CERN-PH-TH/2007-235
BICOCCA-FT-07-16
SISSA 87/2007/EP
LMU-ASC 69/07

On the geometry and the moduli space of β-deformed quiver gauge theories

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Abstract

We consider a class of super-conformal β-deformed $N = 1$ gauge theories dual to string theory on $AdS_5 \times X$ with fluxes, where X is a deformed Sasaki-Einstein manifold. The supergravity backgrounds are explicit examples of Generalised Calabi-Yau manifolds: the cone over X admits an integrable generalised complex structure in terms of which the BPS sector of the gauge theory can be described. The moduli spaces of the deformed toric $N = 1$ gauge theories are studied on a number of examples and are in agreement with the moduli spaces of D3 and D5 static and dual giant probes.

1 Introduction

The super-conformal gauge theories living on D3-branes at singularities generally admit marginal deformations. A particularly interesting case of marginal deformation for theories with $U(1)^3$ global symmetries is the so called -deformation [1]. The most famous example is the -deformation of $\mathcal{N} = 4$ SYM which has been extensively studied both from the field theory point of view and the dual gravity perspective. In particular, in [2], Lunin and Maldacena found the supergravity dual solution, which is a completely regular AdS_5 background. Their construction can be generalised to the super-conformal theories associated with the recently discovered Sasaki-Einstein backgrounds AdS_5 $L^{p,q,r}$ [3]. More generally, all toric quiver gauge theories admit -deformations [4] and, as we will see, have regular gravitational duals. The resulting -deformed theories are interesting both from the point of view of field theory and of the gravity dual.

On the field theory side, we deal with a gauge theory with a deformed moduli space of vacua and a deformed spectrum of BPS operators. The case of $\mathcal{N} = 4$ SYM has been studied in details in the literature [5–7]. In this paper we extend this analysis to a generic toric quiver gauge theory. The moduli space of the -deformed gauge theory presents the same features as in $\mathcal{N} = 4$ case. In particular, its structure depends on the value of the deformation parameter . For generic the deformed theory admits a Coulomb branch which is given by a set of complex lines. For rational there are additional directions corresponding to Higgs branches of the theory.

On the gravity side, the dual backgrounds can be obtained from the original Calabi-Yaus with a continuous T-duality transformation using the general method proposed in [2]. We show that it is possible to study the -deformed background even in the cases where the explicit original Calabi-Yau metric is not known. The toric structure of the original background is enough. Besides the relevance for AdS/CFT, the -deformed backgrounds are also interesting from the geometrical point view. They are Generalised Calabi-Yau manifolds [8,9]: after the deformation the background is no longer complex, but it still admits an integrable generalised complex structure. Actually the -deformed backgrounds represent one of the few explicit known examples of generalised geometry solving the equation of motions of type II supergravity [1]. The extreme simplicity of such backgrounds make it possible to explicitly apply the formalism of Generalised Complex Geometry, which, as we will see, provides an elegant way to study T-duality and brane probes [13–16].

The connection between gravity and field theory is provided by the study of supersymmetric D-brane probes moving on the -deformed background. In this paper we will analyse the case of static D3 and D5 probes, as well as the case of D3 and D5 dual giant gravitons. We will study in details existence and moduli space of such probes. We show that, in the -deformed background, both static D3 probes and D3 dual giants can only live on a set of intersecting complex lines inside

[1] For other non compact examples see [10,11] and for compact ones [12].

the deformed Calabi-Yau, corresponding to the locus where the T^3 toric fibration degenerates to T^1. This is in agreement with the abelian moduli space of the β-deformed gauge theory which indeed consists of a set of lines. Moreover, in the case of rational β, we demonstrate the existence of both static D5 probes and D5 dual giant gravitons with a moduli space isomorphic to the original Calabi-Yau divided by a $Z_n \times Z_n$ discrete symmetry. This statement is the gravity counterpart of the fact that, for rational β, new branches are opening up in the moduli space of the gauge theory [5,6]. Our analysis also generalises the results of [17] where it has been shown that the classical phase space of supersymmetric D3 dual giant gravitons in the undeformed Calabi-Yau background is isomorphic to the Calabi-Yau variety.

The classical way to study probe configuration is to solve the equations of motion coming from the probe Dirac-Born-Infeld action. Generalised Complex Geometry provides an alternative method to approach the problem. As we will explain, a D-brane is characterised by its generalised tangent bundle. The dual probes in the β-deformed geometry can be obtained from the original ones applying T-duality to their generalised tangent bundles. The approach in terms of Generalised Geometry allows also to clarify how the complex structure of the gauge theory is reflected by the gravity dual, which, as we have already mentioned, is not in general a complex manifold.

The study of brane probes we present here can be seen as consisting of two independent and complementary sections, one dealing with the Born-Infeld approach and the other one using Generalised Complex Geometry. We decided to keep the two analysis independent, so that the reader not interested in one of the two can skip the corresponding section.

