2 Scalar one-loop Feynman integrals in arbitrary space–time dimension d – an update

Contribution* by: T. Riemann, J. Usovitsch
Corresponding author: T. Riemann [tordriemann@gmail.com]

2.1 Introduction

The study and use of analyticity of scattering amplitudes was founded by R. Eden, P. Landshoff, D. Olive, and J. Polkinghorn in their famous book *The Analytic S-Matrix* in 1966 [1]. Indeed, as early as 1969, J. Schwinger quoted: "One of the most remarkable discoveries in elementary particle physics has been that of the complex plane, [...] the theory of functions of complex variables plays the role not of a mathematical tool, but of a fundamental description of nature inseparable from physics." [2].

It took many years to make use of analyticity and unitarity, together with renormalizability and gauge invariance of quantum field theory, as a practical tool for the calculation of cross-sections at real colliders. When the analysis of LEP 1 data, around 1989, was prepared, it became evident that the S-matrix language helps to efficiently sort the various perturbative contributions of the Standard Model.

The scattering amplitude for the reaction $e^+e^- \to (Z, \gamma) \to f\bar{f}$ at LEP energies depends on two variables, s and $\cos\theta$, and the integrated cross-section may be described by an analytical function of s with a simple pole, describing mass and width of the Z resonance:

$$A = \frac{R}{s - M_Z^2 + iM_Z\Gamma_Z} + \sum_{i=0}^{\infty} a_i \left(s - M_Z^2 + iM_Z\Gamma_Z\right)^i . \tag{2.1}$$

Here, position $s_0 = M_Z^2 - iM_Z\Gamma_Z$ and residue R of the pole, as well as the background expansion, are of interest. The analytic form of Eq. (2.1) must be respected when deriving a Z amplitude at multiloop accuracy; see Ref. [3] and the references therein.

Shortly after the work by Eden *et al.* [1], physical amplitudes were also proposed for consideration as complex functions of space–time dimension d (dimensional regularisation) [4,5].

In perturbative calculations with dimensional regularisation, Feynman integrals I are complex functions of the space–time dimension $d = 4 - 2\varepsilon$. In fact, they are meromorphic functions of d and may be expanded in Laurent series around poles at, e.g., $d_s = 4+2N_0$, $N_0 \geq 0$. Let J_n be an n-point one-loop Feynman integral, as shown in Fig. C.2.1:

$$J_n \equiv J_n\left(d; \{p_i p_j\}, \left\{m_i^2\right\}\right) = \int \frac{d^d k}{i\pi^{d/2}} \frac{1}{D_1^{\nu_1} D_2^{\nu_2} \cdots D_n^{\nu_n}} \tag{2.2}$$

with

$$D_i = \frac{1}{(k + q_i)^2 - m_i^2 + i\epsilon} \tag{2.3}$$

*This contribution should be cited as:
T. Riemann, J. Usovitsch, Scalar one-loop Feynman integrals in arbitrary space-time dimension d – an update, DOI: 10.23731/CYRM-2020-003.139, in: Theory for the FCC-ee, Eds. A. Blondel, J. Gluza, S. Jadach, P. Janot and T. Riemann,
CERN Yellow Reports: Monographs, CERN-2020-003, DOI: 10.23731/CYRM-2020-003, p. 139.

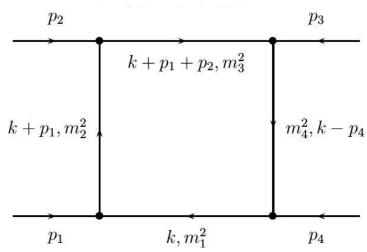

Fig. C.2.1: One-loop Feynman integral

and

$$\nu = \sum_{i=1}^{n} \nu_i, \qquad \sum_{e=1}^{n} p_e = 0. \tag{2.4}$$

The Feynman integrals are analytical functions of d everywhere with exclusion of isolated singular points d_s, where they behave not worse than

$$\frac{A_s}{(d-d_s)^{N_s}}. \tag{2.5}$$

In physics applications, we need the Feynman integrals at a potentially singular point, $d = 4$, so that their general behaviour at non-singular points is not in the original focus. Nevertheless, the question arises:

Can we determine the general d-dependence of a Feynman integral?

For one-loop integrals, the question has been answered recently, in Ref. [6].

At the beginning of systematic cross-section calculations in d dimensions came two seminal papers on one-loop Feynman integrals in dimensional regularisation [7, 8]. Later, many improvements and generalisations were introduced in various respects.

We see several reasons to study the d-dependence of one-loop Feynman integrals and will discuss them briefly in the next subsection.

2.2 Interests in the *d*-dependence of one-loop Feynman integrals

2.2.1 Interest from mathematical physics

There is a general interest in the Feynman integrals as meromorphic functions of space–time dimension d; the easiest case is that of one loop. Early attempts, for the massless case, trace back to Boos and Davydychev [9]. The general one-loop integrals were tackled systematically by Tarasov *et al.* since the 1990s; see, e.g., Refs. [10–13] and references therein. In Refs. [14,15], the class of generalised hypergeometric functions for massive one-loop Feynman integrals with unit indices was determined and studied with a novel approach based on dimensional difference equations.

(a) ${}_2F_1$ Gauss hypergeometric functions are needed for self-energies.
(b) F_1 Appell functions are needed for vertices.
(c) F_S Lauricella–Saran functions are needed for boxes.

Finally, the correct, general massive one-loop one- to four-point functions with unit indices at arbitrary kinematics were determined by Phan and Riemann [6], who also calculated the numerics of the generalised hypergeometric functions.

2.2.2 Interest from tensor reductions of n-point functions in higher space–time dimensions

For many-particle calculations, inverse Gram determinants $1/G(p_i)$ from tensor reductions appear at certain kinematic configurations p_i. These terms $1/G(p_i)$ may diverge, because Gram determinants can exactly vanish: $G(p_i) \equiv 0$. One may perform tensor reductions so that no inverse Gram determinants appear. But then one has to calculate scalar one-loop integrals in higher dimensions, $D = 4 + 2n - 2\epsilon, n > 0$ [16, 17]. In fact, one introduces new scalar integrals [16]. Let us take as an example a rank-5 tensor of an n-point function:

$$\begin{aligned} I_n^{\mu\nu\lambda\rho\sigma} &= \int \frac{\mathrm{d}^d k}{\mathrm{i}\pi^{d/2}} \frac{k^\mu k^\nu k^\lambda k^\rho k^\sigma}{\prod_{j=1}^n c_j} \\ &= -\sum_{i,j,k,l,m=1}^{n} q_i^\mu q_j^\nu q_k^\lambda q_l^\rho q_m^\sigma n_{ijklm} I_{n,ijklm}^{[d+]^5} + \frac{1}{2}\sum_{i,j,k=1}^{n} g^{[\mu\nu} q_i^\lambda q_j^\rho q_k^{\sigma]} n_{ijk} I_{n,ijk}^{[d+]^4} \\ &\quad - \frac{1}{4}\sum_{i=1}^{n} g^{[\mu\nu} g^{\lambda\rho} q_i^{\sigma]} I_{n,i}^{[d+]^3}. \end{aligned} \tag{2.6}$$

The integrals $I_{n,ab\dots}^{[d+]^l}$ are special cases of $I_{n,ab\dots}^{[d+]^l,s}$, defined in $[d+]^l = 4 - 2\varepsilon + 2l$ dimensions, by shrinking line s and raising the powers of propagators (indices) $a, b, \dots$

At this step, the tensor integral is represented by scalar integrals with higher space–time dimensions and higher propagator powers. The publicly available Feynman integral libraries deliver, though, ordinary scalar integrals in $d = 4 - 2\varepsilon$ dimensions, with unit propagator powers. With the usual integration by parts reduction technique [18, 19], one may shift indices, i.e., reduce propagator powers to unity:

$$\nu_j \mathbf{j}^+ I_5 = \frac{1}{\binom{0}{0}_5} \sum_{k=1}^{5} \binom{0j}{0k}_5 \left[d - \sum_{i=1}^{5} \nu_i (\mathbf{k}^- \mathbf{i}^+ + 1) \right] I_5. \tag{2.7}$$

The operators $\mathbf{i}^\pm, \mathbf{j}^\pm, \mathbf{k}^\pm$ act by shifting the indices ν_i, ν_j, ν_k by ± 1.

After this step, one has yet to deal with scalar functions in $d = 4 - 2\varepsilon + 2l$ dimensions. This may be further reduced by applying dimensional reduction formulae invented by Tarasov [10,13]: shift of dimension and index,

$$\nu_j \left(\mathbf{j}^+ I_5^{[d+]} \right) = \frac{1}{()_5} \left[-\binom{j}{0}_5 + \sum_{k=1}^{5} \binom{j}{k}_5 \mathbf{k}^- \right] I_5, \tag{2.8}$$

and shift only of dimension,

$$\left(d - \sum_{i=1}^{5} \nu_i + 1 \right) I_5^{[d+]} = \frac{1}{()_5} \left[\binom{0}{0}_5 - \sum_{k=1}^{5} \binom{0}{k}_5 \mathbf{k}^- \right] I_5. \tag{2.9}$$

The procedure is elegant, but it introduces inverse powers of potentially vanishing Gram determinants in both cases. As a consequence, one has finally to treat the numerical implications in sophisticated ways.

At this stage, one might try an alternative. Perform the reductions of tensor functions to scalar functions with unit indices, but allowing for the use of higher space–time dimensions. This avoids the vanishing inverse Gram problem, but introduces the need of a library of scalar

Feynman integrals in higher dimensions. This idea makes it attractive to derive an algorithm allowing the systematic calculation of scalar one- to n-point functions in arbitrary dimensions, and to implement a numerical solution for it.

