

# Fermion Wavefunctions in Magnetized branes: Theta identities and Yukawa couplings

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## Abstract

Computation of Yukawa couplings, determining superpotentials as well as the Kähler metric, with oblique (non-commuting) fluxes in magnetized brane constructions is an interesting unresolved issue, in view of the importance of such fluxes for obtaining phenomenologically viable models. In order to perform this task, fermion (scalar) wavefunctions on toroidally compactified spaces are presented for general fluxes, parameterized by Hermitian matrices with eigenvalues of arbitrary signatures. We also give explicit mappings among fermion wavefunctions, of different internal chiralities on the tori, which interchange the role of the flux components with the complex structure of the torus. By evaluating the overlap integral of the wavefunctions, we give the expressions for Yukawa couplings among chiral multiplets arising from an arbitrary set of branes (or their orientifold images). The method is based on constructing certain mathematical identities for general Riemann theta functions with matrix valued modular parameter. We briefly discuss an application of the result, for the mass generation of non-chiral fermions, in the  $SU(5)$  GUT model presented by us in arXiv: 0709.2799.

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# 1 Introduction

Close form expressions for Yukawa couplings have been written down for string constructions involving branes at angles [1, 2] or those with magnetized branes [3–17]. In the IIA picture, the interaction is described by the worldsheet instanton contributions from the sum of areas of various triangles that are formed by three  $D6$  branes intersecting at three vertices, forming a triangle. This is due to the fact that the intersection of branes relevant for Yukawa interactions are those which are point-like giving chiral multiplets. Line or surface like intersections, on the other hand, would give rise to interactions of non-chiral matter. In these discussions, the orientation of the branes themselves are parameterized by three angles in the three orthogonal 2-planes, inside  $T^6$ . These results have been further generalized to include Euclidean  $D2$  brane instanton contributions to the Yukawa couplings [18–26], generating up quark and right handed neutrino masses through a Higgs mechanism, in a particular class of models. A limitation on the exercise performed in these papers comes from the factorized structure of the tori, which arises from the orientations of the brane wrappings that are classified by angles in three different  $T^2$  planes, rather than their general orientations in the internal six dimensional space parameterized for instance by the  $SU(3)$  angles in supersymmetric situations.

Similar results for perturbative Yukawa couplings have also been obtained in the magnetized brane picture, based on their gauge theoretic representation [5]. In this case, the interactions are given by the overlap integral of three wavefunctions (contributing to the interaction) along internal directions. The wavefunctions correspond, in the ordinary field theory context, to those belonging to two fermions and a scalar, and are given by Jacobi theta functions, when fluxes are turned on along three diagonal 2-tori. The relationship between the Yukawa interactions in the magnetized brane constructions and those involving  $D6$  branes, have also been established using T-duality rules. However, these exercises have once again been of limited scope due to the fact that explicit expressions are written down only for magnetized branes with fluxes that are diagonal along three  $T^2$ 's.

Technically, the wavefunctions of chiral fields participating in Yukawa interactions are defined in terms of Jacobi theta functions, with a modular parameter identified as a product of the complex structure of the  $T^2$ , with the flux that is turned on along it. The Yukawa interactions are therefore computed for the case when the six dimensional internal

space is of a factorized form:

$$T^2 \times T^2 \times T^2 \in T^6. \quad (1.1)$$

For applications to model building with moduli stabilization though, one in general needs to include both ‘diagonal’ and ‘oblique’ fluxes [10, 12, 13, 27]. Therefore it is imperative that we generalize previous results further and obtain interactions involving branes with oblique fluxes. As stated, in the language of  $D6$  branes such generalizations would amount to intersections of branes with orientations given by  $SU(3)$  rotation angles, resulting to  $N = 1$  supersymmetry in  $D = 4$  with chiral matter. In view of the importance of such fluxes in obtaining realistic particle physics models with stabilized moduli, and to describe the interactions among the chiral fields, we shall study the explicit construction of fermion (and scalar) wavefunctions on compact toroidal spaces with arbitrary constant fluxes.

Scattered results on fermion wavefunctions in presence of constant gauge fluxes, on tori of arbitrary dimensions, exist already in the literature [5, 28]. However, they are of limited use for our purpose. First, any wavefunction obtained through a diagonalization process of the gauge fluxes [28], is not in general suitable for obtaining an overlap integral of wavefunctions. This is because the flux matrices need not commute along different stacks of branes that participate in the interaction through the chiral multiplets, arising from the strings that join these branes and therefore they are not simultaneously diagonalizable.

In [5], a set of wavefunctions was given for constant gauge fluxes. However, once again, explicit results are valid only for those fluxes which satisfy a set of ‘Riemann conditions’, including a positivity criterion on the flux matrices. As the analysis in our paper will clarify, the positivity restrictions on the fluxes is due to the fact that the given wavefunction in [5] corresponds to a specific component of the  $2^n$  dimensional Dirac spinor for a  $2n$ -dimensional torus  $T^{2n}$ . We will show that this restriction is relaxed, if one considers wavefunctions of various chiralities, such that all possible flux matrices are allowed, though in our case we restrict to only those fluxes that are consistent with the requirements of space-time supersymmetry .

In fact, we give explicit solutions for the wavefunctions for arbitrary fluxes, that are well defined globally on the toroidal space. We also give explicit mappings among the wavefunctions of different chiralities, satisfying different consistency criterion. These mappings are shown to relate wavefunctions corresponding to different fluxes and complex structures

of the tori. We in fact reconfirm that our wavefunctions, as well as mappings are indeed correct, by showing that equations of motion also map into each other for the fermion wavefunctions just described, corresponding to different internal chiralities.

Apart from the lack of enough knowledge about the fermion wavefunctions, the limitations on available information about the Yukawa couplings for general gauge fluxes also arose from the technicalities in dealing with general Riemann theta functions that are used for defining the wavefunctions on toroidal spaces. Internal wavefunctions of chiral fermions participating in the interaction are given by a general Riemann theta function whose modular parameter argument is determined in terms of the complex structure of  $T^6$  as well as the ‘oblique’ fluxes that we turn on. Hence, the limitations on available results for Yukawa interactions in the literature, arise due to the intricacies involved in evaluating the overlap integrals of the trilinear product of general Riemann theta functions over the six dimensional internal space. In particular, even for positive chirality wavefunctions along the internal  $T^6$  given in [5], one finds that theta identities [29] need to be further generalized, in order to compute the Yukawa interactions with oblique fluxes. The task goes beyond the identity given in [29], since one needs to evaluate the overlap integral of three wavefunctions, all having different modular parameter matrices as arguments, due to the presence of different fluxes along the three brane stacks involved in generating the Yukawa coupling.

In this paper, first, we generalize the identities used in [5] (available from mathematical literature [29]) for the known positive chirality wavefunctions to those with general Riemann theta functions representing the fermion wavefunctions. This gives an explicit answer for the Yukawa interaction in a close form and generalizes the results of [1, 5]. In particular, we generalize the result further for the positive chirality wavefunction, when general (hermitian) fluxes with all nine parameters are turned on rather than the six components considered before.

On the other hand, as already stated earlier, we give explicit construction of the other  $T^6$  spinor wavefunctions, as well. In these cases too, we obtain the selection rules among chiral multiplets giving nonzero Yukawa couplings. Now, however, the final answer is left as a real finite integration of a theta function, over three toroidal coordinate variables. The integration can be evaluated numerically for any given example. Finally, in the paper, we also briefly discuss the issue of mass generation for non-chiral multiplets given in the

$SU(5)$  GUT model, constructed in [27]. Although the Yukawa couplings can be used for giving the precise masses in our model, where all close string moduli are fixed, we avoid entering into these details in this work.

The plan of the rest of the paper is as follows. In Section 2, we start by discussing the general setup, including the gauge fluxes that can be turned on, in a consistent manner. In Section 3, we review the known results on the Jacobi theta identity given in [29] and present a proof of its validity. We also give an expression for the Yukawa interaction for factorized tori and ‘diagonal’ fluxes using the theta identity. In Section 4, we construct a similar identity, now for the general Riemann theta function. We then use this new mathematical relation for writing down the expression for the Yukawa interaction when oblique fluxes are present and satisfy the ‘Riemann conditions’ of [5]. Results are further generalized to include the most general flux matrices consistent with supersymmetry and ‘Riemann condition’ requirements. In order to relax the later, in Section 5, we present the generalizations to include the wavefunctions of the other internal chiralities, in order to accommodate general fluxes consistent with supersymmetry restrictions. In Section 6, we briefly present an independent analysis of the superpotential and D-terms for the model of [27] in order to show how masses for several non-chiral fermion multiplets can be generated, without evaluating explicitly the superpotential coefficients, which would go beyond the scope of our present work. Conclusions are presented in Section 7. In Appendix A we give the chiral fermion wavefunctions in the presence of constant fluxes. Appendix B contains information on fluxes in terms of windings and Chern numbers, while Appendix C gives some details of our model in [27], needed for the mass generation analysis of Section 6.

## 2 Fluxes

We now start by describing the gauge fluxes that can be turned on along internal tori. A general gauge flux, on  $T^6$  with coordinates  $X^I \equiv (x^i, y^i)$ ,  $i = 1, 2, 3$ , has the form:

$$\begin{aligned}
 F &\equiv p_{IJ} dX^I \wedge dX^J \\
 &= p_{x^i x^j} dx^i \wedge dx^j + p_{y^i y^j} dy^i \wedge dy^j + p_{x^i y^j} dx^i \wedge dy^j + p_{y^i x^j} dy^i \wedge dx^j .
 \end{aligned}
 \tag{2.1}$$

Then using the definition of a general complex structure matrix  $\Omega$ :

$$dz^i = dx^i + \Omega_j^i dy^j, \quad d\bar{z}^i = dx^i + \bar{\Omega}_j^i dy^j, \quad (2.2)$$

we obtain:

$$F = F_{z^i z^j} dz^i \wedge dz^j + F_{z^i \bar{z}^j} (idz^i \wedge d\bar{z}^j) + F_{\bar{z}^i \bar{z}^j} d\bar{z}^i \wedge d\bar{z}^j. \quad (2.3)$$

Choosing the basis  $e^{i\bar{j}}$  of the cohomology  $H^{1,1}$  to be of the form

$$e^{i\bar{j}} = idz^i \wedge d\bar{z}^j, \quad (2.4)$$

we obtain:

$$F_{z^i z^j} = (\bar{\Omega} - \Omega)^{-1T} (\bar{\Omega}^T p_{xx} \bar{\Omega} - \bar{\Omega}^T p_{xy} + p_{xy}^T \bar{\Omega} + p_{yy}) (\bar{\Omega} - \Omega)^{-1} \quad (2.5)$$

and

$$F_{z^i \bar{z}^j} = (-i)(\bar{\Omega} - \Omega)^{-1T} (\bar{\Omega}^T p_{xx} \Omega - \bar{\Omega}^T p_{xy} + p_{xy}^T \Omega + p_{yy}) (\bar{\Omega} - \Omega)^{-1}. \quad (2.6)$$

In addition,  $F_{\bar{z}^i \bar{z}^j}$  is complex conjugate to  $F_{z^i z^j}$  and  $F_{\bar{z}^i z^j} = -F_{z^j \bar{z}^i}$ .

Now, supersymmetry demands all fluxes to be of  $(1, 1)$  form which gives us the condition:

$$(\bar{\Omega}^T p_{xx} \bar{\Omega} - \bar{\Omega}^T p_{xy} + p_{xy}^T \bar{\Omega} + p_{yy}) = 0, \quad (2.7)$$

or equivalently:

$$(\bar{\Omega}^T p_{xx} \Omega - \bar{\Omega}^T p_{xy} + p_{xy}^T \Omega + p_{yy}) = 0. \quad (2.8)$$

Eqs. (2.7) and (2.8) together give two real matrix equations. These equations can then be used to eliminate some of the variables and write the final  $(1, 1)$  form in terms of certain independent variables only.

Using eq. (2.8), eq. (2.6) reduces to the following form,

$$F_{z^i \bar{z}^j} = -i(p_{xx} \Omega - p_{xy}) (\bar{\Omega} - \Omega)^{-1}. \quad (2.9)$$

On the other hand, use of eq. (2.7) in eq. (2.6) gives,

$$F_{z^i \bar{z}^j} = -i(\bar{\Omega} - \Omega)^{-1T} (-\bar{\Omega}^T p_{xx} - p_{xy}^T). \quad (2.10)$$

We also notice that the  $(1, 1)$  form  $F_{z^i \bar{z}^j}$  given in eq. (2.6) satisfies the hermiticity property:

$F_{z^i \bar{z}^j} = F_{z^i \bar{z}^j}^\dagger$ . To explicitly see that, we use eqs. (2.9), (2.10).

$$\begin{aligned} F_{z^i \bar{z}^j}^\dagger &= [(-i(p_{xx} \Omega - p_{xy}) (\bar{\Omega} - \Omega)^{-1})^*]^T \\ &= -i(\bar{\Omega} - \Omega)^{-1T} (-\bar{\Omega}^T p_{xx} - p_{xy}^T) = F_{z^i \bar{z}^j} \end{aligned} \quad (2.11)$$

There are some special cases, however, in which eqs. (2.7) and (2.8) simplify further and the resulting  $F_{z^i \bar{z}^j}$  can be written more compactly. One such case arises when  $p_{xx}$  and  $p_{yy}$  components are turned off. In such a situation  $F_{(2,0)} = 0$  condition (2.8), reduces to:

$$\Omega^T p_{xy} = p_{xy}^T \Omega. \quad (2.12)$$

Thus far, we have concentrated on the spatial components of the gauge fluxes, but ignored the gauge indices. In the magnetized  $D$ -brane construction, gauge quantum numbers arise from the Chan-Paton factors associated with the end points of the open strings for a given stack of branes. The simplest possibility is to consider fluxes with gauge indices given by an  $n \times n$  identity matrix for a stack of  $D$ -branes:

$$F = mI_n, \quad (2.13)$$

with  $m$  an arbitrary integer giving the 1st Chern number. All spatial indices of the gauge flux above have been suppressed, which are given as in eq. (2.1) by the components:  $p_{x^i y^j}$ ,  $p_{x^i x^j}$ ,  $p_{y^i y^j}$ , while their quantized values are expressed in eq. (B.4) for general wrapping of the branes. Actually, eq. (2.13) corresponds to the situation when all the wrapping numbers are trivial:  $n^{x^i} = n^{y^i} = 1$  in the denominators of eq. (B.4).  $F$ , then represents a stack of  $n$  magnetized  $D$ -branes with a  $U(1)^n$  gauge flux. The first Chern number for each of the  $U(1)$  fluxes is equal to  $m$ . Moreover,  $D$ -brane wrapping numbers on the internal directions, are all unity, given by a diagonal embedding of the brane in target space and winding around each 1-cycle once. In most of the paper, we will consider fluxes of the above type.

For multiple stacks of  $n_i$  branes with respective 1st Chern numbers  $m_i$ , the flux matrix is of block diagonal form:

$$F = \begin{pmatrix} m_1 I_{n_1} & & & & \\ & m_2 I_{n_2} & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & m_{n_p} I_{n_p} \end{pmatrix} \quad (2.14)$$

and corresponds to gauge fluxes in the diagonal  $U(1)$ 's of  $U(n_1) \times U(n_2) \times \dots$  gauge group. Chiral fermion wavefunctions, on the other hand, belong to the bifundamental

representations  $(n_a, \bar{n}_b)$  etc. and are determined by the field equations in terms of the difference of fluxes in the respective stacks, parameterized by a matrix  $\mathbf{N}$ :

$$\mathbf{N}^T = p_{xy}^a - p_{xy}^b. \quad (2.15)$$

Using (2.12), it follows that:

$$(\mathbf{N}\Omega)^T = (\mathbf{N}\Omega). \quad (2.16)$$

Moreover, further conditions are imposed on the matrix  $\mathbf{N}$  in order to have well defined bifundamental wavefunctions. These are the so-called Riemann conditions [5] and are written as:

$$\begin{aligned} \mathbf{N}_{\bar{i}j} &\in \mathbf{Z}, \\ (\mathbf{N}.Im\Omega)^T &= \mathbf{N}.Im\Omega, \\ \mathbf{N}.Im\Omega &> 0. \end{aligned} \quad (2.17)$$

The first condition in eq. (2.17) is the integrality of the elements of  $\mathbf{N}$ , that we discuss later on, in the absence of any non-abelian Wilson lines [5], following from the Dirac quantization of fluxes. To understand the last condition of eq. (2.17), one rewrites the  $(1, 1)$  form  $F_{z^i \bar{z}^j}$ , for the case when  $p_{xx} = p_{yy} = 0$ . Indeed using eq. (2.12), one obtains:

$$F_{z^i \bar{z}^j} = -ip_{xy}(\Omega - \bar{\Omega})^{-1}, \quad (2.18)$$

which matches with the expression for  $H$  in eq. (4.73) of [5] upon the identification  $\mathbf{N}^T = \mathbf{p}_{\mathbf{xy}}$  and  $H = \frac{1}{2}\mathbf{N}^T.Im\Omega^{-1}$ . The positivity requirement on  $H$  then arises from the condition that the solutions of the Dirac equation, corresponding to chiral wavefunctions, be normalizable.

Gauge fluxes on branes with higher wrapping numbers can also be given a gauge theoretic interpretation. The method, as stated earlier, is based on a representation of the magnetized brane constructions [5] in terms of fluxes along internal directions in a compactified gauge theory. In this picture, the effect of windings of branes around  $T^6$  is simulated by the rank of the gauge group. In particular, due to the Dirac quantization condition on fluxes, a  $U(n)$  flux on, say  $T^2$ :

$$F = \frac{m}{n}I_n, \quad (2.19)$$

with  $I_n$  being the  $n$ -dimensional identity matrix, and  $(n, m)$  relatively prime, represents a single brane wound  $n$  times around  $T^2$  with flux quantum  $m$  and resulting gauge symmetry



being only  $U(1)$ . On the other hand, if  $m$  is an integer multiple of  $n$  such that  $m = pn$ , then each of the entries in the identity matrix represents a well defined  $U(1)$  flux of quantum  $p$  and the gauge symmetry is  $U(n)$ , given by a stack of  $n$  such magnetized branes, as described in the last paragraph. It turns out that explicit realization of fluxes with  $(n, m)$  relatively prime, needs gauge configurations with non-abelian Wilson lines.

The wavefunctions of the chiral fermion bifundamentals, with both abelian and non-abelian Wilson lines, involved in Yukawa computations, are given in [5] for the case of the factorized tori, eq. (1.1), and diagonal fluxes. For oblique fluxes, we postpone the discussion of non-abelian Wilson lines and rational fluxes to the last part of the paper (Section 7) and for the moment we consider the case of integral fluxes only. This restriction, nevertheless, allows for a rich structure of phenomenological value, since semi-realistic models with three generations of chiral fermions and stabilized moduli can be built even in the context of such integral fluxes, by turning on NS-NS antisymmetric tensor background. For example, a three generation  $SU(5)$  GUT with stabilized moduli given in [27] was constructed with all winding numbers,  $n = 1$ , for different stacks of branes. Also, the presence of a half-integral NS-NS antisymmetric tensor does not modify any of our results, since all the relevant chiral fermion wavefunctions depend on the difference of fluxes along pairs of brane stacks which is always integral.

### 3 Yukawa computation on factorized tori

#### 3.1 Wavefunction

A detail discussion of the chiral fermion wavefunctions in the presence of constant gauge fluxes is given in Appendix A for general tori and fluxes. In the case of factorized tori, eq. (1.1), the six dimensional chiral/anti-chiral wavefunctions are written as a product of wavefunctions on  $T^2$ . To show this explicitly, we present (as in Appendix A) the case of  $T^4$  as an example, with  $T^6$  case working out in a similar fashion. More precisely, considering that on two  $T^2$ 's, fermion wavefunctions

$$\psi^{(1)} = \begin{pmatrix} \psi_+^{(1)} \\ \psi_-^{(1)} \end{pmatrix}, \quad \psi^{(2)} = \begin{pmatrix} \psi_+^{(2)} \\ \psi_-^{(2)} \end{pmatrix}, \quad (3.1)$$

with their internal  $U(n_1) \times U(n_2)$  structure being represented in a manner as in eq. (A.9), satisfy the equations:

$$\begin{aligned}
\bar{\partial}_1 \psi_+^{(1)} + (A^1 - A^2)_{\bar{z}_1} \psi_+^{(1)} &= 0, \\
\partial_1 \psi_-^{(1)} + (A^1 - A^2)_{z_1} \psi_-^{(1)} &= 0, \\
\bar{\partial}_2 \psi_+^{(2)} + (A^1 - A^2)_{\bar{z}_2} \psi_+^{(2)} &= 0, \\
\partial_2 \psi_-^{(2)} + (A^1 - A^2)_{z_2} \psi_-^{(2)} &= 0.
\end{aligned} \tag{3.2}$$

$T^4$  fermion wavefunctions are then constructed through a direct product of  $\psi^1$  and  $\psi^2$  (in the notations of Appendix A):

$$\begin{pmatrix} \Psi_+^1 \\ \Psi_-^2 \\ \Psi_-^1 \\ \Psi_+^2 \end{pmatrix} \equiv \begin{pmatrix} \psi_+^{(1)} \\ \psi_-^{(1)} \end{pmatrix} \otimes \begin{pmatrix} \psi_+^{(2)} \\ \psi_-^{(2)} \end{pmatrix}. \tag{3.3}$$

In particular,

$$\Psi_+^1 \equiv \psi_+^{(1)} \otimes \psi_+^{(2)} \tag{3.4}$$

satisfies precisely the equations (A.11) for chiral fermions on  $T^4$ . We can further extend these results to show that  $T^6$  chiral wavefunctions can also be written as a product of the chiral wavefunctions on three  $T^2$ 's in the decomposition (1.1).

Yukawa interaction on  $T^6$  is then also given by an expression which is a direct product of the interaction terms for the three  $T^2$ 's. Wavefunctions for the chiral fermions on a  $T^2$  (with coordinates  $x, y$ ) are expressed in terms of the basis wavefunctions  $\psi^{j,N}$  [5]:

$$\psi^{j,N}(\tau, z) = \mathcal{N} \cdot e^{i\pi N z \text{Im} z / \text{Im} \tau} \cdot \vartheta \left[ \begin{matrix} \frac{j}{N} \\ 0 \end{matrix} \right] (Nz, N\tau), \quad j = 0, \dots, N-1, \tag{3.5}$$

with  $N$  denoting the difference of the  $U(n_a)$  and  $U(n_b)$  magnetic gauge fluxes as given in eq. (2.15), turned on along the Cartan generators, representing stacks of  $n_a$  and  $n_b$  branes respectively and gives the degeneracy of the chiral fermions:

$$N = m_a - m_b \equiv I_{ab}, \tag{3.6}$$

with  $m_a$  and  $m_b$  being the 1st Chern number of fluxes along stacks  $a$  and  $b$ , with unit windings, as defined through eq. (B.4).

Using such a basis, the chiral and anti-chiral (left and right handed fermions) basis wavefunctions:

$$\psi^j = \begin{pmatrix} \psi_+^j \\ \psi_-^j \end{pmatrix}, \quad (3.7)$$

are given by:

$$\begin{aligned} \psi_+^j &= \psi^{j,N}(\tau, z + \zeta), \quad (\psi_+^j)^* = \psi^{-j,-N}(\bar{\tau}, \bar{z} + \bar{\zeta}), \\ \psi_-^j &= \psi^{j,N}(\bar{\tau}, \bar{z} + \bar{\zeta}), \quad (\psi_-^j)^* = \psi^{-j,-N}(\tau, z + \zeta), \end{aligned} \quad (3.8)$$

and satisfy the equations:

$$\begin{aligned} D\psi_+^j &\equiv \left(\bar{\partial} + \frac{\pi N}{2\text{Im}\tau(z + \zeta)}\right)\psi_+^j = 0, \\ D^\dagger(\psi_+^j)^* &\equiv \left(\partial - \frac{\pi N}{2\text{Im}\tau(z + \zeta)}\right)(\psi_+^j)^* = 0, \\ D^\dagger\psi_-^j &\equiv \left(\partial - \frac{\pi N}{2\text{Im}\tau(z + \zeta)}\right)\psi_-^j = 0, \\ D(\psi_-^j)^* &\equiv \left(\bar{\partial} + \frac{\pi N}{2\text{Im}\tau(z + \zeta)}\right)(\psi_-^j)^* = 0, \end{aligned} \quad (3.9)$$

with  $\zeta$  representing the Wilson lines. In the following we set the Wilson lines  $\zeta = 0$ . Furthermore, expressions of the chiral and anti-chiral solutions, as given in eqs. (3.8) and (3.5), are well defined provided  $N > 0$  for the wavefunctions  $\psi_+^j$  and  $N < 0$  for the wavefunctions  $\psi_-^j$ . In these cases, for  $\psi_+^j$  and  $\psi_-^j$  to be properly normalized:

$$\int_{T^2} dzd\bar{z} \psi_\pm^j (\psi_\pm^k)^* = \delta_{jk}, \quad (3.10)$$

an additional factor

$$\mathcal{N}_j = \left(\frac{2\text{Im}\tau|N|}{\mathcal{A}^2}\right)^{\frac{1}{4}} \quad (3.11)$$

needs to be introduced, with  $\mathcal{A}$  being the area of the  $T^2$ .

In fact, the basis functions (3.5) are also eigenfunctions of the Laplacian. We elaborate on this point more in Section 5.4 and now proceed to make use of these fermion and boson basis functions to determine the Yukawa interaction in the case of factorized tori and ‘diagonal’ fluxes.

### 3.2 Interaction for factorized tori

We now summarize the basic results of [5] regarding the computations of Yukawa interactions. Such four dimensional interaction terms were obtained through a dimensional reduction of the  $D = 10$ ,  $N = 1$  super-Yang-Mills theory to four dimensions in the presence of constant magnetic fluxes. The Yukawa coupling is given by

$$Y_{ijk} = \int_{\mathcal{M}} \psi_i^{a\dagger} \Gamma^m \psi_j^b \phi_{k,m}^c f_{abc}, \quad (3.12)$$

where  $\mathcal{M}$  is the internal space on which the gauge theory has been compactified and  $\psi$  and  $\phi$  are the internal zero mode fluctuations of the gaugino and Yang-Mills fields with  $f_{abc}$  being the structure constants of the higher dimensional gauge group. For the torus compactification that we are discussing, the internal wavefunctions are factorized into those depending on the coordinates of three  $T^2$ 's. In turn, these involve the evaluation of terms of the type:

$$\int_{T^2} dzd\bar{z} \text{Tr} \{ \psi_+ \cdot [\phi_-, \psi_+] \} \quad \text{and} \quad \int_{T^2} dzd\bar{z} \text{Tr} \{ \psi_- \cdot [\phi_+, \psi_-] \}, \quad (3.13)$$

with  $\phi_{\pm}$  being the wavefunctions of the bosonic fluctuations of the ten dimensional gauge fields with helicity  $\pm 1$  along the particular  $T^2$  direction. Similarly  $\psi_{\pm}$  denotes the spinor fluctuations with helicities  $\pm \frac{1}{2}$ . Therefore, In the factorized case of eq. (1.1), the full interaction term is computed as a product of three such integrals. To evaluate these integrals, one uses the wavefunctions (3.1) and basis functions as given in eq. (3.5).

In the language of string construction with magnetized branes,  $N \equiv I_{ab}$  corresponds to the intersection number for the string starting at a stack  $a$  and ending on another one  $b$ . The Yukawa interaction then reads:

$$Y_{ijk} = g\sigma_{abc} \int_{T^2} dzd\bar{z} \psi^{i,I_{ab}}(\tau, z) \cdot \psi^{j,I_{ca}}(\tau, z) \cdot (\psi^{k,I_{cb}}(\tau, z))^* \quad (3.14)$$

with  $I_{bc} < 0$ , corresponding to the fact that when the intersection numbers  $I_{ab}$  and  $I_{ca}$  are positive, then  $I_{bc}$  has to be negative, since  $I_{ab} + I_{bc} + I_{ca} = 0$ . A similar expression exists for  $I_{bc} > 0$  as well. To evaluate this integral, one uses an identity, satisfied by the theta functions appearing in the definition of the basis functions (3.5). The aim of this relation is to establish a connection between the wavefunctions with intersection numbers  $N_1$  and  $N_2$  for bifundamental states in brane intersections  $ab$  and  $ca$  with the one in the intersection  $bc$  with  $N_3 = N_1 + N_2$ . However, in view of the further generalization to the

oblique flux case, we establish this identity explicitly in the next subsection and generalize it further in Section 4.

### 3.3 Jacobi theta function identities

We now explicitly prove the following theta function identity [29] used in [5] for computing the Yukawa couplings:

$$\begin{aligned} \vartheta \begin{bmatrix} \frac{r}{N_1} \\ 0 \end{bmatrix} (z_1, \tau N_1) \cdot \vartheta \begin{bmatrix} \frac{s}{N_2} \\ 0 \end{bmatrix} (z_2, \tau N_2) &= \sum_{m \in \mathbf{Z}_{N_1+N_2}} \vartheta \begin{bmatrix} \frac{r+s+N_1 m}{N_1+N_2} \\ 0 \end{bmatrix} (z_1 + z_2, \tau(N_1 + N_2)) \\ &\times \vartheta \begin{bmatrix} \frac{N_2 r - N_1 s + N_1 N_2 m}{N_1 N_2 (N_1 + N_2)} \\ 0 \end{bmatrix} (z_1 N_2 - z_2 N_1, \tau N_1 N_2 (N_1 + N_2)), \end{aligned} \quad (3.15)$$

where  $\vartheta$  is the Jacobi theta-function:

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (\nu, \tau) = \sum_{l \in \mathbf{Z}} e^{\pi i (a+l)^2 \tau} e^{2\pi i (a+l)(\nu+b)}. \quad (3.16)$$

To proceed with the proof of the above identity, we write its LHS explicitly as:

$$\begin{aligned} \vartheta \begin{bmatrix} \frac{r}{N_1} \\ 0 \end{bmatrix} (z_1, \tau N_1) \cdot \vartheta \begin{bmatrix} \frac{s}{N_2} \\ 0 \end{bmatrix} (z_2, \tau N_2) &= \sum_{l_1 \in \mathbf{Z}} \sum_{l_2 \in \mathbf{Z}} e^{\pi i (\frac{r}{N_1} + l_1)^2 \tau N_1} e^{2\pi i (\frac{r}{N_1} + l_1) z_1} \\ &\cdot e^{\pi i (\frac{s}{N_2} + l_2)^2 \tau N_2} e^{2\pi i (\frac{s}{N_2} + l_2) z_2}. \end{aligned} \quad (3.17)$$

Similarly the RHS of the identity (3.15) can be written as:

$$\begin{aligned} &\sum_{m \in \mathbf{Z}_{N_1+N_2}} \vartheta \begin{bmatrix} \frac{r+s+N_1 m}{N_1+N_2} \\ 0 \end{bmatrix} (z_1 + z_2, \tau(N_1 + N_2)) \\ &\times \vartheta \begin{bmatrix} \frac{N_2 r - N_1 s + N_1 N_2 m}{N_1 N_2 (N_1 + N_2)} \\ 0 \end{bmatrix} (z_1 N_2 - z_2 N_1, \tau N_1 N_2 (N_1 + N_2)) \\ &= \sum_{m \in \mathbf{Z}_{N_1+N_2}} \sum_{l_3 \in \mathbf{Z}} \sum_{l_4 \in \mathbf{Z}} e^{\pi i (\frac{r+s+N_1 m}{N_1+N_2} + l_3)^2 \tau (N_1+N_2)} e^{2\pi i (\frac{r+s+N_1 m}{N_1+N_2} + l_3)(z_1+z_2)} \\ &\times e^{\pi i (\frac{N_2 r - N_1 s + N_1 N_2 m}{N_1 N_2 (N_1 + N_2)} + l_4)^2 \tau N_1 N_2 (N_1 + N_2)} e^{2\pi i (\frac{N_2 r - N_1 s + N_1 N_2 m}{N_1 N_2 (N_1 + N_2)} + l_4)(z_1 N_2 - z_2 N_1)}. \end{aligned} \quad (3.18)$$

Now, to match the  $z_1, z_2$  terms in both sides of eq. (3.15), we first note the identity:

$$\left( \frac{r+s}{N_1+N_2} \right) (z_1 + z_2) + \left( \frac{N_2 r - N_1 s}{N_1 N_2 (N_1 + N_2)} \right) (z_1 N_2 - z_2 N_1) = \left( \frac{r}{N_1} z_1 + \frac{s}{N_2} z_2 \right), \quad (3.19)$$

and find coefficients  $p_1, p_2, q_1, q_2$  such that,

$$(p_1 l_1 + p_2 l_2)(z_1 + z_2) + (q_1 l_1 + q_2 l_2)(z_1 N_2 - z_2 N_1) = (l_1 z_1 + l_2 z_2). \quad (3.20)$$

Eq. (3.20) leads to the following values for  $p_1, p_2, q_1, q_2$  :

$$\begin{aligned} p_1 &= \frac{N_1}{N_1 + N_2}, & p_2 &= \frac{N_2}{N_1 + N_2}, \\ q_1 &= \frac{1}{N_1 + N_2}, & q_2 &= \frac{-1}{N_1 + N_2}. \end{aligned} \quad (3.21)$$

Then the two terms, containing  $z_1, z_2$ , in the RHS of eq. (3.17) can be rewritten as:

$$e^{2\pi i(\frac{r}{N_1} + l_1)z_1} e^{2\pi i(\frac{s}{N_2} + l_2)z_2} = e^{2\pi i(\frac{r+s}{N_1+N_2} + \frac{N_1 l_1}{N_1+N_2} + \frac{N_2 l_2}{N_1+N_2})(z_1+z_2)} e^{2\pi i(\frac{N_2 r - N_1 s}{N_1 N_2 (N_1+N_2)} + \frac{l_1 - l_2}{N_1+N_2})(z_1 N_2 - z_2 N_1)}. \quad (3.22)$$

Similarly, coefficients  $p, q$  satisfying identity:

$$\begin{aligned} p \left[ \frac{r+s}{N_1+N_2} + \frac{N_1 l_1}{N_1+N_2} + \frac{N_2 l_2}{N_1+N_2} \right]^2 + q \left[ \frac{N_2 r - N_1 s}{N_1 N_2 (N_1+N_2)} + \frac{l_1 - l_2}{N_1+N_2} \right]^2 = & (3.23) \\ \left[ \frac{r}{N_1} + l_1 \right]^2 N_1 + \left[ \frac{s}{N_2} + l_2 \right]^2 N_2, \end{aligned}$$

are given as:

$$p = N_1 + N_2, \quad q = N_1 N_2 (N_1 + N_2). \quad (3.24)$$

Using eqs. (3.19), (3.20), (3.22) and (3.24), the RHS of eq. (3.17) (appearing in the LHS of eq. (3.15) ) can be re-written :

$$\begin{aligned} & \sum_{l_1 \in \mathbf{Z}} \sum_{l_2 \in \mathbf{Z}} e^{\pi i(\frac{r}{N_1} + l_1)^2 \tau N_1} e^{2\pi i(\frac{r}{N_1} + l_1)z_1} \cdot e^{\pi i(\frac{s}{N_2} + l_2)^2 \tau N_2} e^{2\pi i(\frac{s}{N_2} + l_2)z_2} = \\ & \sum_{l_1 \in \mathbf{Z}} \sum_{l_2 \in \mathbf{Z}} e^{\pi i(\frac{r+s}{N_1+N_2} + \frac{N_1 l_1}{N_1+N_2} + \frac{N_2 l_2}{N_1+N_2})^2 \tau (N_1+N_2)} e^{2\pi i(\frac{r+s}{N_1+N_2} + \frac{N_1 l_1}{N_1+N_2} + \frac{N_2 l_2}{N_1+N_2})(z_1+z_2)} \cdot \\ & e^{\pi i(\frac{N_2 r - N_1 s}{N_1 N_2 (N_1+N_2)} + \frac{l_1 - l_2}{N_1+N_2})^2 \tau N_1 N_2 (N_1+N_2)} e^{2\pi i(\frac{N_2 r - N_1 s}{N_1 N_2 (N_1+N_2)} + \frac{l_1 - l_2}{N_1+N_2})(z_1 N_2 - z_2 N_1)}. \end{aligned} \quad (3.25)$$

Proving the identity, eq. (3.15), now amounts to showing that the RHS of eq. (3.18) matches precisely with that of eq. (3.25) with  $m$  in eq. (3.18) taking value as  $m = 0, 1, \dots, (N_1 + N_2 - 1)$ . We note:

1. When  $l_1 = l_2$  in eq. (3.25), the terms in the RHS are identical to those in the RHS of eq. (3.18), with  $m = 0, l_4 = 0$ , if we identify  $l_2$  with  $l_3$ .