The paper is organized as follows. In Section 2 we discuss the structure of the β-deformed gauge theory and of its gravity dual, and we characterize it in terms of pure spinors. In Section 3 we study the moduli space of D3 and D5-brane, static probes and dual giant gravitons, on the deformed background using the Born-Infeld action, while in Section 4 we analyse the same configurations using the generalised tangent bundle approach. We will show that, as usual for BPS quantities, the explicit knowledge of the Calabi-Yau metric is not required to extract sensible results. Our analysis thus applies to the most general toric background. In Section 5 we briefly comment about supersymmetric giant gravitons in the deformed background. In Section 6 we explicitly demonstrate through examples and general arguments that the results of Sections 3 and 4 agrees with the field theory analysis which is performed in details. Finally, in the Appendices we collect various technical proofs, arguments and examples.

2 -deformation in toric theories

2.1 -deformed quiver gauge theories

The entire class of super-conformal gauge theories living on D3-branes at toric conical Calabi-Yau singularities admits marginal deformations. The most famous example is the -deformation of $N = 4$ SYM with $SU(N)$ gauge group where the original superpotential

$$_{1\ 2\ 3} \quad _{1\ 3\ 2} \tag{2.1}$$

is replaced by the -deformed one

$$e^{i} \quad _{1\ 2\ 3} \quad e^{i} \quad _{1\ 3\ 2}: \tag{2.2}$$

A familiar argument due to Leigh and Strassler [1] shows that the -deformed theory is conformal for all values of the parameter.

Similarly, a -deformation can be defined for the conifold theory. The gauge theory has gauge group $SU(N) \quad SU(N)$ and bi-fundamental fields $(A_i)^{A}$ and $(B_p)_{A}$ with $;A = 1;:::;N;i;p = 1;2$ transforming in the representations $(2;1)$ and $(1;2)$ of the global symmetry group $SU(2) \quad SU(2)$, respectively, and superpotential

$$A_1B_1A_2B_2 \quad A_1B_2A_2B_1: \tag{2.3}$$

The -deformation corresponds to the marginal deformation where the superpotential is replaced by

$$e^{i} \quad A_1B_1A_2B_2 \quad e^{i} \quad A_1B_2A_2B_1: \tag{2.4}$$

Both theories discussed above possess a $U(1)^3$ geometric symmetry corresponding to the isometries of the internal space, one $U(1)$ is an R-symmetry while the other two act on the fields as flavour global symmetries[2]. The -deformation is strongly related to the existence of such $U(1)^3$ symmetry and has a nice and useful interpretation in terms of non-commutativity in the internal space [2]. The deformation is obtained by selecting in $U(1)^3$ the two flavour symmetries Q_i commuting with the supersymmetry charges and using them to define a modified non-commutative product. This corresponds in field theory to replacing the standard product between two matrix-valued elementary fields f and g by the star-product

$$f \quad g \quad {}^{i}e^{(Q^f \wedge Q^g)} fg \tag{2.5}$$

where $Q^f = (Q_1^f;Q_2^f)$ and $Q^g = (Q_1^g;Q_2^g)$ are the charges of the matter fields under the two $U(1)$ flavour symmetries and

$$(Q^f \wedge Q^g) = (Q_1^f Q_2^g \quad Q_2^f Q_1^g): \tag{2.6}$$

[2] This $U(1)^3$ symmetry can be enhanced to a non abelian one in special cases. For instance it is $SU(4)$ for $N = 4$ SYM and $SU(2) \quad SU(2) \quad U(1)_R$ for the conifold. In addition the conifold possesses a $U(1)_B$ baryonic symmetry. A generic toric quiver, besides the geometric symmetry $U(1)^3 = U(1)_F^2 \quad U(1)_R$, presents several baryonic $U(1)$ symmetries. In this paper we will only be interested in the geometric symmetries of these theories.

The β-deformation preserves the $U(1)^3$ geometric symmetry of the original gauge theory, while other marginal deformations in general further break it.

All the superconformal quiver theories obtained from toric Calabi-Yau singularities have a $U(1)^3$ symmetry corresponding to the isometries of the Calabi-Yau and therefore admit exactly marginal β-deformations. The theories have a gauge group $\prod_{i=1}^{G} SU(N)$, bi-fundamental fields X_i and a bipartite structure which is inherited from the dimer construction [18]. The superpotential contains an even number of terms V naturally divided into $V/2$ terms weighted by a $+1$ sign and $V/2$ terms weighted by a -1 sign

$$\sum_{i=1}^{V/2} W_i(X) - \sum_{i=1}^{V/2} \tilde{W}_i(X) : \tag{2.7}$$

The β-deformed superpotential is obtained by replacing the ordinary product among fields with the star-product (2.5) and, as discussed in Appendix B, can always be written after rescaling fields as [4]

$$e^{i\pi\beta} \sum_{i=1}^{V/2} W_i(\varphi) - e^{-i\pi\beta} \sum_{i=1}^{V/2} W_i(\varphi) \tag{2.8}$$

where β is some rational number. It is obvious how $\mathcal{N} = 4$ SYM and the conifold fit in this picture; other examples will be given in Section 6.

