1)^{(p-1)(q-p-1)+\frac{(p-1)p}{2}} = (-1)^{(p-1)(q-p-1)+[\frac{p}{2}]},$$

and it follows from Lemma 5.2 that

$$P_q^1(\Lambda_R) = (-1)^{(p-1)(q-p-1)+[\frac{p}{2}]} 2^{q-1}.$$

Putting this together with the fact that

$$\#A + \#C = \#C = (p-1)(q-p),$$

that N of (3.3) satisfies

$$N \equiv p - 1 \mod 2, \tag{7.9}$$

and that

$$[\frac{p}{2}] + p - 1 \equiv [\frac{p-1}{2}] \mod 2,$$

we see that (3.4) implies the following result.

THEOREM 7.2. *For $G_{\mathbb{R}} = SO_e(2p, 2q)$, $q \geq p \geq 1$, the constant $c_1^{p,q}$ is*

$$c_1^{p,q} = (-1)^{[\frac{p-1}{2}]} 2^{2p-2}.$$

7.2. *The second real form.* This real form exists for $q \geq p \geq 2$. It corresponds to

$$h_2 = (2, \underbrace{1, \dots, 1}_{p-2}, -1 \,|\, \underbrace{1, \dots, 1}_{p-1}, \underbrace{0, \dots, 0}_{q-p+1}).$$

This real form is conjugate to the first real form by the automorphism σ which acts on $\mathfrak{h}$ by changing the sign of the p-th coordinate, and leaving all other coordinates the same. On the level of $\mathfrak{g}$, this is an outer automorphism, which becomes the standard one if we compose it with the isomorphism $\mathfrak{so}(2p, 2q) \cong \mathfrak{so}(2q, 2p)$. The automorphism σ satisfies the conditions of Lemma 3.4, and we just have to compute the sign. The number n from Proposition 3.4 is, as in Subsection 5.1, equal to $p-1$. The number N_1 is by (7.9) congruent to $p-1$ modulo 2. Finally, N_2 is easily seen to be always even. The conclusion is that there is no sign in Proposition 3.4, so

$$c_2^{p,q} = c_1^{p,q} = (-1)^{[\frac{p-1}{2}]} 2^{2p-2}. \tag{7.10}$$

7.3. *The third real form.* This real form exists for all $q \geq p \geq 1$. It corresponds to

$$h_3 = (\underbrace{1,\dots,1}_{p-1}, 0 \,|\, 2, \underbrace{1,\dots,1}_{p-1}, \underbrace{0,\dots,0}_{q-p}).$$

Since $\mathfrak{l} = \mathfrak{l}_3$ is built from roots that vanish on h_3, we see that

$$\Delta_n^+(\mathfrak{l}) = \{\varepsilon_i - \varepsilon_j \,\big|\, 1 \leq i \leq p-1,\, p+2 \leq j \leq 2p\} \cup \{\varepsilon_p \pm \varepsilon_j \,\big|\, 2p+1 \leq j \leq p+q\}.$$

It follows that for any $A \subseteq \Delta_n^+(\mathfrak{l})$,

$$2\rho(A) = (a_1,\dots,a_{p-1}; x \,|\, 0; -b_1,\dots,-b_{p-1}; y_1,\dots,y_{q-p}), \tag{7.11}$$

with

$$\begin{aligned} &0 \leq a_i, b_j \leq p-1; \qquad \textstyle\sum_i a_i = \sum_j b_j; \\ &0 \leq x \leq 2(q-p); \qquad -1 \leq y_j \leq 1. \end{aligned} \tag{7.12}$$

Furthermore, recall that $\Delta(\mathfrak{p}_1)$ consists of noncompact roots that are 1 on h_3. So

$$\begin{aligned} \Delta(\mathfrak{p}_1) = \{&\varepsilon_i \pm \varepsilon_j \,\big|\, 1 \leq i \leq p-1,\, 2p+1 \leq j \leq p+q\} \\ &\cup \{\varepsilon_j \pm \varepsilon_p \,\big|\, p+2 \leq j \leq 2p\} \cup \{\varepsilon_{p+1} - \varepsilon_i \,\big|\, 1 \leq i \leq p-1\}. \end{aligned}$$

It follows that for any $C \subseteq \Delta(\mathfrak{p}_1)$,

$$2\rho(C) = (c_1,\dots,c_{p-1}; u \,|\, v; d_1,\dots,d_{p-1}; e_1,\dots,e_{q-p}), \tag{7.13}$$

with

$$\begin{aligned} &-1 \leq c_i \leq 2(q-p); \qquad -(p-1) \leq u \leq p-1; \\ &0 \leq v \leq p-1; \qquad 0 \leq d_j \leq 2; \qquad -(p-1) \leq e_j \leq p-1. \end{aligned} \tag{7.14}$$

If we write (7.11) for $A = \Delta_n^+(\mathfrak{l})$, we get

$$\rho_n(\mathfrak{l}) = (p-1,\dots,p-1; 2(q-p) \,|\, 0; -p+1,\dots,-p+1; 0,\dots,0).$$

This is clearly orthogonal to all roots of $\mathfrak{l} \cap \mathfrak{k}$, which are equal to

$$\begin{aligned} \Delta(\mathfrak{l} \cap \mathfrak{k}) = &\{\varepsilon_i - \varepsilon_j \,\big|\, 1 \leq i,j \leq p-1\} \cup \{\varepsilon_i - \varepsilon_j \,\big|\, p+2 \leq i,j \leq 2p\} \\ &\cup \{\varepsilon_i \pm \varepsilon_j \,\big|\, 2p+1 \leq i,j \leq p+q\}. \end{aligned} \tag{7.15}$$

By Proposition 3.1, this means that we can determine the constant $c = c_3^{p,q}$ from the equation (3.4). We will use this for $\lambda = \lambda_0$, where

(7.16)

$$\lambda_0 = (q-\frac{3}{2}, q-\frac{5}{2}\dots, q-p+\frac{1}{2}; q-p+\frac{1}{2}\,|\,p-\frac{3}{2}; \frac{1}{2}, -\frac{1}{2}, \dots, -p+\frac{5}{2};$$
$$q-p-\frac{1}{2}, q-p-\frac{3}{2}, \dots, \frac{1}{2})$$

(out of the 5 groups of coordinates separated by semicolons and the bar, the first and the fourth group are missing if $p = 1$, and the fifth group is missing if $q = p$).