When  $l_1 = l_2 + 1$ , the terms in eq. (3.25) exactly match with those in eq. (3.18)

obtained for the values  $m = 1, l_4 = 0$  with the identification of  $l_2$  with  $l_3$ .

This goes on up to  $l_1 = l_2 + (N_1 + N_2 - 1)$  which corresponds to the case for  $l_3 (= l_2), m = (N_1 + N_2 - 1)$  and  $l_4 = 0$ .

2. The terms obtained in eq. (3.25) for  $l_1 = l_2 + (N_1 + N_2)$  corresponds to  $m = 0, l_4 = 1$  and  $l_2 + N_1$  identified with  $l_3$  in eq. (3.18).

When  $l_1 = l_2 + (N_1 + N_2) + 1$  the terms correspond to the case  $m = 1, l_4 = 1$  and  $l_2 + N_1$  identified with  $l_3$  in eq. (3.18).

This goes on till  $l_1 = l_2 + (N_1 + N_2) + (N_1 + N_2 - 1)$  when they correspond to  $m = (N_1 + N_2 - 1), l_4 = 1$  and  $l_2 + N_1$  identified with  $l_3$  in eq. (3.18).

3. Similarly the terms for  $l_1 = l_2 + 2(N_1 + N_2)$  correspond to the terms for  $m = 0, l_4 = 2$  and  $l_3 = (l_2 + 2N_1)$ . And so on....

We have therefore shown a one-to-one correspondence between the terms in the RHS of eqs. (3.18) and (3.25). The identity eq. (3.15) has thus been proved explicitly.

### 3.4 Application to Yukawa computation for factorized tori

We now make use of the above Jacobi theta identity as well as of the explicit forms of the fermion and scalar wavefunctions, defined in terms of the basis functions in eq. (3.5) to write the expression for the Yukawa interaction term. More precisely, in order to evaluate the Yukawa coupling given in eq. (3.14), one uses the theta identity of eq. (3.15) and the basis function in eq. (3.5) and proceeds by writing down:

$$\begin{aligned} \psi^{i,I_{ab}}(\tau, z) \cdot \psi^{j,I_{ca}}(\tau, z) &= \left( \frac{2Im\tau}{\mathcal{A}^2} \right)^{\frac{1}{2}} (I_{ab}I_{ca})^{\frac{1}{4}} e^{i\pi(N_1+N_2)zImz/Im\tau} \times \\ &\times \vartheta \left[ \begin{array}{c} \frac{i}{N_1} \\ 0 \end{array} \right] (N_1z, N_1\tau) \cdot \vartheta \left[ \begin{array}{c} \frac{j}{N_2} \\ 0 \end{array} \right] (N_2z, N_2\tau), \quad i = 0, \dots, N_1 - 1, \quad j = 0, \dots, N_2 - 1. \end{aligned} \quad (3.26)$$

where we have also made use of the normalization factor,  $\mathcal{N}$  given in eq. (3.11), and identified for a  $T^2$  compactification:

$$N_1 = I_{ab}, \quad N_2 = I_{ca}, \quad (3.27)$$

with

$$I_{ab} = m_a - m_b, \quad etc. \quad (3.28)$$

giving

$$N_3 = (N_1 + N_2) = I_{cb}. \quad (3.29)$$

Now, using the theta identity (3.15), eq. (3.26) can be rewritten in the form:

$$\begin{aligned} \psi^{i,I_{ab}}(\tau, z) \cdot \psi^{j,I_{ca}}(\tau, z) &= \left( \frac{2Im\tau}{\mathcal{A}^2} \right)^{\frac{1}{4}} \left( \frac{I_{ab}I_{ca}}{I_{cb}} \right)^{\frac{1}{4}} \sum_{m \in \mathbf{Z}_{I_{cb}}} \psi^{i+j+I_{ab}m, I_{cb}}(\tau, z) \times \\ &\quad \times \vartheta \left[ \begin{array}{c} \frac{I_{ca}i - I_{ab}j + I_{ab}I_{ca}m}{I_{ab}I_{ca}I_{cb}} \\ 0 \end{array} \right] (0, \tau I_{ab}I_{ca}I_{cb}). \end{aligned} \quad (3.30)$$

The Yukawa interaction (3.14), is then evaluated using the orthogonality property of the wavefunctions given in eq. (3.10) and reads:

$$Y_{ijk} = \sigma_{abc} g \left( \frac{2Im\tau}{\mathcal{A}^2} \right)^{\frac{1}{4}} \left( \frac{I_{ab}I_{ca}}{I_{cb}} \right)^{\frac{1}{4}} \sum_{m \in \mathbf{Z}_{I_{cb}}} \delta_{k, i+j+I_{ab}m} \cdot \vartheta \left[ \begin{array}{c} \frac{I_{ca}i - I_{ab}j + I_{ab}I_{ca}m}{I_{ab}I_{ca}I_{cb}} \\ 0 \end{array} \right] (0, \tau I_{ab}I_{ca}I_{cb}). \quad (3.31)$$

After imposing the Kronecker delta constraint, we obtain:

$$Y_{ijk} = \sigma_{abc} g \left( \frac{2Im\tau}{\mathcal{A}^2} \right)^{\frac{1}{4}} \left( \frac{I_{ab}I_{ca}}{I_{cb}} \right)^{\frac{1}{4}} \vartheta \left[ \begin{array}{c} - \left( \frac{j}{I_{ca}} + \frac{k}{I_{bc}} \right) / I_{ab} \\ 0 \end{array} \right] (0, \tau I_{ab}I_{ca}I_{cb}). \quad (3.32)$$

The final answer can be expressed as :

$$Y_{ijk} = \sigma_{abc} g \left( \frac{2Im\tau}{\mathcal{A}^2} \right)^{\frac{1}{4}} \left( \frac{I_{ab}I_{ca}}{I_{cb}} \right)^{\frac{1}{4}} \vartheta \left[ \begin{array}{c} \delta_{ijk} \\ 0 \end{array} \right] (0, \tau |I_{ab}I_{bc}I_{ca}|), \quad (3.33)$$

with

$$\delta_{ijk} = \frac{i}{I_{ab}} + \frac{j}{I_{ca}} + \frac{k}{I_{bc}}. \quad (3.34)$$

The result can be easily extended to the case of factorized  $T^6$  (1.1) and the interaction is then written in terms of the products of theta functions of the type appearing in eq. (3.33).

We refer the reader to [5] for the details and now go on to the generalization when fluxes of both oblique and diagonal forms are present. Such magnetic fluxes do not respect the factorization and hence involve the wavefunctions written in terms of the general Riemann theta functions.

## 4 General tori and ‘oblique’ fluxes

Let us now consider the more general case where the  $2n$ -dimensional torus is not necessarily factorizable. A generic flat  $2n$ -dimensional torus,  $T^{2n} \simeq \mathbf{C}^n / \Lambda$ , inherits a complex



structure from the covering space  $\mathbf{C}^n$ . Its geometry can hence be described in terms of a Kähler metric and complex structure as

$$\begin{aligned} ds^2 &= h_{\mu\bar{\nu}} dz^\mu d\bar{z}^\nu \\ dz^\mu &= dx^\mu + \tau_\nu^\mu dy^\nu \end{aligned} \tag{4.1}$$

where  $x^\mu, y^\mu \in (0, 1)$ ,  $\mu = 1, \dots, n$ , parametrize the  $2n$  vectors of the lattice  $\Lambda$ . The natural generalization of the Jacobi theta function (3.16) to this higher-dimensional tori is known as Riemann  $\vartheta$ -functions, as defined in eq. (A.15):

$$\vartheta \begin{bmatrix} \vec{a} \\ \vec{b} \end{bmatrix} (\vec{\nu}|\Omega) = \sum_{\vec{l} \in \mathbf{Z}^n} e^{i\pi(\vec{l}+\vec{a}) \cdot \Omega \cdot (\vec{l}+\vec{a})} e^{2\pi i(\vec{l}+\vec{a}) \cdot (\vec{\nu}+\vec{b})}. \tag{4.2}$$

As already elaborated upon earlier, in our case, although the geometry itself may be such that  $T^6$  is factorizable as in eq. (1.1), the fluxes turned on, may violate in general the factorizable structure of the tori. Indeed, the general wavefunctions for bifundamentals given in terms of basis functions (A.14)<sup>1</sup>:

$$\begin{aligned} \psi^{\vec{j}, \mathbf{N}}(\vec{z}, \Omega) &= \mathcal{N} \cdot e^{\{i\pi[\mathbf{N} \cdot \vec{z}] \cdot (\mathbf{N} \cdot \text{Im} \Omega)^{-1} \text{Im}[\mathbf{N} \cdot \vec{z}]\}} \cdot \vartheta \begin{bmatrix} \vec{j} \\ 0 \end{bmatrix} (\mathbf{N} \cdot \vec{z} | \mathbf{N} \cdot \Omega), \\ &= \mathcal{N} \cdot e^{i\pi[\mathbf{N} \cdot \vec{z}] \cdot (\text{Im} \Omega)^{-1} \cdot \text{Im} \vec{z}} \cdot \vartheta \begin{bmatrix} \vec{j} \\ 0 \end{bmatrix} (\mathbf{N} \cdot \vec{z} | \mathbf{N} \cdot \Omega), \end{aligned} \tag{4.3}$$

with  $\mathbf{N}$ 's being the intersection matrices, depend on such fluxes explicitly in terms of its modular parameter argument:  $\mathbf{N}\Omega$ ; this breaks in general the factorized structure, even if the complex structure  $\Omega$  is diagonal. The explicit form of the normalization factor  $\mathcal{N}$  appearing in eq. (4.3) is given eq. (A.17). One needs to obtain an overlap integral of three basis functions of the type (4.3), in order to generalize the results of Yukawa computations given in eqs. (3.14), (3.30) - (3.33).

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<sup>1</sup>See Appendix A for more details on the properties of the wavefunctions and Section 2, as well as Appendix C for discussion on fluxes.

## 4.1 Riemann theta function identity

We now generalize eq. (3.15) to the case of general Riemann theta functions given in eq. (4.2). Explicitly, we consider the LHS of our identity to be given by an expression:

$$\vartheta \begin{bmatrix} \vec{j}_1 \\ 0 \end{bmatrix} (\vec{z}_1 | \mathbf{N}_1 \cdot \Omega) \cdot \vartheta \begin{bmatrix} \vec{j}_2 \\ 0 \end{bmatrix} (\vec{z}_2 | \mathbf{N}_2 \cdot \Omega) \quad (4.4)$$

where  $\Omega$  is an  $n \times n$  complex matrix and  $\mathbf{N}_1, \mathbf{N}_2$  are  $n \times n$  integer-valued symmetric matrices satisfying the constraints (2.17). These constraints, in turn, follow from the convergence of theta series expansion, as well as from the holomorphicity of fluxes: for instance, eq. (2.12) when  $p_{xx}$  and  $p_{yy}$  components of fluxes are zero, with  $x^i, y^i$ , ( $i = 1, 2, 3$ ) denoting the coordinates of three  $T^2$ 's in the decomposition (1.1) and (4.1). Generalization to the case when  $p_{xx}$  and  $p_{yy}$  flux components are also present is discussed in Section 2, as well as later on in subsection 4.7, and is relevant for evaluating the Yukawa couplings in models with moduli stabilization, such as the one of [27].

Initially, we also restrict ourselves to the case when  $\Omega = \tau I_n$  with  $I_n$  being a  $n \times n$  identity matrix, implying that the geometric structure is factorized as in eq. (1.1). However, in Section 4.6, we generalize the results further to the case when  $\Omega$  is an arbitrary matrix satisfying the  $F_{(2,0)} = 0$  supersymmetry condition, as given in eqs. (2.7) and (2.8). Then, using the definition of Riemann  $\vartheta$ -functions (4.2), the expression in eq. (4.4) can be expanded as:

$$\vartheta \begin{bmatrix} \vec{j}_1 \\ 0 \end{bmatrix} (\vec{z}_1 | \mathbf{N}_1 \tau) \cdot \vartheta \begin{bmatrix} \vec{j}_2 \\ 0 \end{bmatrix} (\vec{z}_2 | \mathbf{N}_2 \tau) = \sum_{\vec{l}_1, \vec{l}_2 \in \mathbf{Z}^n} e^{\pi i (\vec{j}_1 + \vec{l}_1) \cdot \mathbf{N}_1 \tau \cdot (\vec{j}_1 + \vec{l}_1)} e^{2\pi i (\vec{j}_1 + \vec{l}_1) \cdot \vec{z}_1} \cdot e^{\pi i (\vec{j}_2 + \vec{l}_2) \cdot \mathbf{N}_2 \tau \cdot (\vec{j}_2 + \vec{l}_2)} e^{2\pi i (\vec{j}_2 + \vec{l}_2) \cdot \vec{z}_2}. \quad (4.5)$$

Now, by defining  $2n$ -dimensional vectors:

$$(\vec{\mathbf{j}} + \vec{\mathbf{l}}) = \begin{pmatrix} \vec{j}_1 + \vec{l}_1 \\ \vec{j}_2 + \vec{l}_2 \end{pmatrix}, \quad \vec{\mathbf{z}} = \begin{pmatrix} \vec{z}_1 \\ \vec{z}_2 \end{pmatrix}, \quad (4.6)$$

and the  $2n \times 2n$  dimensional matrix:

$$\mathbf{Q} = \begin{pmatrix} \mathbf{N}_1 \tau & 0 \\ 0 & \mathbf{N}_2 \tau \end{pmatrix}, \quad (4.7)$$

eq. (4.5) can be re-written as:

$$\vartheta \begin{bmatrix} \vec{j}_1 \\ 0 \end{bmatrix} (\vec{z}_1 | \mathbf{N}_1 \tau) \cdot \vartheta \begin{bmatrix} \vec{j}_2 \\ 0 \end{bmatrix} (\vec{z}_2 | \mathbf{N}_2 \tau) = \sum_{\vec{l} \in \mathbb{Z}^{2n}} e^{\pi i (\vec{j} + \vec{l})^T \cdot \mathbf{Q} \cdot (\vec{j} + \vec{l})} e^{2\pi i (\vec{j} + \vec{l})^T \cdot \vec{z}}. \quad (4.8)$$

Our aim in combining the terms into  $2n$  dimensional vectors and matrices is to generalize the procedure outlined in [29] to our situation, namely when two theta functions appearing in the LHS of the identity (that we propose below) carry independent modular parameter matrices  $\mathbf{N}_1 \tau$  and  $\mathbf{N}_2 \tau$ , which generally may not commute. Note that the results of [29] are insufficient to give such an identity as they involve theta functions whose modular parameter matrices are proportional to each other and therefore commute. In order to proceed, we note that using a transformation matrix:

$$T = \begin{pmatrix} 1 & 1 \\ \alpha \mathbf{N}_1^{-1} & -\alpha \mathbf{N}_2^{-1} \end{pmatrix}, \quad (4.9)$$

$$T^T = \begin{pmatrix} 1 & \mathbf{N}_1^{-1} \alpha^T \\ 1 & -\mathbf{N}_2^{-1} \alpha^T \end{pmatrix}, \quad (4.10)$$

and

$$T^{-1} = (\mathbf{N}_1^{-1} + \mathbf{N}_2^{-1})^{-1} \begin{pmatrix} \mathbf{N}_2^{-1} & \alpha^{-1} \\ \mathbf{N}_1^{-1} & -\alpha^{-1} \end{pmatrix}, \quad (4.11)$$

with  $\alpha$  being an arbitrary matrix (to be determined below) and  $\mathbf{N}_1, \mathbf{N}_2$  being real symmetric matrices, due to the condition (2.17) (for  $\Omega = \tau I_n$ ), one obtains:

$$\mathbf{Q}' \equiv T \cdot \mathbf{Q} \cdot T^T = \begin{pmatrix} (\mathbf{N}_1 + \mathbf{N}_2) \tau & 0 \\ 0 & \alpha (\mathbf{N}_1^{-1} + \mathbf{N}_2^{-1}) \tau \alpha^T \end{pmatrix}. \quad (4.12)$$

In the following we also make use of the identities:

$$(\mathbf{N}_1^{-1} + \mathbf{N}_2^{-1}) = \mathbf{N}_1^{-1} (\mathbf{N}_1 + \mathbf{N}_2) \mathbf{N}_2^{-1} = \mathbf{N}_2^{-1} (\mathbf{N}_1 + \mathbf{N}_2) \mathbf{N}_1^{-1} \quad (4.13)$$

and

$$(\mathbf{N}_1^{-1} + \mathbf{N}_2^{-1})^{-1} = \mathbf{N}_1 (\mathbf{N}_1 + \mathbf{N}_2)^{-1} \mathbf{N}_2 = \mathbf{N}_2 (\mathbf{N}_1 + \mathbf{N}_2)^{-1} \mathbf{N}_1 \quad (4.14)$$

in simplifying certain expressions.

The transformation matrix  $T$  defined above is used to transform the product of theta functions in the LHS of eq. (4.8), in terms of a finite sum over another product of theta's,

now with modular parameter matrices:  $(\mathbf{N}_1 + \mathbf{N}_2)\tau$  and  $\alpha(\mathbf{N}_1^{-1} + \mathbf{N}_2^{-1})\tau\alpha^T$ . Explicitly, we can write the terms appearing in the exponents in the RHS of eq. (4.8) as:

$$(\vec{\mathbf{j}} + \vec{\mathbf{l}})^T \cdot \mathbf{Q} \cdot (\vec{\mathbf{j}} + \vec{\mathbf{l}}) = (\vec{\mathbf{j}} + \vec{\mathbf{l}})^T \cdot (T^{-1}T) \cdot \mathbf{Q} \cdot (T^T(T^{-1})^T) \cdot (\vec{\mathbf{j}} + \vec{\mathbf{l}}) \quad (4.15)$$

$$(\vec{\mathbf{j}} + \vec{\mathbf{l}})^T \cdot \vec{\mathbf{z}} = (\vec{\mathbf{j}} + \vec{\mathbf{l}})^T (T^{-1}T) \cdot \vec{\mathbf{z}}. \quad (4.16)$$

Then using:

$$T \cdot \vec{\mathbf{z}} = \begin{pmatrix} \vec{z}_1 + \vec{z}_2 \\ \alpha \mathbf{N}_1^{-1} \vec{z}_1 - \alpha \mathbf{N}_2^{-1} \vec{z}_2 \end{pmatrix}, \quad (4.17)$$

$$(\vec{\mathbf{j}} + \vec{\mathbf{l}})^T T^{-1} = \begin{pmatrix} (\vec{j}_1 + \vec{l}_1)(\mathbf{N}_1^{-1} + \mathbf{N}_2^{-1})^{-1} \mathbf{N}_2^{-1} + (\vec{j}_2 + \vec{l}_2)(\mathbf{N}_1^{-1} + \mathbf{N}_2^{-1})^{-1} \mathbf{N}_1^{-1} \\ ((\vec{j}_1 + \vec{l}_1) - (\vec{j}_2 + \vec{l}_2))(\mathbf{N}_1^{-1} + \mathbf{N}_2^{-1})^{-1} \alpha^{-1} \end{pmatrix}^T, \quad (4.18)$$

and

$$(T^{-1})^T (\vec{\mathbf{j}} + \vec{\mathbf{l}}) = \begin{pmatrix} \mathbf{N}_2^{-1}(\mathbf{N}_1^{-1} + \mathbf{N}_2^{-1})^{-1}(\vec{j}_1 + \vec{l}_1) + \mathbf{N}_1^{-1}(\mathbf{N}_1^{-1} + \mathbf{N}_2^{-1})^{-1}(\vec{j}_2 + \vec{l}_2) \\ (\alpha^{-1})^T (\mathbf{N}_1^{-1} + \mathbf{N}_2^{-1})^{-1}[(\vec{j}_1 + \vec{l}_1) - (\vec{j}_2 + \vec{l}_2)] \end{pmatrix} \quad (4.19)$$

we can re-write eq. (4.5) as,

$$\begin{aligned} & \vartheta \begin{bmatrix} \vec{j}_1 \\ 0 \end{bmatrix} (\vec{z}_1 | N_1 \tau) \cdot \vartheta \begin{bmatrix} \vec{j}_2 \\ 0 \end{bmatrix} (\vec{z}_2 | N_2 \tau) = \\ & \sum_{\vec{l}_1, \vec{l}_2 \in \mathbf{Z}^n} e^{\pi i \{ [((\vec{j}_1 + \vec{l}_1) \mathbf{N}_1 + (\vec{j}_2 + \vec{l}_2) \mathbf{N}_2) (\mathbf{N}_1 + \mathbf{N}_2)^{-1}] \cdot (\mathbf{N}_1 + \mathbf{N}_2) \tau \cdot [(\mathbf{N}_1 + \mathbf{N}_2)^{-1} (\mathbf{N}_1 (\vec{j}_1 + \vec{l}_1) + \mathbf{N}_2 (\vec{j}_2 + \vec{l}_2))] \}]} \\ & \quad \times e^{2\pi i \{ [((\vec{j}_1 + \vec{l}_1) \mathbf{N}_1 + (\vec{j}_2 + \vec{l}_2) \mathbf{N}_2) (\mathbf{N}_1 + \mathbf{N}_2)^{-1}] \cdot [\vec{\mathbf{z}}_1 + \vec{\mathbf{z}}_2] \} } \times \\ & e^{\pi i \{ [((\vec{j}_1 - \vec{j}_2) + (\vec{l}_1 - \vec{l}_2)) \mathbf{N}_1 (\mathbf{N}_1 + \mathbf{N}_2)^{-1} \mathbf{N}_2 \alpha^{-1}] \cdot [\alpha (\mathbf{N}_1^{-1} (\mathbf{N}_1 + \mathbf{N}_2) \mathbf{N}_2^{-1}) \tau \alpha^T] \cdot [(\alpha^{-1})^T \mathbf{N}_2 (\mathbf{N}_1 + \mathbf{N}_2)^{-1} \mathbf{N}_1 ((\vec{j}_1 - \vec{j}_2) + (\vec{l}_1 - \vec{l}_2))] \}]} \\ & \quad \times e^{2\pi i \{ [((\vec{j}_1 - \vec{j}_2) + (\vec{l}_1 - \vec{l}_2)) \mathbf{N}_1 (\mathbf{N}_1 + \mathbf{N}_2)^{-1} \mathbf{N}_2 \alpha^{-1}] \cdot [\alpha \mathbf{N}_1^{-1} \vec{z}_1 - \alpha \mathbf{N}_2^{-1} \vec{z}_2] \} }. \end{aligned} \quad (4.20)$$

Now, to reexpress the above series expansion in terms of a sum over theta functions with modular parameter matrices:  $\mathbf{N}_1 + \mathbf{N}_2$  and  $\alpha(\mathbf{N}_1^{-1} + \mathbf{N}_2^{-1})\alpha^T$ , we rearrange the series in eq. (4.20) in terms of new summation variables  $\vec{l}_3, \vec{l}_4, \vec{m}$ , whose values and ranges will be assigned later. In the course of going from eq. (4.20) to (4.22) below, however, one needs to make sure that such redefined variables are integers. This requirement constrains the matrix  $\alpha$  whose ‘minimal’ solution will be taken to be

$$\alpha = (\det \mathbf{N}_1 \det \mathbf{N}_2) I. \quad (4.21)$$

We will later on discuss also the possibility of choosing other forms of  $\alpha$  and show that such choices lead to the cyclicity of the superpotential coefficients, as in eqs. (3.33), (3.34).

Using eq. (4.21), the RHS of eq. (4.20) takes the form:

$$\begin{aligned}
& \sum_{\vec{l}_3, \vec{l}_4 \in \mathbf{Z}^n} \sum_{\vec{m}} e^{\pi i[(\vec{j}_1 \mathbf{N}_1 + \vec{j}_2 \mathbf{N}_2 + \vec{m} \mathbf{N}_1)(\mathbf{N}_1 + \mathbf{N}_2)^{-1} + \vec{l}_3] \cdot (\mathbf{N}_1 + \mathbf{N}_2) \tau \cdot [(\mathbf{N}_1 + \mathbf{N}_2)^{-1}(\mathbf{N}_1 \vec{j}_1 + \mathbf{N}_2 \vec{j}_2 + \mathbf{N}_1 \vec{m}) + \vec{l}_3]} \\
& \cdot e^{2\pi i[(\vec{j}_1 \mathbf{N}_1 + \vec{j}_2 \mathbf{N}_2 + \vec{m} \mathbf{N}_1)(\mathbf{N}_1 + \mathbf{N}_2)^{-1} + \vec{l}_3] \cdot [\vec{z}_1 + \vec{z}_2]} \times \\
& e^{\pi i[(\vec{j}_1 - \vec{j}_2 + \vec{m}) \frac{\mathbf{N}_1(\mathbf{N}_1 + \mathbf{N}_2)^{-1} \mathbf{N}_2}{\det \mathbf{N}_1 \det \mathbf{N}_2} + \vec{l}_4] \cdot [(\det \mathbf{N}_1 \det \mathbf{N}_2)^2 \mathbf{N}_1^{-1}(\mathbf{N}_1 + \mathbf{N}_2) \mathbf{N}_2^{-1}] \tau \cdot [\frac{\mathbf{N}_2(\mathbf{N}_1 + \mathbf{N}_2)^{-1} \mathbf{N}_1}{\det \mathbf{N}_1 \det \mathbf{N}_2}(\vec{j}_1 - \vec{j}_2 + \vec{m}) + \vec{l}_4]} \\
& \cdot e^{2\pi i[(\vec{j}_1 - \vec{j}_2 + \vec{m}) \frac{\mathbf{N}_1(\mathbf{N}_1 + \mathbf{N}_2)^{-1} \mathbf{N}_2}{\det \mathbf{N}_1 \det \mathbf{N}_2} + \vec{l}_4] \cdot \det \mathbf{N}_1 \det \mathbf{N}_2 [\mathbf{N}_1^{-1} \vec{z}_1 - \mathbf{N}_2^{-1} \vec{z}_2]}. \tag{4.22}
\end{aligned}$$

This series can now be reexpressed in terms of a finite sum over product of generalized theta functions given in eq. (4.2), leading to a generalization of the identity (3.15) to:

$$\begin{aligned}
& \vartheta \begin{bmatrix} \vec{j}_1 \\ 0 \end{bmatrix} (\vec{z}_1 | \mathbf{N}_1 \tau) \cdot \vartheta \begin{bmatrix} \vec{j}_2 \\ 0 \end{bmatrix} (\vec{z}_2 | \mathbf{N}_2 \tau) = \\
& \sum_{\vec{m}} \vartheta \begin{bmatrix} (\vec{j}_1 \mathbf{N}_1 + \vec{j}_2 \mathbf{N}_2 + \vec{m} \cdot \mathbf{N}_1)(\mathbf{N}_1 + \mathbf{N}_2)^{-1} \\ 0 \end{bmatrix} (\vec{z}_1 + \vec{z}_2 | (\mathbf{N}_1 + \mathbf{N}_2) \tau) \times \\
& \vartheta \begin{bmatrix} [(\vec{j}_1 - \vec{j}_2) + \vec{m}] \frac{\mathbf{N}_1(\mathbf{N}_1 + \mathbf{N}_2)^{-1} \mathbf{N}_2}{\det \mathbf{N}_1 \det \mathbf{N}_2} \\ 0 \end{bmatrix} \\
& ((\det \mathbf{N}_1 \det \mathbf{N}_2)(\mathbf{N}_1^{-1} \vec{z}_1 - \mathbf{N}_2^{-1} \vec{z}_2) | (\det \mathbf{N}_1 \det \mathbf{N}_2)^2 (\mathbf{N}_1^{-1}(\mathbf{N}_1 + \mathbf{N}_2) \mathbf{N}_2^{-1}) \tau), \tag{4.23}
\end{aligned}$$

where  $\vec{m} = \sum_i m_i \vec{e}_i$  are all vectors generated by the basis vectors  $\vec{e}_i$ :

$$\begin{pmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ \cdot \\ \cdot \\ 0 \end{pmatrix} \quad \text{etc.}, \tag{4.24}$$

and lied within the unit-cell defined by the new basis vectors:

$$\vec{e}' = \vec{e}(\det \mathbf{N}_1 \det \mathbf{N}_2)(\mathbf{N}_1^{-1}(\mathbf{N}_1 + \mathbf{N}_2) \mathbf{N}_2^{-1}). \tag{4.25}$$

The above identity already assumes the form  $\Omega = \tau I_n$  for the complex structure of  $T^{2n}$ . As mentioned already, in subsection 4.6 below, we make further generalization to

include arbitrary complex structure  $\Omega$  as well. Also, note that, due to the identities (4.13) and (4.14), the theta functions appearing in the RHS of eq. (4.23) satisfy the constraint (A.16) with respect to their own arguments.

## 4.2 Proof of the identity

We now show the equality of the series expansions (4.20) and (4.22) to establish the identity eq. (4.23). We also show that matrix  $\alpha$  needs to be chosen as in eq. (4.21) for showing the equality of eqs. (4.20) and (4.22) for the case when  $\det \mathbf{N}_1$  and  $\det \mathbf{N}_2$  are relatively prime. In other cases  $\alpha$  can be chosen as the least common multiple of  $\det \mathbf{N}_1$  and  $\det \mathbf{N}_2$ . Here we assume them to be relatively prime, while the remaining cases can be worked out in a similar fashion.

We now follow an exercise similar to the one in Section 3.3, to show that series in eqs. (4.20) and (4.22) precisely match with  $\vec{m}$  restricted to be an integer, provided  $\alpha$  is given by eq. (4.21).

1. When  $\vec{l}_1 = \vec{l}_2$  in eq. (4.20), we have:

$$(\vec{l}_1 \mathbf{N}_1 + \vec{l}_2 \mathbf{N}_2)(\mathbf{N}_1 + \mathbf{N}_2)^{-1} = \vec{l}_2 \quad (4.26)$$

and

$$(\vec{l}_1 - \vec{l}_2) \mathbf{N}_1 (\mathbf{N}_1 + \mathbf{N}_2)^{-1} \mathbf{N}_2 \alpha^{-1} = 0 \quad (4.27)$$

These terms are exactly same if we consider the series given in eq. (4.22) for the values  $\vec{l}_3 (\equiv \vec{l}_2)$ ,  $\vec{l}_4 = 0$  and  $\vec{m} = 0$ , irrespective of the choice for the matrix  $\alpha$ .