The β-deformation drastically reduces the mesonic moduli space of the theory, which is originally isomorphic to the N-fold symmetric product of the internal Calabi-Yau. To see quickly what happens consider the case where the $SU(N)$ groups are replaced by $U(1)$'s – by abuse of language we can refer to this as the $N = 1$ case. Physically, we are considering a mesonic direction in the moduli space where a single D3-brane is moved away from the singularity. In the undeformed theory the D3-brane probes the Calabi-Yau while in the β-deformed theory it can only probe a subvariety consisting of complex lines intersecting at the origin. This can be easily seen in $\mathcal{N} = 4$ and in the conifold case.

For $\mathcal{N} = 4$ SYM the F-term equations read

$$\Phi_i \Phi_j = b\, \Phi_j \Phi_i; \qquad (i;j) = (1;2);(2;3) \text{ or } (3;1) \tag{2.9}$$

where $b = e^{2\pi i\beta}$. Since Φ_i are c-numbers in the $N = 1$ case, these equations are trivially satisfied for $\beta = 0$, implying that the moduli space is given by three unconstrained complex numbers Φ_i, giving a copy of $\mathbb{C}^3$. However, for $\beta \neq 0$ these equations can be satisfied only on the three lines given by the equations $\Phi_j = \Phi_k = 0$ for $j \neq k$. Only one field Φ_i is different from zero at a time.

For the conifold the F-term equations read

$$\begin{aligned} B_1 A_1 B_2 &= b^{-1} B_2 A_1 B_1 ; \\ B_1 A_2 B_2 &= b\, B_2 A_2 B_1 ; \\ A_1 B_1 A_2 &= b\, A_2 B_1 A_1 ; \\ A_1 B_2 A_2 &= b^{-1} A_2 B_2 A_1 : \end{aligned} \tag{2.10}$$

These equations are again trivial for $\beta = 0$ and $N = 1$, the fields becoming commuting c-numbers. The brane moduli space is parametrized by the four gauge invariant mesons

$$x = A_1 B_1\,; \quad y = A_2 B_2\,; \quad z = A_1 B_2\,; \quad w = A_2 B_1 \qquad (2.11)$$

which are not independent but subject to the obvious relation $xy = zw$. This is the familiar description of the conifold as a quadric in C^4. For $\beta \neq 0$, the F-term constraints (2.10) are solved when exactly one field A and one field B are different from zero. This implies that only one meson can be different from zero at a time. The moduli space thus reduces to the four lines

$$y = z = w = 0\,; \qquad x = z = w = 0\,; \qquad x = y = z = 0\,; \qquad x = y = w = 0\,. \qquad (2.12)$$

We will see in Section 3.2 using the dual gravity solutions and in Section 6 using field theory that for all β-deformed toric quivers the abelian mesonic moduli space is reduced to d complex lines, where d is the number of vertices in the toric diagram of the singularity.

Something special happens for β rational. New branches in the moduli space open up. The $\mathcal{N} = 4$ case was originally discussed in [5] and the conifold in [19]. In all cases these branches can be interpreted as one or more branes moving on the quotient of the original Calabi-Yau by a discrete $Z_n \times Z_n$ symmetry. We will describe these branes explicitly in the gravitational duals in Section 3.2. The field theory analysis of these vacua requires a little bit of technical patience and it is deferred to Section 6.

2.2 β-deformed toric manifolds

The general prescription for determining the supergravity dual of a β-deformed theory has been given by Lunin and Maldacena [2]. The original background has a $U(1)^3$ isometry and the prescription amounts to performing a particular T-duality along two $U(1)$ directions commuting with the supersymmetry charges.

For a quiver gauge theory, the undeformed gravity solution is a warped product of 4-dimensional Minkowski times a Calabi-Yau cone over a Sasaki-Einstein manifold

$$ds^2_{10} = e^{2A} ds^2_4 + e^{-2A} ds^2_6\,; \qquad (2.13)$$

where the warp factor is $e^{2A} = r^2$. In all the formulae we are omitting factors of the radius of Anti de Sitter (see footnote 3 at page 9).

In the toric case these Calabi-Yaus have exactly three isometries and the Lunin–Maldacena method can be applied. In [2] the β-deformation of the conifold and of Y^{pq} spaces are explicitly computed using the known metrics for these Sasaki-Einstein spaces. In this paper we consider the general case of a toric Calabi-Yau cone. We will show that, as usual, most computations regarding supersymmetric quantities can be performed without knowing the explicit form of the metric. We will just need the general characterisations of the Calabi-Yau metrics given in [20] which we now review.