Proposition 7.3. *Let $\Lambda = \lambda_0 - 2\rho(A) - 2\rho(C)$, with λ_0 given by (7.16), and with $A \subseteq \Delta_n^+(\mathfrak{l})$ and $C \subseteq \Delta(\mathfrak{p}_1)$. If $P_K(\Lambda) \neq 0$, then*

1. *If $q > p \geq 2$, then*

$$A = \{\varepsilon_p - \varepsilon_j \,|\, 2p+1 \leq j \leq p+q\};$$
$$C = \{\varepsilon_i - \varepsilon_j \,|\, 1 \leq i \leq p-1,\, 2p+1 \leq j \leq p+q\} \cup$$
$$\cup\ \{\varepsilon_j \pm \varepsilon_p \,|\, p+2 \leq j \leq 2p\} \cup \{\varepsilon_{p+1} - \varepsilon_i \,|\, 1 \leq i \leq p-1\};$$
$$\Lambda = (p-\frac{1}{2}, p-\frac{3}{2}, \dots, \frac{3}{2}; \frac{1}{2}\,|\,-\frac{1}{2}; -\frac{3}{2}, -\frac{5}{2}, \dots, -(p-\frac{1}{2});$$
$$q-\frac{1}{2}, q-\frac{3}{2}, \dots, p+\frac{1}{2}).$$

2. *If $q > p = 1$, then A is as in (1), $C = \emptyset$, and*

$$\Lambda = (\frac{1}{2}\,|\,-\frac{1}{2}; q-\frac{1}{2}, q-\frac{3}{2}, \dots, \frac{3}{2}).$$

3. *If $q = p \geq 2$, then*

$$A = \emptyset;$$
$$C = \{\varepsilon_j \pm \varepsilon_p \,|\, p+2 \leq j \leq 2p\} \cup \{\varepsilon_{p+1} - \varepsilon_i \,|\, 1 \leq i \leq p-1\};$$
$$\Lambda = (p-\frac{1}{2}, p-\frac{3}{2}, \dots, \frac{3}{2}; \frac{1}{2}\,|\,-\frac{1}{2}; -\frac{3}{2}, -\frac{5}{2}, \dots, -(p-\frac{1}{2})).$$

4. *If $q = p = 1$, then $A = C = \emptyset$ and $\Lambda = \lambda_0 = (\frac{1}{2}\,|\,-\frac{1}{2})$.*

Proof. By (7.16), (7.11) and (7.13), we have

$$\Lambda = (q-\tfrac{3}{2}-a_1-c_1, \dots, q-p+\tfrac{1}{2}-a_{p-1}-c_{p-1}; q-p+\tfrac{1}{2}-x-u\,|$$
$$|\,p-\tfrac{3}{2}-v; \tfrac{1}{2}+b_1-d_1, -\tfrac{1}{2}+b_2-d_2, \dots, -p+\tfrac{5}{2}+b_{p-1}-d_{p-1};$$
$$q-p-\tfrac{1}{2}-y_1-e_1, q-p-\tfrac{3}{2}-y_2-e_2, \dots, \tfrac{1}{2}-y_{q-p}-e_{q-p}).$$

There are five groups of coordinates separated by semicolons and the bar. If $p = 1$, then the first and the fourth group of coordinates are missing, and if $q = p$, then the fifth group of coordinates is missing.

Using (7.12) and (7.14), we see

(7.17)

$$\Lambda = (\dots | \underbrace{p-\tfrac{3}{2}-v}_{[-\frac{1}{2},p-\frac{3}{2}]}; \underbrace{\tfrac{1}{2}+b_1-d_1}_{[-\frac{3}{2},p-\frac{1}{2}]}, \underbrace{-\tfrac{1}{2}+b_2-d_2}_{[-\frac{5}{2},p-\frac{3}{2}]}, \dots, \underbrace{-p+\tfrac{5}{2}+b_{p-1}-d_{p-1}}_{[-(p-\frac{1}{2}),\frac{3}{2}]};$$
$$\underbrace{q-p-\tfrac{1}{2}-y_1-e_1}_{[q-2p-\frac{1}{2},q-\frac{1}{2}]}, \underbrace{q-p-\tfrac{3}{2}-y_2-e_2}_{[q-2p-\frac{3}{2},q-\frac{3}{2}]}, \dots, \underbrace{\tfrac{1}{2}-y_{q-p}-e_{q-p}}_{[-(p-\frac{1}{2}),p+\frac{1}{2}]}).$$

So $\Lambda_{p+1},\dots,\Lambda_{p+q}$ are q half-integers between $-(p-\frac{1}{2})$ and $q-\frac{1}{2}$. Since $P_K(\Lambda)\neq 0$, no two of them are equal or opposite to each other. If $q>p$, it follows that $q-\frac{1}{2}, q-\frac{3}{2},\dots,p+\frac{1}{2}$ must each be equal to some Λ_i, and the only possibility for that is

$$\Lambda_{2p+1} = q-\frac{1}{2},\ \Lambda_{2p+2} = q-\frac{3}{2},\ \dots,\ \Lambda_{p+q} = p+\frac{1}{2}. \tag{7.18}$$

It follows that $y_1,\dots,y_{q-p}$ are all equal to -1, and that $e_1,\dots,e_{q-p}$ are all equal to $-(p-1)$.

In case $q>p=1$, this implies that A is as stated in the proposition, and it is also clear that $C=\emptyset$. Moreover, it follows that $x=q-p$, and so Λ is as stated in the proposition. Since the case $q=p=1$ is obvious, this proves the proposition for $p=1$ and any $q\geq p$.

We proceed by induction on p. Let $p\geq 2$ and let $q\geq p$ be arbitrary. Assuming that the proposition is true for $p'=q'=p-1$, we will prove it for p and q.

If $q>p$, we get back to (7.18) and see that it implies

$$\begin{aligned} &\varepsilon_p-\varepsilon_j\in A,\ \varepsilon_p+\varepsilon_j\notin A, && 2p+1\leq j\leq p+q;\\ &\varepsilon_i-\varepsilon_j\in C,\ \varepsilon_i+\varepsilon_j\notin C, && 1\leq i\leq p-1,\ 2p+1\leq j\leq p+q. \end{aligned}$$

This implies

$$\begin{aligned} &x = q-p;\\ &q-p-1\leq c_i\leq q-p, \qquad 1\leq i\leq p-1. \end{aligned} \tag{7.19}$$

Note that $c_i=q-p-1$ if $\varepsilon_{p+1}-\varepsilon_i\in C$, and $c_i=q-p$ if $\varepsilon_{p+1}-\varepsilon_i\notin C$.

If $q=p$, then the above discussion does not apply; in this case (7.19) is true, but this information is already contained in (7.12) and (7.14). The following discussion applies equally to $q>p$ and $q=p$, but in the latter case the last group of coordinates should be deleted.