2. In order to see the restriction on the matrix  $\alpha$ , one needs to understand how the nonzero integers  $\vec{l}_4 \neq 0$  in eq. (4.22) are generated from the terms in eq. (4.20). In other words, one needs to make sure that

$$(\vec{l}_1 - \vec{l}_2) \mathbf{N}_1 (\mathbf{N}_1 + \mathbf{N}_2)^{-1} \mathbf{N}_2 \alpha^{-1} \equiv \vec{l}_4 \quad (4.28)$$

is an integer. This in turn is possible only if  $\vec{l}_1$  is of the form:

$$\vec{l}_1 = \vec{l}_2 + \vec{l}_4 \alpha \mathbf{N}_2^{-1} (\mathbf{N}_1 + \mathbf{N}_2) \mathbf{N}_1^{-1}. \quad (4.29)$$

However, since  $\vec{l}_4$ ,  $\mathbf{N}_1$ ,  $\mathbf{N}_2$ , take integral values, the RHS in eq. (4.29) is an integer only if  $\alpha(\mathbf{N}_1^{-1} + \mathbf{N}_2^{-1})$  is an integer. In other words, for  $\det \mathbf{N}_1$  and  $\det \mathbf{N}_2$  relatively

prime,  $\alpha$  needs to be of the form:

$$\alpha = (\det \mathbf{N}_1 \det \mathbf{N}_2)P. \quad (4.30)$$

with  $P$  being an arbitrary invertible integer matrix. ‘Minimal’ choice also demands  $\det P = 1$ , otherwise  $\vec{l}_4$  will not span over *all* integers. Then, since  $P$  is invertible, it is fixed to be the identity matrix. We have therefore established the restriction on  $\alpha$  as in eq. (4.21). At the same time, we have also proved that the series in eqs. (4.20) and (4.22) precisely match for  $\vec{m} = 0$  provided  $\vec{l}_2 + \det \mathbf{N}_1 \det \mathbf{N}_2 \vec{l}_4 \mathbf{N}_2^{-1}$  is identified with  $\vec{l}_3$  in eq.(4.22). Note that  $(\det \mathbf{N}_1 \det \mathbf{N}_2) \mathbf{N}_2^{-1}$  is also integer valued and ensures that such an identification with  $\vec{l}_3$  holds.

3. On the other hand, When  $\vec{l}_1 = \vec{l}_2 + \vec{m}$  in eq. (4.20), we end up with terms like:

$$(\vec{l}_1 \mathbf{N}_1 + \vec{l}_2 \mathbf{N}_2)(\mathbf{N}_1 + \mathbf{N}_2)^{-1} = \vec{l}_2 + \vec{m} \cdot \mathbf{N}_1 (\mathbf{N}_1 + \mathbf{N}_2)^{-1} \quad (4.31)$$

and

$$(\vec{l}_1 - \vec{l}_2) \frac{\mathbf{N}_1 (\mathbf{N}_1 + \mathbf{N}_2)^{-1} \mathbf{N}_2}{\det \mathbf{N}_1 \det \mathbf{N}_2} = \vec{m} \frac{\mathbf{N}_1 (\mathbf{N}_1 + \mathbf{N}_2)^{-1} \mathbf{N}_2}{\det \mathbf{N}_1 \det \mathbf{N}_2} \quad (4.32)$$

These terms can also be obtained in the series (4.22), for the following values of the variables:  $\vec{l}_3 (\equiv \vec{l}_2)$ ,  $\vec{l}_4 = 0$ ,  $\vec{m}$  arbitrary. However, as seen above in eqs. (4.28), (4.29), the sum over  $\vec{m}$  is finite due to the fact that

$$\vec{l}_1 - \vec{l}_2 = \vec{m} = \vec{L} \det \mathbf{N}_1 \det \mathbf{N}_2 \mathbf{N}_2^{-1} (\mathbf{N}_1 + \mathbf{N}_2) \mathbf{N}_1^{-1}, \quad (4.33)$$

for  $\vec{L}$  arbitrary integers, contributes to the values of  $\vec{l}_4$  in the RHS of eq. (4.22) by an amount  $\vec{L}$ , while setting  $\vec{m}$  to zero,  $\vec{l}_3$  is identified with  $\vec{l}_2 + \det \mathbf{N}_1 \det \mathbf{N}_2 \vec{L} \mathbf{N}_2^{-1}$ . In other words, we have shown that the sum over  $\vec{m}$  in (4.22) is over all integrally defined vectors in the unit cell generated by the basis elements:

$$\vec{e}^j = \vec{e} \det \mathbf{N}_1 \det \mathbf{N}_2 \mathbf{N}_2^{-1} (\mathbf{N}_1 + \mathbf{N}_2) \mathbf{N}_1^{-1} \quad (4.34)$$

with  $\vec{e}$  being the elements of the canonical basis (4.24).

We have therefore proved that identity eq. (4.23) holds by explicitly showing a one to one correspondence between the series in eqs. (4.20) and (4.22).

### 4.3 Yukawa expressions for oblique fluxes

We now use the wavefunctions given in eqs. (4.3) and (4.2), to obtain the expression of Yukawa interactions when oblique fluxes, specified by intersection matrices

$$\mathbf{N}_1 = F_a - F_b, \quad \mathbf{N}_2 = F_c - F_a, \quad \mathbf{N}_3 = F_c - F_b. \quad (4.35)$$

are turned on along branes  $a$ ,  $b$  and  $c$ . As already mentioned, in eq. (2.15),  $\mathbf{N}_1$ ,  $\mathbf{N}_2$  and  $\mathbf{N}_3$  are all real symmetric matrices (in the absence of components  $p_{xx}$ ,  $p_{yy}$ ) and in addition the complex structure matrix is chosen to be proportional to the identity:  $\tau I_n$ , with  $\tau$  complex. We then have:

$$\begin{aligned} \psi^{\vec{i}, \mathbf{N}_1}(\vec{z}, \boldsymbol{\Omega} = \tau I_n) \cdot \psi^{\vec{j}, \mathbf{N}_2}(\vec{z}, \boldsymbol{\Omega} = \tau I_n) &= (2^{\frac{n}{2}}) (Vol(T^{2n}))^{-1} (|\det \mathbf{N}_1| \cdot |\det \mathbf{N}_2| (Im\tau)^6)^{\frac{1}{4}} \\ &\times e^{i\pi \mathbf{N}_3 \cdot \vec{z} Im \tau / Im \tau \vartheta} \begin{bmatrix} \vec{i} \\ 0 \end{bmatrix} (\mathbf{N}_1 \cdot \vec{z} | \mathbf{N}_1 \cdot \tau) \cdot \vartheta \begin{bmatrix} \vec{j} \\ 0 \end{bmatrix} (\mathbf{N}_2 \cdot \vec{z} | \mathbf{N}_2 \cdot \tau). \end{aligned} \quad (4.36)$$

Using the Riemann theta identity derived earlier in eq. (4.23), eq. (4.36) can be rewritten as:

$$\begin{aligned} \psi^{\vec{i}, \mathbf{N}_1}(\vec{z}) \cdot \psi^{\vec{j}, \mathbf{N}_2}(\vec{z}) &= \sum_{\vec{m}} (2^{\frac{n}{2}})^{\frac{1}{2}} (Vol(T^{2n}))^{-\frac{1}{2}} \left[ \frac{(|\det \mathbf{N}_1| \cdot |\det \mathbf{N}_2| (Im\tau)^3)}{|\det \mathbf{N}_3|} \right]^{\frac{1}{4}} \times \\ \psi^{(\vec{i}\mathbf{N}_1 + \vec{j}\mathbf{N}_2 + \vec{m}\mathbf{N}_1) \cdot \mathbf{N}_3^{-1}, \mathbf{N}_3}(\vec{z}) \cdot \vartheta &\begin{bmatrix} [(\vec{i} - \vec{j}) + \vec{m}] \frac{\mathbf{N}_1 \mathbf{N}_3^{-1} \mathbf{N}_2}{\det \mathbf{N}_1 \det \mathbf{N}_2} \\ 0 \end{bmatrix} \\ &(0 | (\det \mathbf{N}_1 \det \mathbf{N}_2)^2 (\mathbf{N}_1^{-1} \mathbf{N}_3 \mathbf{N}_2^{-1}) \tau). \end{aligned} \quad (4.37)$$

Note that the integrality condition (A.16) is maintained by  $\psi^{(\vec{i}\mathbf{N}_1 + \vec{j}\mathbf{N}_2 + \vec{m}\mathbf{N}_1) \cdot \mathbf{N}_3^{-1}, \mathbf{N}_3}(\vec{z})$  appearing in the RHS of the above equation, since the expression

$$\left[ (\vec{i}\mathbf{N}_1 + \vec{j}\mathbf{N}_2 + \vec{m}\mathbf{N}_1) \mathbf{N}_3^{-1} \right] \cdot \mathbf{N}_3 \quad (4.38)$$

is always an integer. On the other hand, the sum  $\vec{m}$  in eq. (4.37) is over the integers inside the cell generated by the lattice vectors in eq. (4.34) and total number of them is given by the volume of this compact space. The size of the cell, i.e., its volume matches with those in eq. (3.29) and (3.30) for the  $T^2$  case which is just the number,  $N_3 = I_{cb}$  in eq. (3.29), of chiral states for brane intersection  $bc$ . However, the situation is different for  $T^{2n}$ ,  $n > 1$ . This becomes clear by observing that the size of the cell given in eq. (4.34) is bigger than the number of states ( $\vec{k}$ ) in the intersection  $\mathbf{N}_3$  between the branes  $b$  and



$c$  by a factor  $\det(\det \mathbf{N}_1 \det \mathbf{N}_2 \mathbf{N}_2^{-1} \mathbf{N}_1^{-1})$ . This factor, on the other hand, for  $T^2$  is unity. We therefore notice that the sum  $\vec{m}$  is over many more terms<sup>2</sup> than the actual number of states ( $\vec{k}$ ) in the intersection  $\mathbf{N}_3$  between the branes  $b$  and  $c$ .

The extra factor of terms appearing in eq. (4.37) can be explained by noticing that the sum over terms in eqs. (4.37) and (4.39) is over the states  $\psi^{(\vec{i}\mathbf{N}_1 + \vec{j}\mathbf{N}_2 + \vec{m}\mathbf{N}_1) \cdot \mathbf{N}_3^{-1}, \mathbf{N}_3}(\vec{z})$  that are inside the cell in eq. (4.34) and contribute to the Yukawa coupling by the orthogonality relation eq. (A.18). As any state (with more details given in the subsection-4.4)  $\vec{k}$ , satisfying integrality conditions such as (A.16) is defined only upto the integer lattice shifts, one therefore has appearance of the same states inside the volume of lattice (4.34), multiple times. In other words, for any given state, in the RHS of eqs. (4.37), all those integer vector ( $\vec{m}$ ) shifts also contribute to the sum which satisfy the integrality condition for  $\vec{m}\mathbf{N}_1\mathbf{N}_3^{-1}$  inside the cell (4.34). Explicit solution of this condition is presented later on in section 4.4 in eq. (4.45).

Then, as in the  $T^2$  case, orthonormality of wavefunctions (A.18), implies that the Yukawa coupling, whose explicit form is given in section 4.4, can be ‘formally’ written in a form :

$$Y_{ijk} = g\sigma_{abc} \left(2^{\frac{n}{2}}\right)^{\frac{1}{2}} \left(\text{Vol}(T^{2n})\right)^{-\frac{1}{2}} \left[\frac{(|\det \mathbf{N}_1| \cdot |\det \mathbf{N}_2| (Im\tau)^3)}{|\det \mathbf{N}_3|}\right]^{\frac{1}{4}} \times \sum_{\vec{m} \in \{\vec{e}^i\}} \delta_{\vec{k}, \mathbf{N}_3^{-1}(\mathbf{N}_1 \vec{i} + \mathbf{N}_2 \vec{j} + \mathbf{N}_1 \vec{m})} \\ \times \vartheta \left[ \begin{array}{c} [(\vec{i} - \vec{j}) + \vec{m}] \frac{\mathbf{N}_1 \mathbf{N}_3^{-1} \mathbf{N}_2}{\det \mathbf{N}_1 \det \mathbf{N}_2} \\ 0 \end{array} \right] (0 | (\det \mathbf{N}_1 \det \mathbf{N}_2)^2 (\mathbf{N}_1^{-1} \mathbf{N}_3 \mathbf{N}_2^{-1}) \tau), \quad (4.39)$$

where by the summation index  $\vec{m} \in \{\vec{e}^i\}$ , one means to sum over all integer points inside the lattice generated by  $\vec{e}_1, \vec{e}_2 \cdots \vec{e}_n$  in eq. (4.34) and the Kronecker delta is to identify all the states  $\vec{k}$  upto integer shifts.

The above expression reduces in the case of  $T^2$  flux compactification to eq. (3.32), since the Kronecker delta constraint has a unique solution in such a situation. To compare the two expressions, note that the indices  $i, j, k$  in the factorized case are scaled with respect to the one of general tori, by the factors  $\frac{1}{N_1}$ ,  $\frac{1}{N_2}$  and  $\frac{1}{N_3}$ , respectively. Then, the Kronecker delta constraint in eq. (4.39) precisely matches with the one in eq. (3.31). In the case of general tori, however, the constraint implies that the interaction terms involve the states

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<sup>2</sup>We thank R. Russo and S. Sciuto for invaluable suggestions on this point.

which satisfy the equation

$$\mathbf{N}_3 \vec{k} = (\mathbf{N}_1 \vec{i} + \mathbf{N}_2 \vec{j} + \mathbf{N}_1 \vec{m}) \quad (4.40)$$

among the vectors  $\mathbf{N}_1 \vec{i}$ ,  $\mathbf{N}_2 \vec{j}$ ,  $\mathbf{N}_3 \vec{k}$  for  $\vec{m}$  integers inside the unit cell given in eq. (4.25) and corresponding states  $\vec{k}$  are only defined upto integer lattice shifts. We now find all such solutions of the lattice shifts in the next subsection and present the explicit answer for the Yukawa coupling for general tori.

#### 4.4 Explicit Yukawa coupling expressions

In this subsection we now present the set of terms that contribute to eqs. (4.37) and (4.39). In order to clarify the situation we analyze the correspondence between the chiral multiplet families of states such as the ones appearing in eq. (4.38) and the fluxes along the branes. Our discussion is restricted to  $\mathbf{N}$  being real symmetric matrices, due to the imposition of the Riemann conditions (2.17) for the special complex structure  $\Omega = \tau I_n$  under discussion.

For a given pair of brane-stacks with intersection matrix  $\mathbf{N}$ , the condition eq. (A.16) that a state  $\hat{i}$  needs to satisfy is  $N \cdot \hat{i} = integer$ . The solution of this condition is:  $\hat{i} = \mathbf{N}^{-1} \vec{e}$ , with  $\vec{e}$  being the integer basis vectors in an  $n$ -dimensional space as given in eq. (4.24). The states are therefore generated by the set of  $n$  vectors:  $\hat{i}_i = \vec{e}_i \mathbf{N}^{-1}$ , with subscript  $i = 1, 2 \dots n$  and are  $det(\mathbf{N})$  in number, namely those which are inside the cell generated by  $\vec{e}_i$ 's. Here and in following we also keep in mind that all the chiral multiplet states that we are discussing, are defined only upto the shift by integer lattice vectors  $\vec{e}_i$ 's.

To give an example: for  $n = 2$  (corresponding to  $T^4$ ), with

$$\mathbf{N} = \begin{pmatrix} \alpha & \gamma \\ \gamma & \beta \end{pmatrix}, \quad (4.41)$$

we have the basis vectors for generating the states:

$$\hat{i}_1 = \frac{1}{(\alpha\beta - \gamma^2)} \begin{pmatrix} \beta \\ -\gamma \end{pmatrix}, \quad \hat{i}_2 = \frac{1}{(\alpha\beta - \gamma^2)} \begin{pmatrix} -\gamma \\ \alpha \end{pmatrix}. \quad (4.42)$$

To obtain the degeneracy count, we note that for the above example we have:

$$\vec{e}_1 = \alpha \vec{i}_1 + \gamma \vec{i}_2,$$

$$\vec{e}_2 = \gamma \vec{i}_1 + \beta \vec{i}_2. \quad (4.43)$$

The number of independent states inside the cell with lattice vectors  $\vec{e}_1$  and  $\vec{e}_2$  is then the determinant of the above transformation which is  $\det \mathbf{N}$ . A generic state appearing in eq. (4.40) then has a form:

$$\vec{i} = m_1 \vec{i}_1 + m_2 \vec{i}_2, \quad \vec{j} = n_1 \vec{j}_1 + n_2 \vec{j}_2, \quad \vec{k} = p_1 \vec{k}_1 + p_2 \vec{k}_2. \quad (4.44)$$

with  $\vec{j}_i, \vec{k}_i$  defined in a similar way as in eq. (4.42) with respect to the corresponding intersection matrices. Also, integers  $m_i, n_i, p_i$  label the states of a chiral family in a given brane stack.

We now go on to give explicit solution for the vector  $\vec{m}$  that contribute to the sum of terms in Yukawa coupling expressions (4.37) and (4.39), namely those inside the cell defined in eq. (4.34). The size of the cell, namely the number of states that it contains is equal to  $\det(\det \mathbf{N}_1 \det \mathbf{N}_2 \mathbf{N}_2^{-1} (\mathbf{N}_1 + \mathbf{N}_2) \mathbf{N}_1^{-1})$ , as stated earlier. In a situation with  $2 \times 2$  matrices, for example, it is  $\det \mathbf{N}_1 \det \mathbf{N}_2 \det \mathbf{N}_3$ . For illustration purposes we restrict ourselves to the discussion with  $2 \times 2$  matrices. However, all the results we write below are valid for other situations as well.

Now, restricting to this  $2 \times 2$  case for the simplicity of discussion, we write all possible solutions for  $\vec{m}$  that provide integer solutions for  $\vec{m} \mathbf{N}_1 \mathbf{N}_3^{-1}$ , as appearing in the definition of states in eqs. (4.37), (4.38), and show that they are  $\det \mathbf{N}_1 \det \mathbf{N}_2$  in number. So that the degeneracy of the state matches with  $\det \mathbf{N}_1 \det \mathbf{N}_2 \det \mathbf{N}_3$  given in the last paragraph. To compare, note that for a diagonal flux situation, as in section-3, we have  $m = n_3$  as a single solution of an analogous condition  $mn_1 n_3^{-1} = \text{integer}$ , corresponding to the state degeneracy which is  $n_3$ .

The integer solutions for  $\vec{m} \mathbf{N}_1 \mathbf{N}_3^{-1}$  are:

$$\vec{m} = \vec{p} \det \mathbf{N}_1 \mathbf{N}_3 \mathbf{N}_1^{-1} + \vec{\tilde{p}} \det \mathbf{N}_2 \mathbf{N}_3 \mathbf{N}_2^{-1}, \quad (4.45)$$

where  $\vec{p}$  is all integer vectors within a cell generated by  $\vec{e} \det \mathbf{N}_2 \mathbf{N}_2^{-1}$  and  $\vec{\tilde{p}}$  is all integer vectors within a cell generated by  $\vec{e} \det \mathbf{N}_1 \mathbf{N}_1^{-1}$ . It is easy to see that  $\vec{m}$  satisfies  $\vec{m} \mathbf{N}_1 \mathbf{N}_3^{-1} = \text{integer}$  (by making use of  $\mathbf{N}_1 = \mathbf{N}_3 - \mathbf{N}_2$ ). Together, for every solution of the first term in  $\vec{m}$  we have  $\det \mathbf{N}_1$  solution for the second term and this goes on for  $\det \mathbf{N}_2$  number of terms from the first term. So that total degeneracy of such  $\vec{m}$  is  $\det \mathbf{N}_1 \det \mathbf{N}_2$ , as stated earlier.

About the states:  $\vec{m}$  given in eq. (4.45) defines a periodic set, in the same way as for the  $T^2$  case  $m = n_3$  defines the periodic set of states in the RHS of eqs. (3.30) and (3.31). There the states are explicitly given as  $k = (0), (n_1/n_3), (2n_1/n_3), \dots [(n_3 - 1)n_1/n_3]$  with a periodicity  $n_3$  for this series. Various states inside the cell (4.34) can also be found using eq. (4.40) and making use of the condition:  $\mathbf{N}_1 = \mathbf{N}_3 - \mathbf{N}_2$  as: (also the fact that any state is defined upto integer vectors). The states are:

$$\vec{k} = \vec{p} \det \mathbf{N}_1 \mathbf{N}_3^{-1} + \vec{\tilde{p}} \det \mathbf{N}_2 \mathbf{N}_3^{-1} \text{ etc.} \quad (4.46)$$

and the state degeneracy is  $\det \mathbf{N}_1 \det \mathbf{N}_2 \det \mathbf{N}_3$ .

The Yukawa coupling can now be written in an explicit form given by a sum of  $\det \mathbf{N}_1 \det \mathbf{N}_2$  number of terms, which can be read off from eq. (4.37) directly, with  $\vec{m}$  replaced by

$$\vec{m} + \vec{p} \det \mathbf{N}_1 \mathbf{N}_3 \mathbf{N}_1^{-1} + \vec{\tilde{p}} \det \mathbf{N}_2 \mathbf{N}_3 \mathbf{N}_2^{-1} \quad (4.47)$$

and now such  $\vec{m}$  are the unique solutions of eq. (4.40) where all other solutions defined upto the shifts in  $\vec{m}$  by  $\vec{p} \det \mathbf{N}_1 \mathbf{N}_3 \mathbf{N}_1^{-1} + \vec{\tilde{p}} \det \mathbf{N}_2 \mathbf{N}_3 \mathbf{N}_2^{-1}$  have been identified.

Eq. (4.39) now reads as:

$$Y_{ijk} = g \sigma_{abc} \left(2^{\frac{n}{2}}\right)^{\frac{1}{2}} \left(\text{Vol}(T^{2n})\right)^{-\frac{1}{2}} \left[ \frac{(|\det \mathbf{N}_1| \cdot |\det \mathbf{N}_2| (Im\tau)^3)}{|\det \mathbf{N}_3|} \right]^{\frac{1}{4}} \times \sum_{\vec{p}, \vec{\tilde{p}}} \times \vartheta \left[ \begin{array}{c} \left[ \{(\vec{i} - \vec{j}) + (\vec{k} \mathbf{N}_3 - \vec{i} \mathbf{N}_1 - \vec{j} \mathbf{N}_2) \mathbf{N}_1^{-1}\} \frac{\mathbf{N}_1 \mathbf{N}_3^{-1} \mathbf{N}_2}{\det \mathbf{N}_1 \det \mathbf{N}_2} + (\vec{p} \frac{\mathbf{N}_2}{\det \mathbf{N}_2} + \vec{\tilde{p}} \frac{\mathbf{N}_1}{\det \mathbf{N}_1}) \right] \\ 0 \\ (0 | (\det \mathbf{N}_1 \det \mathbf{N}_2)^2 (\mathbf{N}_1^{-1} \mathbf{N}_3 \mathbf{N}_2^{-1} \tau), \end{array} \right] \quad (4.48)$$

or equivalently:

$$Y_{ijk} = g \sigma_{abc} \left(2^{\frac{n}{2}}\right)^{\frac{1}{2}} \left(\text{Vol}(T^{2n})\right)^{-\frac{1}{2}} \left[ \frac{(|\det \mathbf{N}_1| \cdot |\det \mathbf{N}_2| (Im\tau)^3)}{|\det \mathbf{N}_3|} \right]^{\frac{1}{4}} \times \sum_{\vec{p}, \vec{\tilde{p}}} \times \vartheta \left[ \begin{array}{c} \left[ (-\vec{j} + \vec{k}) \frac{\mathbf{N}_2}{\det \mathbf{N}_1 \det \mathbf{N}_2} + (\vec{p} \frac{\mathbf{N}_2}{\det \mathbf{N}_2} + \vec{\tilde{p}} \frac{\mathbf{N}_1}{\det \mathbf{N}_1}) \right] \\ 0 \\ (0 | (\det \mathbf{N}_1 \det \mathbf{N}_2)^2 (\mathbf{N}_1^{-1} \mathbf{N}_3 \mathbf{N}_2^{-1} \tau). \end{array} \right] \quad (4.49)$$

Note that the sum over  $\vec{m}$  is now broken into sum over  $\vec{p}$  and  $\vec{\tilde{p}}$ . We end this discussion by reminding ourselves once again that  $\vec{p}$  runs over all the states inside the cell generated by  $\vec{e}_1 \det \mathbf{N}_2 \mathbf{N}_2^{-1}$  and  $\vec{e}_2 \det \mathbf{N}_2 \mathbf{N}_2^{-1}$ . Similarly  $\vec{\tilde{p}}$  runs over all the states inside the cell generated by  $\vec{e}_1 \det \mathbf{N}_1 \mathbf{N}_1^{-1}$  and  $\vec{e}_2 \det \mathbf{N}_1 \mathbf{N}_1^{-1}$ .

We now present two explicit examples, one for the oblique situation and the other for the commuting diagonal fluxes. We show that our answer for the diagonal flux is identical to the one for the diagonal yukawa coupling expression given in [5] for  $T^{2n}$ . In fact this holds for any set of fluxes with  $\mathbf{N}_1, \mathbf{N}_2, \mathbf{N}_3$  diagonal. On the other hand, we also show that the set of terms given above in eqs. (4.48) and (4.49) can also be summed up in a number of cases, for the oblique cases as well.

### Example : Oblique flux

For the oblique case, by taking two noncommuting matrices:

$$\mathbf{N}_1 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad \mathbf{N}_2 = \begin{pmatrix} 1 & \\ & 2 \end{pmatrix}, \quad (4.50)$$

we have:

$$(\det \mathbf{N}_1) \mathbf{N}_1^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \quad (\det \mathbf{N}_2) \mathbf{N}_2^{-1} = \begin{pmatrix} 2 & \\ & 1 \end{pmatrix}. \quad (4.51)$$

The set of integer points inside the cell generated by  $\vec{e}_1 \det \mathbf{N}_2 \mathbf{N}_2^{-1} = (2, 0)$  and  $\vec{e}_2 \det \mathbf{N}_2 \mathbf{N}_2^{-1} = (0, 1)$ , are:  $(0, 0)$  and  $(1, 0)$ , as  $\det(\det \mathbf{N}_2 \mathbf{N}_2^{-1}) = 2$ . The set of integer points inside the cell generated by  $\vec{e}_1 \det \mathbf{N}_1 \mathbf{N}_1^{-1} = (2, -1)$  and  $\vec{e}_2 \det \mathbf{N}_1 \mathbf{N}_1^{-1} = (-1, 2)$ , are :  $(0, 0)$ ,  $(1, 0)$  and  $(0, 1)$ , as  $\det(\det \mathbf{N}_1 \mathbf{N}_1^{-1}) = 3$ .<sup>3</sup>

Now, to illustrate our method, we concentrate on finding a particular Yukawa interaction among states:  $\vec{i} = \vec{j} = \vec{k} = (0, 0)$ . This particular Yukawa now has the form, making use of Eq. (4.48) as:

$$Y_{000} = g\sigma_{000} \left(2^{\frac{n}{2}}\right)^{\frac{1}{2}} \left(\text{Vol}(T^{2n})\right)^{-\frac{1}{2}} \left[ \frac{(|\det \mathbf{N}_1| \cdot |\det \mathbf{N}_2| (Im\tau)^3)}{|\det \mathbf{N}_3|} \right]^{\frac{1}{4}} \times \sum_{\vec{p}, \vec{p}'} \vartheta \left[ \begin{array}{c} [(\vec{p} \frac{\mathbf{N}_2}{\det \mathbf{N}_2} + \vec{p}' \frac{\mathbf{N}_1}{\det \mathbf{N}_1})] \\ 0 \end{array} \right] (0 | (\det \mathbf{N}_1 \det \mathbf{N}_2)^2 (\mathbf{N}_1^{-1} \mathbf{N}_3 \mathbf{N}_2^{-1} \tau)),$$

---

<sup>3</sup>Another example with mixed eigenvalues for the matrix  $\mathbf{N}_1$  can be constructed by exchanging the off-diagonal and diagonal entries in eq. (4.50) for  $\mathbf{N}_1$ . Such an example will be relevant for the situation discussed in later sections where intersection matrices with both positive and negative eigenvalues are discussed.

To see what terms in  $\vec{p}$  and  $\vec{p}$  dependent arguments appear in the sum, we write down all the possibilities that arise from the combinations:

$$\left(\vec{p}\frac{\mathbf{N}_2}{\det\mathbf{N}_2} + \vec{p}\frac{\mathbf{N}_1}{\det\mathbf{N}_1}\right) = \vec{p}\begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} + \vec{p}\frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \quad (4.52)$$

with  $\vec{p} = (0, 0), (1, 0)$  and  $\vec{p} = (0, 0), (0, 1), (1, 0)$ . All the six possibilities then imply that in Theta function we get the following explicit sum:

$$\left(\vartheta\begin{bmatrix} [(0, 0)] \\ 0 \end{bmatrix} + \vartheta\begin{bmatrix} [(\frac{1}{2}, 0)] \\ 0 \end{bmatrix} + \vartheta\begin{bmatrix} [(\frac{2}{3}, \frac{1}{3})] \\ 0 \end{bmatrix} + \vartheta\begin{bmatrix} [(\frac{1}{3}, \frac{2}{3})] \\ 0 \end{bmatrix} + \vartheta\begin{bmatrix} [(\frac{1}{6}, \frac{1}{3})] \\ 0 \end{bmatrix} + \vartheta\begin{bmatrix} [(\frac{5}{6}, \frac{2}{3})] \\ 0 \end{bmatrix}\right) (0|(\det\mathbf{N}_1 \det\mathbf{N}_2)^2(\mathbf{N}_1^{-1}\mathbf{N}_3\mathbf{N}_2^{-1}\tau)) \quad (4.53)$$

where a common modular parameter argument of the all the six Theta terms have been written outside of the bracket for saving space. The integer sums of the six terms over integer  $\vec{l}$  are of the forms:

$$\sum_{\vec{l}} e^{i\vec{l}+(q_1, q_2)|(\det\mathbf{N}_1 \det\mathbf{N}_2)^2(\mathbf{N}_1^{-1}\mathbf{N}_3\mathbf{N}_2^{-1}\tau)|\vec{l}+(q_1, q_2)} \quad (4.54)$$

with  $\vec{l} + (q_1, q_2)$  given explicitly as:

$$\vec{l} + (0, 0), \vec{l} + \left(\frac{1}{2}, 0\right), \vec{l} + \left(\frac{2}{3}, \frac{1}{3}\right), \vec{l} + \left(\frac{1}{3}, \frac{2}{3}\right), \vec{l} + \left(\frac{1}{6}, \frac{1}{3}\right), \vec{l} + \left(\frac{5}{6}, \frac{2}{3}\right), \quad (4.55)$$

for the six terms in eq. (4.53). It can also be seen that we can write them as:

$$\begin{pmatrix} l_1 \\ l_2 \end{pmatrix} + \begin{pmatrix} \frac{m}{2} + \frac{2n}{3} \\ \frac{n}{3} \end{pmatrix} \equiv \begin{pmatrix} l_1 \\ l_2 \end{pmatrix} + \frac{1}{6} \begin{pmatrix} 3 & 4 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix} \quad (4.56)$$

with  $m = 0, 1$  and  $n = 0, 1, 2$ . Now, using the inverse of the matrix

$$P = \frac{1}{6} \begin{pmatrix} 3 & 4 \\ 0 & 2 \end{pmatrix}, \quad (4.57)$$

appearing in eq. (4.56):

$$P^{-1} = \begin{pmatrix} 2 & -4 \\ 0 & 3 \end{pmatrix}, \quad (4.58)$$

we can write eq. (4.56) as:

$$\frac{1}{6} \begin{pmatrix} 3 & 4 \\ 0 & 2 \end{pmatrix} \left[ \begin{pmatrix} 2l_1 - 4l_2 \\ 3l_2 \end{pmatrix} + \begin{pmatrix} m \\ n \end{pmatrix} \right] \quad (4.59)$$

with  $m = 0, 1$  and  $n = 0, 1, 2$ .

It can now be seen that as  $l_1, l_2$  vary over all integers, and  $m = 0, 1$  and  $n = 0, 1, 2$ , then the combination of terms in the big square bracket in eq. (4.59) also span over ALL integers. As a result we are able to take the factor of matrix P out by summing over all the six terms, while reducing the six terms in eq.(4.53) to one. The net result is then the arguement of theta function modifies by the factor:

$$(\det \mathbf{N}_1 \det \mathbf{N}_2)^2 (\mathbf{N}_1^{-1} \mathbf{N}_3 \mathbf{N}_2^{-1} \tau) \rightarrow P^T (\det \mathbf{N}_1 \det \mathbf{N}_2)^2 (\mathbf{N}_1^{-1} \mathbf{N}_3 \mathbf{N}_2^{-1} \tau) P \quad (4.60)$$

and final answer for Yukawa coupling is:

$$Y_{000} = g\sigma_{000} \left(2^{\frac{n}{2}}\right)^{\frac{1}{2}} (Vol(T^{2n}))^{-\frac{1}{2}} \left[ \frac{(|\det \mathbf{N}_1| \cdot |\det \mathbf{N}_2| (Im\tau)^3)}{|\det \mathbf{N}_3|} \right]^{\frac{1}{4}} \times \vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0 | P^T (\det \mathbf{N}_1 \det \mathbf{N}_2)^2 (\mathbf{N}_1^{-1} \mathbf{N}_3 \mathbf{N}_2^{-1} \tau) P).$$

We can similarly take care of other nonzero values  $\vec{i}, \vec{j}, \vec{k}$  etc. as well, but details are being left.