2.2.1 The geometry of toric Calabi-Yau cones

The geometry of a toric Calabi-Yau cone is completely determined by d integer vectors $V_\alpha \in Z^3$. In fact there is a very explicit description of toric cones as T^3 fibrations over a rational polyedron described by [20]

$$C = \{y \in R^3 | l_\alpha(y) = V_\alpha^i y_i \geq 0;\ \alpha = 1 \ldots d\} \tag{2.14}$$

where V_α are the inward pointing vectors orthogonal to the facets of the polyedral cone. The T^3 fibration degenerates to T^2 on the facets of the polyedron, $l_\alpha(y) = 0$, and further degenerates to T^1 on the edges (intersections of two facets). As a simple example, the trivial Calabi-Yau C^3 parametrized by three complex variables $Z_i = \sqrt{2y_i}e^{i\phi_i}$ can be considered as a T^3 fibration, parameterised by the three angles ϕ_i, over the first octant in R^3 given by the three equations $y_i \geq 0$. Here $V_1 = (1;0;0)$, $V_2 = (0;1;0)$, and $V_3 = (0;0;1)$. In the following we will make a convenient change of coordinates in order to have the third coordinate of all V_α equal to one. Similarly, the conifold can be described as a T^3 fibration over a polyedron with four sides, as shown in Figure 1.

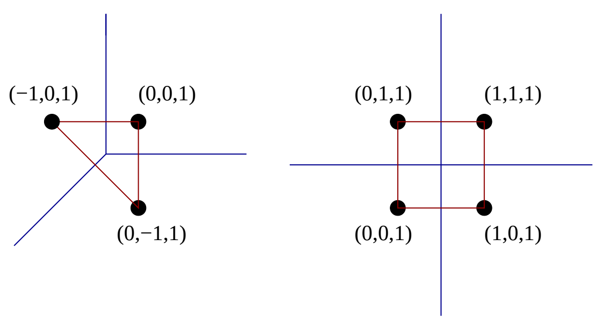

Figure 1: The toric diagram for C^3 and the conifold consisting of the points $V_\alpha = (v_\alpha;1)$ pictured in the plane $z = 1$ in R^3. The vectors V_α determine a rational polyedron in R^3 with three and four sides, respectively, whose projection on the plane $z = 1$ is shown in the Figure.

As shown in [20] the metric on the Calabi-Yau cone can be written as

$$ds_6^2 = g^{ij}dy_i dy_j + g_{ij}d\phi^i d\phi^j \tag{2.15}$$

with g^{ij} the inverse matrix of g_{ij}. Due to the toric condition, g_{ij} only depends on the variables y_i; the metric is a cone if and only if g^{ij} is homogeneous of degree -1 in y. Regularity of the metric implies that near the facets

$$g^{ij} = \sum_{\alpha=1}^{d} \frac{V_\alpha^i V_\alpha^j}{l_\alpha(y)} + \text{regular terms}: \tag{2.16}$$

The Calabi-Yau condition further requires that the vectors V_α lie on a plane. We will choose coordinates where $V_\alpha = (v_\alpha; 1)$. The integer points in the plane, v_α, describe the toric diagram of the Calabi-Yau.

As in [20] we can also use complex coordinates to describe the manifold

$$z^i = x^i + i\phi^i . \tag{2.17}$$

A Kalher metric can be written in terms of a Kalher potential $F(z^i)$. In the toric case F only depends on the real part, x^i, of the coordinates so that, if we define

$$g_{ij} = \frac{\partial^2 F}{\partial x^i \partial x^j} , \tag{2.18}$$

the metric can be written as

$$ds_6^2 = g_{ij} dz^i d\bar{z}^j = g_{ij} dx^i dx^j + g_{ij} d\phi^i d\phi^j . \tag{2.19}$$

There is a nice relation between symplectic and complex coordinates given by

$$y_i = \frac{\partial F}{\partial x^i} \tag{2.20}$$

and, as the notation suggests, the function $g_{ij}(x)$ appearing in the complex coordinates form of the metric is the same as the function $g_{ij}(y)$ appearing in the symplectic form of the metric after changing variables from x to y.

The Kahler form and the holomorphic three-form are given by

$$J_{(0)} = \frac{i}{2} g_{ij} dz^i \wedge d\bar{z}^j , \tag{2.21}$$

$$\Omega_{(0)} = e^{i\psi} \sqrt{\det g_{ij}}\, dz^1 \wedge dz^2 \wedge dz^3 \tag{2.22}$$

$$= e^{x^3 + i\phi^3} dz^1 \wedge dz^2 \wedge dz^3 . \tag{2.23}$$

As shown in [20], the explicit form of $\Omega_{(0)}$ given in (2.23) follows from Ricci-flatness, which implies $\det g_{ij} = e^{2x^3}$, and correlates the phase in $\Omega_{(0)}$ with the complex direction z^3 associated with the third component of the vectors $V_\alpha = (v_\alpha; 1)$.