To be a little more definite, we quote here some unpublished formulae from Refs. [17,20]. The following reduction of a five-point tensor in terms of tensor coefficients E^s_{ijklm}, with line s skipped from the five-point integral, may be used as a starting point:

$$I_5^{\mu\,\nu\,\lambda\rho\sigma} = \sum_{s=1}^{5}\left[\sum_{i,j,k,l,m=1}^{5} q_i^\mu q_j^\nu q_k^\lambda q_l^\rho q_m^\sigma E^s_{ijklm} + \sum_{i,j,k=1}^{5} g^{[\mu\nu} q_i^\lambda q_j^\rho q_k^{\sigma]} E^s_{00ijk} + \sum_{i=1}^{5} g^{[\mu\nu} g^{\lambda\rho} q_i^{\sigma]} E^s_{0000i}\right]. \tag{2.10}$$

The tensor coefficients E^s_{ijklm} are expressed in terms of integrals $I^{[d+]^l,s}_{4,i\cdots}$, e.g.:

$$E^s_{ijklm} = -\frac{1}{\binom{0}{0}_5}\left\{\left[\binom{0l}{sm}_5 n_{ijk} I^{[d+]^4,s}_{4,ijk} + (i\leftrightarrow l) + (j\leftrightarrow l) + (k\leftrightarrow l)\right] + \binom{0s}{0m}_5 n_{ijkl} I^{[d+]^4,s}_{4,ijkl}\right\}. \tag{2.11}$$

No factors $1/G_5 = \left(\right)_5$ appear. Now, in a next step, one may avoid the appearance of inverse sub-Gram determinants $()_4$. Further, the complete dependence on the indices i of the tensor coefficients can be shifted into the integral's pre-factors with signed minors. One can say that the indices *decouple* from the integrals. As an example, we reproduce the four-point part of $I^{[d+]^4}_{4,ijkl}$:

$$\begin{aligned} n_{ijkl} I^{[d+]^4}_{4,ijkl} &= \frac{\binom{0}{i}}{\binom{0}{0}}\frac{\binom{0}{j}}{\binom{0}{0}}\frac{\binom{0}{k}}{\binom{0}{0}}\frac{\binom{0}{l}}{\binom{0}{0}} d(d+1)(d+2)(d+3) I_4^{[d+]^4} \\ &+ \frac{\binom{0i}{0j}\binom{0}{k}\binom{0}{l} + \binom{0i}{0k}\binom{0}{j}\binom{0}{l} + \binom{0j}{0k}\binom{0}{i}\binom{0}{l} + \binom{0i}{0l}\binom{0}{j}\binom{0}{k} + \binom{0j}{0l}\binom{0}{i}\binom{0}{k} + \binom{0k}{0l}\binom{0}{i}\binom{0}{j}}{\binom{0}{0}^3} \\ &\times\ d(d+1) I_4^{[d+]^3} + \frac{\binom{0i}{0l}\binom{0j}{0k} + \binom{0j}{0l}\binom{0i}{0k} + \binom{0k}{0l}\binom{0i}{0j}}{\binom{0}{0}^2} I_4^{[d+]^2} + \cdots \end{aligned} \tag{2.12}$$

In Eq. (2.12), one has to understand the four-point integrals to carry the corresponding index s of Eq. (2.10) and that the signed minors are $\binom{0}{k} \to \binom{0s}{ks}_5$, etc. We arrived at:

(a) no scalar five-point integrals in higher dimensions;
(b) no inverse Gram determinants $()_5$;
(c) four-point integrals without indices;
(d) scalar four-point integrals in higher dimensions appearing as $I_4^{[d+]^2,s}$, etc.;
(e) inverse four-point Gram determinants $\binom{0}{0}_5 \equiv ()_4$.

2.2.3 *Interest from multiloop calculations*

Higher-order loop calculations need higher-order contributions from ϵ-expansions of one-loop terms, typically stemming from the expansions

$$\frac{1}{d-4} = -\frac{1}{2\epsilon} \tag{2.13}$$

and

$$\Gamma(\epsilon) = \frac{a_1}{\epsilon} + a_0 + a_1\epsilon + \cdots \tag{2.14}$$

A seminal paper on the ε-terms of one-loop functions is Ref. [21]. A general analytical solution of the problem of determining the general ε-expansion of Feynman integrals is unsolved so far, even for the one-loop case, although see Refs. [22–25]. The determination of one-loop Feynman integrals as meromorphic functions of d might be a useful preparatory step for determining the pole expansion in d around, e.g., $d = 4$.

2.2.4 Interest from Mellin–Barnes representations

A powerful approach to arbitrary Feynman integrals is based on Mellin–Barnes representations [26, 27]. One-loop integrals with variable, in general non-integer, indices are needed in the context of the *loop-by-loop Mellin–Barnes approach* to multiloop integrals. Details may be found in the literature on the Mathematica package AMBRE [28–35], and in references therein.

A crucial technical problem of the Mellin–Barnes representations arises from the increasing number of dimensions of these representations with an increasing number of physical scales. We will detail this in Section 2.3.1. Thus, there is an unresolved need for low-dimensional one-loop Mellin–Barnes (MB) integrals, with arbitrary indices.

2.3 Mellin–Barnes representations for one-loop Feynman integrals

Two numerical MB approaches are advocated.

2.3.1 AMBRE

There are several ways to take advantage of Mellin–Barnes representations for the calculation of Feynman integrals. One approach is the replacement of massive propagators by Mellin–Barnes integrals over massless propagators, invented by Usyukina [36]. Another approach transforms the Feynman parameter representation with Mellin–Barnes representations into a number of complex path integrals, invented in 1999 by Smirnov for planar diagrams [26] and Tausk for non-planar diagrams [27]. This approach 'automatically' implies a general solution of the infrared problem and has been worked out in the AMBRE approach [28, 32, 34, 35, 37].

The general definitions for a multiloop Feynman integral are

$$J_n^L \equiv J_n^L\left(d; \{p_i p_j\}, \{m_i^2\}\right) = \int \prod_{j=1}^{L} \frac{\mathrm{d}^d k_j}{\mathrm{i}\pi^{d/2}} \frac{1}{D_1^{\nu_1} D_2^{\nu_2} \cdots D_n^{\nu_n}} \tag{2.15}$$

with

$$D_i = \left(\sum_{l=1}^{L} a_{il} k_l + \sum_{e=1}^{E} b_{ie} p_e\right)^2 - m_i^2 + \mathrm{i}\delta, \qquad a_{il},\ b_{ie} \in \{-1, 0, 1\}, \tag{2.16}$$

where m_i are the masses, p_e the external momenta, k_l the loop momenta, $\mathrm{i}\delta$ the Feynman prescription, and, finally, ν_i the complex variables.

With the following Feynman trick, we get a really neat parametric representation:

$$\frac{(-1)^\nu}{\prod_{j=1}^{n}\left(-D_j^{\nu_j}\right)} = \frac{(-1)^\nu \Gamma(\nu) \left(\prod_{j=1}^{n} \int_{\{x_j \geq 0\}} \frac{\mathrm{d}x_j\, x_j^{\nu_j - 1}}{\Gamma(\nu_j)}\right) \delta\left(1 - \sum_{j=1}^{n} x_j\right)}{(-k_l^\mu M_{ll'} k_{l'\mu} + 2k_l^\mu Q_{l\mu} + J - i\delta)^\nu}, \qquad \nu = \sum_{j=1}^{n} \nu_j, \tag{2.17}$$

where

$$M_{ll'} = \sum_{j=1}^{n} a_{jl} a_{jl'} x_j \tag{2.18}$$

is an $L \times L$ symmetric matrix,

$$Q_l^\nu = -\sum_{j=1}^{n} x_j a_{jl} \sum_{e=1}^{E} b_{je} p_e^\nu \tag{2.19}$$

is a vector with L components, and

$$J = -\sum_{j=1}^{n} x_j \left(\sum_{e=1}^{E} b_{je} p_e^\mu \sum_{e'=1}^{E} p_{e'}^\nu b_{je'} g_{\mu\nu} - m_j^2 \right), \tag{2.20}$$

where x_j are the Feynman parameters introduced using the Feynman trick. The metric tensor is $g_{\mu\nu} = \mathrm{diag}(1, -1, \ldots, -1)$.

The Feynman integral can now be written in the Feynman parameter integral representation:

$$J_n^L = (-1)^\nu \Gamma(\nu - LD/2) \left(\prod_{j=1}^{n} \int_{\{x_j \geq 0\}} \frac{\mathrm{d}x_j \, x_j^{\nu_j - 1}}{\Gamma(\nu_j)} \right) \delta \left(1 - \sum_{j=1}^{n} x_j \right) \frac{U(x)^{\nu-(L+1)D/2}}{F(x)^{\nu - LD/2}}, \tag{2.21}$$

where

$$U(x) = \det M, \tag{2.22}$$

$$F(x) = U(x) \left(Q_l^\mu M_{ll'}^{-1} Q_{l'\mu} + J - \mathrm{i}\delta \right). \tag{2.23}$$

From these definitions, it follows that the functions $F(x)$ and $U(x)$ are homogeneous in the Feynman parameters x_i. The function $U(x)$ is of degree L and the function $F(x)$ is of degree $L+1$. The functions $U(x)$ and $F(x)$ are also known as Symanzik polynomials.