Using (7.19) together with the inequalities (7.12) for a_i and (7.14) for u, we see

$$\Lambda = (\underbrace{q-\tfrac{3}{2}-a_1-c_1}_{[-\frac{1}{2},p-\frac{1}{2}]}, \underbrace{q-\tfrac{5}{2}-a_2-c_2}_{[-\frac{3}{2},p-\frac{3}{2}]}, \dots, \underbrace{q-p+\tfrac{1}{2}-a_{p-1}-c_{p-1}}_{[-(p-\frac{3}{2}),\frac{3}{2}]}; \underbrace{\tfrac{1}{2}-u}_{[-(p-\frac{3}{2}),p-\frac{1}{2}]} \mid \dots$$

So $\Lambda_1, \dots, \Lambda_p$ are p half-integers between $-(p-\frac{3}{2})$ and $p-\frac{1}{2}$, such that no two of them are equal or opposite to each other. There are two possibilities:

$$\Lambda_1 = p - \frac{1}{2}, \qquad \text{or} \qquad \Lambda_p = p - \frac{1}{2}.$$

Let us first assume that $\Lambda_p = p - \frac{1}{2}$. Then

$$u = -(p-1)$$

and it follows that

$$\varepsilon_j - \varepsilon_p \in C,\ \varepsilon_j + \varepsilon_p \notin C, \qquad p+2 \le j \le 2p.$$

This implies

$$d_j = 1, \qquad p+2 \le j \le 2p,$$

and we see that (7.17) becomes

$$\Lambda = (\dots \mid \underbrace{p-\tfrac{3}{2}-v}_{[-\frac{1}{2},p-\frac{3}{2}]}; \underbrace{-\tfrac{1}{2}+b_1}_{[-\frac{1}{2},p-\frac{3}{2}]}, \underbrace{-\tfrac{3}{2}+b_2}_{[-\frac{3}{2},p-\frac{5}{2}]}, \dots, \underbrace{-p+\tfrac{3}{2}+b_{p-1}}_{[-(p-\frac{3}{2}),\frac{1}{2}]}; q-\tfrac{1}{2}, q-\tfrac{3}{2}, \dots, p+\tfrac{1}{2}).$$

So $\Lambda_{p+1}, \dots, \Lambda_{2p}$ are p half-integers between $-(p-\frac{3}{2})$ and $p-\frac{3}{2}$, such that no two of them are equal or opposite to each other, and this is impossible.

We conclude that

$$\Lambda_1 = p - \frac{1}{2},$$

and consequently

$$a_1 = 0; \qquad c_1 = q-p-1.$$

It follows that

$$\begin{aligned} &\varepsilon_1 - \varepsilon_j \notin A, \quad p+2 \le j \le 2p; \\ &\varepsilon_{p+1} - \varepsilon_1 \in C, \end{aligned}$$

and therefore

$$\begin{aligned} &0 \le b_j \le p-2, \quad 1 \le j \le p-1; \\ &1 \le v \le p-1. \end{aligned}$$

This implies

$$\Lambda = (\dots \,|\, \underbrace{p-\tfrac{3}{2}-v}_{[-\frac{1}{2},p-\frac{5}{2}]}; \underbrace{\tfrac{1}{2}+b_1-d_1}_{[-\frac{3}{2},p-\frac{3}{2}]}, \underbrace{-\tfrac{1}{2}+b_2-d_2}_{[-\frac{5}{2},p-\frac{5}{2}]}, \dots, \underbrace{-p+\tfrac{5}{2}+b_{p-1}-d_{p-1}}_{[-(p-\frac{1}{2}),\frac{1}{2}]};$$
$$q-\tfrac{1}{2}, q-\tfrac{3}{2}, \dots, p+\tfrac{1}{2}).$$

So $\Lambda_{p+1}, \dots, \Lambda_{2p}$ are p half-integers between $-(p-\frac{1}{2})$ and $p-\frac{3}{2}$, such that no two of them are equal or opposite to each other. It follows that

$$\Lambda_{2p} = -(p-\frac{1}{2}),$$

and consequently

$$b_{p-1} = 0; \qquad d_{p-1} = 2.$$

It follows that

$$\begin{aligned} &\varepsilon_i - \varepsilon_{2p} \notin A, \quad 1 \le i \le p-1; \\ &\varepsilon_{2p} \pm \varepsilon_p \in C, \end{aligned}$$

and therefore

$$\begin{aligned} &0 \le a_i \le p-2, \quad 1 \le i \le p-1; \\ &-(p-2) \le u \le p-2. \end{aligned}$$

We see that

$$\begin{aligned} \Lambda = (p-\tfrac{1}{2}, q-\tfrac{5}{2}-a_2-c_2, \dots, q-p+\tfrac{1}{2}-a_{p-1}-c_{p-1}; q-p+\tfrac{1}{2}-x-u \,| \\ |\, p-\tfrac{3}{2}-v; \tfrac{1}{2}+b_1-d_1, -\tfrac{1}{2}+b_2-d_2, \dots, -(p-\tfrac{7}{2})+b_{p-2}-d_{p-2}, -(p-\tfrac{1}{2}); \\ q-\tfrac{1}{2}, q-\tfrac{3}{2}, \dots, p+\tfrac{1}{2}). \end{aligned}$$

(If $q=p$, the coordinates $q-\frac{1}{2}, \dots, p+\frac{1}{2}$ are not there; if $p=2$ there are no coordinates involving a_i, c_i, b_i or d_i.)

We now consider the subalgebra $\mathfrak{g}' \cong \mathfrak{so}(2p-2, 2p-1)$ of $\mathfrak{g}$ built on coordinates

$$\varepsilon_2, \varepsilon_3, \dots, \varepsilon_p; \varepsilon_{p+1}, \dots, \varepsilon_{2p-1},$$

so the coordinates 1 and $2p, 2p+1, \dots, p+q$ are deleted. We also consider the real form of $\mathcal{O}_{K'}$ given by

$$h_3' = (\underbrace{1, \dots, 1}_{p-2}, 0 \,|\, 2, \underbrace{1, \dots, 1}_{p-2}),$$

with centralizer $\mathfrak{l}' = \mathfrak{l} \cap \mathfrak{g}'$. Then

$$\begin{aligned} &\Delta_n^+(\mathfrak{l}') = \{\varepsilon_i - \varepsilon_j \,\big|\, 2 \le i \le p-1,\, p+2 \le j \le 2p-1\}; \\ &\Delta(\mathfrak{p}_1') = \{\varepsilon_j \pm \varepsilon_p \,\big|\, p+2 \le j \le 2p-1\} \cup \{\varepsilon_{p+1} - \varepsilon_i \,\big|\, 2 \le i \le p-1\}. \end{aligned}$$

We set

$$\begin{aligned} A' &= A \cap \Delta_n^+(\mathfrak{l}') = A \setminus \{\varepsilon_p - \varepsilon_j \,\big|\, 2p+1 \le j \le p+q\}; \\ C' &= C \cap \Delta(\mathfrak{p}_1') \\ &= C \setminus \{\varepsilon_{2p} \pm \varepsilon_p;\ \varepsilon_{p+1} - \varepsilon_1;\ \varepsilon_i - \varepsilon_j \,\big|\, 1 \le i \le p-1,\ 2p+1 \le j \le p+q\} \end{aligned}$$

(if $q = p$, then $A' = A$ and $C' = C \setminus \{\varepsilon_{2p} \pm \varepsilon_p;\ \varepsilon_{p+1} - \varepsilon_1\}$). Then

$$\begin{aligned} 2\rho(A') &= (a_2, \dots, a_{p-1}; 0 \,|\, 0; -b_1, \dots, -b_{p-2}) \\ &= (a_1', \dots, a_{p-2}'; 0 \,|\, 0; -b_1', \dots, -b_{p-2}'); \\ 2\rho(C') &= (c_2 - (q-p), \dots, c_{p-1} - (q-p); u \,|\, v-1; d_1, \dots, d_{p-2}) \\ &= (c_1', \dots, c_{p-2}'; u' \,|\, v'; d_1', \dots, d_{p-2}'), \end{aligned}$$

where we define

$$\begin{aligned} &a_i' = a_{i+1}; \quad b_i' = b_i; \quad c_i' = c_{i+1} - (q-p); \\ &u' = u; \quad v' = v - 1; \quad d_i' = d_i. \end{aligned}$$

The numbers $a_i', b_i', c_i', u', v', d_i'$ satisfy analogues of (7.12) and (7.14).