The R-symmetry of the gauge theory is dual to the Reeb vector of the Sasaki-Einstein space

$$K = \sum_{i=1}^{3} b^i \frac{\partial}{\partial \phi^i} , \tag{2.24}$$

where the components $b^i = 2g^{ij} y_j$ turn out to be constants [20]. Moreover the third component b_3 is set to 3 by the Calabi-Yau condition. The vector $b = (b^i; 3)$ satisfies

$$g_{ij} b^i b^j = r^2 . \tag{2.25}$$

The Reeb vector K is the partner under the complex structure of the dilatation operator $r\partial_r$. Notice that the conical form of the metric is hidden both in the symplectic and complex coordinates. The very same radial coordinate r is given by a non-trivial expression depending on the actual value of the Reeb vector

$$r^2 = 2b^i y_i . \tag{2.26}$$

2.2.2 The β-deformed Calabi-Yau

The β-deformation of toric Calabi-Yaus can be obtained as in [2]. For simplicity we will consider β real in the following. We consider a two-torus in the internal manifold and we perform a T-duality transformation that acts on the complexified Kahler modulus of the two-torus as

$$\tau = B_{T^2} + i\sqrt{\det g_{T^2}} \;\rightarrow\; \frac{\tau}{1+\gamma\tau}\,: \tag{2.27}$$

Here we choose the T^2 in the directions $(\phi_1;\phi_2)$ since the action leaves the holomorphic three-form invariant. The parameter γ in supergravity is proportional to the β-parameter in the gauge theory.

The T-dual metric and B-field can be computed via Buscher rules

$$E = g - B_2 \;\rightarrow\; (dE + c)(aE + b)^{-1} \tag{2.28}$$

by embedding the $O(2;2)$ transformation (2.27) in $O(6;6)$

$$O_{LM} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \mathrm{Id}_6 & \Gamma \\ 0 & \mathrm{Id}_6 \end{pmatrix}\,; \tag{2.29}$$

where the bivector Γ is defined as

$$\Gamma = \begin{pmatrix} 0_3 & 0 & 0 \\ 0 & i\gamma\sigma_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}\,: \tag{2.30}$$

The choice of the two-torus introduces a four plus two splitting in the metric that can be made explicit by rewriting it in the following form

$$ds_6^2 = h_{ab}\,\eta^a_{(0)}\,\eta^b_{(0)} + Z\bar{Z} \qquad a;b = 1;2 \tag{2.31}$$

where $h_{ab} = g_{ab}$ is the metric on the two-torus and we have defined the one-forms

$$\eta^a_{(0)} = dz^a + h^{ac} g_{c3}\, dz^3 \qquad a = 1;2\,; \tag{2.32}$$

$$= (dx^a + h^{ac} g_{c3}\, dx^3) + i(d\phi^a + h^{ac} g_{c3}\, d\phi^3) = X^a + iY^a \tag{2.33}$$

$$Z = e^{i\phi^3}\sqrt{g_{33} - h^{ab} g_{a3} g_{b3}}\, dz^3 = \frac{dw^3}{r^2\sqrt{h}} \tag{2.34}$$

with $h = \det(h_{ab})/r^4$. The subscript (0) is to distinguish these forms from the corresponding one in the T-dual background. We also defined $w_3 = e^{z^3}$. The one form Z parameterises the direction orthogonal to the two-torus and to pass from the first to the second expression in (2.34) we used the identity

$$\det(g_{ij}) = e^{2x^3} = \det(h_{ab})(g_{33} - h^{ab} g_{a3} g_{b3})\,: \tag{2.35}$$

The advantage of writing the metric as in (2.31) is that the T-duality transformation (2.29) results simply in a rescaling of its angular part

$$ds_6^2 = h_{ab}X^aX^b + G\,h_{ab}Y^aY^b + Z\bar{Z} \tag{2.36}$$

by the function

$$G = \frac{1}{1+\gamma^2 h}\,. \tag{2.37}$$

The antisymmetric part of (2.28) gives the NS two-form of the γ-deformed solution

$$B = -\gamma\, h\, G\, Y^1 \wedge Y^2\,. \tag{2.38}$$

The dilaton and the warp factor are

$$e^{\phi} = \sqrt{G}\,; \qquad e^{A} = r\,; \tag{2.39}$$

respectively, while the non-vanishing RR fields are given by[3]

$$F_5 = 4\mathrm{vol}_4 \wedge \frac{dr}{r} + 4G\,\mathrm{vol}_{X_5}\,; \tag{2.40}$$

$$F_3 = -4\gamma\,\omega_2 \wedge d\phi^3 = dC_2\,; \tag{2.41}$$

where $\mathrm{vol}_{X_5} = \iota_{\frac{dr}{r}} \mathrm{vol}_6 = \omega_2 \wedge d\phi^1 \wedge d\phi^2 \wedge d\phi^3$ is the volume form of the undeformed Sasaki-Einstein manifold X_5, and the closed form ω_2 depends only on the x^i coordinates.