At one-loop level, the definition of the Feynman integral simplifies drastically and gives many insights straight away, which we will bring to light in this work:

$$J_n \equiv J_n \left(d; \{p_i p_j\}, \{m_i^2\} \right) = \int \frac{\mathrm{d}^d k}{\mathrm{i}\pi^{d/2}} \frac{1}{D_1^{\nu_1} D_2^{\nu_2} \cdots D_n^{\nu_n}} \tag{2.24}$$

with propagators depending only on one-loop momenta:

$$D_i = \frac{1}{(k + q_i)^2 - m_i^2 + \mathrm{i}\epsilon}, \tag{2.25}$$

with

$$q_i = \sum_{e=1}^{i} p_e. \tag{2.26}$$

We assume here, for brevity,

$$\nu_i = 1, \qquad \sum_{e=1}^{n} p_e = 0. \tag{2.27}$$

If we take the argument of the Dirac delta function to be $1-\sum_{j=1}^{n} x_j$, the Feynman parameter representation for one-loop Feynman integrals simplifies to

$$J_n = (-1)^n \Gamma\left(n-d/2\right) \int_0^1 \prod_{i=1}^{n} \mathrm{d}x_i \delta\left(1-\sum_{j=1}^{n} x_j\right) \frac{1}{F_n(x)^{n-d/2}}. \tag{2.28}$$

Here, the F function is the second Symanzik polynomial, which is just of second degree in the Feynman parameters:

$$F_n(x) = \frac{1}{2}\sum_{i,j} x_i Y_{ij} x_j - i\epsilon. \tag{2.29}$$

The Y_{ij} are elements of the Cayley matrix $Y=(Y_{ij})$,

$$Y_{ij} = Y_{ji} = m_i^2 + m_j^2 - (q_i - q_j)^2. \tag{2.30}$$

Gram and Cayley determinants were introduced by Melrose [38]; see also Ref. [13]. The $(n-1)\times(n-1)$-dimensional Gram determinant $G_n \equiv G_{12\cdots n}$ is

$$G_n = -\begin{vmatrix} (q_1-q_n)^2 & (q_1-q_n)(q_2-q_n) & \dots & (q_1-q_n)(q_{n-1}-q_n) \\ (q_1-q_n)(q_2-q_n) & (q_2-q_n)^2 & \dots & (q_2-q_n)(q_{n-1}-q_n) \\ \vdots & \vdots & \ddots & \vdots \\ (q_1-q_n)(q_{n-1}-q_n) & (q_2-q_n)(q_{n-1}-q_n) & \dots & (q_{n-1}-q_n)^2 \end{vmatrix}. \tag{2.31}$$

The $2^n G_n$ equals, notationally, the G_{n-1} of Ref. [13]. Evidently, the Gram determinant G_n is independent of the propagator masses.

The Cayley determinant $\Delta_n = \lambda_{12\cdots n}$ is composed of the Y_{ij} introduced in Eq. (2.30):

$$\text{Cayley determinant}: \quad \Delta_n = \lambda_n \equiv \lambda_{12\cdots n} = \begin{vmatrix} Y_{11} & Y_{12} & \dots & Y_{1n} \\ Y_{12} & Y_{22} & \dots & Y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{1n} & Y_{2n} & \dots & Y_{nn} \end{vmatrix}. \tag{2.32}$$

We also define the modified Cayley determinant

$$\text{modified Cayley determinant}: \quad ()_n = \begin{vmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & Y_{11} & Y_{12} & \dots & Y_{1n} \\ 1 & Y_{12} & Y_{22} & \dots & Y_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & Y_{1n} & Y_{2n} & \dots & Y_{nn} \end{vmatrix}. \tag{2.33}$$

The determinants Δ_n, $()_n$, and G_n are evidently independent of a common shifting of the momenta q_i.

One may use Mellin–Barnes integrals [39],

$$\frac{1}{(1+z)^\lambda} = \frac{1}{2\pi \mathrm{i}} \int\limits_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}s \, \frac{\Gamma(-s)\,\Gamma(\lambda+s)}{\Gamma(\lambda)} \, z^s = {}_2F_1\left[\begin{matrix}\lambda, b\,; \\ b\,;\end{matrix} -z\right], \tag{2.34}$$

to split the sum $F_n(x)$ in Eq. (2.29) into a product, enabling nested MB integrals to be calculated. For some mathematics behind the derivation, see the corollary at p. 289 in Ref. [40]. Equation (2.34) is valid if $|\mathrm{Arg}(z)| < \pi$. The integration contour must be chosen such that the poles of $\Gamma(-s)$ and $\Gamma(\lambda + s)$ are well-separated. The right-hand side of Eq. (2.34) is identified as Gauss's hypergeometric function.

There are $N_n = n(n+1)/2$ different Y_{ij} for n-point functions, leading to $N_n = [n(n+1)/2-1]$-dimensional Mellin–Barnes integrals when splitting the sum in Eq. (2.29) into a product:

- $N_3 = 5$ MB dimensions for the most general massive vertices;
- $N_4 = 9$ MB dimensions for the most general massive box integrals;
- $N_5 = 14$ MB dimensions for the most general massive pentagon integrals.

The introduction of N_n-dimensional MB integrals allows x integrations to be calculated. The MB integrations must be calculated afterwards, and this raises some mathematical problems with increasing integral dimensions. This is, for Mellin–Barnes integrals numerical applications, one of the most important limiting factors.

For further details of this approach, we refer to the quoted literature on AMBRE and MBnumerics.

2.3.2 MBOneLoop

A completely different approach was initiated in Refs. [6,41]. The idea is based on rewriting the F function in Eq. (2.28) by exploring the factor $\delta(1-\sum x_i)$, which makes the n-fold x-integration to be an integral over an $(n-1)$-simplex.

The δ function allows for the elimination of x_n, just one of the x_i, which creates linear terms in the remaining x_i variables in the F function:

$$F_n(x) = x^{\mathrm{T}} G_n x + 2 H_n^{\mathrm{T}} x + K_n. \tag{2.35}$$

The $F_n(x)$ may be recast into a bilinear form by shifts $x \to (x - y)$,

$$F_n(x) = (x-y)^{\mathrm{T}} G_n (x-y) + r_n - \mathrm{i}\varepsilon \ = \ \Lambda_n(x) + r_n - \mathrm{i}\varepsilon \ = \ \Lambda_n(x) + R_n. \tag{2.36}$$

As a result, there is a separation of F into a homogeneous part $\Lambda_n(x)$,

$$\Lambda_n(x) = (x-y)^{\mathrm{T}} G_n (x-y), \tag{2.37}$$

and an inhomogeneity R_n,

$$R_n = r_n - \mathrm{i}\varepsilon = K_n - H_n^{\mathrm{T}} G_n^{-1} H_n - \mathrm{i}\varepsilon = -\frac{\lambda_n}{g_n} - \mathrm{i}\varepsilon = -\frac{\begin{pmatrix}0\\0\end{pmatrix}_n}{()_n}. \tag{2.38}$$

It is only this inhomogeneity $R_n = r_n \ - \mathrm{i}\varepsilon$ that carries the iε prescription. The $(n-1)$ components y_i of the shift vector y appearing here in $F_n(x)$ are

$$y_i = -\left(G_n^{-1} K_n\right)_i, \qquad i \neq n. \tag{2.39}$$

The following relations are also valid:

$$y_i = \frac{\partial r_n}{\partial m_i^2} = -\frac{1}{g_n}\,\frac{\partial \lambda_n}{\partial m_i^2} = -\frac{\partial_i \lambda_n}{g_n} = \frac{2}{g_n}\begin{pmatrix}0\\i\end{pmatrix}_n, \qquad i = 1, \ldots, n. \tag{2.40}$$

One further notation has been introduced in Eq. (2.40), namely that of *cofactors of the modified Cayley matrix*, also called signed minors in, e.g., [38, 42]

$$\begin{pmatrix} j_1 & j_2 & \cdots & j_m \\ k_1 & k_2 & \cdots & k_m \end{pmatrix}_n . \tag{2.41}$$

The signed minors are determinants, labelled by those rows $j_1, j_2, \ldots, j_m$ and columns $k_1, k_2, \ldots, k_m$ that have been discarded from the definition of the modified Cayley determinant $()_n$, with a sign convention:

$$\begin{aligned} &\text{sign} \begin{pmatrix} j_1 & j_2 & \cdots & j_m \\ k_1 & k_2 & \cdots & k_m \end{pmatrix}_n \\ &= (-1)^{j_1+j_2+\cdots+j_m+k_1+k_2+\cdots+k_m} \times \texttt{Signature}[j_1, j_2, \ldots, j_m] \times \texttt{Signature}[k_1, k_2, \ldots, k_m]. \end{aligned} \tag{2.42}$$

Here, `Signature` (defined like the Wolfram Mathematica command) gives the sign of permutations needed to place the indices in increasing order. The Cayley determinant is a signed minor of the modified Cayley determinant,

$$\Delta_n = \lambda_n = \begin{pmatrix} 0 \\ 0 \end{pmatrix}_n . \tag{2.43}$$

For later use, we also introduce

$$y_n = 1 - \sum_{i-1}^{n-1} y_i \equiv \frac{\partial r_n}{\partial m_n^2}. \tag{2.44}$$

The auxiliary condition $\sum_i^n y_i = 1$ is fulfilled. Further, the notations for the F function are finally independent of the choice of the variable that was eliminated by the use of the δ function in the integrand of Eq. (2.28). Moreover, the inhomogeneity R_n is the only variable carrying the causal iϵ prescription, while, e.g., $\Lambda(x)$ and y_i are, by definition, real quantities. The R_n may be expressed by the ratio of the Cayley determinant (Eq. (2.32)) and the Gram determinant (Eq. (2.31)),

$$R_n = r_{1\cdots n} - \mathrm{i}\epsilon = -\frac{\lambda_{1\cdots n}}{g_{1\cdots n}} - \mathrm{i}\epsilon. \tag{2.45}$$

One may use the Mellin–Barnes relation (Eq. (2.34)) to decompose the integrand of J_n given in Eq. (2.28), as follows:

$$\begin{aligned} J_n &\sim \int \mathrm{d}x \, \frac{1}{[F(x)]^{n-\frac{d}{2}}} \equiv \int \mathrm{d}x \frac{1}{[\Lambda_n(x) + R_n]^{n-\frac{d}{2}}} \equiv \int \mathrm{d}x \frac{R_n^{-(n-\frac{d}{2})}}{\left[1 + \frac{\Lambda_n(x)}{R_n}\right]^{n-\frac{d}{2}}} \\ &= \int \mathrm{d}x \, \frac{R_n^{-(n-\frac{d}{2})}}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}s \, \frac{\Gamma(-s)\,\Gamma\left(n - \frac{d}{2} + s\right)}{\Gamma\left(n - \frac{d}{2}\right)} \left[\frac{\Lambda_n(x)}{R_n}\right]^s , \end{aligned} \tag{2.46}$$

for $|\mathrm{Arg}(\Lambda_n/R_n)| < \pi$. The condition always applies. Further, the integration path in the complex s-plane separates the poles of $\Gamma(-s)$ and $\Gamma(n - d/2 + s)$.