We define λ_0' by (7.16), but for $G_{\mathbb{R}} = SO_e(2p-2, 2p-1)$, i.e.,

$$\lambda_0' = (p - \frac{5}{2}, p - \frac{7}{2}, \dots, \frac{1}{2}; \frac{1}{2}; \,|\, p - \frac{5}{2}; \frac{1}{2}, -\frac{1}{2}, \dots, -(p - \frac{7}{2})).$$

Then A', C' and

$$\Lambda' = \lambda_0' - 2\rho(A') - 2\rho(C')$$

satisfy all conditions of the proposition, but p, q are reduced to $p' = p-1$, $q' = p-1$. Therefore the inductive assumption implies that $A' = \emptyset$, that

$$C' = \{\varepsilon_j \pm \varepsilon_p \,\big|\, p+2 \le j \le 2p-1\} \cup \{\varepsilon_{p+1} - \varepsilon_i \,\big|\, 2 \le i \le p-1\},$$

and that

$$\Lambda' = (p - \frac{3}{2}, \dots, \frac{3}{2}; \frac{1}{2} \,|\, -\frac{1}{2}; -\frac{3}{2}, -\frac{5}{2}, \dots, -(p - \frac{3}{2})).$$

This implies the statement of the proposition for A, C and Λ. ∎

In view of (3.4), to compute the constant $c = c_3^{p,q}$ we need to compute $P_{L\cap K}(\lambda_0)$ and $P_K(\Lambda)$, where λ_0 is given by (7.16), and Λ is given by Proposition 7.3.

To compute $P_{L\cap K}(\lambda_0)$, we note that we described $\mathfrak{l} \cap \mathfrak{k}$ in (7.15); it has up to three factors, two of which are $\mathfrak{u}(p-1)$, and the third is $\mathfrak{so}(2(q-p))$. From the shape of λ_0 it now follows that, in the notation of Lemma 5.2,

$$P_{L\cap K}(\lambda_0) = P^1_{q-p}(\lambda_{q-p}),$$

and in case $q > p$, we see that Lemma 5.2(i) implies that

$$P_{L\cap K}(\lambda_0) = 2^{q-p-1}.$$

If $q = p$, then $P_{L\cap K}(\lambda_0) = 1$, which is not covered by the above formula. (In Lemma 5.2, we could have defined $P_0^1 = 1$ and $\lambda_0 = 0$, but the formula in Lemma 5.2(i) would not work for $p = 0$.)

To compute $P_K(\Lambda)$ for Λ as in Proposition 7.3, we first write $\Lambda = (\Lambda_L \,|\, \Lambda_R)$ and note that

$$P_K(\Lambda) = P_p^1(\Lambda_L)P_q^1(\Lambda_R). \tag{7.20}$$

By Lemma 5.2(i),

$$P_p^1(\Lambda_L) = P_p^1(\lambda_p) = 2^{p-1}. \tag{7.21}$$

To apply Lemma 5.2 also for Λ_R, we must first rearrange coordinates of Λ_R, using the fact that P_q^1 is skew for the Weyl group of $\mathfrak{so}(2q)$, and invariant under sign changes of the variables.

To bring Λ_R to $\mu_q = (q, \dots, 1)$, after removing the signs which does not change the expression, we need to bring coordinates

$$p - \frac{1}{2}, p - \frac{3}{2}, \dots, \frac{3}{2}, \frac{1}{2},$$

in that order, to the right of $p + \frac{1}{2}$. The sign produced in this way is

$$(-1)^{(q-p)+(q-p+1)+\dots+(q-1)} = (-1)^{p(q-p)+[\frac{p}{2}]},$$

and it follows from Lemma 5.2 that

$$P_q^1(\Lambda_R) = (-1)^{p(q-p)+[\frac{p}{2}]}2^{q-1}.$$

Putting this together with (7.20), (7.21), the fact that

$$\#A = q - p; \qquad \#C = 3(p-1) + (p-1)(q-p),$$

and the fact that N of (3.3) satisfies

$$N \equiv p \mod 2 \tag{7.22}$$

we get the following statement.

THEOREM 7.4. *For $G_{\mathbb{R}} = SO_e(2p, 2q)$, the constant $c_3^{p,q}$ corresponding to the third real form of $\mathcal{O}^{\mathbb{C}}$ is*

$$c_3^{p,q} = \begin{cases} (-1)^{[\frac{p}{2}]+1}2^{2p-1}, & q > p; \\ (-1)^{[\frac{p}{2}]+1}2^{2p-2}, & q = p. \end{cases}$$

7.4. *The fourth real form.* This real form exists if $q = p \geq 2$. It corresponds to

$$h_4 = (\underbrace{1, \dots, 1}_{p-1}, 0 \,|\, 2, \underbrace{1, \dots, 1}_{p-2}, -1).$$

This real form is conjugate to the third real form by the automorphism σ which acts on $\mathfrak{h}$ by changing the sign of the last coordinate, and leaving all other coordinates the same. On the level of $\mathfrak{g}$, this is the standard outer automorphism. It satisfies the conditions of Proposition 3.4, and we just have to compute the sign. The number n from Proposition 3.4 is equal to 0, since σ preserves Δ^+. The number N_1 is by (7.22) congruent to p modulo 2. Finally, N_2 is the same as N_1 because σ preserves Δ^+ (or one can do a computation). So there is no sign in Proposition 3.4, and

$$c_4^{p,q} = (-1)^{[\frac{p}{2}]+1} 2^{2p-2} \tag{7.23}$$

8. The case $G_{\mathbb{R}} = SO^*(2n)$, $n \geq 1$

8.1. *The case of even n.* If n is even, the real forms of $\mathcal{O}^{\mathbb{C}}$ correspond to even integers p such that $0 \leq p \leq n$. We denote $n-p$ by q. The h corresponding to p is

$$h_p = (\underbrace{1,\dots,1}_{p} \,|\, \underbrace{-1,\dots,-1}_{q}).$$

Since $\mathfrak{l} = \mathfrak{l}_p$ is built from roots that vanish on h_p, we see that

$$\Delta_n^+(\mathfrak{l}) = \{\varepsilon_i + \varepsilon_{p+j} \,\big|\, 1 \leq i \leq p,\ 1 \leq j \leq q\}.$$