### Example : Diagonal Flux

We take another example, now with diagonal fluxes :

$$\mathbf{N}_1 = \begin{pmatrix} 2 & \\ & 3 \end{pmatrix}, \quad \mathbf{N}_2 = \begin{pmatrix} 5 & \\ & 2 \end{pmatrix}. \quad (4.61)$$

Then:

$$(\det \mathbf{N}_1) \mathbf{N}_1^{-1} = \begin{pmatrix} 3 & \\ & 2 \end{pmatrix}, \quad (\det \mathbf{N}_2) \mathbf{N}_2^{-1} = \begin{pmatrix} 2 & \\ & 5 \end{pmatrix}. \quad (4.62)$$

Set of integer points inside the cell generated by  $\vec{e}_1 \det \mathbf{N}_2 \mathbf{N}_2^{-1} = (2, 0)$  and  $\vec{e}_2 \det \mathbf{N}_2 \mathbf{N}_2^{-1} = (0, 5)$ , are:  $(0, 0), (0, 1), (0, 2), (0, 3), (0, 4), (1, 0), (1, 1), (1, 2), (1, 3), (1, 4)$ , as  $\det(\det \mathbf{N}_2 \mathbf{N}_2^{-1}) =$

10. On the other hand, set of integer points inside the cell generated by  $\vec{e}_1 \det \mathbf{N}_1 \mathbf{N}_1^{-1} = (3, 0)$  and  $\vec{e}_2 \det \mathbf{N}_1 \mathbf{N}_1^{-1} = (0, 2)$ , are:  $(0, 0), (1, 0), (0, 1), (1, 1), (2, 0), (2, 1)$ , as  $\det(\det \mathbf{N}_1 \mathbf{N}_1^{-1}) = 6$ .

We now have:

$$\vec{l} + \left( \vec{p} \frac{\mathbf{N}_2}{\det \mathbf{N}_2} + \vec{\tilde{p}} \frac{\mathbf{N}_1}{\det \mathbf{N}_1} \right) = \vec{l} + \vec{p} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{5} \end{pmatrix} + \vec{\tilde{p}} \begin{pmatrix} \frac{1}{3} \\ \frac{1}{2} \end{pmatrix}, \quad (4.63)$$

which can also be written as:

$$\vec{l} + \left( \vec{p} \frac{\mathbf{N}_2}{\det \mathbf{N}_2} + \vec{\tilde{p}} \frac{\mathbf{N}_1}{\det \mathbf{N}_1} \right) = \begin{pmatrix} l_1 \\ l_2 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ \frac{1}{5} \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} + \begin{pmatrix} \frac{1}{3} \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} \tilde{p}_1 \\ \tilde{p}_2 \end{pmatrix}, \quad (4.64)$$

with  $p_1 = 0, 1, p_2 = 0, 1, 2, 3, 4, \tilde{p}_1 = 0, 1, 2, \tilde{p}_2 = 0, 1$ .

By taking a factor of  $\frac{\mathbf{N}_1 \mathbf{N}_2}{\det \mathbf{N}_1 \det \mathbf{N}_2}$  out, the above equation can also be rewritten as:

$$\frac{\mathbf{N}_1 \mathbf{N}_2}{\det \mathbf{N}_1 \det \mathbf{N}_2} \left[ \vec{l} + \left( \vec{p} \frac{\mathbf{N}_2}{\det \mathbf{N}_2} + \vec{\tilde{p}} \frac{\mathbf{N}_1}{\det \mathbf{N}_1} \right) \right] = \begin{pmatrix} \frac{1}{6} \\ \frac{1}{10} \end{pmatrix} \left[ \begin{pmatrix} 6l_1 \\ 10l_2 \end{pmatrix} + \begin{pmatrix} 3p_1 \\ 2p_2 \end{pmatrix} + \begin{pmatrix} 2\tilde{p}_1 \\ 5\tilde{p}_2 \end{pmatrix} \right] \quad (4.65)$$

with  $p_1 = 0, 1, p_2 = 0, 1, 2, 3, 4, \tilde{p}_1 = 0, 1, 2, \tilde{p}_2 = 0, 1$ . It can again be checked explicitly that it leads to ALL integer variables inside the square bracket. The net result of summing over different terms in the diagonal case therefore is the appearance of the matrix outside the square bracket:  $\frac{\mathbf{N}_1 \mathbf{N}_2}{\det \mathbf{N}_1 \det \mathbf{N}_2}$ . When multiplying the modular parameter argument as appearing in eq. (4.48), from both left and the right, this precisely reproduces a modified modular parameter which matches with the known diagonal flux solution for Yukawa coupling in [5]. This holds for the diagonal flux in general, not restricted to the example above.

## 4.5 arbitrary- $\alpha$

The results, obtained so far in this section, are derived for a particular choice of  $\alpha$  given in the eq. (4.21). However, all the results can be re-derived for arbitrary  $\alpha$ , appearing in eq. (4.9) etc.. For the factorized case, we saw in that the Yukawa coupling expression (3.32) can be recast into a symmetric form in eq. (3.33) (apart from the prefactor), where the arguments of the Jacobi theta functions are invariant under a cyclic change:  $a \rightarrow b \rightarrow c$ . This is due to the cyclic property of the superpotential coefficients obtained by a third derivative of the superpotential  $W_{ijk}$ . The prefactor does not obey in general this



symmetry, since it depends on the wave function normalizations (Kähler metric). Here, we show a similar cyclic property in the non-factorized case, given above in the Yukawa coupling expression (4.49), by making different choices of the matrix  $\alpha$  in eq. (4.21). Note that different choices of this matrix provide equivalent expressions for the wavefunctions, and in turn Yukawa couplings, since they are related through a change of variables inside the theta sum. The  $\alpha$  matrix can be chosen appropriately so that the redefined variables in eqs. (4.29) and (4.33) are well defined integers. Below we present a few examples with different choices of  $\alpha$ , to demonstrate the cyclicity mentioned above.

Eq. (4.22), for arbitrary  $\alpha$ , can be written as:

$$\begin{aligned} & \sum_{\vec{l}_3, \vec{l}_4 \in \mathbf{Z}^n} \sum_{\vec{m}} \left( e^{\pi i [(\vec{j}_1 \mathbf{N}_1 + \vec{j}_2 \mathbf{N}_2 + \vec{m} \mathbf{N}_1)(\mathbf{N}_1 + \mathbf{N}_2)^{-1} + \vec{l}_3] \cdot (\mathbf{N}_1 + \mathbf{N}_2) \tau \cdot [(\mathbf{N}_1 + \mathbf{N}_2)^{-1} (\mathbf{N}_1 \vec{j}_1 + \mathbf{N}_2 \vec{j}_2 + \mathbf{N}_1 \vec{m}) + \vec{l}_3]} \right. \\ & \quad \left. \times e^{2\pi i [(\vec{j}_1 \mathbf{N}_1 + \vec{j}_2 \mathbf{N}_2 + \vec{m} \mathbf{N}_1)(\mathbf{N}_1 + \mathbf{N}_2)^{-1} + \vec{l}_3] \cdot [\vec{z}_1 + \vec{z}_2]} \right) \\ \times & \left( e^{\pi i [(\vec{j}_1 - \vec{j}_2 + \vec{m}) \mathbf{N}_1 (\mathbf{N}_1 + \mathbf{N}_2)^{-1} \mathbf{N}_2 \alpha^{-1} + \vec{l}_4] \cdot [\alpha \mathbf{N}_1^{-1} (\mathbf{N}_1 + \mathbf{N}_2) \mathbf{N}_2^{-1} \tau] \alpha^T \cdot [(\alpha^{-1})^T \mathbf{N}_2 (\mathbf{N}_1 + \mathbf{N}_2)^{-1} \mathbf{N}_1 (\vec{j}_1 - \vec{j}_2 + \vec{m}) + \vec{l}_4]} \right. \\ & \quad \left. \times e^{2\pi i [(\vec{j}_1 - \vec{j}_2 + \vec{m}) \mathbf{N}_1 (\mathbf{N}_1 + \mathbf{N}_2)^{-1} \mathbf{N}_2 \alpha^{-1} + \vec{l}_4] \cdot [\alpha \mathbf{N}_1^{-1} \vec{z}_1 - \mathbf{N}_2^{-1} \vec{z}_2]} \right), \end{aligned} \quad (4.66)$$

provided  $\vec{l}_4$ , defined in eq. (4.28), is an integer vector, and so is  $\vec{m}$  given in eq. (4.33). In addition the unit-cell, within which  $\vec{m}$  lie, is now defined by the basis vectors :

$$\vec{e}^j = \vec{e}^j \alpha (\mathbf{N}_1^{-1} (\mathbf{N}_1 + \mathbf{N}_2) \mathbf{N}_2^{-1}). \quad (4.67)$$

Moreover, eq. (4.23) takes the form:

$$\begin{aligned} & \vartheta \begin{bmatrix} \vec{j}_1 \\ 0 \end{bmatrix} (\vec{z}_1 | \mathbf{N}_1 \tau) \cdot \vartheta \begin{bmatrix} \vec{j}_2 \\ 0 \end{bmatrix} (\vec{z}_2 | \mathbf{N}_2 \tau) = \\ & \sum_{\vec{m}} \vartheta \begin{bmatrix} (\vec{j}_1 \mathbf{N}_1 + \vec{j}_2 \mathbf{N}_2 + \vec{m} \cdot \mathbf{N}_1)(\mathbf{N}_1 + \mathbf{N}_2)^{-1} \\ 0 \end{bmatrix} (\vec{z}_1 + \vec{z}_2 | (\mathbf{N}_1 + \mathbf{N}_2) \tau) \\ & \quad \times \vartheta \begin{bmatrix} [(\vec{j}_1 - \vec{j}_2) + \vec{m}] \mathbf{N}_1 (\mathbf{N}_1 + \mathbf{N}_2)^{-1} \mathbf{N}_2 \alpha^{-1} \\ 0 \end{bmatrix} \\ & \quad (\alpha (\mathbf{N}_1^{-1} \vec{z}_1 - \mathbf{N}_2^{-1} \vec{z}_2) | \alpha (\mathbf{N}_1^{-1} (\mathbf{N}_1 + \mathbf{N}_2) \mathbf{N}_2^{-1} \tau) \alpha^T). \end{aligned} \quad (4.68)$$

It is then easy to see, all equations from (4.36) to (4.39) go through for arbitrary  $\alpha$ , giving the following expression for the Yukawa couplings:

$$Y_{ijk} = g \sigma_{abc} \left(2^{\frac{n}{2}}\right)^{\frac{1}{2}} (Vol(T^{2n}))^{-\frac{1}{2}} \left[ \frac{(|\det \mathbf{N}_1| \cdot |\det \mathbf{N}_2| (Im \tau)^3)}{|\det \mathbf{N}_3|} \right]^{\frac{1}{4}} \times \sum_{\vec{m}}$$

$$\vartheta \begin{bmatrix} (-\vec{j} + \vec{k})\mathbf{N}_2\alpha^{-1} + \vec{m}\mathbf{N}_1\mathbf{N}_3^{-1}\mathbf{N}_2\alpha^{-1} \\ 0 \end{bmatrix} (0|\alpha(\mathbf{N}_1^{-1}\mathbf{N}_3\mathbf{N}_2^{-1}\tau)\alpha^T). \quad (4.69)$$

where the sum  $\vec{m}$  is now over all the integer solutions of  $\vec{m}\mathbf{N}_1\mathbf{N}_3^{-1}$  in the cell given in eq. (4.67). Explicit contributions to this sum, of course, will depend on the exact form of  $\alpha$ . In subsection 4.4, we have presented the case of  $\alpha = \det\mathbf{N}_1\det\mathbf{N}_2$ .

We now study how the above expression (4.69) reduces for another choice of  $\alpha$ , such as:

$$\alpha = \mathbf{N}_3^{-1}\mathbf{N}_1(\det\mathbf{N}_2.\det\mathbf{N}_3). \quad (4.70)$$

Note, for this choice of  $\alpha$ , that the degeneracy of states in the cell given in eq. (4.67) is  $\det(\det\mathbf{N}_3\det\mathbf{N}_2\mathbf{N}_2^{-1})$ . As a result, for the case of  $2 \times 2$  matrices for example, one now expects the sum over  $\vec{m}$  to run over  $\det\mathbf{N}_2\det\mathbf{N}_3$  values. Explicit solutions are now given as:

$$\vec{m} = \vec{p}\det\mathbf{N}_2\mathbf{N}_3\mathbf{N}_2^{-1} + \vec{\tilde{p}}\det\mathbf{N}_3, \quad (4.71)$$

where  $\vec{p}$  is all integer vectors within a cell generated by  $\vec{e}\det\mathbf{N}_3\mathbf{N}_3^{-1}$  and  $\vec{\tilde{p}}$  is all integer vectors within a cell generated by  $\vec{e}\det\mathbf{N}_2\mathbf{N}_2^{-1}$ . It is again easy to see that  $\vec{m}$  satisfies  $\vec{m}\mathbf{N}_1\mathbf{N}_3^{-1} = \text{integer}$  (by making use of  $\mathbf{N}_1 = \mathbf{N}_3 - \mathbf{N}_2$ ).

The characteristic of the  $\vartheta$ -function in eq. (4.69), becomes:

$$\begin{aligned} (-\vec{j} + \vec{k})\mathbf{N}_2\alpha^{-1} &= (-\vec{k}\mathbf{N}_1 + \vec{i}\mathbf{N}_1 + \vec{m}\mathbf{N}_1) \frac{\mathbf{N}_1^{-1}\mathbf{N}_3}{(\det\mathbf{N}_2.\det\mathbf{N}_3)} \\ &= \frac{(-\vec{k} + \vec{i})\mathbf{N}_3}{(\det\mathbf{N}_2.\det\mathbf{N}_3)}, \end{aligned} \quad (4.72)$$

where in the first equality we have made use of eq. (4.40). Also we have,

$$\begin{aligned} \alpha(\mathbf{N}_1^{-1}\mathbf{N}_3\mathbf{N}_2^{-1}\tau)\alpha^T &= (\mathbf{N}_3^{-1}\mathbf{N}_1)(\mathbf{N}_1^{-1}\mathbf{N}_3\mathbf{N}_2^{-1})(\mathbf{N}_1\mathbf{N}_3^{-1})\tau(\det\mathbf{N}_2.\det\mathbf{N}_3)^2 \\ &= (\mathbf{N}_2^{-1}\mathbf{N}_1\mathbf{N}_3^{-1}\tau)(\det\mathbf{N}_2.\det\mathbf{N}_3)^2. \end{aligned} \quad (4.73)$$

The Yukawa couplings then read (following the exercise performed in subsection 4.4):

$$\begin{aligned} Y_{ijk} &= g\sigma_{abc} \left(2^{\frac{n}{2}}\right)^{\frac{1}{2}} \left(\text{Vol}(T^{2n})\right)^{-\frac{1}{2}} \left[ \frac{(|\det\mathbf{N}_1| \cdot |\det\mathbf{N}_2| (Im\tau)^3)}{|\det\mathbf{N}_3|} \right]^{\frac{1}{4}} \times \\ \sum_{\vec{p}, \vec{\tilde{p}}} \vartheta &\begin{bmatrix} (-\vec{k} + \vec{i}) \frac{\mathbf{N}_3}{\det\mathbf{N}_2\det\mathbf{N}_3} + (\vec{p} \frac{\mathbf{N}_3}{\det\mathbf{N}_3} + \vec{\tilde{p}} \frac{\mathbf{N}_2}{\det\mathbf{N}_2}) \\ 0 \end{bmatrix} (0|(\det\mathbf{N}_2\det\mathbf{N}_3)^2(\mathbf{N}_2^{-1}\mathbf{N}_1\mathbf{N}_3^{-1})\tau), \end{aligned} \quad (4.74)$$

where the summation over indices  $\vec{p}$  and  $\vec{\tilde{p}}$  is explained earlier after eq. (4.71). We can also explicitly obtain the sums, as done for various examples in the last subsection.

Now, a comparison of eqs. (4.49) and (4.74) shows a symmetry between the  $\vartheta$ -function characteristics in these cases, including the summation variables  $\vec{p}$  and  $\vec{\tilde{p}}$ . It is obvious that the replacement  $\vec{i} \rightarrow \vec{j}, \vec{j} \rightarrow \vec{k}, \vec{k} \rightarrow \vec{i}$  and  $\mathbf{N}_1 \rightarrow \mathbf{N}_2, \mathbf{N}_2 \rightarrow \mathbf{N}_3, \mathbf{N}_3 \rightarrow \mathbf{N}_1$  in eq. (4.49) results eq. (4.74). We have thus established that just as in the factorized case, for oblique fluxes too, one can show the cyclicity property of the Yukawa superpotential coefficients, as naively expected.

## 4.6 General complex structure

In the previous subsections 4.1 - 4.3, we have confined ourselves to the complex structure matrix  $\Omega = \tau I_n$  for a  $2n$  dimensional torus. This implies the restriction to orthogonal tori, a solution which is already used in many phenomenologically interesting models. However, the results are easily generalized to complex structure with arbitrary  $\Omega$ . More precisely, to write down an identity generalizing eq. (4.23) one starts with the product expression given in eq. (4.4) and rescales  $\mathbf{N}_1, \mathbf{N}_2$  in eqs. (4.7) - (4.25) to  $\mathbf{N}_1\Omega/\tau, \mathbf{N}_2\Omega/\tau$ . At the same time, the matrix  $\alpha$  in eq. (4.21) is also rescaled :

$$\alpha \rightarrow \tilde{\alpha} = \det \mathbf{N}_1 \det \mathbf{N}_2 \Omega / \tau = \frac{\alpha \Omega}{\tau}. \quad (4.75)$$

Moreover, one needs to take into account that in relations such as (4.10) earlier, we have made use of the property  $\mathbf{N}^T = \mathbf{N}$ , which is true for the complex structure of the form:  $\tau I_n$ . Replacements:  $\mathbf{N}\tau \rightarrow \mathbf{N}\Omega$  are, however, to be done in the original expression.

Explicitly, under the changes mentioned, the transformation matrix  $T$  in eq. (4.9) remains unchanged, while its transposition in eq. (4.10) is now written as:

$$T^T = \begin{pmatrix} 1 & \mathbf{N}_1^{-1T} \alpha^T \\ 1 & -\mathbf{N}_2^{-1T} \alpha^T \end{pmatrix}. \quad (4.76)$$

Also, (4.11) is unchanged, whereas  $\mathbf{Q}'$  in eq. (4.12) goes over to

$$\mathbf{Q}' \equiv T \cdot \mathbf{Q} \cdot T^T = \begin{pmatrix} (\mathbf{N}_1 + \mathbf{N}_2)\Omega & 0 \\ 0 & \alpha(\mathbf{N}_1^{-1} + \mathbf{N}_2^{-1})\Omega^T \alpha^T \end{pmatrix}, \quad (4.77)$$

where we have made use of the fact that both  $(\mathbf{N}_1 + \mathbf{N}_2)\Omega$  and  $(\mathbf{N}_1^{-1} + \mathbf{N}_2^{-1})\Omega^T$  are symmetric matrices, due to the condition (2.12), with  $\mathbf{N}$  defined in eq. (2.15). Then

expressions (4.17) and (4.18) remain unchanged, while (4.19) is modified to:

$$(T^{-1})^T(\vec{\mathbf{j}} + \vec{\mathbf{l}}) = \begin{pmatrix} \mathbf{N}_2^{-1T}(\mathbf{N}_1^{-1T} + \mathbf{N}_2^{-1T})^{-1}(\vec{j}_1 + \vec{l}_1) + \mathbf{N}_1^{-1T}(\mathbf{N}_1^{-1T} + \mathbf{N}_2^{-1T})^{-1}(\vec{j}_2 + \vec{l}_2) \\ (\alpha^{-1})^T(\mathbf{N}_1^{-1T} + \mathbf{N}_2^{-1T})^{-1}[(\vec{j}_1 + \vec{l}_1) - (\vec{j}_2 + \vec{l}_2)] \end{pmatrix} \quad (4.78)$$

The identity (4.23) then takes the form:

$$\begin{aligned} \vartheta \begin{bmatrix} \vec{j}_1 \\ 0 \end{bmatrix} (\vec{z}_1 | \mathbf{N}_1 \Omega) \cdot \vartheta \begin{bmatrix} \vec{j}_2 \\ 0 \end{bmatrix} (\vec{z}_2 | \mathbf{N}_2 \Omega) = \\ \sum_{\vec{m}} \vartheta \begin{bmatrix} (\vec{j}_1 \mathbf{N}_1 + \vec{j}_2 \mathbf{N}_2 + \vec{m} \cdot \mathbf{N}_1)(\mathbf{N}_1 + \mathbf{N}_2)^{-1} \\ 0 \end{bmatrix} (\vec{z}_1 + \vec{z}_2 | (\mathbf{N}_1 + \mathbf{N}_2) \Omega) \times \\ \vartheta \begin{bmatrix} [(\vec{j}_1 - \vec{j}_2) + \vec{m}] \frac{\mathbf{N}_1(\mathbf{N}_1 + \mathbf{N}_2)^{-1} \mathbf{N}_2}{\det \mathbf{N}_1 \det \mathbf{N}_2} \\ 0 \end{bmatrix} \\ ((\det \mathbf{N}_1 \det \mathbf{N}_2)(\mathbf{N}_1^{-1} \vec{z}_1 - \mathbf{N}_2^{-1} \vec{z}_2) | (\det \mathbf{N}_1 \det \mathbf{N}_2)^2 (\mathbf{N}_1^{-1} (\mathbf{N}_1 + \mathbf{N}_2) \mathbf{N}_2^{-1} \Omega^T)), \end{aligned} \quad (4.79)$$

leading to the expression for the Yukawa interaction:

$$\begin{aligned} Y_{ijk} = \sigma_{abcg} \left(2^{\frac{n}{2}}\right)^{\frac{1}{2}} (Vol(T^{2n}))^{-\frac{1}{2}} \left[ \frac{(|\det \mathbf{N}_1| \cdot |\det \mathbf{N}_2| |\det \Omega|)}{|\det \mathbf{N}_3|} \right]^{\frac{1}{4}} \times \sum_{\vec{p}, \vec{p}} \\ \vartheta \begin{bmatrix} (-\vec{j} + \vec{k}) \frac{\mathbf{N}_2}{\det \mathbf{N}_1 \det \mathbf{N}_2} + (\vec{p} \frac{\mathbf{N}_2}{\det \mathbf{N}_2} + \vec{p} \frac{\mathbf{N}_1}{\det \mathbf{N}_1}) \\ 0 \end{bmatrix} (0 | (\det \mathbf{N}_1 \det \mathbf{N}_2)^2 (\mathbf{N}_1^{-1} \mathbf{N}_3 \mathbf{N}_2^{-1} \Omega^T)). \end{aligned} \quad (4.80)$$

We leave the rest of the details, which readers can work out.

## 4.7 Hermitian intersection matrices

In subsections 4.1, 4.2, 4.3, we have assumed that intersection matrices  $\mathbf{N}_1, \mathbf{N}_2$  etc. are real symmetric. As explained, this restriction originates from the case when fluxes  $p_{xx}, p_{yy}$  are zero and the intersection matrix  $\mathbf{N}$  is represented by the real matrix  $p_{xy}$  in eq. (2.15), which is symmetric whenever the complex structure is of the canonical form:  $\Omega = iI_d$ . Moreover, the Yukawa coupling expression was generalized nicely in the last subsection to the case of arbitrary complex structure, as well.

In this subsection we discuss the case when fluxes  $p_{xx}$  and  $p_{yy}$  are also present, in addition to those of the type  $p_{xy}$  and  $p_{yx}$ . Furthermore, all these fluxes are constrained by

the conditions (2.7) and (2.8) giving a resulting (1, 1) - form flux which can be represented by the Hermitian matrix (2.9), (2.10). We explicitly present the case of  $\Omega = iI_d$  solution ( $I_d$  :  $d$ -dimensional Identity matrix), which is particularly simple, since in this case due to constraints (B.1), the Hermitian flux has the simple final form of eq. (B.2). The generalization to arbitrary complex structure  $\Omega$  can also be done, but is left as an exercise to the reader.

Wavefunctions on  $T^6$ , as given in eq. (4.3), satisfy the following field equations (A.12) and (A.13):<sup>4</sup>

$$\bar{\partial}_i \chi_+^{ab} + (A^1 - A^2)_{\bar{z}_i} \chi_+^{ab} = 0, \quad (i = 1, 2, 3). \quad (4.81)$$

We now show that the solution for the above equation, together with proper periodicity requirements on  $T^6$ , is given by the basis elements:

$$\begin{aligned} \psi^{\vec{j}, \mathbf{N}}(\vec{z}) &= \mathcal{N}_{\vec{j}} \cdot f(z, \bar{z}) \cdot \hat{\Theta}(z, \bar{z}) \\ &= \mathcal{N}_{\vec{j}} \cdot e^{i\pi[(\mathbf{N}_{\mathbf{R}} - i\mathbf{N}_{\mathbf{I}}) \cdot \vec{z}] \cdot \text{Im} \vec{z}} \cdot \vartheta \left[ \begin{array}{c} \vec{j} \\ 0 \end{array} \right] (\mathbf{N}_{\mathbf{R}} \cdot \vec{z} | \mathbf{N}_{\mathbf{R}} \cdot iI_3) \end{aligned} \quad (4.82)$$

where  $\mathbf{N}_{\mathbf{R}}$  is a real, symmetric matrix.

The wavefunction given in eq. (4.82) satisfies the Dirac equations (4.81) for the following gauge potentials.

$$(A^1 - A^2)_{\bar{z}_j} = \left(\frac{\pi}{2}\right) z_j (\mathbf{N}_{\mathbf{R}} - i\mathbf{N}_{\mathbf{I}})_{i\bar{j}}, \quad (4.83)$$

which exactly matches with eq. (A.13) for the complex structure  $\Omega = iI_3$ . The intersection matrix is therefore given by :

$$\mathbf{N} = \mathbf{N}_{\mathbf{R}} - i\mathbf{N}_{\mathbf{I}}, \quad (4.84)$$

where we identify,

$$\mathbf{N}_{\mathbf{R}} = p_{xy}^a - p_{xy}^b, \quad \mathbf{N}_{\mathbf{I}} = p_{xx}^a - p_{xx}^b. \quad (4.85)$$

The wavefunction described in eq. (4.82) can be re-written in terms of the real coordinates  $\vec{x}$  and  $\vec{y}$  as well as matrices  $\mathbf{N}_{\mathbf{R}}$ ,  $\mathbf{N}_{\mathbf{I}}$ . By a slight abuse of notation, below, only for this subsection, we use  $\mathbf{N}_{\mathbf{R}} = p_{xy}$ ,  $\mathbf{N}_{\mathbf{I}} = p_{xx}$ , by setting  $p^b$ 's to zero in eq. (4.85) and suppressing the superscript  $a$  in  $p^a$ . Such a notational change, helps to make comparison

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<sup>4</sup>See Appendix A for details.

of the transformation rules we derive for the wavefunction written above in eq. (4.82) with general transition functions, consistent with the gauge transformations along the  $2n$  non-contractible cycles of  $T^n$ , given in [5]. These transition functions are written in equations (4.40), (4.41) of [5] for the fields that transform in fundamental representation rather than as bifundamentals. Hence, the notation changes above are meant to make the expressions consistent with the ones of [5].

The wavefunction (4.82), in the real coordinates  $\vec{x}$  and  $\vec{y}$ , then reads:

$$\begin{aligned} \psi^{\vec{j}, \mathbf{N}}(\vec{z}) &= \mathcal{N}_{\vec{j}} \cdot e^{i\pi[(x^i \cdot p_{x^i y^j} \cdot y^j) + i(-x^i \cdot p_{x^i x^j} \cdot y^j + y^i \cdot p_{x^i y^j} \cdot y^j)]} \\ &\cdot \sum_{l_i \in \mathbf{Z}^n} e^{i\pi(i)[(l_i + j_i) \cdot p_{x^i y^j} \cdot (l_j + j_j)]} e^{2i\pi[(l_i + j_i) \cdot p_{x^i y^j} \cdot (x^j + iy^j)]}. \end{aligned} \quad (4.86)$$

This expression in terms of real coordinates is useful in comparing the transformation properties of the wavefunction over  $T^6$  with the one in [5]. The transformation properties, as derived from eq. (4.82), are given by,

$$\begin{aligned} \psi^{\vec{j}, \mathbf{N}}(\vec{z} + \vec{n}) &= e^{i\pi([\mathbf{N} \cdot \vec{n}] \cdot \text{Im} \vec{z})} \cdot \psi^{\vec{j}, \mathbf{N}}(\vec{z}), \\ \psi^{\vec{j}, \mathbf{N}}(\vec{z} + i\vec{n}) &= e^{-i\pi([\mathbf{N}^t \cdot \vec{n}] \cdot \text{Re} \vec{z})} \cdot \psi^{\vec{j}, \mathbf{N}}(\vec{z}), \end{aligned} \quad (4.87)$$

provided that

- $(\mathbf{N}_{\mathbf{R}})_{i\vec{j}} \equiv p_{x^i y^j} \in \mathbf{Z}$ , i.e  $\mathbf{N}_{\mathbf{R}}$  is integrally quantized,
- $\vec{j}$  satisfies  $\vec{j} \cdot \mathbf{N}_{\mathbf{R}} \in \mathbf{Z}^n$ .

We therefore notice that the integer quantization is imposed only on the symmetric part  $\mathbf{N}_{\mathbf{R}}$  of the intersection matrix from the periodicity of the wavefunction as well. However, Dirac quantization already imposes both  $p_{xy}$  and  $p_{xx}$  to be integral for unit windings, as discussed in Section 2.

Using eq. (4.86), the expressions (4.87) can be re-written in terms of real coordinates as:

$$\psi^{\vec{j}, \mathbf{N}}(\vec{x} + \vec{n} + i\vec{y}) = e^{i\pi[n_i(p_{x^i y^j} - ip_{x^i x^j})y^j]} \cdot \psi^{\vec{j}, \mathbf{N}}(\vec{x} + i\vec{y}), \quad (4.88)$$

$$\psi^{\vec{j}, \mathbf{N}}(\vec{x} + i[\vec{y} + \vec{n}]) = e^{-i\pi[n_i(p_{x^j y^i} - ip_{x^j x^i})x^j]} \cdot \psi^{\vec{j}, \mathbf{N}}(\vec{x} + i\vec{y}). \quad (4.89)$$

In order to see that eqs. (4.88) and (4.89) are the proper transformation properties of the fermion wavefunction over  $T^6$ , let us compare them with the the transition functions eq.

(4.41) of [5] given for a fundamental representation in six real coordinates  $X_I$ ,  $I = 1, \dots, 6$ , as used in our eq. (2.1) as well. After changing variables first to the coordinates  $x^i, y^i$ ,  $i = 1, 2, 3$  and then making coordinate transformation to  $z^i, i\bar{z}^i$ , as described in Section 2, the general transition function is given by,

$$\chi(x_i, y_i) = e^{i\pi[(m_i + in_i) \cdot F_{i\bar{j}}(y^j + ix^j) + (im_i + n_i) \cdot F_{i\bar{j}}(x^j + iy^j)]}. \quad (4.90)$$

In correspondence to the transformation along the 1-cycles, the integer parameters on  $x_i$  and  $y_i$  are denoted as  $m_i$  and  $n_i$  respectively. One then has two cases:

Case -I : When  $n_i = 0$ , i.e  $\vec{x} \longrightarrow (\vec{x} + \vec{m})$ , eq. (4.90) reduces to

$$\begin{aligned} \chi(x_i, y_i) &= e^{i\pi\{[m_i \cdot F_{i\bar{j}} \cdot y^j - m_i \cdot F_{i\bar{j}} \cdot y^j] + i[m_i \cdot F_{i\bar{j}} \cdot x^j + m_i \cdot F_{i\bar{j}} \cdot x^j]\}}, \\ &= e^{2i\pi(m_i \cdot F_{i\bar{j}} \cdot y^j)}, \end{aligned} \quad (4.91)$$

where we used the hermiticity property of  $F$ . Using the expression (B.2) in eq. (4.91), we recover the transformation given in eq. (4.88).

Case -II : When  $m_i = 0$  i.e  $\vec{y} \longrightarrow (\vec{y} + \vec{n})$ , eq. (4.90) takes the form,

$$\begin{aligned} \chi(x_i, y_i) &= e^{i\pi\{[-n_i F_{i\bar{j}} x^j + n_i F_{i\bar{j}} x^j] + i[n_i F_{i\bar{j}} y^j + n_i F_{i\bar{j}} y^j]\}}, \\ &= e^{-2i\pi[n_i \cdot F_{i\bar{j}} \cdot x^j]}. \end{aligned} \quad (4.92)$$

Again, using eq. (B.2) in eq. (4.92), we reproduce the transformation (4.89).