As a result of Eq. (2.46), the Feynman parameter integral of J_n becomes homogeneous:

$$\kappa_n = \int \mathrm{d}x \, \left[\frac{\Lambda_n(x)}{R_n}\right]^s = \prod_{j=1}^{n-1} \int_0^{1-\sum_{i=j+1}^{n-1} x_i} \mathrm{d}x_j \, \left[\frac{\Lambda_n(x)}{R_n}\right]^s \equiv \int \mathrm{d}S_{n-1} \, \left[\frac{\Lambda_n(x)}{R_n}\right]^s . \tag{2.47}$$

To reformulate this integral, one may introduce the differential operator $\hat{P}_n$ [43, 44],

$$\frac{\hat{P}_n}{s} \left[\frac{\Lambda_n(x)}{R_n}\right]^s \equiv \sum_{i=1}^{n-1} \frac{1}{2s}(x_i - y_i)\frac{\partial}{\partial x_i} \left[\frac{\Lambda_n(x)}{R_n}\right]^s = \left[\frac{\Lambda_n(x)}{R_n}\right]^s , \tag{2.48}$$

into Eq. (2.47):

$$K_n = \frac{1}{s} \int \mathrm{d}S_{n-1} \, \hat{P}_n \left[\frac{\Lambda_n(x)}{R_n}\right]^s = \frac{1}{2s} \sum_{i=1}^{n-1} \prod_{k=1}^{n-1} \int_0^{u_k} \mathrm{d}x'_k \, (x_i - y_i) \frac{\partial}{\partial x_i} \left[\frac{\Lambda_n(x)}{R_n}\right]^s . \tag{2.49}$$

After calculating one of the x integrations—by partial integration, eliminating in this way the corresponding differential, and applying a Barnes relation [39] (see Ref. [45]), one arrives at a recursion relation in the number of internal lines n:

$$J_n(d, \{q_i, m_i^2\}) = \frac{-1}{2\pi\mathrm{i}} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}s \frac{\Gamma(-s)\Gamma(\frac{d-n+1}{2} + s)\Gamma(s+1)}{2\Gamma(\frac{d-n+1}{2})} \left(\frac{1}{R_n}\right)^s \times \sum_{k=1}^{n} \left(\frac{1}{R_n}\frac{\partial r_n}{\partial m_k^2}\right) \mathbf{k}^- J_n\left(d + 2s; \{q_i, m_i^2\}\right) . \tag{2.50}$$

The operator $\mathbf{k}^-$, introduced in Eq. (2.7), will reduce an n-point Feynman integral J_n to a sum of $(n-1)$-point integrals J_{n-1} by shrinking propagators D_k from the original n-point integral. The starting term is the one-point function, or tadpole,

$$J_1(d; m^2) = \int \frac{\mathrm{d}^d k}{\mathrm{i}\pi^{d/2}} \frac{1}{k^2 - m^2 + \mathrm{i}\varepsilon} = -\frac{\Gamma(1-d/2)}{R_1^{1-d/2}}, \tag{2.51}$$

$$R_1 = m^2 - \mathrm{i}\varepsilon. \tag{2.52}$$

The cases $G_n = 0$ and $\lambda_n = r_n = 0$ are discussed in Section 2.5.

Equation (2.50) is the master integral for one-loop n-point functions in space–time dimension d, representing them by n integrals over $(n-1)$-point functions with a shifted dimension $d + 2s$. The recursion was first published in Ref. [41]. This implies a series of Mellin–Barnes representations for arbitrary massive one-loop n-point integrals with $(n-1)$ Mellin–Barnes integral dimensions. This linear increase in the MB dimension is highly advantageous compared with the number of MB integral dimensions in the AMBRE approach (increasing as n^2 with the number n of scales).

Based on Eq. (2.50), one has now several opportunities to proceed.

(i) Evaluate the MB integral in a direct numerical way.

(ii) Derive ε-expansions for the Feynman integrals.

(iii) Apply the Cauchy theorem for deriving sums and determine analytical expressions in terms of known special functions.

The first approach is based on AMBRE/MBOneLoop, the middle one is not yet finished, and the last approach was applied in Ref. [41] for massive vertex integrals and in Ref. [6] for massive box integrals.

A few comments are in order.

1. Any four-point integral, e.g., is, in the recursion, a three-fold Mellin–Barnes integral, whereas, with AMBRE, one gets for, e.g., box integrals up to nine-fold MB integrals.
2. Euclidean and Minkowskian integrals converge equally well; see Refs. [46, 47].
3. There appear to be no numerical problems due to vanishing Gram determinants. For a few details, see Table C.2.6 and Ref. [48].

2.4 The basic scalar one-loop functions

2.4.1 Massive two-point functions

From the recursion relation (Eq. (2.50)), taken at $n = 2$ and using Eq. (2.51) with $d \to d+2s$ for the one-point functions under the integral, one gets the following Mellin–Barnes representation:

$$J_2(d; q_1, m_1^2, q_2, m_2^2) = \frac{e^{\epsilon\gamma_E}}{2\pi i}\int_{-i\infty}^{+i\infty} ds\, \frac{\Gamma(-s)\,\Gamma\left(\frac{d-1}{2}+s\right)\,\Gamma(s+1)}{2\,\Gamma\left(\frac{d-1}{2}\right)} R_2^s \times\left[\frac{1}{r_2}\frac{\partial r_2}{\partial m_2^2}\frac{\Gamma\left(1-\frac{d+2s}{2}\right)}{(m_1^2)^{1-\frac{d+2s}{2}}} + (m_1^2 \leftrightarrow m_2^2)\right]. \quad (2.53)$$

One may close the integration contour of the MB integral in Eq. (2.53) to the right, apply the Cauchy theorem, and collect the residua originating from two series of zeros of arguments of Γ functions at $s = m$ and $s = m - d/2 - 1$ for $m \in \mathbb{N}$. The first series stems from the MB-integration kernel, the other one from the dimensionally shifted one-point functions. And then one may sum up analytically in terms of Gauss' hypergeometric functions.

The two-point function, with $R_2 \equiv R_{12}$, becomes

$$J_2(d; Q^2, m_1^2, q_2, m_2^2) = -\frac{\Gamma\left(2-\frac{d}{2}\right)}{(d-2)}\frac{\Gamma\left(\frac{d}{2}-1\right)}{\Gamma\left(\frac{d}{2}\right)}\frac{\partial_2 R_2}{R_2}\left[(m_1^2)^{\frac{d}{2}-1}\,{}_2F_1\left[\begin{matrix}1, \frac{d}{2}-\frac{1}{2}; \\ \frac{d}{2};\end{matrix}\frac{m_1^2}{R_2}\right] + \frac{R_2^{\frac{d}{2}-1}}{\sqrt{1-\frac{m_1^2}{R_2}}}\sqrt{\pi}\frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d}{2}-\frac{1}{2}\right)}\right] + (m_1^2 \leftrightarrow m_2^2). \quad (2.54)$$

Equation (2.54) is valid for $|m_1^2/r_{12}| < 1$, $|m_2^2/r_{12}| < 1$ and $\mathcal{R}\mathrm{e}((d-2)/2) > 0$. The result is in agreement with Eq. (53) of Ref. [15].

The iterative determination of higher-point functions proceeds analogously. Closing the integration contours to the right or to the left will cover different kinematic regions in the invariants R_n.

2.4.2 Massive three-point functions

The Mellin–Barnes integral for the massive vertex is a sum of three terms [49]:

$$J_3 = J_{123} + J_{231} + J_{312}, \tag{2.55}$$

using the representation for, e.g., J_{123},

$$J_{123}(d, \{q_i, m_i^2\}) = -\frac{e^{\epsilon\gamma_E}}{2\pi i} \int_{-i\infty}^{+i\infty} ds \frac{\Gamma(-s)\,\Gamma(\frac{d-2+2s}{2})\Gamma(s+1)}{2\,\Gamma(\frac{d-2}{2})} R_3^{-s} \times \frac{1}{r_3}\frac{\partial r_3}{\partial m_3^2}\, J_2(d+2s; q_1, m_1^2, q_2, m_2^2). \tag{2.56}$$

After applying the Cauchy theorem and summing up, one gets an analytical representation. The integrated massive vertex has been published in Ref. [41]. We quote here the representation given in Ref [6]:

$$\begin{aligned}
J_{123} = {} & \Gamma\left(2-\frac{d}{2}\right) \frac{\partial_3 r_3}{r_3}\frac{\partial_2 r_2}{r_2}\frac{r_2}{2\sqrt{1-m_1^2/r_2}} \\
& \left[-R_2^{d/2-2}\frac{\sqrt{\pi}}{2}\frac{\Gamma\left(\frac{d}{2}-1\right)}{\Gamma\left(\frac{d}{2}-\frac{1}{2}\right)}\, {}_2F_1\left[\begin{matrix}\frac{d-2}{2}, 1\,; R_2\\ \frac{d-1}{2}\,;\; R_3\end{matrix}\right] + R_3^{d/2-2}\, {}_2F_1\left[\begin{matrix}1, 1\,; R_2\\ 3/2\,; R_3\end{matrix}\right]\right] \\
& + \Gamma\left(2-\frac{d}{2}\right) \frac{\partial_3 r_3}{r_3}\frac{\partial_2 r_2}{r_2}\frac{m_1^2}{4\sqrt{1-m_1^2/r_2}} \\
& \left[+\frac{2(m_1^2)^{d/2-2}}{d-2} F_1\left(\frac{d-2}{2}; 1, \frac{1}{2}; \frac{d}{2}; \frac{m_1^2}{R_3}, \frac{m_1^2}{R_2}\right) - R_3^{d/2-2} F_1\left(1; 1, \frac{1}{2}; 2; \frac{m_1^2}{R_3}, \frac{m_1^2}{R_2}\right)\right] \\
& + (m_1^2 \leftrightarrow m_2^2),
\end{aligned}$$

with the short notation

$$R_3 = R_{123}, \qquad R_2 = R_{12}, \tag{2.57}$$

etc. For $d \to 4$, the bracket expressions vanish so that their product with the prefactor $\Gamma(2-d/2)$ stays finite in this limit, as it must come out for a massive vertex function. For some numerics, see Tables C.2.1, C.2.2, C.2.3, and C.2.4.