It follows that for any $A \subseteq \Delta_n^+(\mathfrak{l})$,

$$2\rho(A) = (a_1,\dots,a_p \,|\, b_1,\dots,b_q), \tag{8.1}$$

with

$$0 \leq a_i \leq q, \qquad 0 \leq b_j \leq p, \qquad \textstyle\sum_i a_i = \sum_j b_j. \tag{8.2}$$

In particular,

$$\rho_n(\mathfrak{l}) = (q,\dots,q \,|\, p,\dots,p),$$

and this is clearly orthogonal to the roots of $\mathfrak{l} \cap \mathfrak{k}$, which are given by

$$\Delta^+(\mathfrak{l} \cap \mathfrak{k}) = \{\varepsilon_i - \varepsilon_j \,\big|\, 1 \leq i < j \leq p\} \cup \{\varepsilon_{p+i} - \varepsilon_{p+j} \,\big|\, 1 \leq i < j \leq q\}. \tag{8.3}$$

So the constants $c = c_p^n$ can be calculated from (3.4). Since it is clear that in the present case

$$\Delta(\mathfrak{p}_1) = \emptyset,$$

(3.4) becomes

$$\sum_{A \subseteq \Delta_n^+(\mathfrak{l})} (-1)^{\#A} P_K(\lambda - 2\rho(A)) = c P_{L\cap K}(\lambda). \tag{8.4}$$

We take $\lambda = \lambda_0$, where

$$\lambda_0 = (n, n-1, \dots, q+1 \,|\, n, n-1, \dots, p+1) \tag{8.5}$$

(if p is 0 or n, then there is only one group of coordinates in the above expression, and $\lambda_0 = (n, n-1, \dots, 1)$).

Since λ_0 differs from $\rho_{\mathfrak{l}\cap\mathfrak{k}}$ by a weight orthogonal to all roots of $\mathfrak{l} \cap \mathfrak{k}$,

$$P_{L\cap K}(\lambda_0) = 1.$$

So to compute c_p^n we have to compute the left side of (8.4). The following proposition describes the relevant A and the corresponding Λ.

PROPOSITION 8.1. *Let $\Lambda = \lambda_0 - 2\rho(A)$, with λ_0 given by* (8.5)*, and with $A \subseteq \Delta_n^+(\mathfrak{l})$.*

Suppose that for some A the corresponding Λ satisfies $P_K(\Lambda) \neq 0$. Then:

1. *If $p = 0$ or $q = 0$, then $A = \emptyset$ and $\Lambda = \lambda_0 = (n, n-1, \dots, 1)$.*
2. *If $p, q > 0$, let $r = \frac{p}{2}$ and $s = \frac{q}{2}$. Then there is a shuffle*
$$1 \le i_1 < \dots < i_r \le r+s; \qquad 1 \le j_1 < \dots < j_s \le r+s$$
of $1, 2, \dots, r+s$ such that
$$A = \{\alpha_{u,v}, \beta_{u,v} \,\big|\, 1 \le u \le r,\, 1 \le v \le s\},$$
where
$$\alpha_{u,v} = \varepsilon_{p+1-u} + \varepsilon_{n+1-v};$$
$$\beta_{u,v} = \begin{cases} \varepsilon_{p+1-u} + \varepsilon_{p+v}, & i_u < j_v; \\ \varepsilon_u + \varepsilon_{n+1-v}, & i_u > j_v. \end{cases}$$
The corresponding Λ is
$$\Lambda = (n+1-i_1, \dots, n+1-i_r, i_r, \dots, i_1 \\ \quad |n+1-j_1, \dots, n+1-j_s, j_s, \dots, j_1).$$

PROOF. The situation is combinatorially exactly the same as for $G_{\mathbb{R}} = \mathrm{Sp}(2n, \mathbb{R})$, with n, p and q even. Therefore the proof of Proposition 6.1 applies verbatim; the only difference is that the present proof is simpler because p and q are even. ∎

The complete parallel with the case of $G_{\mathbb{R}} = \mathrm{Sp}(2n, \mathbb{R})$, with n, p and q even extends also to the computation of $P_K(\Lambda)$ for any Λ from Proposition 8.1, and the constant c_p^n. The only difference is that in the present case, N of (3.3) is
$$N = \binom{p}{2} + pq \equiv \frac{p}{2} \mod 2,$$
so the sign is now $(-1)^{\frac{p}{2}}$. We conclude that the following theorem holds.

THEOREM 8.2. *Let $G_{\mathbb{R}} = SO^*(2n)$, with $n \ge 2$ even. Let p, $0 \le p \le n$, be an even integer. Let $r = \frac{p}{2}$ and let $s = \frac{n-p}{2}$. Then the constant c_p^n for the real form of $\mathcal{O}^{\mathbb{C}}$ corresponding to p is*
$$c_p^n = (-1)^{\frac{p}{2}} \binom{r+s}{r}.$$

8.2. *The case of odd n.* For odd n the real forms of $\mathcal{O}^{\mathbb{C}}$ correspond to even integers p such that $0 \le p \le n-1$. We denote $n-1-p$ by q, so q is another even integer. The h corresponding to p is
$$h_p = (\underbrace{1, \dots, 1}_{p} \,|\, 0; \underbrace{-1, \dots, -1}_{q}).$$

Since $\mathfrak{l} = \mathfrak{l}_p$ is built from roots that vanish on h_p, we see that

$$\Delta_n^+(\mathfrak{l}) = \{\varepsilon_i + \varepsilon_{p+1+j} \,\big|\, 1 \le i \le p,\ 1 \le j \le q\}.$$

It follows that for any $A \subseteq \Delta_n^+(\mathfrak{l})$,

$$2\rho(A) = (a_1, \dots, a_p \,|\, 0; b_1, \dots, b_q), \tag{8.6}$$

with

$$0 \le a_i \le q, \qquad 0 \le b_j \le p, \qquad \textstyle\sum_i a_i = \sum_j b_j. \tag{8.7}$$

In particular,

$$\rho_n(\mathfrak{l}) = (q, \dots, q \,|\, 0; p, \dots, p),$$

and this is clearly orthogonal to the roots of $\mathfrak{l} \cap \mathfrak{k}$, which are given by

$$\Delta^+(\mathfrak{l}\cap\mathfrak{k}) = \{\varepsilon_i - \varepsilon_j \,\big|\, 1 \le i < j \le p\} \cup \{\varepsilon_{p+1+i} - \varepsilon_{p+1+j} \,\big|\, 1 \le i < j \le q\}. \tag{8.8}$$

So the constants $c = c_p^n$ can be calculated from (3.4).