2.3 The γ-deformed pure spinors

Recently it has been shown that a unifying formalism to treat $\mathcal{N} = 1$ compactifications with non trivial background fluxes is provided by Generalised Complex Geometry. For a detailed discussion of pure spinors, Generalised Complex Geometry and its applications to string theory see [12,21,22]; here we will very briefly summarise what we will need in the following section.

The idea is, given a manifold, to study objects defined on the sum of the tangent and cotangent bundles, $T \oplus T^*$. We can for instance define spinors on $T \oplus T^*$: these will be $SO(6,6)$ spinors and have a representation in terms of differential forms of mixed degree, $\Lambda^\bullet(T^*)$. We call pure the spinors that are annihilated by half of the generators of $Cliff(6,6)$. They are represented by sum of even and odd forms, $\Phi_\pm$, corresponding to the positive and negative chirality, respectively.

[3] In all the formulae for the background we are understanding factors of the AdS_5 radius, L, which is given by: $L^4 = 4\pi^4 g_s N \alpha'^2/\mathrm{Vol}(X_5)$, where N is the number of D3-branes and X_5 is the undeformed Sasaki-Einstein manifold. In particular the metric ds^2_{10} has a factor of L^2, the NS flux H a factor of L^4, F_3 and F_5 a factor of L^4/g_s and G should be defined as: $G^{-1} = 1 + \gamma^2 L^4 h$. Our formulae are in the string frame and we will set $\alpha' = 1$.

The relevance for supergravity lies in the observation that such pure spinors can be obtained as tensor products of ordinary spinors. More precisely, if we decompose the type IIB ten-dimensional supersymmetry parameters as

$$\varepsilon^i = \zeta_+ \otimes \eta^i_+ + \zeta_- \otimes \eta^i_- \, ; \tag{2.42}$$

where ζ_+ ($\zeta_- = \zeta_+^*$) and η^i_+ ($\eta^i_- = \eta^{i*}_+$) are positive chirality spinors in four and six dimensions, the pure spinors are defined as

$$\Phi_+ = \eta^1_+ \otimes \eta^{2\dagger}_+ \, ; \tag{2.43}$$

$$\Phi_- = \eta^1_+ \otimes \eta^{2\dagger}_- \, . \tag{2.44}$$

The spinors constructed this way define an $SU(3) \times SU(3)$ structure on $T \oplus T^*$ [4]. By introducing an inner product between forms (Mukai pairing)

$$\langle A, B\rangle \equiv (A \wedge \sigma(B))|_{top} \qquad \sigma(A_n) = (-)^{Int[n/2]} \, ; \tag{2.45}$$

we can define the norm of the pure spinors as

$$\langle \Phi_+, \bar{\Phi}_+ \rangle = \langle \Phi_-, \bar{\Phi}_- \rangle = \frac{i}{8}\|\Phi\|^2 \mathrm{vol}_6 = \frac{i}{8}\|\eta_1\|^2\|\eta_2\|^2 \mathrm{vol}_6 \, . \tag{2.46}$$

It is convenient to introduce normalised twisted spinors

$$\hat{\Phi}_\pm = e^{-\phi} e^{-B} \hat{\Phi}_\pm = \frac{8i}{\|\Phi\|} e^{-\phi} e^{-B} \hat{\Phi}_\pm \, . \tag{2.47}$$

All the NS content of the background (internal metric, B field and dilaton) can be extracted from $\hat{\Phi}_\pm$. Moreover the twisted pure spinors are those transforming nicely under T-duality.

Using the above definition as bispinors, it is possible to rewrite the supersymmetry conditions for type IIB supergravity as differential equations for the pure spinors $\hat{\Phi}_\pm$

$$d(e^{3A-\phi}\hat{\Phi}_-) = 0 \, ; \tag{2.48}$$

$$d(e^{2A-\phi}\,\mathrm{Im}\,\hat{\Phi}_+) = 0 \, ; \tag{2.49}$$

$$d(e^{4A-\phi}\,\mathrm{Re}\,\hat{\Phi}_+) = e^{4A} e^{-B} * \lambda(F) \, . \tag{2.50}$$

Here the $*$ is with respect to the six dimensional internal metric $e^{-2A} ds^2_6$ and F is the sum of the internal magnetic fields $F = F_1 + F_3 + F_5$. It is related to the ten-dimensional RR fields as $F^{(10)} = F + \mathrm{vol}_4 \wedge \lambda(*F)$. The ten-dimensional Bianchi identity $(d - H\wedge)F^{(10)} = 0$ yields the Bianchi identity and the equations of motion for F: $(d - H\wedge)F = 0$ and $(d + H\wedge)(e^{4A} * F) = 0$, respectively. Notice that the equations of motion follow automatically from (2.50).