2.4.3 Massive four-point functions

Finally, we reproduce the box integral, as a three-dimensional Mellin–Barnes representation:

$$\begin{aligned}
J_4(d; \{p_i^2\}, s, t, \{m_i^2\}) = {} & \left(\frac{-1}{4\pi i}\right)^4 \frac{1}{\Gamma(\frac{d-3}{2})} \sum_{k_1,k_2,k_3,k_4=1}^{4} D_{k_1k_2k_3k_4} \left(\frac{1}{r_4}\frac{\partial r_4}{\partial m_{k_4}^2}\right) \\
& \left(\frac{1}{r_{k_3k_2k_1}}\frac{\partial r_{k_3k_2k_1}}{\partial m_{k_3}^2}\right)\left(\frac{1}{r_{k_2k_1}}\frac{\partial r_{k_2k_1}}{\partial m_{k_2}^2}\right)(m_{k_1}^2)^{d/2-1} \\
& \int_{-i\infty}^{+i\infty} dz_4 \int_{-i\infty}^{+i\infty} dz_3 \int_{-i\infty}^{+i\infty} dz_2 \left(\frac{m_{k_1}^2}{R_4}\right)^{z_4}\left(\frac{m_{k_1}^2}{R_{k_3k_2k_1}}\right)^{z_3}\left(\frac{m_{k_1}^2}{R_{k_2k_1}}\right)^{z_2} \\
& \Gamma(-z_4)\Gamma(z_4+1)\frac{\Gamma(z_4+\frac{d-3}{2})}{\Gamma(z_4+\frac{d-2}{2})}\Gamma(-z_3)\Gamma(z_3+1)\frac{\Gamma(z_3+z_4+\frac{d-2}{2})}{\Gamma(z_3+z_4+\frac{d-1}{2})} \times \cdots
\end{aligned} \tag{2.58}$$

Table C.2.1: Numerics for a vertex, $d = 4 - 2\epsilon$. Input quantities suggest that, according to Eq. (73) in Ref. [15], one has to set $b_3 = 0$. Although b_3 of Ref. [15] deviates from our vanishing value, it has to be set to zero, $b_3 \to 0$. The results of both calculations for J_3 agree for this case.

$[p_i^2]$, $[m_i^2]$	[+100, +200, +300], [10, 20, 30]	
G_{123}	$-160\,000$	
λ_{123}	$-8\,860\,000$	
m_i^2/r_{123}	$-0.180\,587,\ -0.361\,174,\ -0.541\,761$	
m_i^2/r_{12}	$-0.975\,61,\ -1.951\,22,\ -2.926\,83$	
m_i^2/r_{23}	$-0.398\,01,\ -0.796\,02,\ -1.194\,03$	
m_i^2/r_{31}	$-0.180\,723,\ -0.361\,446,\ -0.542\,169$	
$\sum J$ terms [15]	$(0.019\,223\,879 - 0.007\,987\,267$ i)	
$\sum b_3$ terms (TR)	0	
J_3 (TR)	$(0.019\,223\,879 - 0.007\,987\,267$ i)	
b_3 term [15]	$(-0.089\,171\,509 + 0.069\,788\,641$ i)	$(0.022\,214\,414)$/eps
$b_3 + \sum J$ terms	$(-0.012\,307\,377 - 0.009\,301\,346$ i)	
J_3 (OT)	$\sum J$ terms, b_3 term $\to 0$, OK	
MB suite		
$(-1)\times$fiesta3 [50]	$-(0.012\,307 + 0.009\,301$ i)	$+\ (8 \times 10^{-6} + 0.000\,01$ i) $\pm (1 + i)10^{-4}$)
LoopTools [51]	$0.019\,223\,88 - 0.007\,987\,267$ i	

Table C.2.2: Numerics for a vertex, $d = 4 - 2\epsilon$. Input quantities suggest that, according to Eq. (73) in Ref. [15], one has to set $b_3 = 0$. Further, we have set in the numerics for Eq. (75) of Ref. [15] so that the root of the Gram determinant is $\sqrt{-g_{123} + i\varepsilon}$, which seems counterintuitive for a 'momentum'-like function. Both results agree if we *do not* set Tarasov's $b_3 \to 0$. Table courtesy of Ref. [52].

$[p_i^2]$, $[m_i^2]$	[−100, +200, −300], [10, 20, 30]	
G_{123}	$480\,000$	
λ_3	$-19\,300\,000$	
m_i^2/r_3	$0.248\,705,\ 0.497\,409,\ 0.746\,114$	
m_i^2/r_{12}	$0.248\,447,\ 0.496\,894,\ 0.745\,342$	
m_i^2/r_{23}	$-0.398\,01,\ -0.796\,02,\ -1.194\,03$	
m_i^2/r_{31}	$0.104\,895,\ 0.209\,79,\ 0.314\,685$	
$\sum J$ terms	$(-0.012\,307\,377 - 0.056\,679\,689$ i)	$(0.012\,825\,498$ i)/eps
$\sum b_3$ terms	$(0.047\,378\,343$ i)	$(-0.012\,825\,498$ i)/eps
J_3(TR)	$(-0.012\,307\,377 - 0.009\,301\,346$ i)	
b_3 term	$(0.047\,378\,343$ i)	$(-0.012\,825\,498$ i)/eps
$b_3 + \sum J$ terms	$(-0.012\,307\,377 - 0.009\,301\,346$ i)	
J_3(OT)	$\sum J$ terms, b_3 term$\to$0, gets wrong!	
MB suite		
(-1)*fiesta3	$-(0.012\,307 + 0.009\,301$ i)	$(8 \times 10^{-6} + 0.000\,01$ i) $\pm (1 + i)10^{-4}$)
LoopTools/FF, ϵ^0	$-0.012\,307\,377\,367\,78 - 0.009\,301\,346\,170$ i	

Table C.2.3: Numerics for a vertex in space–time dimension $d = 4 - 2\epsilon$. Causal $\varepsilon = 10^{-20}$. Agreement with Ref. [15]. Table courtesy Ref. [52].

p_i^2	$-100,-200,-300$	
m_i^2	10,20,30	
G_{123}	$-160\,000$	
λ_{123}	$15\,260\,000$	
m_i^2/r_{123}	0.104 849, 0.209 699, 0.314 548	
m_i^2/r_{12}	0.248 447, 0.496 894, 0.745 342	
m_i^2/r_{23}	0.133 111, 0.266 223, 0.399 334	
m_i^2/r_{31}	0.104 895, 0.209 79, 0.314 685	
$\sum J$ terms	$(0.093\,387\,7 - 0\text{ i})$	$-(0.022\,214\,4 - 0\text{ i})$/eps
$\sum b$ terms	$-0.101\,249$	$+0.022\,214\,4$/eps
J_3(TR)	$(-0.007\,861\,55 - 0\text{ i})$	
b_3	$(-0.101\,249 + 0\text{ i})$	$(0.022\,214\,4 + 0\text{ i})$/eps
b_3+J terms	$(-0.007\,861\,546 + 0\text{ i})$	
J_3(OT)	$b_3 + J$ terms $\to$ OK	
MB suite	$-0.007\,862\,014,\ 5.002\,549\,159 \times 10^{-6},\ 0$	
(-1)*fiesta3	$-0.007\,862$	$6 \times 10^{-6} + 6 \times 10^{-6}\text{ i} \pm (1+\text{i})10^{-10}$
LoopTools/FF, ϵ^0	$-0.007\,861\,546\,132\,290\,822\,90$	

Table C.2.4: Numerics for a vertex in space–time dimension $d = 4-2\epsilon$. Causal $\varepsilon = 10^{-20}$. Input quantities suggest that, according to Eq. (73) in Ref. [15], one has to set $b_3 = 0$. Agreement, owing to setting $b_3 = 0$ there. Table courtesy Ref. [52].

p_i^2	$+100,\ -200,\ +300$	
m_i^2	10, 20, 30	
G_{123}	$480\,000$	
λ_{123}	$4\,900\,000$	
m_i^2/r_{123}	$-0.979\,592,\ -1.959\,18,\ -2.938\,78$	
m_i^2/r_{12}	-0.975 61, $-1.951\,22,\ -2.926\,83$	
m_i^2/r_{23}	0.133 111, 0.266 223, 0.399 334	
m_i^2/r_{31}	$-0.180\,723,\ -0.361\,446,\ -0.542\,169$	
$\sum J$ terms	$(0.006\,243\,624 - 0.018\,272\,524\text{ i})$	
$\sum b_3$ terms	0	
J_3(TR)	$(0.006\,243\,624 - 0.018\,272\,524\text{ i})$	
b_3 term	$(0.040\,292\,491 + 0.029\,796\,253\text{ i})$	$(-0.012\,825\,498\text{ i})$/eps
$b_3 + \sum J$ terms	$(-0.012\,307\,377 - 0.009\,301\,346\text{ i})$	$(4 \times 10^{-18} - 6 \times 10^{-18}\text{ i})$/eps
J_3(OT)	$\sum J$ terms, b_3 term$\to$0, OK	
MB suite		
$(-1)\times$fiesta3	$-(-0.006\,322 + 0.014\,701\text{ i})$	$+\ (0.000\,012 + 0.000\,014\text{ i}) \pm (1+\text{i})10^{-2}$
LoopTools/FF, ϵ^0	$0.006\,243\,624\,78 - 0.018\,272\,524\,0\text{ i}$	

Table C.2.5: Comparison of the box integral J_4 defined in Eq. (2.60) with the LoopTools function `D0`$(p_1^2, p_2^2, p_3^2, p_4^2, (p_1+p_2)^2, (p_2+p_3)^2, m_1^2, m_2^2, m_3^2, m_4^2)$ [51, 55] at $m_2^2 = m_3^2 = m_4^2 = 0$. Further numerical references are from the packages K.H.P_D0 (PHK, unpublished) and MBOneLoop [56, 57]. External invariants: $(p_1^2 = \pm 1, p_2^2 = \pm 5, p_3^2 = \pm 2, p_4^2 = \pm 7, s = \pm 20, t = \pm 1)$. Table from Ref. [6], licence: https://creativecommons.org/licenses/by/4.0/.