It can also be easily seen that the basis wavefunctions given in eqs. (4.82) and (4.86) satisfy the orthonormality condition

$$\int_{T^{2n}} (\psi^{\vec{k}, \mathbf{N}})^\dagger \psi^{\vec{j}, \mathbf{N}} = \delta_{\vec{j}, \vec{k}}, \quad (4.93)$$

by fixing the normalization constant to

$$\mathcal{N}_{\vec{j}} = (2^n |\det \mathbf{N}_{\mathbf{R}}|)^{1/4} \cdot \text{Vol}(T^{2n})^{-1/2}, \quad \forall j. \quad (4.94)$$

We have therefore confirmed that the wavefunction written in (4.82) is not only a solution of the field equation, but also has the correct periodicity properties on the torus under the gauge transformation. Now, regarding the Yukawa interaction, since only  $\mathbf{N}_{\mathbf{R}}$ , which is real symmetric matrix, appears in the  $\hat{\Theta}(z, \bar{z})$  part of the wavefunction (4.82), all the theta function identities described in Sections 4.1, 4.2 hold for this new  $\hat{\Theta}(z, \bar{z})$ . Similarly,

as in the expression (4.49), the Yukawa coupling  $Y_{ijk}$  now has the following form,

$$Y_{ijk} = g\sigma_{abc} \left(2^{\frac{n}{2}}\right)^{\frac{1}{2}} (Vol(T^{2n}))^{-\frac{1}{2}} \left[ \frac{(|\det \mathbf{N}_{\mathbf{R}}^1| |\det \mathbf{N}_{\mathbf{R}}^2|)}{|\det \mathbf{N}_{\mathbf{R}}^3|} \right]^{\frac{1}{4}} \times \sum_{\vec{p}, \vec{p}'} \vartheta \left[ \begin{array}{c} (-\vec{j} + \vec{k}) \frac{\mathbf{N}_{\mathbf{R}}^2}{\det \mathbf{N}_{\mathbf{R}}^1 \det \mathbf{N}_{\mathbf{R}}^2} + (\vec{p} \frac{\mathbf{N}_{\mathbf{R}}^2}{\det \mathbf{N}_{\mathbf{R}}^2} + \vec{p}' \frac{\mathbf{N}_{\mathbf{R}}^1}{\det \mathbf{N}_{\mathbf{R}}^1}) \\ 0 \end{array} \right] (0 | (\det \mathbf{N}_{\mathbf{R}}^1 \det \mathbf{N}_{\mathbf{R}}^2)^2 (\mathbf{N}_{\mathbf{R}}^1)^{-1} \mathbf{N}_{\mathbf{R}}^3 \mathbf{N}_{\mathbf{R}}^2)^{-1} \tau) (4.95)$$

with  $\vec{p}$  running over all the states inside the cell generated by  $\vec{e}_1 \det \mathbf{N}_{\mathbf{R}}^2 \mathbf{N}_{\mathbf{R}}^2{}^{-1}$  and  $\vec{e}_2 \det \mathbf{N}_{\mathbf{R}}^2 \mathbf{N}_{\mathbf{R}}^2{}^{-1}$ . Similarly  $\vec{p}'$  runs over all the states inside the cell generated by  $\vec{e}_1 \det \mathbf{N}_{\mathbf{R}}^1 \mathbf{N}_{\mathbf{R}}^1{}^{-1}$  and  $\vec{e}_2 \det \mathbf{N}_{\mathbf{R}}^1 \mathbf{N}_{\mathbf{R}}^1{}^{-1}$ .

## 4.8 Constraints on the results in section-4 and further generalization

To summarize, in this section we have given a close form expression for the Yukawa couplings in the magnetized brane constructions, when in general both oblique and diagonal fluxes are present along the branes. However, the results of this section are somewhat restrictive, since the basis wavefunctions used for the computations are well defined only when the intersection matrices satisfy a positivity condition given in eq. (2.17) for arbitrary complex structure  $\Omega$ . A similar positivity criterion, for the case when  $p_{x^i x^j}$  and  $p_{y^i y^j}$  are nonzero, can be written using the wavefunction (4.82), as well; it implies simply the positivity of  $\mathbf{N}_{\mathbf{R}}$ .

On the other hand, in realistic string model building, one may need intersection matrices that are not necessarily positive definite. The simplest examples correspond simply to diagonal intersection matrices, having some positive and some negative elements along the diagonal. In such a factorized torus case, there is a unique prescription, to define the basis functions corresponding to the negative elements in the intersection matrix, as given in [5], consisting of taking complex conjugates of the wavefunctions for the positive elements. Such a prescription also works, in the case of oblique + diagonal fluxes, when some intersection matrices are ‘negative-definite’ rather than being positive definite. One can then take a complete complex conjugation over all the coordinates, in order to obtain a well defined wavefunction.

Such a process, however, does not work when oblique fluxes are present and intersection



matrices have mixed eigenvalues. Note that a diagonal flux of the type  $F_{z^i \bar{z}^i}$  preserves its  $(1, 1)$ -form structure, under the interchange  $z^i \rightarrow \bar{z}^i$ , required by supersymmetry. This is, however, no longer true when oblique fluxes are present, since off diagonal elements of a  $(1, 1)$ -form flux, say  $F_{z^1 \bar{z}^2}$ , does not remain of the  $(1, 1)$  form when complex conjugation is taken only along  $z^1$  or  $z^2$ .

In order to cure the problem, one needs to construct new basis functions. We present the results of our investigation in the next section, where we first restrict to the case of a  $T^4$  compactification, for simplicity. The complications arising from the oblique nature of the fluxes are manifest in the  $T^4$  example as well, though it is possible to generalize the result to the full  $T^6$ , which is discussed in Section 5.8.

## 5 Negative-chirality fermion wavefunction

As already mentioned, the basis wavefunctions given in eq. (4.3), used for deriving the Yukawa coupling expression in eq. (4.80), are constrained by the Riemann conditions (2.17), which imply in particular the positive-definiteness of the matrix  $\mathbf{NIm}\Omega$ .

Now, first restricting to  $T^4$ , we will show that the basis function (4.3) corresponds to the positive chirality spinor on  $T^4$ . On the other hand, to accommodate intersection matrices, having two eigenvalues of opposite signature, one needs to find out the basis function corresponding to negative chirality spinor. The need to use such basis functions, for intersection matrices with mixed eigenvalues, can be easily seen in the case when the  $T^4$  factorizes into  $T^2 \times T^2$  and one turns on only non-oblique (diagonal) fluxes. In this case, the intersection matrix has one positive diagonal element along the first  $T^2$  and one negative diagonal element along the second one. Good basis functions are then products of two  $T^2$  wavefunctions of opposite chiralities [5], and the total wavefunction on  $T^4$  is of negative chirality.

Our task therefore amounts to searching for the basis functions corresponding to negative chirality spinors on  $T^4$  with oblique fluxes. Search for fermion wavefunctions in the presence of arbitrary fluxes (in general oblique) has been pursued in [28]. However, the resulting wavefunctions are presented in terms of diagonalized coordinates and eigenvalues of fluxes. Any such solution is however unsuitable for the Yukawa computation, both for the purpose of extracting the selection rules of the type given in eq. (4.40), as well as

in actual evaluation, since the diagonalized coordinates become ‘stack dependent’ and inherent nonlinearities involved in the diagonalization process appear in the wavefunctions, prohibiting the derivation of Yukawa couplings in a concrete form.

In this section, we are able to write both the positive and negative chirality basis functions in a ‘unified’ fashion, by showing that all basis functions have a form similar to the one given in eq. (4.3). However, the complex structure  $\Omega$  appearing in eq. (4.3) for a positive chirality wavefunction needs to be replaced by an ‘effective’ modular parameter matrix  $\tilde{\Omega} = \hat{\Omega}\Omega$ , in order to accommodate the negative chirality wavefunctions, where  $\hat{\Omega}$  is given in terms of the elements of the intersection matrices (as explicitly obtained later). We also show that our results reduce to the ones in [5] for the case of diagonal fluxes.

First, in the next subsection we present new basis functions, relevant for the situation when the intersection matrices are neither positive nor negative definite. In a later subsection, we show how the negative chirality spinor basis functions can be identified with the positive chirality ones given in eq. (4.3), with an effective modular parameter, defined in terms of the fluxes. We verify this fact by mapping the wavefunctions into each other, as well as, by showing explicitly that the relevant field equations transform into each other through such a mapping. As a result, we are able to absorb the complications associated in the diagonalization process of the modular parameter matrix, and the final wavefunction once again has an identical form as given in eq. (4.3), however, with a flux dependent modular parameter argument.

## 5.1 Construction of the wavefunction

In this subsection, as mentioned earlier, we discuss the case of 4-tori, though  $T^6$  generalization can be analyzed in a similar manner. We first also restrict ourselves to the situation with canonical complex structure:  $\Omega = iI_2$  and  $\Omega = iI_3$  for  $T^4$  and  $T^6$  respectively, where  $I_d$  represents the  $d$ -dimensional identity matrix. The generalization to arbitrary  $\Omega$  is given in subsections 5.6 - 5.8. Now, in order to avoid the restriction to the positivity condition (2.17), we present an explicit solution of a wavefunction of negative chirality satisfying both the equations of motion, as well as the periodicity requirements on  $T^4$ .

Going back to the positive chirality wavefunctions, note that the two equations for the component  $\chi_+^1$  in eq. (A.11) (derived from the original Dirac equation (A.6)) can be simultaneously solved, since when acting on  $\chi_+^1$  with two covariant derivatives, we have:

$[D_{\bar{1}}, D_{\bar{2}}] \sim F_{\bar{1}\bar{2}}^{ab}$  and the RHS is zero, since all the  $(0, 2)$  components of the gauge fluxes are zero in order to maintain supersymmetry. The superscript  $ab$  in this relation implies that we need to take the difference of fluxes in brane stacks  $a$  and  $b$  due to the combination  $A^a - A^b$  that appears in eq. (A.11) for the bifundamental wavefunction. Same is true for the two  $\chi_+^2$  equations, since  $(2, 0)$  components of the fluxes are zero as well. On the other hand, the relevant equations for the negative chirality spinors are:

$$D_1\chi_-^2 + D_2\chi_-^1 = 0, \quad (5.1)$$

and

$$\bar{D}_2\chi_-^2 - \bar{D}_1\chi_-^1 = 0. \quad (5.2)$$

When only one of the two components  $\chi_-^{1,2}$  is excited at a time,  $\chi_-^{1,2}$  satisfy:  $\bar{D}_1\chi_-^1 = D_2\chi_-^1 = 0$  or  $D_1\chi_-^2 = \bar{D}_2\chi_-^2 = 0$ . But none of these sets of equations can be consistently solved when oblique fluxes are present, since  $[D_1, \bar{D}_2] \sim F_{1\bar{2}} \neq 0$ .

The two negative chirality components  $\chi_-^{1,2}$  therefore need to be mixed up in order to obtain a solution of the relevant Dirac equations, when oblique fluxes are present. In other words, we need to simultaneously excite both  $\chi_-^{1,2}$ . Then, taking

$$\chi_-^1 = \alpha\psi, \quad \chi_-^2 = \beta\psi, \quad (5.3)$$

equations (5.1) and (5.2) become:

$$(\beta\bar{D}_2 - \alpha\bar{D}_1)\psi = 0, \quad (5.4)$$

and

$$(\beta D_1 + \alpha D_2)\psi = 0. \quad (5.5)$$

In order for these two equations to have simultaneous solution, one obtains the condition:

$$-\alpha\beta F_{1\bar{1}}^{ab} - \alpha^2 F_{2\bar{1}}^{ab} + \beta^2 F_{1\bar{2}}^{ab} + \alpha\beta F_{2\bar{2}}^{ab} = 0, \quad (5.6)$$

where  $F_{i\bar{j}}^{ab} \equiv \mathbf{N}_{i\bar{j}}$  is again the difference of fluxes in brane stacks  $a$  and  $b$  and  $\mathbf{N}_{i\bar{j}}$  is the same hermitian intersection matrix, eq. (4.84), used in writing the positive chirality wavefunction and Yukawa couplings in eq. (4.3), and other parts of Section 4. When  $p_{x^i x^j} = 0$ , and  $\Omega = iI_3$ ,  $\mathbf{N}$  reduces to the real symmetric matrix as in eq. (2.15).

Fortunately, equation (5.6) has arbitrary solutions of the type:

$$F^{ab} \equiv \mathbf{N} \equiv \hat{N}_{1\bar{1}} \begin{pmatrix} 1 & -q \\ -q & q^2 \end{pmatrix} + \tilde{N}_{2\bar{2}} \begin{pmatrix} q^2 & q \\ q & 1 \end{pmatrix}, \quad (5.7)$$

with  $q = \frac{\beta}{\alpha}$  and  $\hat{N}_{1\bar{1}}, \tilde{N}_{2\bar{2}}$  being arbitrary integers whose notation will become clear later (see eq. (5.26) below). The RHS of the above relation is a general parameterization of a  $2 \times 2$  symmetric matrix, since the two terms can be written as

$$F^{ab} \equiv \mathbf{N} \equiv \hat{N}_{1\bar{1}} \begin{pmatrix} 1 \\ -q \end{pmatrix} \begin{pmatrix} 1 & -q \end{pmatrix} + \tilde{N}_{2\bar{2}} \begin{pmatrix} q \\ 1 \end{pmatrix} \begin{pmatrix} q & 1 \end{pmatrix}. \quad (5.8)$$

After having shown the possible existence of the solution of the type (5.3), we proceed to find the explicit form of the wavefunction  $\psi$  by applying the allowed orthogonal transformations on the wavefunction of the negative chirality fermion on a  $T^4$  which is factorized into  $T^2 \times T^2$ . To obtain the explicit form of this orthogonal transformation, we start by writing the coordinate  $T^4$  coordinate,  $X^M = z^i, \bar{z}^i$  ( $i = 1, 2$ ), in the spinor basis. We note, for the choice of Dirac Gamma matrices (in a real basis) given in eqs. (A.1), (A.2) that

$$\Gamma^M X_M = \begin{pmatrix} & \bar{z}_1 & \bar{z}_2 \\ z_1 & & \bar{z}_2 \\ z_2 & & -\bar{z}_1 \\ & z_2 & -\bar{z}_1 \end{pmatrix}, \quad (5.9)$$

with  $z_i = x_i + iy_i$  and  $\bar{z}_i = x_i - iy_i$ , ( $i = 1, 2$ ), which factorizes into  $2 \times 2$  blocks providing the basis on which  $SU(2)$ 's in the Lorentz group :  $SU(2)_L \times SU(2)_R \sim SO(1, 3)$  act. We get  $x^i$  in the spinor basis in the form of a  $2 \times 2$  matrix:

$$X_{\alpha\dot{\alpha}} = \begin{pmatrix} \bar{z}_1 & \bar{z}_2 \\ z_2 & -z_1 \end{pmatrix}. \quad (5.10)$$

Now to understand the transformation properties of the fermions on  $T^4$ , we consider the following transformations on  $X_{\alpha\dot{\alpha}}$ :

$$\begin{pmatrix} e^{i\theta_1} & 0 \\ 0 & e^{-i\theta_1} \end{pmatrix} \begin{pmatrix} \bar{z}_1 & \bar{z}_2 \\ z_2 & -z_1 \end{pmatrix} \begin{pmatrix} e^{-i\theta_2} & 0 \\ 0 & e^{-i\theta_2} \end{pmatrix} = \begin{pmatrix} e^{i(\theta_1-\theta_2)} \bar{z}_1 & e^{i(\theta_1+\theta_2)} \bar{z}_2 \\ e^{-i(\theta_1+\theta_2)} z_2 & -e^{-i(\theta_1-\theta_2)} z_1 \end{pmatrix} \quad (5.11)$$

We learn from eq. (5.11) that when  $T^4$  factorizes into  $T^2 \times T^2$ , the transformations of the positive and negative chirality fermions on the two  $T^2$ 's can be read off from the transformation rules of  $z_1$  and  $z_2$  given above. Indeed, the transformation rules for the fermions  $\psi_{\pm}^{(i)}$  on the two  $T^2$ 's, denoted by indices  $i = 1, 2$  are:

$$\begin{aligned}\psi_+^{(1)} &\longrightarrow e^{-i\frac{(\theta_1-\theta_2)}{2}}\psi_+^{(1)}; & \psi_-^{(1)} &\longrightarrow e^{i\frac{(\theta_1-\theta_2)}{2}}\psi_-^{(1)}, \\ \psi_+^{(2)} &\longrightarrow e^{-i\frac{(\theta_1+\theta_2)}{2}}\psi_+^{(2)}; & \psi_-^{(2)} &\longrightarrow e^{i\frac{(\theta_1+\theta_2)}{2}}\psi_-^{(2)}.\end{aligned}\tag{5.12}$$

In this case, as described in the section 3.1, the  $T^4$  fermion wavefunctions can be written as a direct product of the ones on two  $T^2$ 's as in eq. (3.3). We obtain the transformation of  $T^4$  wavefunctions (eq. (3.3)):

$$\begin{aligned}\Psi_+^1 &\longrightarrow e^{-i\theta_1}\Psi_+^1, & \Psi_+^2 &\longrightarrow e^{i\theta_1}\Psi_+^2, \\ \Psi_-^1 &\longrightarrow e^{i\theta_2}\Psi_-^1, & \Psi_-^2 &\longrightarrow e^{-i\theta_2}\Psi_-^2.\end{aligned}\tag{5.13}$$

It follows that a left transformation ( $\theta_1 \neq 0, \theta_2 = 0$ ) acts independently on (left handed) positive chirality wavefunctions, and a right transformation ( $\theta_1 = 0, \theta_2 \neq 0$ ) acts on the negative-chirality (right handed) wavefunctions. Now, consider the following complex transformation on vectors in spinor basis:

$$\begin{pmatrix} \bar{z}_1 & \bar{z}_2 \\ z_2 & -z_1 \end{pmatrix} \longrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \bar{z}_1 & \bar{z}_2 \\ z_2 & -z_1 \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix}\tag{5.14}$$

Case-I: For  $e = h = 1, f = g = 0, c = -b, a = d$ , i.e a left transformation results in the following orthogonal coordinate transformation,

$$z_1 \longrightarrow az_1 + b\bar{z}_2; \quad z_2 \longrightarrow az_2 - b\bar{z}_1.\tag{5.15}$$

Case-II: Similarly, for  $a = d = 1, c = b = 0, h = e, f = -g$ , i.e a right transformation leads to

$$z_1 \longrightarrow ez_1 - fz_2; \quad z_2 \longrightarrow ez_2 + fz_1.\tag{5.16}$$

In order to maintain the holomorphicity of the gauge fluxes, one therefore needs to make use of the later transformation, in order to generate a general wavefunction, starting

with the one which corresponds to the diagonal (non-oblique) flux. In addition, we need to maintain the integrality of the fluxes, as we make such orthogonal transformations. However, in our case, we do not make use of any specific form of the transformation and rather use the above analysis as a guide for writing down a general solution. We then verify the equations of motion directly, in order to confirm that the solution we propose is indeed the correct one.

## 5.2 New wavefunction

We now use the transformation (5.16) to obtain the wavefunction associated with the negative chirality fermion bifundamentals, starting with a wavefunction associated with a negative chirality spinor for a diagonal flux. In the notations of eq. (A.5), it corresponds to exciting only the negative chirality component

$$\begin{pmatrix} \Psi_-^2 \\ \Psi_-^1 \end{pmatrix} = \begin{pmatrix} \psi \\ 0 \end{pmatrix}. \quad (5.17)$$

We ignore the explicit form of  $\psi$ , except to note that after the transformation (5.16), one generates

$$\begin{pmatrix} \psi \\ 0 \end{pmatrix} \longrightarrow \begin{pmatrix} \Psi_-^2 \\ \Psi_-^1 \end{pmatrix} = \begin{pmatrix} \beta\psi \\ \alpha\psi \end{pmatrix}, \quad (5.18)$$

while  $(\Psi_+^1, \Psi_+^2)$  remain zero. In the gauge sector, such wavefunctions are parameterized in the bifundamental representations by:

$$\Psi_{ab} = \begin{pmatrix} C_{n_a} & \chi_{ab} \\ & C_{n_b} \end{pmatrix}, \quad (5.19)$$

as also given in eq. (A.9). For negative chirality components, the equations to be satisfied by the various components are: (see eq. (A.10))

$$\begin{aligned} \partial_1 \chi_-^2 + \partial_2 \chi_-^1 + (A^1 - A^2)_{z_1} \chi_-^2 + (A^1 - A^2)_{z_2} \chi_-^1 &= 0, \\ \bar{\partial}_2 \chi_-^2 - \bar{\partial}_1 \chi_-^1 + (A^1 - A^2)_{\bar{z}_2} \chi_-^2 - (A^1 - A^2)_{\bar{z}_1} \chi_-^1 &= 0. \end{aligned} \quad (5.20)$$

We now show that the solution to eqs. (5.20), together with proper periodicity requirements on  $T^4$ , is given by the basis elements:

$$\psi^{\vec{j}, \vec{\mathbf{N}}, \vec{\mathbf{N}}} = \mathcal{N} \cdot f(z, \bar{z}) \cdot \hat{\Theta}(z, \bar{z}) \quad (5.21)$$

where,

$$f(z, \bar{z}) = e^{i\pi[(\hat{\mathbf{N}}_{i\bar{j}}z_i \text{Im}z_j) - (\tilde{\mathbf{N}}_{i\bar{j}}\bar{z}_i \text{Im}\bar{z}_j)]} , \quad (5.22)$$

$$\hat{\Theta}(z, \bar{z}) = \sum_{m_1, m_2 \in \mathbf{Z}^n} e^{\pi i(i)[(m_i+j_i)\mathbf{M}_{i\bar{j}}(m_j+j_j)]} e^{2\pi i[(m_i+j_i)\hat{\mathbf{N}}_{i\bar{j}}z_j] e^{2\pi i(m_i+j_i)\tilde{\mathbf{N}}_{i\bar{j}}\bar{z}_j}} , \quad (5.23)$$

with

$$\mathbf{M}_{i\bar{j}} = \hat{\mathbf{N}}_{i\bar{j}} - \tilde{\mathbf{N}}_{i\bar{j}} \quad (5.24)$$

where both  $\hat{\mathbf{N}}$ ,  $\tilde{\mathbf{N}}$  are real, symmetric matrices, given earlier in eq. (5.7), and so also is  $\mathbf{M}$  ( $\mathbf{M}_{i\bar{j}} = \mathbf{M}_{\bar{j}i}$ ). We retain, however, both types of indices:  $i$  and  $\bar{j}$  to incorporate real as well as complex components of the (1, 1)-form fluxes  $F_{i\bar{j}}$ . Also, an extra factor of  $i$  in the exponent of  $\hat{\Theta}(z, \bar{z})$  corresponds to the fact that we are working with the canonical complex structure :  $\Omega = iI_2$  for the present example of the fermion wavefunction on  $T^4$ .

The wavefunction (5.21) satisfies the Dirac equations (5.20) for the following gauge potentials:

$$\begin{aligned} (A^1 - A^2)_{\bar{z}_1} &= (\hat{\mathbf{N}}_{1\bar{1}} + \tilde{\mathbf{N}}_{1\bar{1}})z_1 + (\hat{\mathbf{N}}_{1\bar{2}} + \tilde{\mathbf{N}}_{1\bar{2}})z_2 \\ (A^1 - A^2)_{z_2} &= (\hat{\mathbf{N}}_{1\bar{2}} + \tilde{\mathbf{N}}_{1\bar{2}})z_1 + (\hat{\mathbf{N}}_{2\bar{2}} + \tilde{\mathbf{N}}_{2\bar{2}})z_2. \end{aligned} \quad (5.25)$$

The intersection matrix  $\mathbf{N}$  is therefore given by:

$$\mathbf{N} = \hat{\mathbf{N}} + \tilde{\mathbf{N}}, \quad (5.26)$$

as appearing previously in eqs. (5.6), (5.7). Also, we have imposed the following constraints, in order to retain the holomorphicity of gauge potentials:

$$\frac{\alpha}{\beta} = \frac{-\hat{\mathbf{N}}_{1\bar{1}}}{\hat{\mathbf{N}}_{1\bar{2}}} = \frac{-\hat{\mathbf{N}}_{1\bar{2}}}{\hat{\mathbf{N}}_{2\bar{2}}} = \frac{\tilde{\mathbf{N}}_{1\bar{2}}}{\tilde{\mathbf{N}}_{1\bar{1}}} = \frac{\tilde{\mathbf{N}}_{2\bar{2}}}{\tilde{\mathbf{N}}_{1\bar{2}}} = \frac{1}{q}. \quad (5.27)$$

Note that the ratios of the matrix elements of  $\hat{\mathbf{N}}$  and  $\tilde{\mathbf{N}}$  are identical to those given in eq. (5.7). We have therefore explicitly shown that the solution given in eqs. (5.21) - (5.23) satisfies the equations of motion. The transformation properties of this wavefunction (5.21) along the four 1-cycles of  $T^4$ , are given by:

$$\begin{aligned} \psi^{\vec{j}, \hat{\mathbf{N}}, \tilde{\mathbf{N}}}(\vec{z} + \vec{n}) &= e^{i\pi([\hat{\mathbf{N}} \cdot \vec{n}] \cdot \text{Im} \vec{z} - [\tilde{\mathbf{N}} \cdot \vec{n}] \cdot \text{Im} \vec{z})} \cdot \psi^{\vec{j}, \hat{\mathbf{N}}, \tilde{\mathbf{N}}}(\vec{z}), \\ \psi^{\vec{j}, \hat{\mathbf{N}}, \tilde{\mathbf{N}}}(\vec{z} + i\vec{n}) &= e^{-i\pi([\hat{\mathbf{N}} \cdot \vec{n}] \cdot \text{Re} \vec{z} + [\tilde{\mathbf{N}} \cdot \vec{n}] \cdot \text{Re} \vec{z})} \cdot \psi^{\vec{j}, \hat{\mathbf{N}}, \tilde{\mathbf{N}}}(\vec{z}), \end{aligned} \quad (5.28)$$

provided that

- $\mathbf{N}_{\vec{j}} \equiv (\hat{\mathbf{N}} + \tilde{\mathbf{N}})_{i\vec{j}} \in \mathbf{Z}$ , i.e.  $(\hat{\mathbf{N}} + \tilde{\mathbf{N}})$  is integrally quantized,
- $\vec{j}$  satisfies:  $\vec{j} \cdot (\hat{\mathbf{N}} + \tilde{\mathbf{N}}) \in \mathbf{Z}^n$ .

We therefore notice that the integer quantization is imposed only on the intersection matrix  $\mathbf{N}$  given in eq. (5.26) and does not necessarily hold for the matrix  $\mathbf{M}$  in eq. (5.24). Explicitly, we have:

$$\begin{aligned}\mathbf{N} &= \hat{\mathbf{N}} + \tilde{\mathbf{N}} = \hat{\mathbf{N}}_{1\bar{1}} \begin{pmatrix} 1 & -q \\ -q & q^2 \end{pmatrix} + \tilde{\mathbf{N}}_{2\bar{2}} \begin{pmatrix} q^2 & q \\ q & 1 \end{pmatrix}, \\ \mathbf{M} &= \hat{\mathbf{N}} - \tilde{\mathbf{N}} = \hat{\mathbf{N}}_{1\bar{1}} \begin{pmatrix} 1 & -q \\ -q & q^2 \end{pmatrix} - \tilde{\mathbf{N}}_{2\bar{2}} \begin{pmatrix} q^2 & q \\ q & 1 \end{pmatrix},\end{aligned}\tag{5.29}$$

where the first eq. in (5.29) is identical to the solutions in eq. (5.7).

Note that the wavefunction given in eqs. (5.21), (5.22) and (5.23) is now well defined, as the series expansion in eq. (5.23) is now convergent. To show this, we note the following relation:

$$\det \mathbf{N} = -\det \mathbf{M} = \hat{\mathbf{N}}_{1\bar{1}} \tilde{\mathbf{N}}_{2\bar{2}} (1 + q^2)^2.\tag{5.30}$$

As a result, in the case when  $\det \mathbf{N}$  is negative (when  $\mathbf{N}$  has two eigenvalues of opposite signatures),  $\det \mathbf{M} > 0$ . So, the series (5.23) is now convergent when the two eigenvalues are of positive signature, since it is the quadratic part, in the summation index in theta series, that dominates in the exponent of this expansion. An overall complex conjugation will be required, for the case when two eigenvalues are negative rather than positive.

### 5.3 Normalization

Now that we have found a basis of wavefunctions, classified by the index  $j_i$  in the exponent in (5.23), we proceed to show its orthonormality. The wavefunctions described in eqs. (5.21), (5.22), (5.23) can be re-written in terms of the real coordinates  $\vec{x}$  and  $\vec{y}$  as follows:

$$\psi^{\vec{j}, \mathbf{N}, \mathbf{M}} = \mathcal{N}_{\vec{j}} \cdot e^{i\pi[\vec{x} \cdot \mathbf{N} \cdot \vec{y} + i\vec{y} \cdot \mathbf{M} \cdot \vec{y}]} \sum_{\vec{m} \in \mathbf{Z}^n} e^{\pi i(i)[(\vec{m} + \vec{j}) \cdot \mathbf{M} \cdot (\vec{m} + \vec{j})]} e^{2\pi i[(\vec{m} + \vec{j}) \cdot \mathbf{N} \cdot \vec{x} + i(\vec{m} + \vec{j}) \cdot \mathbf{M} \cdot \vec{y}]}.\tag{5.31}$$

Then the following orthonormality conditions are satisfied:

$$\int_{T^4} (\psi^{\vec{k}, \mathbf{N}, \mathbf{M}})^* \psi^{\vec{j}, \mathbf{N}, \mathbf{M}} = \delta_{\vec{j}, \vec{k}}.\tag{5.32}$$



To verify the orthogonality relation and obtain the normalization factor, we note that, in terms of the wavefunctions (5.31) we have:

$$\begin{aligned}
(\psi^{\vec{k}, \mathbf{N}, \mathbf{M}})^* \psi^{\vec{j}, \mathbf{N}, \mathbf{M}} &= \mathcal{N}_{\vec{k}} \cdot e^{-i\pi[\vec{x} \cdot \mathbf{N} \cdot \vec{y} - i\vec{y} \cdot \mathbf{M} \cdot \vec{y}]} \sum_{\vec{l} \in \mathbf{Z}^n} e^{\pi i(i)[(\vec{l} + \vec{k}) \cdot \mathbf{M} \cdot (\vec{l} + \vec{k})]} \cdot e^{-2\pi i[(\vec{l} + \vec{k}) \cdot \mathbf{N} \cdot \vec{x} - i(\vec{l} + \vec{k}) \cdot \mathbf{M} \cdot \vec{y}]} \\
&\mathcal{N}_{\vec{j}} \cdot e^{i\pi[\vec{x} \cdot \mathbf{N} \cdot \vec{y} + i\vec{y} \cdot \mathbf{M} \cdot \vec{y}]} \sum_{\vec{m} \in \mathbf{Z}^n} e^{\pi i(i)[(\vec{m} + \vec{j}) \cdot \mathbf{M} \cdot (\vec{m} + \vec{j})]} \cdot e^{2\pi i[(\vec{m} + \vec{j}) \cdot \mathbf{N} \cdot \vec{x} + i(\vec{m} + \vec{j}) \cdot \mathbf{M} \cdot \vec{y}]} \\
&= \mathcal{N}_{\vec{j}} \mathcal{N}_{\vec{k}} \cdot e^{-2\pi(\vec{y} \cdot \mathbf{M} \cdot \vec{y})} \sum_{\vec{m}, \vec{l} \in \mathbf{Z}^n} e^{\pi i(i)[(\vec{m} + \vec{j}) \cdot \mathbf{M} \cdot (\vec{m} + \vec{j})]} \cdot e^{\pi i(i)[(\vec{l} + \vec{k}) \cdot \mathbf{M} \cdot (\vec{l} + \vec{k})]} \\
&e^{2\pi i[(\vec{m} + \vec{j}) - (\vec{l} + \vec{k})] \cdot \mathbf{N} \cdot \vec{x}} \cdot e^{2\pi i(i)[(\vec{m} + \vec{j}) + (\vec{l} + \vec{k})] \cdot \mathbf{M} \cdot \vec{y}}.
\end{aligned} \tag{5.33}$$

The integration over  $\vec{x}$  in eq. (5.32) imposes the condition  $\vec{j} = \vec{k}$  and equality on the summation indices  $\vec{m} = \vec{l}$ . In particular, the condition  $\vec{j} = \vec{k}$  gives our orthogonality condition (5.32). One can now obtain the normalization factor by performing the integration:

$$\begin{aligned}
\int_0^1 d^2 \vec{y} &\left[ e^{-2\pi \vec{y} \cdot \mathbf{M} \cdot \vec{y}} \sum_{\vec{m} \in \mathbf{Z}^n} e^{-2\pi(\vec{m} + \vec{j}) \cdot \mathbf{M} \cdot (\vec{m} + \vec{j})} \cdot e^{-4\pi(\vec{m} + \vec{j}) \cdot \mathbf{M} \cdot \vec{y}} \right] \\
&= \int_0^1 d^2(\vec{y}) \left[ \sum_{\vec{m} \in \mathbf{Z}^n} e^{-2\pi((\vec{m} + \vec{j}) + \vec{y}) \cdot \mathbf{M} \cdot ((\vec{m} + \vec{j}) + \vec{y})} \right].
\end{aligned} \tag{5.34}$$

One can integrate over  $\vec{y}$ , using

$$\begin{aligned}
\int_0^1 d^2 \vec{y} \left[ \sum_{\vec{m} \in \mathbf{Z}^n} e^{-2\pi((\vec{m} + \vec{j}) + \vec{y}) \cdot \mathbf{M} \cdot ((\vec{m} + \vec{j}) + \vec{y})} \right] &= \sum_{\vec{m} \in \mathbf{Z}^n} \int_0^1 d^2 \vec{y} \left[ e^{-2\pi[(\vec{m} + \vec{j}) + \vec{y}] \cdot \mathbf{M} \cdot ((\vec{m} + \vec{j}) + \vec{y})} \right] \\
&= \int_{-\infty}^{\infty} d^2 \vec{y}' \left[ e^{-2\pi \vec{y}' \cdot \mathbf{M} \cdot \vec{y}'} \right]
\end{aligned} \tag{5.35}$$

The integration (5.35) fixes then the normalization constant to

$$\mathcal{N}_{\vec{j}} = (2|\det \mathbf{M}|)^{1/4} \cdot \text{Vol}(T^4)^{-1/2}, \quad \forall j. \tag{5.36}$$

## 5.4 Eigenfunctions of the Laplace equation

The wavefunctions (5.21) not only represent zero modes of the Dirac operator, but are also eigenfunctions of the Laplacian. In order to see this, we start with computing the Dirac operator in four dimensions. In our notations:

$$\Gamma^\mu \partial_\mu = \begin{pmatrix} \bar{\partial}_1 & \bar{\partial}_2 & & \\ \partial_1 & & \bar{\partial}_2 & \\ \partial_2 & & & -\bar{\partial}_1 \\ & \partial_2 & -\partial_1 & \end{pmatrix}, \tag{5.37}$$

which leads to

$$\begin{aligned}
(\mathcal{D})^2 &= \begin{pmatrix} \bar{D}_1 D_1 + \bar{D}_2 D_2 & & & \\ & D_1 \bar{D}_1 + \bar{D}_2 D_2 & & \\ & & D_2 \bar{D}_2 + \bar{D}_1 D_1 & \\ & & & D_1 \bar{D}_1 + D_2 \bar{D}_2 \end{pmatrix} \\
&= \Delta + \begin{pmatrix} F_{1\bar{1}} + F_{2\bar{2}} & & & \\ & -F_{1\bar{1}} + F_{2\bar{2}} & & \\ & & F_{1\bar{1}} - F_{2\bar{2}} & \\ & & & -(F_{1\bar{1}} + F_{2\bar{2}}) \end{pmatrix}. \tag{5.38}
\end{aligned}$$

The Dirac equation  $\mathcal{D}\Psi = 0$ , with  $\Psi$  given in eq. (3.3), implies that such basis functions are also eigenfunctions of the Laplacian  $\Delta$ . The question whether massless scalars exist, depends on whether some combination of fluxes appearing in eq. (5.38) vanish. Of course, their existence is guaranteed in the supersymmetric case.