[4] The pure spinors must obey the $SU(3) \times SU(3)$ compatibility conditions $\langle \Phi_-, X \cdot \Phi_+\rangle = \langle \bar{\Phi}_-, X \cdot \Phi_+\rangle = 0$ for any element $X = x + \xi$ of $T \oplus T^*$, where x and ξ are a vector and a one-form, respectively.

The pure spinor satisfying $d(e^{3A}\hat\Phi_-) = 0$, defines a twisted generalised Calabi–Yau [21,22]. Thus one can interpret the closure of the pure spinor coming from the supersymmetry variations as the generalisation to the flux case of the standard Calabi–Yau condition for fluxless compactifications: all $N = 1$ vacua are Generalised Calabi–Yau manifolds [9].

The explicit form of the pure spinors depends on how the internal supersymmetry parameters ϵ^i are related to the globally defined spinors on the manifold. For the toric Calabi–Yau manifolds there is one globally defined (in this case covariantly constant) spinor, η_+, so that one can choose

$$\epsilon^1_+ = e^{A/2}\eta_+ ; \qquad \epsilon^2_+ = i e^{A/2}\eta_+ ; \tag{2.51}$$

and the pure spinors are given in terms of the Kalher form and holomorphic three–form

$$\hat\Phi^{(0)}_- = e^{3A}\,\Omega_{(0)} = e^{3A}\, dz^1\wedge dz^2\wedge dw^3 ; \tag{2.52}$$

$$\hat\Phi^{(0)}_+ = e^{-ie^{-2A}J_{(0)}} = e^{1/2\, e^{-2A} g_{i\bar j} dz^i\wedge d\bar z^j} . \tag{2.53}$$

In the Calabi–Yau background the dilaton and the NS two–form are zero, so that there is no difference between twisted and untwisted spinors.

We now want to construct the pure spinors corresponding to the β-deformed backgrounds as the T-duals of the Calabi–Yau ones. As shown in [23] the T-duality transformation (2.29) on the pure spinors is given by

$$\hat\Phi^{(0)} \to \hat\Phi = e^{\beta}\,\hat\Phi^{(0)} = (1+\beta)\,\hat\Phi^{(0)} ; \tag{2.54}$$

where β is a bivector associated with the two $U(1)$ isometries, ϕ_1 and ϕ_2, of the Calabi–Yau. It acts on the pure spinor by contractions[5]

$$\beta = \gamma\,\partial_{\phi_1}\wedge\partial_{\phi_2} = \gamma\,\partial_{\phi_1}\lrcorner\,\partial_{\phi_2}\lrcorner . \tag{2.56}$$

Applying (2.56) to (2.53) and (2.52) we obtain a new pair of pure spinors (here we have undone the twist)

$$\Phi_- = -\sqrt{G}\, e^{3A}\, dw^3\wedge e^{\frac{1}{\gamma}dz^1\wedge dz^2 + B} ; \tag{2.57}$$

$$\Phi_+ = \sqrt{G}\, e^{-ie^{-2A}J_{(0)} - \gamma h X^1\wedge X^2 + B} ; \tag{2.58}$$

---

[5] A generator of $O(6;6)$ acts linearly on the elements of $T\oplus T^*$. If we define a generic element of $T\oplus T^*$ as $(X;\xi)$, with X a vector and ξ a one form, we have

$$\begin{pmatrix} X \\ \xi \end{pmatrix} \to \begin{pmatrix} A & \beta \\ B & -A^T \end{pmatrix}\begin{pmatrix} X \\ \xi \end{pmatrix} ; \tag{2.55}$$

where A is an $SO(6)$ element, $A = A^n_m\, dx^m \otimes \partial_{x^n}$, B is a two–form $B = \frac{1}{2}B_{mn}dx^m\wedge dx^n$, and β is a bivector $\beta = \frac{1}{2}\beta^{mn}\partial_{x^m}\wedge\partial_{x^n}$. Then $O(6;6)$ element corresponding to the β-deformation, (2.29), is just the bivector and and thus acts as in (2.56) on a generic differential form.

where $B = -\gamma h G Y^1 \wedge Y^2$ is the NS two-form of the β-deformed background[6]. The usual $SU(3)\times SU(3)$ compatibility conditions between $\hat\Phi_-$ and $\hat\Phi_+$ continue to hold since the Mukai pairing is invariant under a general $SO(6;6)$ transformation.