The set $\Delta(\mathfrak{p}_1)$ consisting of noncompact roots that are 1 on h_p is

$$\Delta(\mathfrak{p}_1) = \{\varepsilon_i + \varepsilon_{p+1} \,\big|\, 1 \le i \le p\} \cup \{-\varepsilon_{p+1} - \varepsilon_{p+1+j} \,\big|\, 1 \le j \le q\}.$$

So for any $C \subseteq \Delta(\mathfrak{p}_1)$,

$$2\rho(C) = (c_1, \dots, c_p \,|\, d; -e_1, \dots, -e_q), \tag{8.9}$$

with

$$0 \le c_i \le 1; \qquad -q \le d \le p; \qquad 0 \le e_j \le 1; \qquad d = \textstyle\sum_i c_i - \sum_j e_j. \tag{8.10}$$

To compute the constant c_p^n using (3.4), we take $\lambda = \lambda_0$, where

$$\begin{aligned}
&\lambda_0 = (n, n-1, \dots, q+2 \,|\, p+1; n-1, n-2, \dots, p+1), \qquad p, q > 0;\\
&\lambda_0 = (\,|\, 1; n-1, n-2, \dots, 1), \qquad p = 0, q > 0;\\
&\lambda_0 = (n, n-1, \dots, 2 \,|\, p+1), \qquad p > 0, q = 0;\\
&\lambda_0 = (\,|\, 1), \qquad p = q = 0.
\end{aligned} \tag{8.11}$$

Using (8.8), we see that λ_0 differs from $\rho_{\mathfrak{l}\cap\mathfrak{k}}$ by a weight orthogonal to all roots of $\mathfrak{l} \cap \mathfrak{k}$, and hence

$$P_{L\cap K}(\lambda_0) = 1. \tag{8.12}$$

So to compute c_p^n we have to compute the left side of (3.4). The following proposition describes the relevant A and C, and the corresponding Λ.

PROPOSITION 8.3. *Let $\Lambda = \lambda_0 - 2\rho(A) - 2\rho(C)$, with λ_0 given by* (8.11), *and with $A \subseteq \Delta_n^+(\mathfrak{l})$, $C \subseteq \Delta(\mathfrak{p}_1)$ as above.*

Suppose that for some A and C the corresponding Λ satisfies $P_K(\Lambda) \ne 0$. Let $r = \frac{p}{2}$ and $s = \frac{q}{2}$. Then there is an (r, s) shuffle

$$1 \le i_1 < \cdots < i_r \le r+s; \qquad 1 \le j_1 < \cdots < j_s \le r+s$$

of $1, 2, \dots, r+s$ such that

$$A = \{\alpha_{u,v}, \beta_{u,v} \,\big|\, 1 \le u \le r,\ 1 \le v \le s\},$$

where

$$\alpha_{u,v} = \varepsilon_{p+1-u} + \varepsilon_{n+1-v};$$

$$\beta_{u,v} = \begin{cases} \varepsilon_{p+1-u} + \varepsilon_{p+1+v}, & i_u < j_v; \\ \varepsilon_u + \varepsilon_{n+1-v}, & i_u > j_v, \end{cases}$$

and

$$C = \{\varepsilon_i + \varepsilon_{p+1} \,\big|\, r+1 \leq i \leq p\} \cup \{-\varepsilon_{p+1} - \varepsilon_{p+1+j} \,\big|\, 1 \leq j \leq s\}.$$

The corresponding Λ *is*

$$\Lambda = (n+1-i_1, \dots, n+1-i_r, i_r, \dots, i_1 \,|\, r+s+1; n+1-j_1, \dots, n+1-j_s, j_s, \dots, j_1).$$

If $p = 0$*, then the shuffle is necessarily trivial, i.e., there are no* i_u *and* $(j_1, \dots, j_s) = (1, \dots, s)$*. This means that*

$$\begin{aligned} A &= \emptyset; \\ C &= \{-\varepsilon_{p+1} - \varepsilon_{p+1+j} \,\big|\, 1 \leq j \leq s\}; \\ \Lambda &= (\,|\, s+1; n, \dots, n+1-s, s, \dots, 1) \\ &= (\,|\, s+1; 2s+1, \dots, s+2, s, \dots, 1). \end{aligned}$$

Similarly, if $q = 0$ *then* $(i_1, \dots, i_r) = (1, \dots, r)$*, there are no* j_v*, and*

$$\begin{aligned} A &= \emptyset; \\ C &= \{\varepsilon_i + \varepsilon_{p+1} \,\big|\, r+1 \leq i \leq p\}; \\ \Lambda &= (n, \dots, n+1-r, r, \dots, 1 \,|\, r+1) \\ &= (2r+1, \dots, r+2, r, \dots, 1 \,|\, r+1). \end{aligned}$$

Finally, if $p = q = 0$*, i.e.,* $n = 1$*, then the shuffle contains no* i_u *or* j_v*,* $A = C = \emptyset$*, and* $\Lambda = \lambda_0 = (\,|\, 1)$*.*

Proof. The statement is obviously true if $n = 1$, i.e., if $p = q = 0$. We proceed by induction on n.

So let us assume that $n \geq 3$ is odd, and let $0 \leq p \leq n-1$ be an even integer. We assume that the statement is true for $n-2$ and for any even integer p' between 0 and $n-3$.

Using the definitions and the inequalities (8.7), we see that

$$\begin{aligned} \Lambda = (&\underbrace{n - a_1 - c_1}_{[p,n]}, \underbrace{n-1-a_2-c_2}_{[p-1,n-1]}, \dots, \underbrace{q+2-a_p-c_p}_{[1,q+2]} \,|\, \underbrace{p+1-d}_{[1,n]}; \\ &\underbrace{n-1-b_1+e_1}_{[q,n]}, \underbrace{n-2-b_2+e_2}_{[q-1,n-1]}, \dots, \underbrace{p+1-b_q+e_q}_{[1,p+2]}). \end{aligned}$$

So the coordinates of Λ are n integers between 1 and n, and assuming that $P_K(\Lambda) \neq 0$, they have to be different from each other, i.e., Λ has to be a permutation of $(n, \dots, 1)$. In particular, some Λ_i must be equal to n and there are three possibilities:

$$\Lambda_1 = n \qquad \text{or} \qquad \Lambda_{p+1} = n \qquad \text{or} \qquad \Lambda_{p+2} = n. \tag{8.13}$$

Assume first that $\Lambda_1 = n$; this is only possible if $p > 0$, i.e., $p \geq 2$. Then

$$a_1 = 0, \qquad c_1 = 0,$$

and it follows that

$$\begin{aligned} &\varepsilon_1 + \varepsilon_{p+1+j} \notin A, \qquad 1 \leq j \leq q; \\ &\varepsilon_1 + \varepsilon_{p+1} \notin C. \end{aligned}$$

This implies that

$$\begin{aligned} &0 \leq b_j \leq p-1, \qquad 1 \leq j \leq q; \\ &-q \leq d \leq p-1, \end{aligned}$$

and so

$$\begin{aligned} \Lambda = (n, \underbrace{n-1-a_2-c_2}_{[p-1,n-1]}, \dots, \underbrace{q+2-a_p-c_p}_{[1,q+2]} \,|\, \underbrace{p+1-d}_{[2,n]}; \underbrace{n-1-b_1+e_1}_{[q+1,n]}, \\ \underbrace{n-2-b_2+e_2}_{[q,n-1]}, \dots, \underbrace{p+1-b_q+e_q}_{[2,p+2]}). \end{aligned}$$