$(p_1^2, p_2^2, p_3^2, p_4^2, s, t)$	Four-point integral
$(-,-,-,-,-,-)$	$d = 4, \quad m_1^2 = 100$
J_4	0.009 178 67
LoopTools	0.009 178 670 7
MBOneLoop	0.009 178 670 7
$(+,+,+,+,+,+)$	$d = 4, \quad m_1^2 = 100$
J_4	−0.011 592 7 − 0.000 406 03 i
LoopTools	−0.011 591 7 − 0.000 406 02 i
MBOneLoop	−0.011 591 736 9 − 0.000 406 024 3 i
$(-,-,-,-,-,-)$	$d = 5, \quad m_1^2 = 100$
J_4	0.009 268 95
K.H.P_D0	0.009 268 88
MBOneLoop	0.009 268 948 8
$(+,+,+,+,+,+)$	$d = 5, \quad m_1^2 = 100$
J_4	−0.002 728 89 + 0.012 648 8 i
K.H.P_D0	(−)
MBOneLoop	−0.002 728 424 2 + 0.012 648 813 4 i
$(-,-,-,-,-,-)$	$d = 5, \quad m_1^2 = 100 - 10\,\mathrm{i}$
J_4	0.009 200 65 + 0.000 782 308 i
K.H.P_D0	0.009 200 6 + 0.000 782 301 i
MBOneLoop	0.009 200 648 1 + 0.000 782 309 0 i
$(+,+,+,+,+,+)$	$d = 5, \quad m_1^2 = 100 - 10\,\mathrm{i}$
J_4	−0.003 987 25 + 0.012 067 i
K.H.P_D0	−0.003 987 23 + 0.012 069 i
MBOneLoop	−0.003 986 770 2 + 0.012 067 038 8 i

$$\cdots \times \Gamma\left(z_2 + z_3 + z_4 + \frac{d-1}{2}\right) \Gamma\left(-z_2 - z_3 - z_4 - \frac{d+2}{2}\right)$$
$$\Gamma(-z_2)\Gamma(z_2 + 1).$$

Equation (2.58) can be treated using the Mathematica packages MB and MBnumerics of the MBsuite, replacing AMBRE with a derivative of MBnumerics: MBOneLoop [46, 53]. For numerical examples, see Table C.2.5.

After applying the Cauchy theorem and summing the residues, we get [6, 54]:

$$J_4 = J_{1234} + J_{2341} + J_{3412} + J_{4123}, \tag{2.59}$$

with $R_4 = R_{1234}, R_3 = R_{123}, R_2 = R_{12}$, etc.:

$$J_{1234} = \Gamma\left(2-\frac{d}{2}\right)\frac{\partial_4 r_4}{r_4}\Bigg\{\left[\frac{b_{123}}{2}\left(-R_3^{d/2-2}\,{}_2F_1\left[\begin{matrix}\frac{d-3}{2},1;R_2\\ \frac{d}{2}-1;R_3\end{matrix}\right]+R_4^{d/2-2}\sqrt{\pi}\frac{\Gamma\left(\frac{d}{2}-1\right)}{\Gamma\left(\frac{d}{2}-\frac{3}{2}\right)}{}_2F_1(d\to 4)\right)\right]$$
$$+\left[+\frac{R_2^{d/2-2}}{d-3}F_1\left(\frac{d-3}{2};1,\frac{1}{2};\frac{d-1}{2};\frac{R_2}{R_4},\frac{R_2}{R_3}\right)-R_4^{d/2-2}F_1\left(d\to 4\right)\right]$$
$$+\frac{m_1^2}{8}\frac{\Gamma\left(\frac{d}{2}-1\right)}{\Gamma\left(\frac{d}{2}-\frac{3}{2}\right)}\frac{\partial_3 r_3}{r_3}\frac{\partial_2 r_2}{r_2}\frac{r_3}{r_3-m_1^2}\frac{r_2}{r_2-m_1^2}$$
$$\left[-(m_1^2)^{d/2-2}\frac{\Gamma\left(\frac{d}{2}-3/2\right)}{\Gamma\left(\frac{d}{2}\right)}F_S\left(\frac{d}{2}-\frac{3}{2},1,1,1,1,\frac{d}{2},\frac{d}{2},\frac{d}{2},\frac{d}{2},\frac{m_1^2}{R_4},\frac{m_1^2}{m_1^2-R_3},\frac{m_1^2}{m_1^2-R_2}\right)\right.$$
$$\left.+R_4^{d/2-2}\sqrt{\pi}F_S(d\to 4)\right]+(m_1^2\leftrightarrow m_2^2)\Bigg\}. \tag{2.60}$$

For $d \to 4$, all three contributions in square brackets approach zero, so that the massive J_4 gets finite in this limit, as it should do. Table C.2.5 contains numerical examples.

2.5 The cases of vanishing Cayley determinant $\lambda_n = 0$ and of vanishing Gram determinant $G_n = 0$

We refer here to two important special cases, where the general derivations cannot be applied.

In the case of vanishing Cayley determinant, $\lambda_n = 0$, we cannot introduce the inhomogeneity $R_n = -\lambda_n/G_n$ into the Symanzik polynomial F_n. Let us assume that it is $G_n \neq 0$, so that $r_n = 0$. A useful alternative representation to Eq. (2.50) is known from the literature, see e.g., Eq. (3) in Ref. [15]:

$$J_n(d) = \frac{1}{d-n-1}\sum_{k=1}^{n}\frac{\partial_k \lambda_n}{G_n}\mathbf{k}^- J_n(d-2). \tag{2.61}$$

Another special case is a vanishing Gram determinant, $G_n = 0$. Here again, one may use Eq. (3) of Ref. [15] and the result is (for $\lambda_n \neq 0$):

$$J_n(d) = -\sum_{k=1}^{n}\frac{\partial_k \lambda_n}{2\lambda_n}\mathbf{k}^- J_n(d). \tag{2.62}$$

The representation was, for the special case of the vertex function, also given in Eq. (46) of Ref. [58].

For the vertex function, a general study of the special cases has been conducted, as reported in Ref. [59].

2.6 A massive four-point function with vanishing Gram determinant

As a very interesting non-trivial example, we have restudied the numerics of a massive four-point function with a small or vanishing Gram determinant [46,48,49,53]. The original example has been taken from Appendix C of Ref. [17].

Table C.2.6: The Feynman integral $J_4(12-2\epsilon,1,5,1,1)$ compared with numbers from Ref. [17]. The $I_{4,2222}^{[d+]^4}$ is the scalar integral, where propagator 2 has index $\nu_2 = 1+(1+1+1+1) = 5$, and the other propagators have index 1. The integral corresponds to D_{1111} in the notation of LoopTools [51]. For $x=0$, the Gram determinant vanishes. We see an agreement of about 10 to 11 relevant digits. The deviations of the two calculations seem to stem from a limited accuracy of the Padé approximations used in Ref. [17]. Table courtesy Refs. [48, 53].

x	Value for $4!\times\ J_4(12-2\epsilon,1,5,1,1)$	
0	$(2.059\,692\,897\,30 + 1.555\,949\,101\,18\ \mathrm{i})10^{-10}$	[17]
0	$(2.059\,692\,897\,30 + 1.555\,949\,101\,18\ \mathrm{i})10^{-10}$	MBOneLoop+Kira+MBnumerics
10^{-8}	$(2.059\,692\,893\,42 + 1.555\,949\,091\,87\ \mathrm{i})10^{-10}$	[17]
10^{-8}	$(2.059\,692\,893\,63 + 1.555\,949\,091\,87\ \mathrm{i})10^{-10}$	MBOneLoop+Kira+MBnumerics
10^{-4}	$(2.059\,656\,094\,97 + 1.555\,856\,053\,43\ \mathrm{i})10^{-10}$	[17]
10^{-4}	$(2.059\,656\,094\,89 + 1.555\,856\,053\,43\ \mathrm{i})10^{-10}$	MBOneLoop+Kira+MBnumerics

The sample outcome is shown in Table C.2.6. The new iterative Mellin–Barnes representations deliver very precise numerical results for, e.g., box functions, including cases of small or vanishing Gram determinants. The software used is MBOneLoop [60]. The notational correspondences are, e.g.,

$$J_4(12-2\epsilon,1,5,1,1) \to I_{4,2222}^{[d+]^4} = D_{1111}.$$

2.7 Calculation of Gauss hypergeometric function ${}_2F_1$, Appell function F_1, and Saran function F_S at arbitrary kinematics

Little is known about the precise numerical calculation of generalised hypergeometric functions at arbitrary arguments. Numerical calculations of specific Gauss hypergeometric functions ${}_2F_1$, Appell functions F_1 (Eq. (1) of Ref. [61]), and Lauricella–Saran functions F_S (Eq. (2.9) of Ref. [62]) are needed for the scalar one-loop Feynman integrals:

$$ {}_2F_1(a,b;c;x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{k!\ (c)_k}\ x^k, \tag{2.63}$$

$$F_1(a;b,b';c;y,z) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n}{m!\ n!\ (c)_{m+n}}\ y^m z^n, \tag{2.64}$$

$$F_S(a_1,a_2,a_2;b_1,b_2,b_3;c,c,c;x,y,z) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_m(a_2)_{n+p}(b_1)_m(b_2)_n(b_3)_p}{m!\ n!\ p!\ (c)_{m+n+p}}\ x^m y^n z^p. \tag{2.65}$$

The $(a)_k$ is the Pochhammer symbol. The specific cases needed here are discussed in the appendices of Ref. [6]. Here, we repeat only few definitions.