## 5.5 Mapping of basis functions from positive to negative chirality

We now show that the basis for the negative chirality wavefunction, given in eqs. (5.21), (5.22), (5.23) can in fact be obtained by a mapping from the basis of the positive chirality wavefunction given in eq. (4.3). We also present the mapping between the corresponding field equations. Our mapping reduces to the ones in [5] for the case of factorized tori.

More precisely, we show that our negative chirality wavefunction, given in eqs. (5.21), (5.22), (5.23), as well as (5.31) (for a trivial modular parameter matrix :  $\Omega = iI_2$ ) is identical to the positive chirality wavefunction (4.3) for a ‘nontrivial’ (flux dependent) modular parameter matrix  $\Omega = i\hat{\Omega}$ . Explicitly,  $\hat{\Omega}$  is given in terms of the ratios ( $q$ ) of flux components. This result gives a ‘unified’ picture of all the relevant basis functions. Later on, in Section 5.7, we show that a similar mapping holds for nontrivial complex structure on  $T^4$ , by examining the equations of motion.

Let us write down explicitly the wavefunction (4.3) for complex structure with arbitrary

$\Omega (= i\hat{\Omega})$ .

$$\begin{aligned} \psi^{\vec{j}, \mathbf{N}'}(\vec{z}, \Omega) &= \mathcal{N} \cdot e^{i\pi[(\vec{x}+i\hat{\Omega}\vec{y})\cdot\mathbf{N}'\hat{\Omega}^{-1}\cdot\hat{\Omega}\vec{y}]} \cdot \sum_{\vec{m} \in \mathbf{Z}^n} e^{i\pi[(\vec{m}+\vec{j})\cdot i\mathbf{N}'\hat{\Omega}\cdot(\vec{m}+\vec{j})]} e^{2i\pi[(\vec{m}+\vec{j})(\mathbf{N}'\vec{x}+i\mathbf{N}'\hat{\Omega}\cdot\vec{y})]} \\ &\sim e^{i\pi[\vec{x}\cdot\mathbf{N}'\cdot\vec{y}+i\hat{\Omega}\vec{y}\cdot\mathbf{N}'\cdot\vec{y}]} \cdot \sum_{\vec{m} \in \mathbf{Z}^n} e^{i\pi[(\vec{m}+\vec{j})\cdot i\mathbf{N}'\hat{\Omega}\cdot(\vec{m}+\vec{j})]} e^{2i\pi[(\vec{m}+\vec{j})(\mathbf{N}'\vec{x}+i\mathbf{N}'\hat{\Omega}\cdot\vec{y})]}, \end{aligned} \quad (5.39)$$

where  $\mathbf{N}$  is changed to  $\mathbf{N}'$  to show a distinction between the two wavefunctions for the purpose of defining the mapping as given below. Next consider the negative chirality wavefunction (5.31), written in terms of real coordinates  $\vec{x}$  and  $\vec{y}$ ,

$$\psi^{\vec{j}, \mathbf{N}, \mathbf{M}} \sim e^{i\pi[\vec{x}\cdot\mathbf{N}\cdot\vec{y}+i\vec{y}\cdot\mathbf{M}\cdot\vec{y}]} \sum_{\vec{m} \in \mathbf{Z}^n} e^{\pi i(i)[(\vec{m}+\vec{j})\cdot\mathbf{M}\cdot(\vec{m}+\vec{j})]} e^{2\pi i[(\vec{m}+\vec{j})\cdot\mathbf{N}\cdot\vec{x}+i(\vec{m}+\vec{j})\cdot\mathbf{M}\cdot\vec{y}]}. \quad (5.40)$$

It is now easy to check that the above equations (5.39) and (5.40) precisely match with the following identification :

$$\begin{aligned} \mathbf{N} &= \hat{\mathbf{N}} + \tilde{\mathbf{N}} = \mathbf{N}', \\ \mathbf{M} &= \hat{\mathbf{N}} - \tilde{\mathbf{N}} = \mathbf{N}'\hat{\Omega} \Rightarrow \hat{\Omega} = \mathbf{N}^{-1}\mathbf{M}, \end{aligned} \quad (5.41)$$

with  $\Omega = i\hat{\Omega}$ , and  $\hat{\Omega}$  is a real matrix. For the  $\mathbf{N}$  and  $\mathbf{M}$ , defined in eq. (5.29),  $\mathbf{N}^{-1}$  and  $\hat{\Omega}$  are given by;

$$\mathbf{N}^{-1} = \frac{1}{(1+q^2)^2} \left[ \frac{1}{\hat{\mathbf{N}}_{1\bar{1}}} \begin{pmatrix} 1 & -q \\ -q & q^2 \end{pmatrix} + \frac{1}{\tilde{\mathbf{N}}_{2\bar{2}}} \begin{pmatrix} q^2 & q \\ q & 1 \end{pmatrix} \right], \quad (5.42)$$

$$\hat{\Omega} = \frac{1}{(1+q^2)} \begin{pmatrix} 1-q^2 & -2q \\ -2q & q^2-1 \end{pmatrix} = (\hat{\Omega})^{-1}. \quad (5.43)$$

We have therefore shown explicitly that the positive chirality basis wavefunction (4.3), known earlier in the literature, can be mapped to the negative chirality wavefunctions that we have constructed in eqs. (5.21)-(5.23), (5.31). Such a map also confirms the validity of our construction for the negative chirality basis functions, presented using basic principles, such as equations of motion as well as periodicity requirement. In fact, in the next subsection, the same mapping is also obtained through comparison of the relevant equations of motion, which further confirms our results for the construction of the basis functions. Note that for  $q = 0$  or  $q \rightarrow \infty$ , corresponding to the case when both matrices

$\mathbf{N}$  and  $\mathbf{M}$  in eq. (5.29) are diagonal, we have:

$$\hat{\Omega} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \text{or} \quad \hat{\Omega} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (5.44)$$

respectively. As a result, one reproduces the known mapping of the wavefunctions between positive and negative chirality spinors in the case when  $T^4$  is factorized into  $T^2 \times T^2$  [5].

## 5.6 Mapping the equations of motion

In order to derive a similar mapping of the equations of motion, we show below that the covariant derivative operators appearing in eqs. (A.12) for the positive chirality wavefunction, with a nontrivial complex structure ( $i\hat{\Omega}$ ), are equivalent to the derivative operators appearing in eqs. (5.4), (5.5) for the negative chirality wavefunction (with complex structure  $\Omega = iI_2$ ). The mapping of corresponding gauge potentials can also be shown in the same manner, since they have similar dependence on the complex structure as the derivative operator. Note that the complex structure appears in the wavefunctions as modular parameter matrices. We therefore reconfirm the mapping between the two wavefunctions by comparing the equations of motion as well.

We now examine the Dirac equations for both cases. For the first one, with arbitrary  $\Omega (= i\hat{\Omega})$ , we have

$$\vec{z} = \vec{x} + i\hat{\Omega}\vec{y}; \quad \vec{\bar{z}} = \vec{x} - i\hat{\Omega}\vec{y} \quad \Rightarrow \quad \vec{x} = \frac{\vec{z} + \vec{\bar{z}}}{2}; \quad \vec{y} = (\hat{\Omega})^{-1} \left( \frac{\vec{z} - \vec{\bar{z}}}{2i} \right),$$

which implies

$$\begin{aligned} \frac{\partial}{\partial z_i} &= \frac{1}{2} \left( \frac{\partial}{\partial x^i} - i(\hat{\Omega})_{ji}^{-1} \frac{\partial}{\partial y^j} \right), \\ \frac{\partial}{\partial \bar{z}_i} &= \frac{1}{2} \left( \frac{\partial}{\partial x^i} + i(\hat{\Omega})_{ji}^{-1} \frac{\partial}{\partial y^j} \right). \end{aligned} \quad (5.45)$$

Then, the Dirac equation for the positive chirality wavefunction is:

$$\bar{D}_{\bar{z}_i} \psi^{\vec{j}, \mathbf{N}'}(\vec{z}, \Omega) \equiv \frac{1}{2} \left( D_{x^i} + i(\hat{\Omega})_{ji}^{-1} D_{y^j} \right) \psi^{\vec{j}, \mathbf{N}'}(\vec{z}, \Omega) = 0, \quad i, j = 1, 2. \quad (5.46)$$

On the other hand, for the negative chirality solution (5.20), with complex structure  $\Omega = iI_2$ , the relevant derivative operators are:

$$(\beta D_1 + \alpha D_2) \psi^{\vec{j}, \mathbf{N}, \mathbf{M}} = 0; \quad (\beta \bar{D}_2 - \alpha \bar{D}_1) \psi^{\vec{j}, \mathbf{N}, \mathbf{M}} = 0. \quad (5.47)$$

These equations, using the definitions  $z^i = x^i + iy^i$ ,  $\bar{z}_i = x^i - iy^i$ , can be rewritten as:

$$\begin{aligned} \frac{1}{2} \left\{ D_{x^1} + i \left( \frac{\alpha^2 - \beta^2}{\alpha^2 + \beta^2} D_{y^1} - \frac{2\alpha\beta}{\alpha^2 + \beta^2} D_{y^2} \right) \right\} \psi^{\vec{j}, \mathbf{N}, \mathbf{M}} &= 0, \\ \frac{1}{2} \left\{ D_{x^2} + i \left( \frac{-2\alpha\beta}{\alpha^2 + \beta^2} D_{y^1} + \frac{\beta^2 - \alpha^2}{\alpha^2 + \beta^2} D_{y^2} \right) \right\} \psi^{\vec{j}, \mathbf{N}, \mathbf{M}} &= 0. \end{aligned} \quad (5.48)$$

Now using  $\frac{\beta}{\alpha} = q$  from eq. (5.27) and comparing the equations (5.46) and (5.48), one finds that they precisely match for the following complex structure:

$$(\hat{\Omega})^{-1} = \frac{1}{(1+q^2)} \begin{pmatrix} 1 - q^2 & -2q \\ -2q & q^2 - 1 \end{pmatrix}, \quad (5.49)$$

which is exactly the same as eq. (5.43). Thus, the wavefunctions as well as the Dirac equations for both cases match exactly. This mapping can be generalized further, as given in subsection 5.8 below.

## 5.7 Mapping for arbitrary complex structure $\Omega$

In this subsection, we generalize the mapping between the equations of motion associated with the positive and negative chirality wavefunction to the case of  $T^4$  compactification with arbitrary complex structure  $\Omega$ . Now, the negative chirality basis functions satisfy:

$$\begin{aligned} \frac{1}{2} \left\{ D_{x^1} + i(\Omega)_{i1}^{-1} \left( \frac{\alpha^2 - \beta^2}{\alpha^2 + \beta^2} D_{y^i} \right) - i(\Omega)_{i2}^{-1} \left( \frac{2\alpha\beta}{\alpha^2 + \beta^2} D_{y^i} \right) \right\} \psi^{\vec{j}, \mathbf{N}, \mathbf{M}} &= 0 \\ \frac{1}{2} \left\{ D_{x^2} + i(\Omega)_{i1}^{-1} \left( \frac{-2\alpha\beta}{\alpha^2 + \beta^2} D_{y^i} \right) + i(\Omega)_{i2}^{-1} \left( \frac{\beta^2 - \alpha^2}{\alpha^2 + \beta^2} D_{y^i} \right) \right\} \psi^{\vec{j}, \mathbf{N}, \mathbf{M}} &= 0, \end{aligned} \quad (5.50)$$

which can be identified with the equations satisfied by the positive chirality wavefunction with  $\tilde{\Omega} = \hat{\Omega}\Omega$ , as can be seen through the decomposition:

$$\begin{aligned} \frac{\partial}{\partial z_i} &= \frac{1}{2} \left( \frac{\partial}{\partial x^i} - i(\tilde{\Omega})_{ji}^{-1} \frac{\partial}{\partial y^j} \right), \\ \frac{\partial}{\partial \bar{z}_i} &= \frac{1}{2} \left( \frac{\partial}{\partial x^i} + i(\tilde{\Omega})_{ji}^{-1} \frac{\partial}{\partial y^j} \right). \end{aligned} \quad (5.51)$$

Thus, eq. (4.3) with  $\tilde{\Omega} = \hat{\Omega}\Omega$ , with  $\hat{\Omega}$  given in eq. (5.49), provides the negative chirality solution for arbitrary complex structure  $\Omega$ , where both ‘oblique’ and diagonal fluxes are turned on.

## 5.8 Generalization for the $T^6$ - case

In this subsection, we generalize the results obtained so far for negative chirality fermions on  $T^4$  to the more general  $T^6$  case. We only consider the wavefunctions that are well defined with two positive and one negative eigenvalues of the  $3 \times 3$  Hermitian intersection matrices, since these will complete the list of well defined wavefunctions, once complex conjugations are taken into account. For the case of  $T^6$ , the relevant equations, obtained by generalization of eqs. (5.4) and (5.5) to be examined, are:

$$(\alpha \bar{D}_1 - \beta_i \bar{D}_i) \psi = 0, \quad (5.52)$$

and

$$(\alpha D_i + \beta_i D_1) \psi = 0. \quad (5.53)$$

Note that in these equations and below, the indices  $i, j = 1, 2$  (used for the  $T^4$  with wavefunctions of positive chirality). In order for the above two equations to have simultaneous solution, one obtains the condition :

$$\alpha^2 F_{i1}^{ab} + \alpha \beta_i F_{11}^{ab} - \alpha \beta_j F_{ij}^{ab} - \beta_i \beta_j F_{1j}^{ab} = 0, \quad (5.54)$$

where  $F^{ab} \equiv \mathbf{N}$  is the difference of fluxes in brane stacks  $a$  and  $b$ . The general solution of this equation is of the following type:

$$F^{ab} \equiv \mathbf{N} \equiv \hat{N} \begin{pmatrix} 1 & -(\vec{q})^T \\ -\vec{q} & \vec{q}(\vec{q})^T \end{pmatrix} + \begin{pmatrix} (\vec{q})^T \tilde{N} \vec{q} & \vec{q}^T \tilde{N} \\ \tilde{N} \vec{q} & \tilde{N} \end{pmatrix}, \quad (5.55)$$

where  $\tilde{N}$  is a  $2 \times 2$  matrix and  $\hat{N}$  is a number. Also,  $\vec{q}$  is the two-dimensional ( $2d$ ) vector defined as:

$$\vec{q} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \quad (5.56)$$

with  $q_i = \frac{\beta_i}{\alpha}$ .

Now, after showing the possible existence of the solution by defining  $F^{ab}$  in (5.55), for the negative chirality wavefunction on  $T^6$ , we proceed to present a mapping between the equations of motion for negative chirality and positive chirality wavefunctions on  $T^6$ . As described before in section 5.6. Here also we show that the covariant derivative operators appearing in eqs. (A.12), for the positive chirality wavefunction, with a nontrivial complex

structure are equivalent to the derivative operators appearing in eqs. (5.52), (5.53) for the negative chirality wavefunction (with complex structure  $\Omega = iI_3$ ) and the corresponding gauge potentials map in the same manner.

For the positive chirality case, with arbitrary  $\Omega (= i\hat{\Omega})$  and eqs. (5.45), (5.45), the Dirac equation reads:

$$\bar{D}_{\bar{z}_\mu} \psi^{\vec{j}, \mathbf{N}'}(\vec{z}, \Omega) \equiv \frac{1}{2} \left( D_{x^\mu} + i(\hat{\Omega})_{\nu\mu}^{-1} D_{y^\nu} \right) \psi^{\vec{j}, \mathbf{N}'}(\vec{z}, \Omega) = 0, \quad \mu, \nu = 1, 2, 3. \quad (5.57)$$

On the other hand, for the negative chirality solution, with complex structure  $\Omega = iI_3$ , the relevant derivative operators, given in eqs. (5.52), (5.53), take the form:

$$\begin{aligned} \frac{1}{2} \left\{ (\alpha^2 \delta_{ij} + \beta_i \beta_j) D_{x^j} - i(2\alpha\beta_i) D_{y^1} + i(\beta_i \beta_j - \alpha^2 \delta_{ij}) D_{y^j} \right\} \psi^{\vec{j}, \mathbf{N}, \mathbf{M}} &= 0, \\ \frac{1}{2} \left\{ (\beta_i^2 + \alpha^2) D_{x^1} + i(\alpha^2 - \beta_i^2) D_{y^1} - i(2\beta_i \alpha) D_{y^i} \right\} \psi^{\vec{j}, \mathbf{N}, \mathbf{M}} &= 0. \end{aligned} \quad (5.58)$$

Now, defining new  $2 \times 2$  matrices,

$$A_{ij} = (\alpha^2 \delta_{ij} + \beta_i \beta_j), \quad B_{ij} = (\beta_i \beta_j - \alpha^2 \delta_{ij}),$$

and

$$P_i = (2\alpha\beta_i), \quad (5.59)$$

eqs. (5.58) can be re-written as:

$$\begin{aligned} \frac{1}{2} \left\{ D_{x^i} - i(A^{-1}P)_i D_{y^1} + i(A^{-1}B)_{ij} D_{y^j} \right\} \psi^{\vec{j}, \mathbf{N}, \mathbf{M}} &= 0 \\ \frac{1}{2} \left\{ D_{x^1} + i\left(\frac{\alpha^2 - \beta_i^2}{\beta_i^2 + \alpha^2}\right) D_{y^1} - i\left(\frac{2\alpha\beta_i}{\beta_i^2 + \alpha^2}\right) D_{y^i} \right\} \psi^{\vec{j}, \mathbf{N}, \mathbf{M}} &= 0. \end{aligned} \quad (5.60)$$

A comparison of equations (5.57) and (5.60) implies that they precisely match for the following complex structure:

$$\begin{aligned} (\hat{\Omega})_{11}^{-1} &= \left( \frac{\alpha^2 - \beta_i^2}{\beta_i^2 + \alpha^2} \right), \quad (\hat{\Omega})_{1i}^{-1} = (-A^{-1}P)_i, \\ (\hat{\Omega})_{i1}^{-1} &= -\left( \frac{2\alpha\beta_i}{\beta_i^2 + \alpha^2} \right), \quad (\hat{\Omega})_{ij}^{-1} = (A^{-1}B)_{ij}. \end{aligned} \quad (5.61)$$

This expression for the complex structure generalizes the one derived earlier in eq. (5.43) for the  $T^4$  case. The results are also easily generalizable to arbitrary complex structure  $\Omega$  following the discussions in subsection 5.7 for the special case of  $T^4$  (see eq. (5.51)).

## 5.9 Computation of Yukawa couplings

Now that we have derived both the fermionic and bosonic internal wavefunctions and expressed them as an orthonormal basis, we compute the Yukawa couplings using the basis wavefunctions (5.31). We also point out how the results derived below reduce to the ones in section 4.

Starting with basis functions described in eq. (5.31), for the case of the canonical complex structure  $\Omega = iI_2$  (in the  $T^4$  case), we have:

$$\begin{aligned} \psi^{\vec{i}, \mathbf{N}_1, \mathbf{M}_1}(\vec{z}) \cdot \psi^{\vec{j}, \mathbf{N}_2, \mathbf{M}_2}(\vec{z}) &= \mathcal{N}_{\vec{i}} \cdot \mathcal{N}_{\vec{j}} \cdot e^{i\pi[\vec{x} \cdot (\mathbf{N}_1 + \mathbf{N}_2) \cdot \vec{y} + i\vec{y} \cdot (\mathbf{M}_1 + \mathbf{M}_2) \cdot \vec{y}]} \\ &\cdot \sum_{\vec{l}_1, \vec{l}_2 \in \mathbf{Z}^n} e^{\pi i(i)[(\vec{l}_1 + \vec{i}) \cdot \mathbf{M}_1 \cdot (\vec{l}_1 + \vec{i}) + (\vec{l}_2 + \vec{j}) \cdot \mathbf{M}_2 \cdot (\vec{l}_2 + \vec{j})]} \\ &\cdot e^{2\pi i[(\vec{l}_1 + \vec{i}) \cdot \mathbf{N}_1 + (\vec{l}_2 + \vec{j}) \cdot \mathbf{N}_2] \cdot \vec{x}} e^{2\pi i(i)[(\vec{l}_1 + \vec{i}) \cdot \mathbf{M}_1 + (\vec{l}_2 + \vec{j}) \cdot \mathbf{M}_2] \cdot \vec{y}} \end{aligned} \quad (5.62)$$

This expression can be re-written as:

$$\begin{aligned} \psi^{\vec{i}, \mathbf{N}_1, \mathbf{M}_1}(\vec{z}) \cdot \psi^{\vec{j}, \mathbf{N}_2, \mathbf{M}_2}(\vec{z}) &= \mathcal{N}_{\vec{i}} \cdot \mathcal{N}_{\vec{j}} \cdot e^{i\pi[\vec{x} \cdot (\mathbf{N}_1 + \mathbf{N}_2) \cdot \vec{y} + i\vec{y} \cdot (\mathbf{M}_1 + \mathbf{M}_2) \cdot \vec{y}]} \\ &\cdot \sum_{\vec{l}_1, \vec{l}_2 \in \mathbf{Z}^n} e^{\pi i(i)(\vec{\mathbf{I}}^T \cdot \hat{\mathbf{Q}} \cdot \vec{\mathbf{I}})} e^{2\pi i(\vec{\mathbf{I}}^T \cdot \mathbf{Q} \cdot \vec{\mathbf{X}})} e^{2\pi i(i)(\vec{\mathbf{I}}^T \cdot \hat{\mathbf{Q}} \cdot \vec{\mathbf{Y}})}, \end{aligned} \quad (5.63)$$

where we defined the  $4d$ -vectors:

$$\vec{\mathbf{I}} = \begin{pmatrix} \vec{i} + \vec{l}_1 \\ \vec{j} + \vec{l}_2 \end{pmatrix}, \quad \vec{\mathbf{X}} = \begin{pmatrix} \vec{x} \\ \vec{x} \end{pmatrix}, \quad \vec{\mathbf{Y}} = \begin{pmatrix} \vec{y} \\ \vec{y} \end{pmatrix}, \quad (5.64)$$

and the  $4d$ -matrices:

$$\mathbf{Q} = \begin{pmatrix} \mathbf{N}_1 & 0 \\ 0 & \mathbf{N}_2 \end{pmatrix}, \quad \hat{\mathbf{Q}} = \begin{pmatrix} \mathbf{M}_1 & 0 \\ 0 & \mathbf{M}_2 \end{pmatrix}. \quad (5.65)$$

Using the transformation matrix  $T$ , defined in eq. (4.9), and eqs. (4.10)-(4.14), we explicitly write the terms appearing in the exponents in the RHS of eq. (5.63) as:

$$\begin{aligned} (\vec{\mathbf{I}})^T \cdot \hat{\mathbf{Q}} \cdot (\vec{\mathbf{I}}) &= (\vec{\mathbf{I}})^T \cdot (T^{-1}T) \cdot \hat{\mathbf{Q}} \cdot (T^T(T^{-1})^T) \cdot (\vec{\mathbf{I}}), \\ (\vec{\mathbf{I}}^T \cdot \mathbf{Q} \cdot \vec{\mathbf{X}}) &= \vec{\mathbf{I}}^T \cdot (T^{-1}T) \cdot \mathbf{Q} \cdot (T^T(T^{-1})^T) \cdot \vec{\mathbf{X}}, \\ (\vec{\mathbf{I}}^T \cdot \hat{\mathbf{Q}} \cdot \vec{\mathbf{Y}}) &= \vec{\mathbf{I}}^T \cdot (T^{-1}T) \cdot \hat{\mathbf{Q}} \cdot (T^T(T^{-1})^T) \cdot \vec{\mathbf{Y}}. \end{aligned} \quad (5.66)$$



Then using:

$$\mathbf{Q}' \equiv T \cdot \mathbf{Q} \cdot T^T = \begin{pmatrix} (\mathbf{N}_1 + \mathbf{N}_2) & 0 \\ 0 & \alpha(\mathbf{N}_1^{-1} + \mathbf{N}_2^{-1})\alpha^T \end{pmatrix}, \quad (5.67)$$

$$\hat{\mathbf{Q}}' \equiv T \cdot \hat{\mathbf{Q}} \cdot T^T = \begin{pmatrix} (\mathbf{M}_1 + \mathbf{M}_2) & (\mathbf{M}_1\mathbf{N}_1^{-1} - \mathbf{M}_2\mathbf{N}_2^{-1})\alpha^T \\ \alpha(\mathbf{N}_1^{-1}\mathbf{M}_1 - \mathbf{N}_2^{-1}\mathbf{M}_2) & \alpha(\mathbf{N}_1^{-1}\mathbf{M}_1\mathbf{N}_1^{-1} + \mathbf{N}_2^{-1}\mathbf{M}_2\mathbf{N}_2^{-1})\alpha^T \end{pmatrix},$$

$$(\vec{\mathbf{1}})^T T^{-1} = \begin{pmatrix} (\vec{i} + \vec{l}_1)(\mathbf{N}_1^{-1} + \mathbf{N}_2^{-1})^{-1}\mathbf{N}_2^{-1} + (\vec{j} + \vec{l}_2)(\mathbf{N}_1^{-1} + \mathbf{N}_2^{-1})^{-1}\mathbf{N}_1^{-1} \\ [(\vec{i} + \vec{l}_1) - (\vec{j} + \vec{l}_2)](\mathbf{N}_1^{-1} + \mathbf{N}_2^{-1})^{-1}\alpha^{-1} \end{pmatrix}^T, \quad (5.68)$$

and

$$(T^{-1})^T(\vec{\mathbf{1}}) = \begin{pmatrix} \mathbf{N}_2^{-1}(\mathbf{N}_1^{-1} + \mathbf{N}_2^{-1})^{-1}(\vec{i} + \vec{l}_1) + \mathbf{N}_1^{-1}(\mathbf{N}_1^{-1} + \mathbf{N}_2^{-1})^{-1}(\vec{j} + \vec{l}_2) \\ (\alpha^{-1})^T(\mathbf{N}_1^{-1} + \mathbf{N}_2^{-1})^{-1}[(\vec{i} + \vec{l}_1) - (\vec{j} + \vec{l}_2)] \end{pmatrix}, \quad (5.69)$$

$$(T^{-1})^T(\vec{\mathbf{X}}) = \begin{pmatrix} \vec{x} \\ 0 \end{pmatrix}; \quad (T^{-1})^T(\vec{\mathbf{Y}}) = \begin{pmatrix} \vec{y} \\ 0 \end{pmatrix}, \quad (5.70)$$

we can re-write eq. (5.63) as

$$\begin{aligned} & \psi^{\vec{i}, \mathbf{N}_1, \mathbf{M}_1}(\vec{z}) \cdot \psi^{\vec{j}, \mathbf{N}_2, \mathbf{M}_2}(\vec{z}) = \mathcal{N}_i \cdot \mathcal{N}_j \cdot e^{i\pi[\vec{x} \cdot (\mathbf{N}_1 + \mathbf{N}_2) \cdot \vec{y} + i\vec{y} \cdot (\mathbf{M}_1 + \mathbf{M}_2) \cdot \vec{y}]} \times \\ & \sum_{\vec{l}_1, \vec{l}_2 \in \mathbf{Z}^n} e^{\pi i(i) \{ [(\vec{l}_1 + \vec{i})\mathbf{N}_1 + (\vec{l}_2 + \vec{j})\mathbf{N}_2](\mathbf{N}_1 + \mathbf{N}_2)^{-1} \cdot (\mathbf{M}_1 + \mathbf{M}_2) \cdot \{(\mathbf{N}_1 + \mathbf{N}_2)^{-1}(\mathbf{N}_1(\vec{i} + \vec{l}_1) + \mathbf{N}_2(\vec{j} + \vec{l}_2))\} \}} \times \\ & e^{2\pi i(i) \{ [(\vec{l}_1 + \vec{i})\mathbf{N}_1 + (\vec{l}_2 + \vec{j})\mathbf{N}_2](\mathbf{N}_1 + \mathbf{N}_2)^{-1} \cdot (\mathbf{N}_1 + \mathbf{N}_2)\vec{x} \}} \cdot e^{2\pi i(i) \{ [(\vec{l}_1 + \vec{i})\mathbf{N}_1 + (\vec{l}_2 + \vec{j})\mathbf{N}_2](\mathbf{N}_1 + \mathbf{N}_2)^{-1} \cdot (\mathbf{M}_1 + \mathbf{M}_2)\vec{y} \}} \times \\ & e^{2\pi i(i) \{ [(\vec{i} + \vec{l}_1) - (\vec{j} + \vec{l}_2)](\mathbf{N}_1^{-1} + \mathbf{N}_2^{-1})^{-1}\alpha^{-1} \cdot \alpha(\mathbf{N}_1^{-1}\mathbf{M}_1 - \mathbf{N}_2^{-1}\mathbf{M}_2) \cdot \vec{y} \}} \times \\ & e^{\pi i(i) \{ [(\vec{l}_1 + \vec{i})\mathbf{N}_1 + (\vec{l}_2 + \vec{j})\mathbf{N}_2](\mathbf{N}_1 + \mathbf{N}_2)^{-1} \cdot (\mathbf{M}_1\mathbf{N}_1^{-1} - \mathbf{M}_2\mathbf{N}_2^{-1})\alpha^T \{ (\alpha^{-1})^T\mathbf{N}_2(\mathbf{N}_1 + \mathbf{N}_2)^{-1}\mathbf{N}_1[(\vec{i} - \vec{j}) + (\vec{l}_1 - \vec{l}_2)] \}} \}} \times \\ & e^{\pi i(i) \{ [(\vec{i} - \vec{j}) + (\vec{l}_1 - \vec{l}_2)]\mathbf{N}_1(\mathbf{N}_1 + \mathbf{N}_2)^{-1}\mathbf{N}_2\alpha^{-1} \cdot [\alpha(\mathbf{N}_1^{-1}\mathbf{M}_1 - \mathbf{N}_2^{-1}\mathbf{M}_2)](\mathbf{N}_1 + \mathbf{N}_2)^{-1}(\mathbf{N}_1(\vec{i} + \vec{l}_1) + \mathbf{N}_2(\vec{j} + \vec{l}_2)) \}} \times \\ & e^{\pi i(i) \{ [(\vec{i} - \vec{j}) + (\vec{l}_1 - \vec{l}_2)]\mathbf{N}_1(\mathbf{N}_1 + \mathbf{N}_2)^{-1}\mathbf{N}_2\alpha^{-1} \cdot [\alpha(\mathbf{N}_1^{-1}\mathbf{M}_1\mathbf{N}_1^{-1} + \mathbf{N}_2^{-1}\mathbf{M}_2\mathbf{N}_2^{-1})\alpha^T][(\alpha^{-1})^T\mathbf{N}_2(\mathbf{N}_1 + \mathbf{N}_2)^{-1}\mathbf{N}_1[(\vec{i} - \vec{j}) + (\vec{l}_1 - \vec{l}_2)]] \}} \end{aligned} \quad (5.71)$$