We see that there is exactly one place where 1 can be, i.e.,

$$\Lambda_p = 1.$$

This implies

$$a_p = q, \qquad c_p = 1,$$

and therefore

$$\begin{aligned} &\varepsilon_p + \varepsilon_{p+1+j} \in A, \qquad 1 \leq j \leq q; \\ &\varepsilon_p + \varepsilon_{p+1} \in C. \end{aligned}$$

It follows that

$$1 \leq b_j \leq p-1, \qquad 1 \leq j \leq q, \qquad\qquad -q+1 \leq d \leq p-1$$

and so

$$\begin{aligned} \Lambda = (n, \underbrace{n-1-a_2-c_2}_{[p-1,n-1]}, \dots, \underbrace{q+3-a_{p-1}-c_{p-1}}_{[2,q+3]}, 1 \,|\, \underbrace{p+1-d}_{[2,n-1]}; \\ \underbrace{n-1-b_1+e_1}_{[q+1,n-1]}, \underbrace{n-2-b_2+e_2}_{[q,n-2]}, \dots, \underbrace{p+1-b_q+e_q}_{[2,p+1]}). \end{aligned}$$

Let now $\mathfrak{g}' \cong \mathfrak{so}^*(2(n-2))$ be the subalgebra of $\mathfrak{g}$ built on coordinates $2, \dots, p-1, p+1, \dots, n$, and let $\mathfrak{l}' = \mathfrak{l} \cap \mathfrak{g}'$. We consider the real form of the corresponding $\mathcal{O}^{\mathbb{C}}$ given by $h = h_{p-2}$.

Then

$$\begin{aligned}\Delta_n^+(\mathfrak{l}') &= \Delta_n^+(\mathfrak{l}) \setminus \{\varepsilon_1+\varepsilon_{p+1+j}, \varepsilon_p+\varepsilon_{p+1+j} \,\big|\, 1\le j\le q\};\\ \Delta(\mathfrak{p}_1') &= \Delta(\mathfrak{p}_1)\setminus\{\varepsilon_1+\varepsilon_{p+1},\varepsilon_p+\varepsilon_{p+1}\},\end{aligned}$$

and we set

$$\begin{aligned}A' &= A\setminus\{\varepsilon_p+\varepsilon_{p+1+j}\,\big|\,1\le j\le q\};\\ C' &= C\setminus\{\varepsilon_p+\varepsilon_{p+1}\}.\end{aligned}$$

We define λ_0 as in (8.11), but with n replaced by $n-2$ and p replaced by $p-2$. Then Λ' corresponding to A' and C' can be obtained from Λ by deleting coordinates Λ_1 and Λ_p, and decreasing all the other coordinates by 1. More precisely, deleting the first and the p-th coordinate, we have

$$\begin{aligned}2\rho(A') &= (a_2,\dots,a_{p-1}\,|\,0;b_1-1,\dots,b_q-1) = (a_1',\dots,a_{p-2}'\,|\,0;b_1',\dots,b_q');\\ 2\rho(C') &= (c_2,\dots,c_{p-1}\,|\,d-1;-e_1\dots,-e_q)\\ &= (c_1',\dots,c_{p-2}'\,|\,d';-e_1'\dots,-e_q');\\ \Lambda' &= (n-2-a_1'-c_1',\dots,q+2-a_{p-2}'-c_{p-2}'\,|\,p-1-d';\\ &\qquad\qquad n-3-b_1'+e_1',\dots,p-1-b_q'+e_q')\\ &= (\Lambda_2-1,\dots,\Lambda_{p-1}-1\,|\,\Lambda_{p+1}-1,\dots,\Lambda_n-1).\end{aligned}$$

We now see that Λ is a permutation of $(n,\dots,1)$ if and only if Λ' is a permutation of $(n-2,\dots,1)$. By inductive assumption, this is equivalent to A' and Λ' being defined by a shuffle as in the statement of the proposition, and this clearly implies the same statement for A and Λ.

The second possibility in (8.13) is $\Lambda_{p+1}=n$, which implies $d=-q$ and hence

$$\begin{aligned}-\varepsilon_{p+1}-\varepsilon_{p+1+j}\in C,&\qquad 1\le j\le q;\\ \varepsilon_i+\varepsilon_{p+1}\notin C,&\qquad 1\le i\le p.\end{aligned}$$

This implies

$$\begin{aligned}c_i=0,&\qquad 1\le i\le p;\\ e_j=1,&\qquad 1\le j\le q,\end{aligned}$$

and so

$$\Lambda = (\underbrace{n-a_1}_{[p+1,n]},\underbrace{n-1-a_2}_{[p,n-1]},\dots,\underbrace{q+2-a_p}_{[2,q+2]}\,|\,n;\underbrace{n-b_1}_{[q+1,n]},\underbrace{n-1-b_2}_{[q,n-1]},\dots,\underbrace{p+2-b_q}_{[2,p+2]}).$$

We see that there is no place where 1 could be, so this case is impossible if $P_K(\Lambda)\neq 0$.

The third possibility in (8.13) is $\Lambda_{p+2} = n$; this is possible only if $q > 0$, i.e., $q \geq 2$. It follows that $b_1 = 0, e_1 = 1$, and so

$$\begin{aligned} &\varepsilon_i + \varepsilon_{p+2} \notin A, \qquad 1 \leq i \leq p; \\ &-\varepsilon_{p+1} - \varepsilon_{p+2} \in C. \end{aligned}$$

This implies

$$0 \leq a_i \leq q-1, \qquad 1 \leq i \leq p; \qquad\qquad -q \leq d \leq p-1,$$

and hence

$$\begin{aligned} \Lambda = (&\underbrace{n - a_1 - c_1}_{[p+1,n]}, \underbrace{n-1-a_2-c_2}_{[p,n-1]}, \dots, \underbrace{q+2-a_p-c_p}_{[2,q+2]} \,|\, \underbrace{p+1-d}_{[2,n]}; \\ &n, \underbrace{n-2-b_2+e_2}_{[q-1,n-1]}, \dots, \underbrace{p+1-b_q+e_q}_{[1,p+2]}). \end{aligned}$$

We see that there is exactly one place where 1 can be, i.e., $\Lambda_n = 1$. This implies $b_q = p, e_q = 0$ and therefore

$$\varepsilon_i + \varepsilon_n \in A, \qquad 1 \leq i \leq p; \qquad\qquad -\varepsilon_{p+1} - \varepsilon_n \notin C.$$

It follows that

$$1 \leq a_i \leq q-1, \qquad 1 \leq i \leq p; \qquad\qquad -q+1 \leq d \leq p-1,$$

and so

$$\begin{aligned} \Lambda = (&\underbrace{n - a_1 - c_1}_{[p+1,n-1]}, \underbrace{n-1-a_2-c_2}_{[p,n-2]}, \dots, \underbrace{q+2-a_p-c_p}_{[2,q+1]} \,|\, \underbrace{p+1-d}_{[2,n-1]}; \\ &n, \underbrace{n-2-b_2+e_2}_{[q-1,n-1]}, \underbrace{n-3-b_3+e_3}_{[q-2,n-2]}, \dots, \underbrace{p+2-b_{q-1}+e_{q-1}}_{[2,p+3]}, 1). \end{aligned}$$