One approach to the numerics of ${}_2F_1$, F_1, and F_S may be based on Mellin–Barnes representations. For the Gauss function ${}_2F_1$ and the Appell function F_1, Mellin–Barnes representations have been known for some time. See Eq. (1.6.1.6) in Ref. [63],

$$ {}_2F_1(a,b;c;z) = \frac{1}{2\pi \mathrm{i}}\ \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}s\ (-z)^s\ \frac{\Gamma(a+s)\Gamma(b+s)\Gamma(-s)}{\Gamma(c+s)}, \tag{2.66}$$

and Eq. (10) in Ref. [61], which is a two-dimensional MB integral:

$$F_1(a;b,b';c;x,y) = \frac{1}{2\pi \mathrm{i}} \frac{\Gamma(c)}{\Gamma(a)\Gamma(b')} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}t \; (-y)^t \; {}_2F_1(a+t,b;c+t,x) \frac{\Gamma(a+t)\Gamma(b'+t)\Gamma(-t)}{\Gamma(c+t)}. \tag{2.67}$$

For the Lauricella–Saran function F_S, the following, new, three-dimensional MB integral was given in Ref. [6]:

$$F_\mathrm{S}(a_1,a_2,a_2;b_1,b_2,b_3;c,c,c;x,y,z) = \frac{1}{2\pi \mathrm{i}} \frac{\Gamma(c)}{\Gamma(a_1)\Gamma(b_1)} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}t (-x)^t \frac{\Gamma(a_1+t)\Gamma(b_1+t)\Gamma(-t)}{\Gamma(c+t)} \times F_1(a_2;b_2,b_3;c+t;y,z). \tag{2.68}$$

The numerics of the Gauss hypergeometric function are generally known in all detail.

For the Appell function F_1, the numerical mean value integration of the one-dimensional integral representation of Ref. [64] may be advocated, being quoted in Eq. (9) of Ref. [61]:

$$F_1(a;b,b';c;x,y) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 \mathrm{d}u \frac{u^{a-1}(1-u)^{c-a-1}}{(1-xu)^b(1-yu)^{b'}}. \tag{2.69}$$

We need three specific cases, taken at $d \geq 4$. For vertices, e.g.,

$$F_1^v(d) \equiv F_1\left(\frac{d-2}{2};1,\frac{1}{2};\frac{d}{2};x_c,y_c\right) = \frac{1}{2}(d-2)\int_0^1 \frac{\mathrm{d}u \; u^{\frac{d}{2}-2}}{(1-x_c u)\sqrt{1-y_c u}}. \tag{2.70}$$

Integrability is violated at $u=0$ if not $\Re\mathrm{e}(d) > 2$. The stability of numerics is well-controlled, as exemplified in Table C.2.7.

For the calculation of the four-point Feynman integrals, one also needs the Lauricella–Saran function F_S [62]. Saran defines F_S as a three-fold sum (Eq. (2.65)), see Eq. (2.9) in Ref. [62]. Saran derives a three-fold integral representation in Eq. (2.15) and a two-fold integral in Eq. (2.16). We recommend use of the following representation, given on p. 304 of [62]:

$$F_\mathrm{S}(a_1,a_2,a_2;b_1,b_2,b_3;c,c,c,x,y,z) = \frac{\Gamma(c)}{\Gamma(a_1)\Gamma(c-a_1)} \int_0^1 \mathrm{d}t \frac{t^{c-a_1-1}(1-t)^{a_1-1}}{(1-x+tx)^{b_1}} F_1(a_2;b_2,b_3;c-a_1;ty,tz). \tag{2.71}$$

For the box integrals, one needs the specific case

$$F_\mathrm{S}^b(d) = F_\mathrm{S}\left(\frac{d-3}{2},1,1;1,1,\frac{1}{2};\frac{d}{2},\frac{d}{2},\frac{d}{2},x_c,y_c,z_c\right) = \frac{\Gamma(\frac{d}{2})}{\Gamma(\frac{d-3}{2})\Gamma(\frac{3}{2})} \int_0^1 \mathrm{d}t \frac{\sqrt{t}(1-t)^{\frac{d-5}{2}}}{(1-x_c+x_c t)} F_1(1;1,\frac{1}{2};\frac{3}{2};y_c t, z_c t). \tag{2.72}$$

Equation (2.72) is valid if $\Re\mathrm{e}(d) > 3$.

Acknowledgement

We would like to thank J. Gluza for a careful reading of the manuscript.

Table C.2.7: The Appell function F_1 of the massive vertex integrals as defined in Eq. (2.70). As a proof of principle, only the constant term of the expansion in $d = 4 - 2\varepsilon$ is shown, $F_1(1;1,\frac{1}{2};2;x,y)$. Upper values from general numerics of appendices of Ref. [6]; lower values from setting $d = 4$ and the use of analytical formulae. Table courtesy of Ref. [6] under licence http://creativecommons.org/licenses/by/4.0/.

$x - i\varepsilon_x$	$y - i\varepsilon_y$	$F_1(1;1,\frac{1}{2};2;x,y)$	
$+11.1 - 10^{-12}$i	$+12.1 - 10^{-12}$i	−0.175 044 248 073 5	−0.054 228 129 473 2 i
		−0.175 044 248 073 518 778 844 982 899 12	−0.054 228 129 473 304 027 882 097 641 167 i
$+11.1 - 10^{-12}$i	$+12.1 + 10^{-12}$i	+1.710 854 529 324 4	+0.054 228 129 473 2 i
		+1.710 854 529 324 335 571 348 382 041 75	+0.054 228 129 471 482 173 815 892 709 24 i
$+11.1 + 10^{-12}$i	$+12.1 - 10^{-12}$i	+1.710 854 530 411 4	−0.054 228 129 473 2 i
		+1.710 854 529 324 335 571 348 382 041 75	−0.054 228 129 471 482 173 815 892 709 24 i
$+11.1 + 10^{-12}$ i	$+12.1 + 10^{-12}$ i	−0.175 044 248 073 5	+0.054 228 129 473 3 i
		−0.175 044 248 073 518 778 844 982 899 12	+0.054 228 129 473 304 027 882 097 641 167 i
$+12.1 - 10^{-15}$ i	$+11.1 - 10^{-15}$ i	−0.170 082 716 648 4	−0.051 868 484 603 7 i
$+12.1 - 10^{-10}$ i	$+11.1 - 10^{-15}$ i	−0.170 082 716 648 000 581 011 657 492 79	−0.051 868 484 604 656 749 765 565 256 21 i
$+12.1 - 10^{-15}$ i	$+11.1 + 10^{-15}$ i	−0.170 082 716 648 4	−1.754 420 290 995 5 i
		−0.170 082 716 648 440 256 472 688 173 99	−1.754 420 290 995 576 887 358 425 620 38 i
$+12.1 + 10^{-15}$ i	$+11.1 - 10^{-15}$ i	−0.170 082 716 648 4	+1.754 420 290 995 5 i
		−0.170 082 716 648 440 256 472 688 173 99	+1.754 420 290 995 576 887 358 425 620 38 i
$+12.1 + 10^{-15}$ i	$+11.1 + 10^{-15}$ i	−0.170 082 716 648 4	+0.051 868 484 603 7 i
$+12.1 - 10^{-10}$ i	$+11.1 - 10^{-15}$ i	−0.170 082 716 648 000 581 011 657 492 79	+0.051 868 484 604 656 749 765 565 256 21 i
$+11.1 - 10^{-15}$ i	−12.1	−0.053 370 514 651 8	−0.195 769 211 155 7 i
		−0.053 370 514 651 899 444 733 494 011 52	−0.195 769 211 155 733 985 388 920 833 693 i
$+11.1 + 10^{-15}$ i	−12.1	−0.053 370 514 651 8	+0.195 769 211 155 7 i
		−0.053 370 514 651 899 444 733 494 011 52	+0.195 769 211 155 733 985 388 920 833 693 i
−11.1	$+12.1 - 10^{-12}$ i	+0.106 086 408 466 2	−0.144 744 070 008 2 i
		+0.106 086 408 476 510 642 871 335 275 99	−0.144 744 070 021 333 407 167 349 619 088 i
−11.1	$+12.1 + 10^{-12}$ i	+0.106 086 408 466 2	+0.144 744 070 008 2 i
		+0.106 086 408 476 510 642 871 335 275 99	+0.144 744 070 021 333 407 167 349 619 088 i
−12.1	−11.1	+0.122 456 767 687 224 028	
		+0.122 456 767 687 224 025 065 133 951 61	

References

[1] R.J. Eden *et al.*, *The Analytic S-Matrix* (Cambridge University Press, Cambridge, 1966).

[2] J. Schwinger, *Particles, Sources, and Fields*, vol. 1 (CRC Press, Boca Raton, 1970).

[3] A. Blondel *et al.*, Standard Model theory for the FCC-ee Tera-Z stage, CERN (CERN Yellow Rep. Monogr. 3, Geneva, Switzerland), arXiv:1809.01830, doi:10.23731/CYRM-2019-003

[4] C.G. Bollini and J.J. Giambiagi, *Nuovo Cim.* **B12** (1972) 20. doi:10.1007/BF02895558

[5] G. 't Hooft and M.J.G. Veltman, *Nucl. Phys.* **B44** (1972) 189. doi:10.1016/0550-3213(72)90279-9

[6] K.H. Phan and T. Riemann, *Phys. Lett.* **B791** (2019) 257. arXiv:1812.10975, doi:10.1016/j.physletb.2019.02.044

[7] G. Passarino and M. Veltman, *Nucl. Phys.* **B160** (1979) 151. doi:10.1016/0550-3213(79)90234-7

[8] G. 't Hooft and M. Veltman, *Nucl. Phys.* **B153** (1979) 365. doi:10.1016/0550-3213(79)90605-9

[9] E.E. Boos and A.I. Davydychev, *Moscow Univ. Phys. Bull.* **42N3** (1987) 6 [*Vestn. Mosk. Univ. Fiz. Astron.* **28N3** (1987) 8]. http://www.higgs.de/~davyd/preprints/bd_vmu.pdf

[10] O. Tarasov, *Phys. Rev.* **D54** (1996) 6479. arXiv:hep-th/9606018, doi:10.1103/PhysRevD.54.6479

[11] O. Tarasov, *Nucl. Phys.* **B502** (1997) 455. arXiv:hep-ph/9703319, doi:10.1016/S0550-3213(97)00376-3

[12] O.V. Tarasov, *Acta Phys. Pol.* **B29** (1998) 2655. arXiv:hep-ph/9812250, http://www.actaphys.uj.edu.pl/fulltext?series=Reg&vol=29&page=2655

[13] J. Fleischer *et al.*, *Nucl. Phys.* **B566** (2000) 423. arXiv:hep-ph/9907327, doi:10.1016/S0550-3213(99)00678-1