Now, in a similar exercise as the one performed earlier in sections 4.2, 4.3, 4.4, we rearrange the series in eq. (5.71) in terms of new summation variables  $\vec{l}_3, \vec{l}_4, \vec{m}$ , whose values and ranges are assigned as in these sections.<sup>5</sup> With the value of  $\alpha = (\det \mathbf{N}_1 \det \mathbf{N}_2)I$ ,

<sup>5</sup>For details see sections 4.1, 4.2, 4.3, 4.4.

defined in eq. (4.21), eq. (5.71) takes the form:

$$\begin{aligned}
\psi^{\vec{i}, \mathbf{N}_1, \mathbf{M}_1}(\vec{z}) \cdot \psi^{\vec{j}, \mathbf{N}_2, \mathbf{M}_2}(\vec{z}) &= \mathcal{N}_{\vec{i}} \cdot \mathcal{N}_{\vec{j}} \cdot e^{i\pi[\vec{x} \cdot (\mathbf{N}_1 + \mathbf{N}_2) \cdot \vec{y} + i\vec{y} \cdot (\mathbf{M}_1 + \mathbf{M}_2) \cdot \vec{y}]} \\
&\sum_{\vec{l}_3, \vec{l}_4 \in \mathbf{Z}^n} \sum_{\vec{m}} e^{\pi i(i)[(\vec{i}\mathbf{N}_1 + \vec{j}\mathbf{N}_2 + \vec{m}\mathbf{N}_1)(\mathbf{N}_1 + \mathbf{N}_2)^{-1} + \vec{l}_3] \cdot (\mathbf{M}_1 + \mathbf{M}_2) \cdot [(\mathbf{N}_1 + \mathbf{N}_2)^{-1}(\mathbf{N}_1\vec{i} + \mathbf{N}_2\vec{j} + \mathbf{N}_1\vec{m}) + \vec{l}_3]} \times \\
&e^{2\pi i(i)[(\vec{i}\mathbf{N}_1 + \vec{j}\mathbf{N}_2 + \vec{m}\mathbf{N}_1)(\mathbf{N}_1 + \mathbf{N}_2)^{-1} + \vec{l}_3] \cdot (\mathbf{N}_1 + \mathbf{N}_2) \vec{x}} \cdot e^{2\pi i(i)[(\vec{i}\mathbf{N}_1 + \vec{j}\mathbf{N}_2 + \vec{m}\mathbf{N}_1)(\mathbf{N}_1 + \mathbf{N}_2)^{-1} + \vec{l}_3] \cdot (\mathbf{M}_1 + \mathbf{M}_2) \vec{y}} \times \\
&e^{2\pi i(i)[(\vec{i} - \vec{j} + \vec{m}) \frac{\mathbf{N}_1(\mathbf{N}_1 + \mathbf{N}_2)^{-1} \mathbf{N}_2}{\det \mathbf{N}_1 \det \mathbf{N}_2} + \vec{l}_4] \cdot [(\det \mathbf{N}_1 \det \mathbf{N}_2)(\mathbf{N}_1^{-1} \mathbf{M}_1 - \mathbf{N}_2^{-1} \mathbf{M}_2)] \cdot \vec{y}} \times \\
&e^{\pi i(i)[(\vec{i}\mathbf{N}_1 + \vec{j}\mathbf{N}_2 + \vec{m}\mathbf{N}_1)(\mathbf{N}_1 + \mathbf{N}_2)^{-1} + \vec{l}_3] \cdot [(\det \mathbf{N}_1 \det \mathbf{N}_2)(\mathbf{M}_1 \mathbf{N}_1^{-1} - \mathbf{M}_2 \mathbf{N}_2^{-1})] \cdot [\frac{\mathbf{N}_2(\mathbf{N}_1 + \mathbf{N}_2)^{-1} \mathbf{N}_1}{\det \mathbf{N}_1 \det \mathbf{N}_2}(\vec{i} - \vec{j} + \vec{m}) + \vec{l}_4]} \times \\
&e^{\pi i(i)[(\vec{i} - \vec{j} + \vec{m}) \frac{\mathbf{N}_1(\mathbf{N}_1 + \mathbf{N}_2)^{-1} \mathbf{N}_2}{\det \mathbf{N}_1 \det \mathbf{N}_2} + \vec{l}_4] \cdot [(\det \mathbf{N}_1 \det \mathbf{N}_2)(\mathbf{N}_1^{-1} \mathbf{M}_1 - \mathbf{N}_2^{-1} \mathbf{M}_2)] \cdot [(\mathbf{N}_1 + \mathbf{N}_2)^{-1}(\mathbf{N}_1\vec{i} + \mathbf{N}_2\vec{j} + \mathbf{N}_1\vec{m}) + \vec{l}_3]} \times \\
&e^{\pi i(i)[(\vec{i} - \vec{j} + \vec{m}) \frac{\mathbf{N}_1(\mathbf{N}_1 + \mathbf{N}_2)^{-1} \mathbf{N}_2}{\det \mathbf{N}_1 \det \mathbf{N}_2} + \vec{l}_4] \cdot [(\det \mathbf{N}_1 \det \mathbf{N}_2)^2(\mathbf{N}_1^{-1} \mathbf{M}_1 \mathbf{N}_1^{-1} + \mathbf{N}_2^{-1} \mathbf{M}_2 \mathbf{N}_2^{-1})] \cdot [\frac{\mathbf{N}_2(\mathbf{N}_1 + \mathbf{N}_2)^{-1} \mathbf{N}_1}{\det \mathbf{N}_1 \det \mathbf{N}_2}(\vec{i} - \vec{j} + \vec{m}) + \vec{l}_4]}
\end{aligned} \tag{5.72}$$

Using from eq.(5.31):

$$\begin{aligned}
(\psi^{\vec{k}, \mathbf{N}_3, \mathbf{M}_3})^* &= \mathcal{N}_{\vec{k}} \cdot e^{-i\pi[\vec{x} \cdot \mathbf{N}_3 \cdot \vec{y} - i\vec{y} \cdot \mathbf{M}_3 \cdot \vec{y}]} \\
&\times \sum_{\vec{l}'_3 \in \mathbf{Z}^n} e^{\pi i(i)[(\vec{l}'_3 + \vec{k}) \cdot \mathbf{M}_3 \cdot (\vec{l}'_3 + \vec{k})]} \cdot e^{-2\pi i[(\vec{l}'_3 + \vec{k}) \cdot \mathbf{N}_3 \cdot \vec{x} - i(\vec{l}'_3 + \vec{k}) \cdot \mathbf{M}_3 \cdot \vec{y}]},
\end{aligned} \tag{5.73}$$

we can then proceed to calculate the Yukawa coupling:

$$Y_{ijk} = \sigma_{abc} g \int_{T^4} dz_i d\bar{z}_i \cdot \psi^{\vec{i}, \mathbf{N}_1, \mathbf{M}_1} \cdot \psi^{\vec{j}, \mathbf{N}_2, \mathbf{M}_2} \cdot (\psi^{\vec{k}, \mathbf{N}_3, \mathbf{M}_3})^* \quad (i = 1, 2). \tag{5.74}$$

Consider first the integration over  $\vec{x}$ :

$$\int d^2 \vec{x} e^{i\pi\{\vec{x} \cdot (\mathbf{N}_1 + \mathbf{N}_2) - \mathbf{N}_3\} \cdot \vec{y}} \sum_{\vec{l}_3, \vec{l}_4, \vec{l}'_3 \in \mathbf{Z}^n} \sum_{\vec{m}} e^{2\pi i[(\vec{i}\mathbf{N}_1 + \vec{j}\mathbf{N}_2 + \vec{m}\mathbf{N}_1)(\mathbf{N}_1 + \mathbf{N}_2)^{-1} + \vec{l}_3] \cdot (\mathbf{N}_1 + \mathbf{N}_2) \vec{x}} e^{-2\pi i[(\vec{l}'_3 + \vec{k}) \cdot \mathbf{N}_3 \cdot \vec{x}]} \tag{5.75}$$

which implies, using  $(\mathbf{N}_1 + \mathbf{N}_2) = \mathbf{N}_3$ , the following conditions:

- equality of the summation indices  $\vec{l}_3 = \vec{l}'_3$ ,
- the relation  $(\vec{i}\mathbf{N}_1 + \vec{j}\mathbf{N}_2 + \vec{m}\mathbf{N}_1)(\mathbf{N}_3)^{-1} = \vec{k}$ .

Note that  $(\mathbf{N}_1 + \mathbf{N}_2) = \mathbf{N}_3$  is a valid condition in a triple intersection since  $I_{ab} + I_{bc} = I_{ac}$ , with complex conjugation taking care of the fact that  $I_{ac} = -I_{ca}$ , which changes the signs of  $\mathbf{N}_3$  and  $\mathbf{M}_3$ . Also, as in section 4.3, 4.4, for any given solution of the above constraint equation for  $\vec{i}, \vec{j}, \vec{k}, \vec{m}$ , other solutions inside the cell of eq. (4.34) that are shifted by  $\vec{m}$ 's satisfying  $\vec{m}\mathbf{N}_1\mathbf{N}_3^{-1}$ : integer are also allowed. In view of this, as in eq. (4.47), we break

the sum over  $\vec{m}$  into two parts, one corresponding to  $\vec{m}$ , which is a given specific solution of eq. (4.40) and the other ones as given by sum over integer variables  $\vec{p}$  and  $\vec{\bar{p}}$  whose ranges are as defined in eq. (4.45).

Imposing the constraints from the  $\vec{x}$  integration, we obtain:

$$\begin{aligned}
Y_{ijk} &= \sigma_{abc} g \cdot \mathcal{N}_{\vec{i}} \cdot \mathcal{N}_{\vec{j}} \cdot \mathcal{N}_{\vec{k}} \tag{5.76} \\
&\int d^2 \vec{y} \{ e^{-\pi [\vec{y} \cdot (\mathbf{M}_1 + \mathbf{M}_2 + \mathbf{M}_3) \cdot \vec{y}]} \sum_{\vec{l}_3, \vec{l}_4 \in \mathbf{Z}^n} \sum_{\vec{p}, \vec{\bar{p}}} e^{\pi i(i) [\vec{k} + \vec{l}_3] \cdot (\mathbf{M}_1 + \mathbf{M}_2) \cdot [\vec{k} + \vec{l}_3]} \times \\
&e^{\pi i(i) [\vec{k} + \vec{l}_3] \cdot [(\det \mathbf{N}_1 \det \mathbf{N}_2) (\mathbf{M}_1 \mathbf{N}_1^{-1} - \mathbf{M}_2 \mathbf{N}_2^{-1})] \cdot [\frac{\mathbf{N}_2 (\mathbf{N}_1 + \mathbf{N}_2)^{-1} \mathbf{N}_1}{\det \mathbf{N}_1 \det \mathbf{N}_2} (\vec{i} - \vec{j} + \vec{m}) + \vec{l}_4]} \times \\
&e^{\pi i(i) [(\vec{i} - \vec{j} + \vec{m}) \frac{\mathbf{N}_1 (\mathbf{N}_1 + \mathbf{N}_2)^{-1} \mathbf{N}_2}{\det \mathbf{N}_1 \det \mathbf{N}_2} + \vec{l}_4] \cdot [(\det \mathbf{N}_1 \det \mathbf{N}_2) (\mathbf{N}_1^{-1} \mathbf{M}_1 - \mathbf{N}_2^{-1} \mathbf{M}_2)] \cdot [\vec{k} + \vec{l}_3]} \times \\
&e^{\pi i(i) [(\vec{i} - \vec{j} + \vec{m}) \frac{\mathbf{N}_1 (\mathbf{N}_1 + \mathbf{N}_2)^{-1} \mathbf{N}_2}{\det \mathbf{N}_1 \det \mathbf{N}_2} + \vec{l}_4] \cdot [(\det \mathbf{N}_1 \det \mathbf{N}_2)^2 (\mathbf{N}_1^{-1} \mathbf{M}_1 \mathbf{N}_1^{-1} + \mathbf{N}_2^{-1} \mathbf{M}_2 \mathbf{N}_2^{-1})] \cdot [\frac{\mathbf{N}_2 (\mathbf{N}_1 + \mathbf{N}_2)^{-1} \mathbf{N}_1}{\det \mathbf{N}_1 \det \mathbf{N}_2} (\vec{i} - \vec{j} + \vec{m}) + \vec{l}_4]} \\
&\times e^{\pi i(i) [\vec{k} + \vec{l}_3] \cdot (\mathbf{M}_1 + \mathbf{M}_2) \cdot \vec{y}} \cdot e^{\pi i(i) [(\vec{i} - \vec{j} + \vec{m}) \frac{\mathbf{N}_1 (\mathbf{N}_1 + \mathbf{N}_2)^{-1} \mathbf{N}_2}{\det \mathbf{N}_1 \det \mathbf{N}_2} + \vec{l}_4] \cdot [(\det \mathbf{N}_1 \det \mathbf{N}_2) (\mathbf{M}_1 \mathbf{N}_1^{-1} - \mathbf{M}_2 \mathbf{N}_2^{-1})] \cdot \vec{y}} \},
\end{aligned}$$

where the range of the sum over  $\vec{p}, \vec{\bar{p}}$  is as used in eq. (4.45) in section 4.3.

The above expression for the Yukawa interaction can be written as following:

$$\begin{aligned}
Y_{ijk} &= \sigma_{abc} g \cdot (2^3)^{\frac{1}{4}} (|\det \mathbf{M}_1| \cdot |\det \mathbf{M}_2| \cdot |\det \mathbf{M}_3|)^{\frac{1}{4}} (Vol(T^4))^{-\frac{3}{2}} \\
&\int d^2 \vec{y} \{ e^{-\pi [\vec{y} \cdot (\mathbf{M}_1 + \mathbf{M}_2 + \mathbf{M}_3) \cdot \vec{y}]} \sum_{\vec{l}_3, \vec{l}_4 \in \mathbf{Z}^n} \sum_{\vec{p}, \vec{\bar{p}}} e^{\pi i(i) [\vec{\mathbf{K}} + \vec{\mathbf{L}}] \cdot \hat{\mathbf{Q}}' \cdot [\vec{\mathbf{K}} + \vec{\mathbf{L}}]} e^{2\pi i(i) [\vec{\mathbf{K}} + \vec{\mathbf{L}}] \cdot \vec{\mathbf{Y}}'} \\
&= \sigma_{abc} g \cdot (2^3)^{\frac{1}{4}} (|\det \mathbf{M}_1| \cdot |\det \mathbf{M}_2| \cdot |\det \mathbf{M}_3|)^{\frac{1}{4}} (Vol(T^4))^{-\frac{3}{2}} \times \\
&\sum_{\vec{p}, \vec{\bar{p}}} \int d^2 \vec{y} \{ e^{-\pi [\vec{y} \cdot (\mathbf{M}_1 + \mathbf{M}_2 + \mathbf{M}_3) \cdot \vec{y}]} \cdot \vartheta \left[ \begin{array}{c} \vec{\mathbf{K}} \\ 0 \end{array} \right] (\vec{\mathbf{Y}}' | i \hat{\mathbf{Q}}') \} \tag{5.77}
\end{aligned}$$

where we defined new  $4d$ -vectors:

$$\vec{\mathbf{L}} = \begin{pmatrix} \vec{l}_3 \\ \vec{l}_4 \end{pmatrix}, \quad \vec{\mathbf{K}} = \begin{pmatrix} \vec{k} \\ [(\vec{i} - \vec{j} + \vec{m})] [\frac{\mathbf{N}_1 (\mathbf{N}_1 + \mathbf{N}_2)^{-1} \mathbf{N}_2}{\det \mathbf{N}_1 \det \mathbf{N}_2}] \end{pmatrix}, \tag{5.78}$$

$$\vec{\mathbf{Y}}' = \begin{pmatrix} (\mathbf{M}_1 + \mathbf{M}_2) \vec{y} \\ [(\det \mathbf{N}_1 \det \mathbf{N}_2) (\mathbf{M}_1 \mathbf{N}_1^{-1} - \mathbf{M}_2 \mathbf{N}_2^{-1})] \cdot \vec{y} \end{pmatrix} \tag{5.79}$$

and the  $4d$ -matrix:

$$\hat{\mathbf{Q}}' = \begin{pmatrix} (\mathbf{M}_1 + \mathbf{M}_2) & (\det \mathbf{N}_1 \det \mathbf{N}_2) (\mathbf{M}_1 \mathbf{N}_1^{-1} - \mathbf{M}_2 \mathbf{N}_2^{-1}) \\ ((\det \mathbf{N}_1 \det \mathbf{N}_2) (\mathbf{N}_1^{-1} \mathbf{M}_1 - \mathbf{N}_2^{-1} \mathbf{M}_2) & (\det \mathbf{N}_1 \det \mathbf{N}_2)^2 (\mathbf{N}_1^{-1} \mathbf{M}_1 \mathbf{N}_1^{-1} + \mathbf{N}_2^{-1} \mathbf{M}_2 \mathbf{N}_2^{-1}) \end{pmatrix}$$

with  $\vec{k}$  appearing in eq. (5.78) restricted by the Kronecker delta relation written above, as following from the  $x$  integration, in eq. (5.75) and the range of the sum over  $\vec{p}, \vec{\bar{p}}$  is as used in eq. (4.45) in section 4.3, we skip the details regarding them.

In fact, the form of the result (5.77) is valid for all basis functions, whether corresponding to positive or negative chirality wavefunctions, since the negative chirality wavefunction (5.31), written for the complex structure  $\Omega = iI_2$  and used in obtaining the final answer for Yukawa coupling in eq. (5.77), reduces to the one for positive chirality wavefunction for the same complex structure when  $\mathbf{M}$  is set to  $\mathbf{N}$  (see eq. (4.3) for the general form of the positive chirality wavefunction). For such a choice:  $\mathbf{M}_i = \mathbf{N}_i$ ,  $\hat{\mathbf{Q}}'$  has a factorized block form and the vector  $\vec{\mathbf{Y}}'$  in eq. (5.79) now has a form:

$$\vec{\mathbf{Y}}' = \begin{pmatrix} (\mathbf{N}_1 + \mathbf{N}_2)\vec{y} \\ 0 \end{pmatrix}. \quad (5.81)$$

The theta function in eq. (5.77) then factorizes and the final answer reduces to the form given in eqs. (4.39), (4.49) for the choice  $\tau = i$  corresponding to the complex structure of our choice in the negative chirality wavefunction (5.21).

The Yukawa coupling expression (5.77) can be further generalized to other situations. First, although the above analysis was very specific to the case of  $T^4$  due to our choice of wavefunction in eq. (5.31), the generalization to the  $T^6$  is straightforward. Mapping between matrices  $\mathbf{N}$  and  $\mathbf{M}$  is identical and follows from the definition of  $\hat{\Omega}$  in subsection 5.8. The final answer is identical to the one given in eq. (5.77).

Further generalization to the situation of arbitrary complex structure should also be possible, using the wavefunctions that emerge due to the mappings obtained in subsection (5.7) and scaling procedure presented in section (4.6) for the positive chirality wavefunctions. One, however, also needs to examine the symmetry property of the matrices  $\mathbf{N}\hat{\Omega}\Omega$  etc., appearing in the definition of the wavefunction. We leave further details for future work.

## 6 Mass generation for non-chiral fermions

In this section, we briefly discuss one of the applications of the results derived in the paper, for giving mass to the non-chiral gauge non-singlet states of the magnetized brane model

discussed in [27]. This is a three generation  $SU(5)$  supersymmetric grand unified (GUT) model in simple toroidal compactifications of type I string theory with magnetized  $D9$  branes. The final gauge group is just  $SU(5)$  and the chiral gauge non-singlet spectrum consists of three families with the quantum numbers of quarks and leptons, transforming in the  $\mathbf{10} + \bar{\mathbf{5}}$  representations of  $SU(5)$ . Brane stacks with oblique fluxes played a central role in this construction, in order to stabilize all close string moduli, in a manner restricting the chiral matter content to precisely that of  $SU(5)$  GUT. Another interesting feature of this model is that it is free from any chiral exotics that often appear in such brane constructions. However, the model contains extra non-chiral matter that is expected to become massive at a high scale, close to that of  $SU(5)$  breaking.

The results of the previous sections can be used for examining the issue of the mass generation for these non-chiral multiplets in a supersymmetric ground state. The aim is to analyze the D and F term conditions, and show that a ground state allowing masses for the above matter multiplets is possible. The exercise will further fine tune our  $SU(5)$  GUT model to the ones used in conventional grand unification.

Although, we will not be evaluating any of the Yukawa couplings explicitly, which using our results of the previous sections is in principle possible to do, the aim of the exercise below is to show that indeed one can give masses to non-chiral matter. Our procedure involves the analysis of both the F and D-term supersymmetry conditions. In the context of our previous work [27], we like to remind the reader that certain charged scalar vacuum expectation values (VEVs) were turned on in order to restore supersymmetry in some of the “hidden” branes sector. These charged scalar VEVs gave a nontrivial solution to the D-term conditions, but left the F-terms identically zero in the vacuum. In the following, on the other hand, our aim is to find out the possibility for a large number of scalars in various chiral multiples to acquire expectation values. For this, we need to examine both the F and D conditions, as already mentioned.

The model in [27] is described by twelve stacks of branes, namely  $U_5, U_1, O_1 \dots, O_8, A,$  and  $B$ . The magnetic fluxes are chosen to generate the required spectrum, to stabilize all the geometric moduli and to satisfy the RR-tadpole conditions as well. The fluxes for all the stacks are summarized in Appendix C. The fluxes for stacks  $U_5, U_1, A, B$  are purely diagonal whereas stacks  $O_1 \dots, O_8$  carry in general both oblique and diagonal fluxes. All 36 closed string moduli are fixed in a  $\mathcal{N} = 1$  supersymmetric vacuum, apart from the

dilaton, in a way that the  $T^6$ -torus metric becomes diagonal with the six internal radii given in terms of the integrally quantized magnetic fluxes.

The two brane stacks  $U_5$  and  $U_1$  give the particle spectrum of  $SU(5)$  GUT. We solve the condition  $I_{U_5 U_1} + I_{U_5 U_1^*} = -3$  for the presence of three generations of chiral fermions transforming in  $\bar{\mathbf{5}}$  of  $SU(5)$  and continue with the solution  $I_{U_5 U_1} = 0$ ,  $I_{U_5 U_1^*} = -3$ . The intersection of  $U_5$  with  $U_1$  is non-chiral since  $I_{U_5 U_1}$  vanishes. The corresponding non-chiral massless spectrum consists of four pairs of  $\mathbf{5} + \bar{\mathbf{5}}$ , which we would like to give mass<sup>6</sup>. Obviously, we would like to keep massless at least one pair of electroweak higgses but this requires a detailed phenomenological analysis that goes beyond the scope of this paper. Here, we would like only to show how to use the results of the previous sections in order to give masses to unwanted non chiral states that often appear in intersecting brane constructions.

So, we have the following non-chiral fields where the superscript refers to the two stacks between which the open string is stretched and the subscript denotes the charges under the respective  $U(1)$ 's:  $(\phi_{+-}^{U_5 U_1}, \phi_{-+}^{U_5 U_1}, 4)$ , with numbers in the brackets denoting the corresponding multiplicities. Similarly, the intersections of the  $U_5$  stack with the two extra branes  $A, B$  and their images are non-chiral, giving rise to the extra  $\mathbf{5} + \bar{\mathbf{5}}$  pairs:  $(\phi_{+-}^{U_5 A}, \phi_{-+}^{U_5 A}, 149)$ ,  $(\phi_{++}^{U_5 A^*}, \phi_{--}^{U_5 A^*}, 146)$ ,  $(\phi_{+-}^{U_5 B}, \phi_{-+}^{U_5 B}, 51)$ ,  $(\phi_{++}^{U_5 B^*}, \phi_{--}^{U_5 B^*}, 16)$ . A common feature of all these states is that they arise in non-chiral intersections, where the two brane stacks involved have diagonal fluxes and are parallel in one of the three tori. It is then straightforward to give masses by moving, say, the  $U_5$  stack away from the others along these tori. In the language of  $D9$  branes, this amounts to turn on corresponding open string Wilson lines.

On the other hand, analysis of the particle spectrum on the intersections of the stack  $U_5$  with the oblique branes  $O_a$  and  $O_a^*$ , satisfying the condition  $I_{U_5 a} + I_{U_5 a^*} = 0$ , for  $a = 1, \dots, 8$ , leads to  $4 \times (23 + 14) = 148$  pairs of  $(\mathbf{5} + \bar{\mathbf{5}})$  representations of  $SU(5)$ :

$$I_{U_5 O_a} = -23, \quad I_{U_5 O_a^*} = 23, \quad a = 1, \dots, 4, \quad (6.1)$$

$$I_{U_5 O_a} = -14, \quad I_{U_5 O_a^*} = 14, \quad a = 5, \dots, 8. \quad (6.2)$$

We then have the following chiral multiplets,  $(\phi_{-+}^{U_5 O_a}, 23)$ ,  $(\phi_{++}^{U_5 O_a^*}, 23)$ ,  $(\phi_{-+}^{U_5 O_b}, 14)$ ,  $(\phi_{++}^{U_5 O_b^*}, 14)$  ( $a = 1, \dots, 4$ ,  $b = 5, \dots, 8$ ). In order to examine the mass generation for these fields, one needs to write down the superpotential terms involving the above chiral multiplets, as

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<sup>6</sup>For details, see Section 3.7 of [27].

well as those coming from the brane stacks  $O_1, \dots, O_8$  and their orientifold images. The list of the later, involving purely oblique stacks, is given in Appendix C.

Now, using the results in Appendix C in eqs. (C.13) and (C.14), one can analyze the associated superpotential and D-terms and look for supersymmetric minima. The relevant superpotential reads:

$$\begin{aligned}
W &= \sum_{ijk} W_{O_1}^{ijk} (\phi_{+-}^{O_1 U_5})^i (\phi_{+++}^{U_5 O_3^*})^j (\phi_{--}^{O_3^* O_1})^k + \sum_{ijk} W_{O_2}^{ijk} (\phi_{+-}^{O_2 U_5})^i (\phi_{+++}^{U_5 O_4^*})^j (\phi_{--}^{O_4^* O_2})^k \\
&+ \sum_{ijk} W_{O_3}^{ijk} (\phi_{+-}^{O_3 U_5})^i (\phi_{+++}^{U_5 O_8^*})^j (\phi_{--}^{O_8^* O_3})^k + \sum_{ijk} W_{O_4}^{ijk} (\phi_{+-}^{O_4 U_5})^i (\phi_{+++}^{U_5 O_7^*})^j (\phi_{--}^{O_7^* O_4})^k \quad (6.3) \\
&+ \sum_{ijk} W_{O_5}^{ijk} (\phi_{+-}^{O_5 U_5})^i (\phi_{+++}^{U_5 O_6^*})^j (\phi_{--}^{O_6^* O_5})^k + \sum_{ijk} W_{O_7}^{ijk} (\phi_{+-}^{O_7 U_5})^i (\phi_{+++}^{U_5 O_8^*})^j (\phi_{--}^{O_8^* O_7})^k
\end{aligned}$$

where the sum over  $i, j, k$  runs over the ‘‘flavor’’ indices. The couplings  $W_{O_i}^{ijk}$ , given in eq. (6.3), can be read off from our results in the previous sections. In addition to the complex structure, these also depend on the first Chern numbers of the branes in each triangle.

The F-flatness conditions  $\langle F_i \rangle = \langle \mathcal{D}_{\phi_i} W \rangle = 0$  (at zero superpotential,  $W = 0$ ), imply that for each ‘‘triangle’’ at least two fields must have a zero VEV in order to form a supersymmetric vacuum [30]. In this theory, there exists indeed a supersymmetric vacuum where six charged fields remain unconstrained by the F-flatness conditions. Let’s choose them to be  $(\phi_{--}^{O_3^* O_1})$ ,  $(\phi_{--}^{O_4^* O_2})$ ,  $(\phi_{--}^{O_8^* O_3})$ ,  $(\phi_{--}^{O_7^* O_4})$ ,  $(\phi_{--}^{O_6^* O_5})$ ,  $(\phi_{--}^{O_8^* O_7})$  (they are neutral under the  $U(1)$  of the  $U(5)$ ). The remaining fields appearing in the superpotential acquire a mass from the F-term potential only if these unconstrained scalars possess a non-vanishing VEV. Indeed, their masses read:

$$\begin{aligned}
M_{\phi_{u_5 o_1}}^2 &\sim M_{\phi_{u_5 o_3^*}}^2 \sim \langle |\phi_{o_3^* o_1}|^2 \rangle, \quad M_{\phi_{u_5 o_2}}^2 \sim M_{\phi_{u_5 o_4^*}}^2 \sim \langle |\phi_{o_4^* o_2}|^2 \rangle, \\
M_{\phi_{u_5 o_7'}}^2 &\sim M_{\phi_{u_5 o_8^*}}^2 \sim \langle |\phi_{o_8^* o_7'}|^2 \rangle, \quad M_{\phi_{u_5 o_4}}^2 \sim M_{\phi_{u_5 o_7^*}}^2 \sim \langle |\phi_{o_7^* o_4}|^2 \rangle, \quad (6.4) \\
M_{\phi_{u_5 o_5}}^2 &\sim M_{\phi_{u_5 o_6^*}}^2 \sim \langle |\phi_{o_6^* o_5}|^2 \rangle,
\end{aligned}$$

where  $\phi_{u_5 o_7'}$  denotes linear combinations of  $\phi_{u_5 o_7}$  with  $\phi_{u_5 o_3}$  and  $\phi_{o_8^* o_7'}$  denotes linear combinations of  $\phi_{o_8^* o_7}$  with  $\phi_{o_8^* o_3}$ . Thus, the leftover massless states from the intersection of  $U_5$  with the oblique branes are 60 pairs of  $\mathbf{5} + \bar{\mathbf{5}}$ :  $\phi_{u_5 o_a^*}$  for  $a = 1, 2, 5$  of positive chirality together with the negative chirality states  $\phi_{u_5 o_a}$  for  $a = 6, 7$ , as well as 23 linear combinations of  $\phi_{u_5 o_3}$  with  $\phi_{u_5 o_7}$ , and 14  $\phi_{u_5 o_4}$ .

However, switching on non-zero VEVs for these fields, modifies the existing D-term conditions for the stacks of branes  $O_1, \dots, O_8$ . Recall that, in [27], the stacks  $U_5, O_1 \dots O_8$

satisfy the supersymmetry conditions in the absence of charged scalar VEVs, but VEVs for the fields  $\phi_{-+}^{U_1 A}$ ,  $\phi_{++}^{U_1 B^*}$  and  $\phi_{+-}^{AB}$  are switched on, for the same supersymmetry to be preserved by the stacks  $U_1$ ,  $A$  and  $B$ .<sup>7</sup> The D-terms for each  $U(1)$  factor of the eight branes  $O_1, \dots, O_8$  read

$$\begin{aligned}
D_{O_1} &= -|\phi^{O_1 O_3^*}|^2, \quad D_{O_2} = -|\phi^{O_2 O_4^*}|^2 \\
D_{O_3} &= -|\phi^{O_1 O_3^*}|^2 - |\phi^{O_3 O_8^*}|^2, \quad D_{O_4} = -|\phi^{O_2 O_4^*}|^2 - |\phi^{O_4 O_7^*}|^2 \\
D_{O_5} &= -|\phi^{O_5 O_6^*}|^2, \quad D_{O_6} = -|\phi^{O_5 O_6^*}|^2 \\
D_{O_7} &= -|\phi^{O_4 O_7^*}|^2 - |\phi^{O_7 O_8^*}|^2, \quad D_{O_8} = -|\phi^{O_3 O_8^*}|^2 - |\phi^{O_7 O_8^*}|^2
\end{aligned} \tag{6.5}$$

We can regain the supersymmetry conditions  $D_a = 0, \forall a = 1, \dots, 8$  with  $\xi_a(F^a, J) = 0$ , by switching on VEVs for the following fields:  $(\phi_{++}^{O_1 O_5^*})$ ,  $(\phi_{++}^{O_2 O_7^*})$ ,  $(\phi_{++}^{O_3 O_7^*})$ ,  $(\phi_{++}^{O_3 O_4^*})$ ,  $(\phi_{++}^{O_4 O_8^*})$ ,  $(\phi_{++}^{O_6 O_8^*})$ , provided these fields do not modify the superpotential (6.3). The modified D-terms read:

$$\begin{aligned}
D_{O_1} &= -|\phi^{O_1 O_3^*}|^2 + |\phi^{O_1 O_5^*}|^2 \\
D_{O_2} &= -|\phi^{O_2 O_4^*}|^2 + |\phi^{O_2 O_7^*}|^2 \\
D_{O_3} &= -|\phi^{O_1 O_3^*}|^2 - |\phi^{O_3 O_8^*}|^2 + |\phi^{O_3 O_4^*}|^2 + |\phi^{O_3 O_7^*}|^2 \\
D_{O_4} &= -|\phi^{O_2 O_4^*}|^2 - |\phi^{O_4 O_7^*}|^2 + |\phi^{O_3 O_4^*}|^2 + |\phi^{O_4 O_8^*}|^2 \\
D_{O_5} &= -|\phi^{O_5 O_6^*}|^2 + |\phi^{O_1 O_5^*}|^2 \\
D_{O_6} &= -|\phi^{O_5 O_6^*}|^2 + |\phi^{O_6 O_8^*}|^2 \\
D_{O_7} &= -|\phi^{O_4 O_7^*}|^2 - |\phi^{O_7 O_8^*}|^2 + |\phi^{O_2 O_7^*}|^2 + |\phi^{O_3 O_7^*}|^2 \\
D_{O_8} &= -|\phi^{O_3 O_8^*}|^2 - |\phi^{O_7 O_8^*}|^2 + |\phi^{O_6 O_8^*}|^2 + |\phi^{O_4 O_8^*}|^2
\end{aligned} \tag{6.6}$$

The supersymmetry conditions  $D_a = 0, \forall a = 1, \dots, 8$  with  $\xi_a(F^a, J) = 0$  can be simultaneously satisfied if and only if the VEVs for all these fields appearing in the expressions (6.6), have the same value, say  $v^2$ . Moreover we can restrict  $v \ll 1$  in string units, as required by the validity of the approximation for inclusion of charged scalar fields in the D-term.