We now reason in the same way as in the first case, and conclude that the proposition follows from the inductive assumption for $n-2$ with p staying the same and q being replaced by $q-2$. ∎

To finish the computation of the constant c_p^n, we first note that for every A and C described in Proposition 8.3

$$\#A = 2rs; \qquad \#C = r + s. \tag{8.14}$$

On the other hand, since Λ is a permutation of $(n, \dots, 1)$, $P_K(\Lambda)$ is equal to ± 1. To compute the sign, we need to find the parity of the permutation bringing Λ to $(n, \dots, 1)$. As in type C, we find this parity by counting the number of inversions in Λ when compared with $(n, \dots, 1)$. We know from Proposition 8.3 that

$$\begin{aligned} \Lambda = (&n+1-i_1, \dots, n+1-i_r, i_r, \dots, i_1 \\ &|r+s+1; n+1-j_1, \dots, n+1-j_s, j_s, \dots, j_1). \end{aligned}$$

Clearly $i_r, \ldots, i_1$ are in inversion with $n+1-j_1, \ldots, n+1-j_s$; that is rs inversions. Arguing as in type C, we get further rs inversions from the groups

$$n+1-i_1, \ldots, n+1-i_r \qquad \text{and} \qquad n+1-j_1, \ldots, n+1-j_s,$$

and

$$i_r, \ldots, i_1 \qquad \text{and} \qquad j_s, \ldots, j_1.$$

Finally, the coordinate

$$\Lambda_{p+1} = r+s+1$$

is in inversion with

$$i_r, \ldots, i_1 \qquad \text{and} \qquad n+1-j_1, \ldots, n+1-j_s.$$

So the total number of inversions is $2rs+r+s$, and combined with (8.14) this implies that the nonzero contributions to the sum in (6.3), which we know come from A, C and Λ as in Proposition 8.3, are all equal to

$$(-1)^{\#A+\#C} P_K(\Lambda) = 1.$$

Furthermore, the number N of (3.3) satisfies

$$N \equiv \frac{p}{2} \mod 2.$$

Since the number of nonzero summands is by Proposition 8.3 equal to the number of (r,s)-shuffles of $r+s$, i.e., to $\binom{r+s}{r}$, we have proved the following theorem.

THEOREM 8.4. *Let $G_{\mathbb{R}} = SO^*(2n)$, for an odd $n \geq 1$, and let p, $0 \leq p \leq n-1$, be an even integer. Let*

$$r = \frac{p}{2}, \qquad s = \frac{n-1-p}{2}.$$

Then the constant c_p^n for the real form of $\mathcal{O}^{\mathbb{C}}$ corresponding to p is

$$c_p^n = (-1)^{\frac{p}{2}} \binom{r+s}{r}.$$

For the convenience of the reader, below is a table that gives the value of the constant for each real form in every case.

$G_{\mathbb{R}}$ $K_{\mathbb{R}}$		$\mathcal{O}^{\mathbb{C}}$	Real forms	Constants
$SU(p,q)$ $S(U(p)\times U(q))$	$q\geq p\geq 1$	$[2^p,1^{q-p}]$	$(\underbrace{1,\cdots,1}_{k},\underbrace{-1,\cdots,-1}_{p-k},$ $\underbrace{1,\cdots,1}_{p-k},\underbrace{0,\cdots,0}_{q-p},\underbrace{-1,\cdots,-1}_{k})$ $k=0,1,\cdots,p$	$(-1)^{k(p+q-k)}\binom{p}{k}$
$SO_e(2p,2q+1)$ $SO(2p)\times SO(2q+1)$	$q\geq p\geq 1$	$[3,2^{2p-2},1^{2(q-p+1)}]$	$(2,\underbrace{1,\cdots,1}_{p-1},$ $\underbrace{1,\cdots,1}_{p-1},\underbrace{0,\cdots,0}_{q-p+1})$	$(-1)^{[(p/2)]+1}2^{2p-2}$
			$(2,\underbrace{1,\cdots,-1}_{p-1},$ $\underbrace{1,\cdots,1}_{p-1},\underbrace{0,\cdots,0}_{q-p+1})$	$(-1)^{[(p/2)]}2^{2p-2}$
		3rd real form only if $q>p-1$	$(\underbrace{1,\cdots,1}_{p-1},0,$ $2,\underbrace{1,\cdots,1}_{p-1},\underbrace{0,\cdots,0}_{q-p})$	0
$\mathrm{Sp}(2n,\mathbb{R})$ $U(n)$	$n\geq 1$	$[2^n]$	$(\underbrace{1,\cdots,1}_{k},\underbrace{-1,\cdots,-1}_{n-k})$ $k=0,1,\cdots,n$	$(-1)^{[(k+1)/2]}\binom{r+s}{r}$ (n odd) or (n and k even) 0 n even and k odd $r=[\frac{k}{2}]$ and $s=[\frac{n-k}{2}]$
$SO_e(2p,2q)$ $SO(2p)\times SO(2q)$	$q\geq p\geq 1$	$[3,2^{2p-2},1^{2(q-p)+1}]$	$(2,\underbrace{1,\cdots,1}_{p-1},$ $\underbrace{1,\cdots,1}_{p-1},\underbrace{0,\cdots,0}_{q-p+1})$	$(-1)^{[(p-1)/2]}2^{2p-2}$
			$(2,\underbrace{1,\cdots,-1}_{p-1},$ $\underbrace{1,\cdots,1}_{p-1},\underbrace{0,\cdots,0}_{q-p+1})$	$(-1)^{[(p-1)/2]}2^{2p-2}$
			$(\underbrace{1,\cdots,1}_{p-1},0,$ $2,\underbrace{1,\cdots,1}_{p-1},\underbrace{0,\cdots,0}_{q-p})$	$(-1)^{[p/2]+1}2^{2p-1}$ if $q>p$ $(-1)^{[p/2]+1}2^{2p-2}$ if $q=p$
		fourth real form only if $q=p$	$(\underbrace{1,\cdots,1}_{p-1},0,$ $2,\underbrace{1,\cdots,1}_{p-1},\underbrace{0,\cdots,0}_{q-p})$	$(-1)^{[p/2]+1}2^{2p-2}$
$SO^*(2n)$ $U(n)$	$n\geq 1$	$[2^n]$ n even	$(\underbrace{1,\cdots,1}_{p},\underbrace{-1,\cdots,-1}_{n-p})$ $0\leq p\leq n$ with p even	$(-1)^{p/2}\binom{n/2}{p/2}$
		$[2^{n-1},1^2]$ n odd	$(\underbrace{1,\cdots,1}_{p},0,\underbrace{-1,\cdots,-1}_{n-1-p})$ $0\leq p\leq n-1$ with p even	$(-1)^{p/2}\binom{(n-1)/2}{p/2}$

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