[14] O. Tarasov, *Nucl. Phys. Proc. Suppl.* **89** (2000) 237. arXiv:hep-ph/0102271, doi:10.1016/S0920-5632(00)00849-5

[15] J. Fleischer *et al.*, *Nucl. Phys.* **B672** (2003) 303. arXiv:hep-ph/0307113, doi:10.1016/j.nuclphysb.2003.09.004

[16] A.I. Davydychev, *Phys. Lett.* **B263** (1991) 107. http://www.higgs.de/~davyd/preprints/tensor1.pdf, doi:10.1016/0370-2693(91)91715-8

[17] J. Fleischer and T. Riemann, *Phys. Rev.* **D83** (2011) 073004. arXiv:1009.4436, doi:10.1103/PhysRevD.83.073004

[18] F.V. Tkachov, *Phys. Lett.* **100B** (1981) 65. doi:10.1016/0370-2693(81)90288-4

[19] K. Chetyrkin and F. Tkachov, *Nucl. Phys.* **B192** (1981) 159. doi:10.1016/0550-3213(81)90199-1

[20] J. Fleischer *et al.*, Replacing 1-loop tensor reduction by contractions, 11th Int. Symposium on Radiative Corrections—Applications of Quantum Field Theory to Phenomenology—RADCOR, Lumley Castle, UK, 2013, https://conference.ippp.dur.ac.uk/event/341/session/8/contribution/56/material/slides/0.pdf

[21] U. Nierste *et al.*, *Z. Phys.* **C57** (1993) 605. doi:10.1007/BF01561479

[22] G. Passarino, *Nucl. Phys.* **B619** (2001) 257. arXiv:hep-ph/0108252, doi:10.1016/S0550-3213(01)00528-4

[23] A. Ferroglia *et al.*, *Nucl. Phys. Proc. Suppl.* **117** (2003) 206. doi:10.1016/S0920-5632(03)90526-3

[24] A. Ferroglia *et al.*, *Nucl. Instrum. Meth.* **A502** (2003) 391. doi:10.1016/S0168-9002(03)00450-9

[25] A. Ferroglia *et al.*, *Nucl. Phys.* **B650** (2003) 162. arXiv:hep-ph/0209219, doi:10.1016/S0550-3213(02)01070-2

[26] V.A. Smirnov, *Phys. Lett.* **B460** (1999) 397. arXiv:hep-ph/9905323, doi:10.1016/S0370-2693(99)00777-7

[27] J. Tausk, *Phys. Lett.* **B469** (1999) 225. arXiv:hep-ph/9909506, doi:10.1016/S0370-2693(99)01277-0

[28] J. Gluza *et al.*, *Comput. Phys. Commun.* **177** (2007) 879. arXiv:0704.2423, doi:10.1016/j.cpc.2007.07.001

[29] J. Gluza *et al.*, *PoS* **ACAT08** (2008) 124. arXiv:0902.4830, doi:10.22323/1.070.0124

[30] K. Kajda and I. Dubovyk, AMBRE 2.2 (12 Sep 2015), a Mathematica package representing Feynman integrals by Mellin–Barnes integrals, http://prac.us.edu.pl/~gluza/ambre/

[31] J. Gluza *et al.*, *Nucl. Phys. Proc. Suppl.* **205–206** (2010) 147. arXiv:1006.4728, doi:10.1016/j.nuclphysbps.2010.08.034

[32] I. Dubovyk *et al.*, *J. Phys. Conf. Ser.* **608** (2015) 012070. doi:10.1088/1742-6596/608/1/012070

[33] I. Dubovyk *et al.*, *PoS* **LL2016** (2016) 034. arXiv:1607.07538, doi10.22323/1.260.0034

[34] J. Usovitsch, Ph.D. thesis, Humboldt-Universität zu Berlin, 2018. doi:10.18452/19484

[35] I. Dubovyk, Ph.D. thesis, Universität Hamburg, 2019. http://ediss.sub.uni-hamburg.de/volltexte/2019/10081

[36] N.I. Usyukina, *Theor. Math. Phys.* **22** (1975) 210. doi:10.1007/BF01037795

[37] K. Kajda, Ph.D. thesis, University of Silesia, 2009. http://inspirehep.net/record/1419883/files/phdKajda.pdf

[38] D.B. Melrose, *Nuovo Cim.* **40** (1965) 181. doi:10.1007/BF028329

[39] E.W. Barnes, *Proc. London Math. Soc.* **s2-6** (1908) 141. doi:10.1112/plms/s2-6.1.141

[40] E. Whittaker and G. Watson, *A Course of Modern Analysis* (Cambridge University Press, Cambridge, 1927).

[41] K.H. Phan *et al.*, *Acta Phys. Pol.* **B48** (2017) 2313. arXiv:1711.05510, doi:10.5506/APhysPolB.48.2313

[42] T. Regge and G. Barucchi, *Nuovo Cim.* **34** (1964) 106. doi:10.1007/BF02725874

[43] I. Bernshtein, *Funct. Anal. Appl.* **5** (1971) 89. doi:10.1007/BF01076413

[44] V.A. Golubeva and V.Z. Énol'skii, *Math. Notes Acad. Sci USSR* **23** (1978) 63. doi:10.1007/BF01104888

[45] G.N. Watson, A Treatise on the Theory of Bessel Functions (Cambridge University Press, Cambridge 1922). https://www.forgottenbooks.com/de/books/ATreatiseontheTheoryofBesselFunctions_10019747

[46] J. Usovitsch *et al.*, MBnumerics: numerical integration of Mellin–Barnes integrals in physical regions, LL2018, St. Goar, Germany, 2018 2018, https://indico.desy.de/indico/event/16613/session/4/contribution/22/material/slides/0.pdf

[47] J. Usovitsch *et al.*, *PoS* **LL2018** (2018) 046. arXiv:1810.04580, doi:10.22323/1.303.0046

[48] J. Usovitsch and T. Riemann, New approach to Mellin–Barnes integrals for massive one-loop Feynman integrals, Standard Model Theory for the FCC-ee Tera-Z Stage, (CERN-2019-003), section E.6. arXiv:1809.01830, doi:10.23731/CYRM-2019-003

[49] T. Riemann, Scalar 1-loop Feynman integrals in arbitrary space–time dimension D, 14th Workshop on Calculations for Modern and Future Colliders (CALC2018), 2018, JINR, Dubna, Russia, https://indico.jinr.ru/conferenceOtherViews.py?showSession=all&showDate=all&view=standard&fr=no&confId=418, http://indico.jinr.ru/getFile.py/access?contribId=77&sessionId=5&resId=0&materialId=slides&confId=418

[50] A.V. Smirnov, *Comput. Phys. Commun.* **185** (2014) 2090. arXiv:1312.3186, doi:10.1016/j.cpc.2014.03.015

[51] T. Hahn and M. Perez-Victoria, *Comput. Phys. Commun.* **118** (1999) 153. arXiv:hep-ph/9807565, doi:10.1016/S0010-4655(98)00173-8, http://www.feynarts.de/looptools

[52] K.H. Phan *et al.*, Scalar one-loop vertex integrals as meromorphic functions of space–time dimension d, 41st International Conference of Theoretical Physics: Matter to the Deepest, Podlesice, Poland, 2017. http://indico.if.us.edu.pl/event/4/contribution/32/material/slides/0.pdf

[53] J. Usovitsch, Numerical evaluation of Mellin–Barnes integrals in Minkowskian regions, FCC-ee Mini Workshop Precision EW and QCD calculations for the FCC Studies: Methods and Techniques, 2018, CERN, Geneva, Switzerland. https://indico.cern.ch/event/669224/contributions/2805454/attachments/1581984/2500208/Usovitsch.pdf

[54] T. Riemann *et al.*, Scalar one-loop Feynman integrals in arbitrary space–time dimension, LL2018, St. Goar, Germany, 2018, https://indico.desy.de/indico/event/16613/session/12/contribution/24/material/slides/0.pdf

[55] G. van Oldenborgh, *Comput. Phys. Commun.* **66** (1991) 1. doi:10.1016/0010-4655(91)90002-3

[56] J. Usovitsch *et al.*, The MBnumerics project, Standard Model Theory for the FCC-ee Tera-Z stage, (CERN-2019-003), section E.2. arXiv:1809.01830, doi:10.23731/CYRM-2019-003

[57] T. Riemann, 1-loop Feynman integrals at arbitrary space–time d, 11th FCC-ee Workshop: Theory and Experiments, CERN, Geneva, Switzerland, 2019, https://indico.cern.ch/event/766859/contributions/3252618/attachments/1776255/2887926/riemann11FCCeeWS2019.pdf

[58] G. Devaraj and R.G. Stuart, *Nucl. Phys.* **B519** (1998) 483. arXiv:hep-ph/9704308, doi:10.1016/S0550-3213(98)00035-2

[59] K.H. Phan and D.T. Tran, *PTEP*, **2019** (2019) 063B01 arXiv:1904.07430, doi:10.1093/ptep/ptz050

[60] J. Usovitsch, MBOneLoop, a Mathematica/Fortran package for the numerical calculation of multiple MB-integral representations for one-loop Feynman integrals at arbitrary kinematics, http://prac.us.edu.pl/~gluza/ambre/

[61] P. Appell, Sur les fonctions hypergéométriques de plusieurs variables, les polynômes d'Hermite et autres fonctions sphériques dans l'hyperespace, *Mémorial Sci Math* **3** (1925) 82. http://www.numdam.org/item?id=MSM_1925__3__1_0

[62] S. Saran, *Acta Math.* **93** (1955) 293. doi:10.1007/BF02392525

[63] L. Slater, *generalised Hypergeometric Functions* (Cambridge University Press, Cambridge, 1966). doi:10.1002/zamm.19660460536

[64] E. Picard, *Ann Sci l'É.N.S. 2* **10** (1881) 305. http://www.numdam.org/item?id=ASENS_1881_2_10__305_0

[65] J. Gluza *et al.*, *Eur. Phys. J.* **C71** (2011) 1516. arXiv:1010.1667, doi:10.1140/epjc/s10052-010-1516-y