We have therefore shown the mass generation for a large set of non-chiral fields as given in eq. (6.4). It is possible, that remaining ones can also be made massive by incorporating non perturbative instanton contributions to the superpotential. However, we leave this

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<sup>7</sup>For details see Section 5 of [27].



exercise for the moment. We also do not give any superpotential couplings, in terms of fluxes, as given explicitly in the previous sections.

## 7 Discussions and Conclusions

In this concluding section, we first comment on the case of magnetized branes with higher winding numbers. The form of the wrapping matrices [13] for  $D9$  branes on  $T^6$  was discussed in our earlier papers [12, 27]. They are real  $6 \times 6$  matrices giving the embedding of the brane along spatial internal directions. The situation where worldvolume coordinates are identified with the spatial coordinates corresponds to  $W$  being diagonal. Then, for example, for a canonical complex structure  $\Omega = iI_3$ , the spatial components of the flux matrices are of the form given in eqs. (B.3), (B.4), (B.5). Taking into account the gauge indices, one obtains a block diagonal matrix structure for the fluxes, that reduces in the case of factorized tori to the form:

$$F = \begin{pmatrix} \frac{m_i^a}{n_i^a} I_{N^a} & \\ & \frac{m_i^b}{n_i^b} I_{N^b} \end{pmatrix}, \quad (7.1)$$

with  $a$  and  $b$  representing the brane-stacks and  $i$  denotes the  $i$ 'th  $T^2$ . Also  $m_i^{a,b}$  are the first Chern numbers, as given in eqs. (B.3) and (B.4), whereas  $n_i^{a,b}$  are the product of the winding numbers along various 1-cycles of  $(T^2)^3 \in T^6$ . Also,  $N^a$  and  $N^b$  are the number of branes in stacks  $a$  and  $b$  respectively and the above expression has a straightforward generalization when many such brane stacks are involved.

In [5], a gauge theoretic picture of the magnetic fluxes along brane stacks with higher winding numbers ( $> 1$ ) was given. For instance, consider the simplest choice  $N^a = N^b = 1$ . In this case, the configuration of the brane stacks  $a$  and  $b$  with one  $D$ -brane each, having wrapping numbers  $n^a, n^b$  and 1st Chern numbers  $m^a, m^b$ , is given by a flux matrix associated with a  $U(n^a + n^b)$  gauge group with flux having the internal (gauge) components:

$$F = \begin{pmatrix} \frac{m_i^a}{n_i^a} I_{n_i^a} & \\ & \frac{m_i^b}{n_i^b} I_{n_i^b} \end{pmatrix}, \quad (7.2)$$

along the  $i$ 'th  $T^2$  and  $m_i^a, n_i^a$  etc. are relatively prime.

Given the  $U(n^a + n^b)$  flux in eq. (7.2), the fermion wavefunctions associated with bifundamentals were constructed in [5]. The new feature is that, to have proper periodicity

property for these fermion wavefunctions, non-abelian Wilson lines need to be turned on. In turn, these non-abelian Wilson lines mix up  $n_i^a \times n_i^b$  components and the set of periodicity constraints only allows the bifundamentals belonging to the representations of the gauge group:  $U(P_i^a) \times U(P_i^b)$ , with  $P_i^a = g.c.d.(m_i^a, n_i^a)$ . In our example above we have  $P_i^a = P_i^b = 1$ .

The case of oblique fluxes brings in extra complexities in the analysis due to the presence of six independent 1-cycles along which non-abelian Wilson line actions need to be fixed. Given the action of these Wilson lines, one can then proceed to obtain the wavefunctions as well as the Yukawa couplings. However, unlike the factorized situation in [5], one finds that the action of non-abelian Wilson lines on the wavefunction, is dependent on the particular model, or more precisely, on the details of the oblique fluxes that are turned on. Further analysis along this line is, though cumbersome, possible. We now conclude our paper with the following remarks.

In this work, we have been able to explicitly generalize the Yukawa coupling expressions to the situation when the worldvolume fluxes that are responsible for moduli stabilization, chiral mass generation, supersymmetry breaking to  $N = 1$  etc., do not respect the factorization of  $T^6$  into  $(T^2)^3$ . For the factorized tori, the mappings of the Yukawa couplings, superpotentials and Kähler potential between the type IIB and IIA expressions was discussed in [5]. In the IIA case, the results are obtained through a ‘diagonal’ wrapping of the  $D6$  branes in three  $T^2$ 's.

It will be interesting to map our IIB expressions, given in this paper to the IIA side and find the corresponding intersecting brane picture. As stated earlier, such a IIA construction will require putting the branes along general  $SU(3)$  rotation angles and then obtain the area of the triangles corresponding to the intersections of three branes giving chiral multiplets.

Supersymmetry breaking is of course an important issue in model building. Though generally, for magnetized branes, one encounters instabilities in such a situation, it should be however possible to obtain non-supersymmetric magnetized brane constructions for a rich variety of fluxes accompanied by orientifold planes which can possibly project out tachyons that may be generated during the process of supersymmetry breaking.

The recent developments in writing the instanton induced superpotential terms are also encouraging, for the purpose of examining the supersymmetry breaking as well as

up-quark mass generations in a GUT setting. In this context, it has been shown that the magnetized branes too can give rise to interesting superpotentials through the lift of fermion zero modes when fluxes are turned on.

Recently, there have been interesting developments in deriving particle models and interactions from string theory, using the non-perturbative picture of F-theory [31–38], with a geometric picture of 7-brane intersection curves inside del Pezzo surfaces giving the chiral spectrum as well as Yukawa interactions, including those of the spinors of  $SO(10)$  GUT, and thus generating observable masses for both up and down type quarks. It is also interesting to note that F-theory results are reproducible in a globally consistent IIB string theory, taking into account the instanton generated superpotential terms [39]. It will be of interest to see the implications of these results on the construction of  $SU(5)$  GUT in [27], as well as on the Yukawa interactions discussed in this paper.

Finally, it will be interesting to explore the generalization of our results to higher-point functions (computing couplings of higher dimensional effective operators) [40] and make explicit comparisons of our results with those in [14, 15], where the situation with diagonal intersection matrices  $\mathbf{N}_i$ , but non-factorized complex structure, is addressed through a computation of twist field correlations. However, one then needs to examine the effect of supersymmetry conditions (2.7) and (2.8) to see if the interaction indeed remains nontrivial in a supersymmetric set up.

We hope to return to all issues above at a later stage.

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# A Wavefunction

We first present the construction of chiral fermion wavefunctions on tori and give their representation in terms of theta functions. For definiteness we first discuss the case of 4-tori, though  $T^6$  chiral multiplet structure can be analyzed in a similar manner. To be explicit, for the moment we restrict ourselves to the canonical complex structure:  $\Omega = iI_2$  and  $\Omega = iI_3$  for  $T^4$  and  $T^6$  respectively, where  $I_d$  represents a  $d$ -dimensional identity matrix. The general complex structure is restored while writing the wavefunctions as well as interaction vertices.

To obtain the Dirac wavefunctions in  $T^4$ , we start by writing four Dirac Gamma matrices (in a complex basis) :

$$\Gamma^{z_1} = \sigma^z \times \sigma^3 = \begin{pmatrix} 0 & 2 & & \\ 0 & 0 & & \\ & & 0 & -2 \\ & & 0 & 0 \end{pmatrix}, \quad \Gamma^{z_2} = I \times \sigma^z = \begin{pmatrix} & & 2 & 0 \\ & & 0 & 2 \\ 0 & 0 & & \\ 0 & 0 & & \end{pmatrix}, \quad (\text{A.1})$$

where the information about the complex structure in the above expression is hidden in the fact that we have used the definitions:  $z_i = x_i + iy_i$  in writing these Dirac matrices. Similarly,

$$\Gamma^{\bar{z}_1} = \sigma^{\bar{z}} \times \sigma^3 = \begin{pmatrix} 0 & 0 & & \\ 2 & 0 & & \\ & & 0 & 0 \\ & & -2 & 0 \end{pmatrix}, \quad \Gamma^{\bar{z}_2} = I \times \sigma^{\bar{z}} = \begin{pmatrix} & & 0 & 0 \\ & & 0 & 0 \\ 2 & 0 & & \\ 0 & 2 & & \end{pmatrix}. \quad (\text{A.2})$$

They satisfy the anti-commutation relations:

$$\{\Gamma^{z_i}, \Gamma^{z_j}\} = 0, \quad \{\Gamma^{\bar{z}_i}, \Gamma^{\bar{z}_j}\} = 0, \quad \{\Gamma^{z_i}, \Gamma^{\bar{z}_j}\} = 4\delta_{ij} \quad (\text{A.3})$$

with  $i, j = 1, 2$ . In the above basis  $\Gamma^5$  takes the form:

$$\Gamma^5 = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix} \quad (\text{A.4})$$

with 4-component Dirac wavefunctions having the form:

$$\Psi = \begin{pmatrix} \Psi_+^1 \\ \Psi_-^2 \\ \Psi_-^1 \\ \Psi_+^2 \end{pmatrix}. \quad (\text{A.5})$$

In such a decomposition of  $\psi$ , Dirac equations for fermions in the adjoint representation are of the form:

$$\begin{aligned} \bar{\partial}_1 \Psi_+^1 + \partial_2 \Psi_+^2 + [A_{\bar{z}_1}, \Psi_+^1] + [A_{z_2}, \Psi_+^2] &= 0, \\ \bar{\partial}_2 \Psi_+^1 - \partial_1 \Psi_+^2 + [A_{\bar{z}_2}, \Psi_+^1] - [A_{z_1}, \Psi_+^2] &= 0, \\ \partial_1 \Psi_-^2 + \partial_2 \Psi_-^1 + [A_{z_1}, \Psi_-^2] + [A_{z_2}, \Psi_-^1] &= 0, \\ \bar{\partial}_2 \Psi_-^2 - \bar{\partial}_1 \Psi_-^1 + [A_{\bar{z}_2}, \Psi_-^2] - [A_{\bar{z}_1}, \Psi_-^1] &= 0. \end{aligned} \quad (\text{A.6})$$

In a generic model, chiral fermions arise either from the string starting at a brane stack- $a$  and ending at another brane stack- $b$  (or its image  $b^*$ ) or from strings starting at a brane stack  $a$  and ending at its image  $a^*$ . We already showed the correspondence between a stack of magnetized branes and flux quanta in supersymmetric gauge theory, in eq. (2.19). The correspondence is easily generalized when several stacks of branes are present. Explicitly, in a construction with  $P$  number of stacks of branes, with number of branes being  $n_i$  for the  $i$ 'th stack, the flux (for a given target space component  $(i\bar{j})$ ) takes a form:

$$F_{i\bar{j}} = \begin{pmatrix} F^1 I_{n_1} & & & & \\ & F^2 I_{n_2} & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & F^{n_p} I_{n_p} \end{pmatrix}, \quad (\text{A.7})$$

with  $I_{n_i}$  being the  $n_i$ -dimensional identity matrix and we have hidden the  $i\bar{j}$  indices in the RHS of eq. (A.7) in constants  $F^i$  that are all integrally quantized, as given earlier explicitly in eqs. (2.13) and (2.14). The corresponding gauge potentials will also then

have a block diagonal structure:

$$A_i = \begin{pmatrix} A_i^1 I_{n_1} & & & \\ & A_i^2 I_{n_2} & & \\ & & \ddots & \\ & & & A_i^{n_p} I_{n_p} \end{pmatrix}. \quad (\text{A.8})$$

Now, in order to understand the wavefunctions associated with chiral fermion bifundamentals, in such a representation of the brane stacks, we consider the flux matrix  $F_{i\bar{j}}$  in eq. (A.7) and gauge potential in eq. (A.8) with only two blocks ( $P = 2$ ). The chiral fermion bilinears between stack- $a$  and stack- $b$  are then represented by:

$$\Psi_{ab} = \begin{pmatrix} C_{n_a} & \chi_{ab} \\ & C_{n_b} \end{pmatrix}, \quad (\text{A.9})$$

with  $C_{n_a}, C_{n_b}$  being constant matrices of dimensions  $n_a$  and  $n_b$  respectively. We can easily derive the equation satisfied by the various Dirac components, as given in eq. (A.5), for  $\chi_{ab}$  such that  $\psi_{ab}$  satisfies the Dirac equation (A.6). We obtain:

$$\begin{aligned} \bar{\partial}_1 \chi_+^1 + \partial_2 \chi_+^2 + (A^1 - A^2)_{\bar{z}_1} \chi_+^1 + (A^1 - A^2)_{z_2} \chi_+^2 &= 0, \\ \bar{\partial}_2 \chi_+^1 - \partial_1 \chi_+^2 + (A^1 - A^2)_{\bar{z}_2} \chi_+^1 - (A^1 - A^2)_{z_1} \chi_+^2 &= 0, \\ \partial_1 \chi_-^2 + \partial_2 \chi_-^1 + (A^1 - A^2)_{z_1} \chi_-^2 + (A^1 - A^2)_{z_2} \chi_-^1 &= 0, \\ \bar{\partial}_2 \chi_-^2 - \bar{\partial}_1 \chi_-^1 + (A^1 - A^2)_{\bar{z}_2} \chi_-^2 - (A^1 - A^2)_{\bar{z}_1} \chi_-^1 &= 0, \end{aligned} \quad (\text{A.10})$$

with subscript  $a, b$  being dropped from  $\chi_{ab}$  to make the expressions simpler. We will, however, restore the indices at a later stage while evaluating the overlap of three such wave functions from different intersections. In particular, for the chiral components,  $\chi_+^1$  equations reduce to:

$$\begin{aligned} \bar{\partial}_1 \chi_+^1 + (A^1 - A^2)_{\bar{z}_1} \chi_+^1 &= 0, \\ \bar{\partial}_2 \chi_+^1 + (A^1 - A^2)_{\bar{z}_2} \chi_+^1 &= 0. \end{aligned} \quad (\text{A.11})$$

The generalization of eq. (A.11) to the  $T^6$  case is straightforward and can be written as:

$$\bar{D}_i \chi_+^{ab} \equiv \bar{\partial}_i \chi_+^{ab} + (A^1 - A^2)_{\bar{z}_i} \chi_+^{ab} = 0, \quad (i = 1, 2, 3). \quad (\text{A.12})$$

Eq. (A.12) matches with eq. (4.65) of [5] for  $\Omega = iI_3$ , with the identification:

$$(A^1 - A^2)_{\bar{z}_i} \equiv \frac{\pi}{2} \left( [\mathbf{N} \cdot (\tilde{\mathbf{z}} + \tilde{\zeta})] \cdot (\mathbf{Im} \Omega)^{-1} \right)_i, \quad (\text{A.13})$$

with  $\vec{\zeta}$  being the complex constants representing the Wilson lines and  $N$  is the difference of fluxes between the two stacks  $a$  and  $b$  (see eq. (2.15)), having constant fluxes  $F^1$  and  $F^2$ , giving the fermion bilinears in the representation  $(n_1, \bar{n}_2)$ .

Such a solution for eq. (A.12) and (A.13) is given in [5] for arbitrary complex structure  $\Omega$  by the basis elements:

$$\psi^{\vec{j}, \mathbf{N}}(\vec{z}, \Omega) = \mathcal{N} \cdot e^{\{i\pi[\mathbf{N} \cdot \vec{z}] \cdot (\mathbf{N} \cdot \text{Im} \Omega)^{-1} \text{Im}[\mathbf{N} \cdot \vec{z}]\}} \cdot \vartheta \begin{bmatrix} \vec{j} \\ 0 \end{bmatrix} (\mathbf{N} \cdot \vec{z}, \mathbf{N} \cdot \Omega), \quad (\text{A.14})$$

with general definition of Riemann theta function:

$$\vartheta \begin{bmatrix} \vec{a} \\ \vec{b} \end{bmatrix} (\vec{\nu} | \Omega) = \sum_{\vec{m} \in \mathbf{Z}^n} e^{\pi(\vec{m} + \vec{a}) \cdot \Omega \cdot (\vec{m} + \vec{a})} e^{2\pi i(\vec{m} + \vec{a}) \cdot (\vec{\nu} + \vec{b})}. \quad (\text{A.15})$$

and  $\mathbf{N}$  satisfying the constraints given in eqs. (2.17) as well as:

$$\vec{j} \cdot \mathbf{N} \in \mathbf{Z}^n, \quad (\text{A.16})$$

implying that  $\vec{j} \cdot \mathbf{N}$  is an  $n$ -dimensional vector with integer entries. Also, the normalization factor  $\mathcal{N}$  in eq. (A.14) is given by:

$$\mathcal{N} = (2^n |\det \mathbf{N}| \cdot \det(\text{Im} \Omega))^{\frac{1}{4}} (\text{Vol}(T^{2n}))^{-\frac{1}{2}}. \quad (\text{A.17})$$

Then wavefunctions satisfy the orthonormality relations:

$$\int_{T^{2n}} (\psi^{\vec{j}, \mathbf{N}})^* \psi^{\vec{k}, \mathbf{N}} = \delta_{\vec{j}, \vec{k}}. \quad (\text{A.18})$$

These results are useful in determining the interaction terms in Section 4. However, to have well-defined wavefunctions,  $\mathbf{N}$ 's must satisfy the Riemann conditions given in eq. (2.17).

## B More information on fluxes

In general, the  $(1, 1)$  form flux  $F_{z^i \bar{z}^j}$  given by a hermitian matrix in eq. (2.6) is constrained by two equations (2.7) and (2.8) which mix the matrix components  $p_{xx}$ ,  $p_{yy}$  and  $p_{xy}$  for general  $\Omega$ . However, for a canonical complex structure, corresponding to orthogonal tori, the constraints simplify and are written in the matrix form:

$$p_{xx} = p_{yy}, \quad p_{xy}^T = p_{xy}. \quad (\text{B.1})$$

Fluxes of such types have been used in [27] for constructing an  $SU(5)$  GUT with stabilized moduli and in Section 6 we apply the Yukawa couplings computation results to show the mass generation for extra non-chiral states in the model of [27]. In this case, the  $(1, 1)$  form flux  $F_{z^i \bar{z}^j}$ , for  $(\Omega = iI_3)$ , reduces to:

$$F_{z^i \bar{z}^j} = \frac{1}{2}(p_{xy} - ip_{xx}) \quad (\text{B.2})$$

Explicitly, the hermitian flux matrix  $F$  in eq. (A.7) is given as:

$$F = \begin{pmatrix} p_{x^1 y^1} & p_{x^1 y^2} + ip_{x^1 x^2} & p_{x^1 y^3} + ip_{x^3 x^1} \\ p_{x^1 y^2} - ip_{x^1 x^2} & p_{x^2 y^2} & p_{x^2 y^3} + ip_{x^2 x^3} \\ p_{x^3 y^1} - ip_{x^3 x^1} & p_{x^2 y^3} - ip_{x^2 x^3} & p_{x^3 y^3} \end{pmatrix}. \quad (\text{B.3})$$

For magnetized branes in [12, 27], we used the quantization rule for  $p$ 's:

$$p_{x^i y^j} = \frac{m_{x^i y^j}}{n^{x^i} n^{y^j}}, \quad p_{x^i x^j} = \frac{m_{x^i x^j}}{n^{x^i} n^{x^j}}, \quad p_{x^i y^j} = \frac{m_{x^i y^j}}{n^{y^i} n^{y^j}}, \quad (\text{B.4})$$

where  $m_{x^i y^j}$ ,  $m_{x^i x^j}$ ,  $m_{y^i y^j}$  are the first Chern numbers along the corresponding 2-cycles and  $n^{x^i}$ ,  $n^{y^i}$  etc. are the wrapping numbers along the 1-cycles  $x^i$ ,  $y^i$ . However, for the model [27], we have used only integral fluxes corresponding to  $n^{x^i} = n^{y^i} = 1$ .

An additional modification comes when nonzero NS-NS  $B$ -field background is turned on along some 2-cycle. In this case, the first Chern number along the particular 2-cycle (for  $n^{x^i} = n^{y^i} = 1$ ) is shifted by:

$$m_{x^i y^j} \rightarrow \tilde{m}_{x^i y^j} = m_{x^i y^j} + \frac{1}{2}, \text{ etc.} \quad (\text{B.5})$$

In the model that we discussed in [27], we turn on nonzero NS-NS  $B$ -field, ( $B = \frac{1}{2}$ ), along the 2-cycles diagonally in the three  $T^2$ 's. Resulting fluxes are then half-integral. However, as already mentioned earlier, in writing the wavefunctions of chiral fermions  $\chi_{ab}$  in bifundamentals, the relevant quantities are the difference of fluxes in the two stacks, or the two diagonal blocks in the gauge theory picture. In addition to the  $D$ -branes, an orientifold model also contains image  $D$ -branes with fluxes of opposite signature than the ones present in the original brane. In such cases, the corresponding wavefunctions  $\chi_{ab^*}$  will obey similar equations as that of  $\chi_{ab}$ , but with the addition of the gauge potentials  $A^a + A^b$  rather than their difference as in eq. (A.12). The relevant matrix  $\mathbf{N}$  which will now be the addition of fluxes in the two stacks, rather than their difference, will once again be integral.



We also learnt from the second equation in (2.17) that  $(\mathbf{N.Im}\Omega)$  is a symmetric matrix. However, as explained in eqs. (2.6) in the general situation and in (B.2) for  $\Omega = iI_3$ , fluxes are in general hermitian when components of all types:  $p_{xx}$ ,  $p_{yy}$  and  $p_{xy}$  are present.

## C Fluxes for the stacks $U_5, U_1, A, B, O_1, \dots, O_8$

In this Appendix, we write all the fluxes in the complex coordinate basis  $(z, \bar{z})$  with  $z = x + iy$  for our GUT model in [27] and used in Section 6 for the non-chiral mass generation.

$$F^{U_5} = -\frac{i}{2} \begin{pmatrix} dz_1 & dz_2 & dz_3 \end{pmatrix} \begin{pmatrix} -\frac{3}{2} & & \\ & -\frac{1}{2} & \\ & & \frac{1}{2} \end{pmatrix} \begin{pmatrix} d\bar{z}_1 \\ d\bar{z}_2 \\ d\bar{z}_3 \end{pmatrix}, \quad (\text{C.1})$$

$$F^{U_1} = -\frac{i}{2} \begin{pmatrix} dz_1 & dz_2 & dz_3 \end{pmatrix} \begin{pmatrix} -\frac{3}{2} & & \\ & \frac{3}{2} & \\ & & \frac{1}{2} \end{pmatrix} \begin{pmatrix} d\bar{z}_1 \\ d\bar{z}_2 \\ d\bar{z}_3 \end{pmatrix}, \quad (\text{C.2})$$

$$F^{O_1} = -\frac{i}{2} \begin{pmatrix} dz_1 & dz_2 & dz_3 \end{pmatrix} \begin{pmatrix} \frac{5}{2} & 4 & 3 \\ 4 & \frac{1}{2} & 1 \\ 3 & 1 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} d\bar{z}_1 \\ d\bar{z}_2 \\ d\bar{z}_3 \end{pmatrix}, \quad (\text{C.3})$$

$$F^{O_2} = -\frac{i}{2} \begin{pmatrix} dz_1 & dz_2 & dz_3 \end{pmatrix} \begin{pmatrix} \frac{5}{2} & 4 & -3 \\ 4 & \frac{1}{2} & -1 \\ -3 & -1 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} d\bar{z}_1 \\ d\bar{z}_2 \\ d\bar{z}_3 \end{pmatrix}, \quad (\text{C.4})$$

$$F^{O_3} = -\frac{i}{2} \begin{pmatrix} dz_1 & dz_2 & dz_3 \end{pmatrix} \begin{pmatrix} \frac{5}{2} & -4 & -3i \\ -4 & \frac{1}{2} & i \\ 3i & -i & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} d\bar{z}_1 \\ d\bar{z}_2 \\ d\bar{z}_3 \end{pmatrix}, \quad (\text{C.5})$$

$$F^{O_4} = -\frac{i}{2} \begin{pmatrix} dz_1 & dz_2 & dz_3 \end{pmatrix} \begin{pmatrix} \frac{5}{2} & -4 & 3i \\ -4 & \frac{1}{2} & -i \\ -3i & i & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} d\bar{z}_1 \\ d\bar{z}_2 \\ d\bar{z}_3 \end{pmatrix}, \quad (\text{C.6})$$

$$F^{O_5} = -\frac{i}{2} \begin{pmatrix} dz_1 & dz_2 & dz_3 \end{pmatrix} \begin{pmatrix} -\frac{25}{2} & -2i & -i \\ 2i & \frac{1}{2} & 1 \\ i & 1 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} d\bar{z}_1 \\ d\bar{z}_2 \\ d\bar{z}_3 \end{pmatrix}, \quad (\text{C.7})$$

$$F^{O_6} = -\frac{i}{2} \begin{pmatrix} dz_1 & dz_2 & dz_3 \end{pmatrix} \begin{pmatrix} -\frac{25}{2} & -2i & i \\ 2i & \frac{1}{2} & -1 \\ -i & -1 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} d\bar{z}_1 \\ d\bar{z}_2 \\ d\bar{z}_3 \end{pmatrix}, \quad (\text{C.8})$$

$$F^{O_7} = -\frac{i}{2} \begin{pmatrix} dz_1 & dz_2 & dz_3 \end{pmatrix} \begin{pmatrix} -\frac{25}{2} & 2i & -1 \\ -2i & \frac{1}{2} & i \\ -1 & -i & \frac{1}{2} \end{pmatrix} \begin{pmatrix} d\bar{z}_1 \\ d\bar{z}_2 \\ d\bar{z}_3 \end{pmatrix}, \quad (\text{C.9})$$

$$F^{O_8} = -\frac{i}{2} \begin{pmatrix} dz_1 & dz_2 & dz_3 \end{pmatrix} \begin{pmatrix} -\frac{25}{2} & 2i & 1 \\ -2i & \frac{1}{2} & -i \\ 1 & i & \frac{1}{2} \end{pmatrix} \begin{pmatrix} d\bar{z}_1 \\ d\bar{z}_2 \\ d\bar{z}_3 \end{pmatrix}, \quad (\text{C.10})$$

$$F^A = -\frac{i}{2} \begin{pmatrix} dz_1 & dz_2 & dz_3 \end{pmatrix} \begin{pmatrix} \frac{295}{2} & & \\ & \frac{1}{2} & \\ & & \frac{1}{2} \end{pmatrix} \begin{pmatrix} d\bar{z}_1 \\ d\bar{z}_2 \\ d\bar{z}_3 \end{pmatrix}, \quad (\text{C.11})$$

$$F^B = -\frac{i}{2} \begin{pmatrix} dz_1 & dz_2 & dz_3 \end{pmatrix} \begin{pmatrix} \frac{3}{2} & & \\ & \frac{33}{2} & \\ & & \frac{1}{2} \end{pmatrix} \begin{pmatrix} d\bar{z}_1 \\ d\bar{z}_2 \\ d\bar{z}_3 \end{pmatrix}. \quad (\text{C.12})$$

Using the above fluxes, one can find out the chiral multiplets in the model. This has been done for the brane intersections involving stacks -  $U_5, U_1$ . A computation of the chiral fermion multiplicities on the intersections  $O_i - O_j$  and  $O_i - O_j^*$ , for  $i, j = 1, \dots, 8$ , implies the existence of following fields in the non-chiral spectrum of the model. They are:  $(\phi_{+-}^{O_1O_2}, \phi_{-+}^{O_1O_2}, 40)$ ,  $(\phi_{+-}^{O_1O_3}, \phi_{-+}^{O_1O_3}, 84)$ ,  $(\phi_{+-}^{O_1O_4}, \phi_{-+}^{O_1O_4}, 84)$ ,  $(\phi_{+-}^{O_1O_5}, 20)$ ,  $(\phi_{+-}^{O_1O_6}, \phi_{-+}^{O_1O_6}, 49)$ ,  $(\phi_{+-}^{O_1O_7}, 6)$ ,  $(\phi_{+-}^{O_1O_8}, 14)$ ,  $(\phi_{+-}^{O_2O_3}, \phi_{-+}^{O_2O_3}, 84)$ ,  $(\phi_{+-}^{O_2O_4}, \phi_{-+}^{O_2O_4}, 84)$ ,  $(\phi_{+-}^{O_2O_5}, \phi_{-+}^{O_2O_5}, 49)$ ,  $(\phi_{+-}^{O_2O_6}, 20)$ ,  $(\phi_{+-}^{O_2O_7}, 14)$ ,  $(\phi_{+-}^{O_2O_8}, 6)$ ,  $(\phi_{+-}^{O_3O_4}, \phi_{-+}^{O_3O_4}, 40)$ ,  $(\phi_{+-}^{O_3O_5}, 14)$ ,  $(\phi_{+-}^{O_3O_6}, 6)$ ,  $(\phi_{+-}^{O_3O_7}, 20)$ ,  $(\phi_{+-}^{O_3O_8}, \phi_{-+}^{O_3O_8}, 49)$ ,  $(\phi_{+-}^{O_4O_5}, 6)$ ,  $(\phi_{+-}^{O_4O_6}, 14)$ ,  $(\phi_{+-}^{O_4O_7}, \phi_{-+}^{O_4O_7}, 49)$ ,  $(\phi_{+-}^{O_4O_8}, 20)$ ,  $(\phi_{+-}^{O_5O_6},$

$$\begin{aligned}
& (\phi_{-+}^{O_5O_6}, 8), (\phi_{+-}^{O_5O_7}, \phi_{-+}^{O_5O_7}, 20), (\phi_{+-}^{O_5O_8}, \phi_{-+}^{O_5O_8}, 20), (\phi_{+-}^{O_6O_7}, \phi_{-+}^{O_6O_7}, 20), (\phi_{+-}^{O_6O_8}, \phi_{-+}^{O_6O_8}, 20), \\
& (\phi_{+-}^{O_7O_8}, \phi_{-+}^{O_7O_8}, 8), (\phi_{++}^{O_1O_2^*}, 59), (\phi_{--}^{O_1O_3^*}, 33), (\phi_{--}^{O_1O_4^*}, 33), (\phi_{++}^{O_1O_5^*}, 86), (\phi_{--}^{O_1O_6^*}, 10), (\phi_{++}^{O_1O_7^*}, \\
& 24), (\phi_{++}^{O_1O_8^*}, 52), (\phi_{--}^{O_2O_3^*}, 33), (\phi_{--}^{O_2O_4^*}, 33), (\phi_{--}^{O_2O_5^*}, 10), (\phi_{++}^{O_2O_6^*}, 86), (\phi_{++}^{O_2O_7^*}, 52), (\phi_{++}^{O_2O_8^*}, \\
& 24), (\phi_{++}^{O_3O_4^*}, 59), (\phi_{++}^{O_3O_5^*}, 52), (\phi_{++}^{O_3O_6^*}, 24), (\phi_{++}^{O_3O_7^*}, 86), (\phi_{--}^{O_3O_8^*}, 10), (\phi_{++}^{O_4O_5^*}, 24), (\phi_{++}^{O_4O_6^*}, \\
& 52), (\phi_{--}^{O_4O_7^*}, 10), (\phi_{++}^{O_4O_8^*}, 86), (\phi_{--}^{O_5O_6^*}, 41), (\phi_{++}^{O_5O_7^*}, 23), (\phi_{++}^{O_5O_8^*}, 23), (\phi_{++}^{O_6O_7^*}, 23), (\phi_{++}^{O_6O_8^*}, \\
& 23), (\phi_{--}^{O_7O_8^*}, 41). \tag{C.13}
\end{aligned}$$

As a result of a similar analysis for the remaining stacks  $A$  and  $B$ , we have also the following fields:

$$\begin{aligned}
& (\phi_{+-}^{U_5A}, \phi_{-+}^{U_5A}, 149), (\phi_{++}^{U_5A^*}, \phi_{--}^{U_5A^*}, 146), (\phi_{+-}^{U_5B}, \phi_{-+}^{U_5B}, 51), (\phi_{++}^{U_5B^*}, \phi_{--}^{U_5B^*}, 16), (\phi_{+-}^{U_1A}, \phi_{-+}^{U_1A}, \\
& 149), (\phi_{+-}^{U_1B}, \phi_{-+}^{U_1B}, 45), (\phi_{+-}^{AB}, \phi_{-+}^{AB}, 2336), (\phi_{++}^{U_1B^*}, \phi_{--}^{U_1B^*}, 18), (\phi_{+-}^{U_1A^*}, 292), (\phi_{+-}^{AB^*}, 149). \tag{C.14}
\end{aligned}$$

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