The expression for the closed pure spinor, (2.57), has a nice interpretation in terms of the generalised Darboux theorem [22]. The pure spinors (2.57), (2.58) are of type $(1;0)$ and determine a splitting into four coordinates of symplectic type and two of complex type. The closure condition $d(e^{3A}\hat\Phi_-) = 0$ implies the existence of symplectic-complex coordinates $(\xi^i; z)$, $i = 1, \ldots, 4$ with

$$e^{3A}\hat\Phi_- = e^{ik_0 + \tilde B}\wedge dz\,; \tag{2.63}$$

where $k_0 = d\xi^1\wedge d\xi^2 + d\xi^3\wedge d\xi^4$ is the natural symplectic form and $\tilde B$ is a potential for H, $d\tilde B = H$ [22]. The symplectic coordinates predicted by the theorem are easily identified from equation (2.57)

$$\frac{1}{\gamma}dz^1\wedge dz^2 + B \sim \frac{i}{\gamma}(dx^1\wedge d\xi^2 - dx^2\wedge d\xi^1) + \tilde B \tag{2.64}$$

with the real and imaginary parts of the original complex coordinates of the Calabi-Yau $(x^i; \xi^i)$; $\tilde B = B + \frac{1}{\gamma}(dx^1\wedge dx^2 - d\xi^1\wedge d\xi^2)$. We see that, although the β-deformed manifold looks very complicated and it is not even a complex manifold, the generalised geometry selects coordinates that are trivially related to the original complex coordinates of the Calabi-Yau. As a consequence, all questions about supersymmetric and BPS quantities in the β-deformed background can be still analysed in terms of the original complex coordinates. This is not completely unexpected, since the β-deformed $N = 1$ gauge theory has a natural complex structure for all values of β.

In terms of the pure spinors it is straightforward to check that the T-dual background is still supersymmetric. If we assume that $\Phi^{1,2}$ are supersymmetry-preserving

[6] It is a straightforward computation to show that these pure spinors are equivalent to the dielectric ones in [11]

$$\Phi_- = -(\sin 2\varphi\, e^{i(\theta_+ + \theta_-)} e^{A}\, z)\wedge e^{i\frac{\mathrm{Re}\,\omega}{\sin 2\varphi\, e^{2A}} - \cot 2\varphi\frac{\mathrm{Im}\,\omega}{e^{2A}}}\,; \tag{2.59}$$

$$\Phi_+ = \cos 2\varphi\left(-ie^{-2A}j - \frac{\cos 2\varphi}{2}e^{-2A}j^2 + \sin 2\varphi\, e^{-2A}\,\mathrm{Im}\,\omega\right)e^{\frac{z\bar z}{2e^{2A}}}$$

with $\sin 2\varphi = \gamma\sqrt{h}\sqrt{G}$, $\cos 2\varphi = \sqrt{G}$. The $SU(2)$ structure

$$j = \frac{i}{2}(\epsilon^1\wedge\bar\epsilon^1 + \epsilon^2\wedge\bar\epsilon^2)\,; \tag{2.60}$$

$$\omega = i\sqrt{h}\,\epsilon^1\wedge\epsilon^2\,; \tag{2.61}$$

is defined in terms of the vielbein adapted to the β-deformed metric (2.36)

$$\epsilon^i = X^i + i\sqrt{G}\,Y^i : \tag{2.62}$$

As before, the analogous quantities with superscript (0) refer to the original Calabi-Yau metric.

isometries, $\mathcal{L}_{\partial_{\phi_{1,2}}}\hat\Phi = 0$, then $\mathcal{L}_{\partial_{\phi_1}}(\iota_{\partial_{\phi_2}}\hat\Phi) = 0$ and

$$d(\beta\,\hat\Phi) = d(\iota_{\partial_{\phi_1}}\iota_{\partial_{\phi_2}}\hat\Phi) = -\iota_{\partial_{\phi_1}} d(\iota_{\partial_{\phi_2}}\hat\Phi) = \iota_{\partial_{\phi_1}}\iota_{\partial_{\phi_2}} d\hat\Phi = \beta\, d\hat\Phi : \tag{2.65}$$

Thus for a $\hat\Phi$ which is invariant along ϕ^1, ϕ^2

$$d(e^{-\gamma\beta}\hat\Phi) = e^{-\gamma\beta}\, d\hat\Phi : \tag{2.66}$$

Then from (2.66) it follows that the T-dual spinors satisfy the supersymmetry conditions, (2.48)–(2.50), if the original ones do. The T-dualised RR fields can be computed from $e^{-B}\wedge(F) = e^{-\gamma\beta}\, e^{-B^{(0)}}\wedge(F^{(0)})$. For the β-deformation of the quiver theories, this gives in particular

$$F_5 = \star d(4A) = G\,F_5^{(0)} ; \tag{2.67}$$

$$F_3 = -\gamma\,(\beta \llcorner F_5) ; \qquad F_1 = 0 : \tag{2.68}$$

One can check that these are the same as in (2.40) and (2.41) and satisfy (2.50) with the pure spinor given by (2.58).

Finally, it is also easy to verify that the topology of the β-transformed background is the same as that of the original one, which was assumed to be smooth. The only points where one can have topology changes are the edges of the symplectic cone $\mathcal{C}$, where the circles defined by $\phi^{1,2}$ shrink to zero. These are precisely the points where the bivector β vanishes. To see this we can use the definition of the toric manifold as a T^3 fibration over the symplectic cone $\mathcal{C}$ [20]. On the α-th facet of the cone $\mathcal{C}$ a given combination of the three angles ϕ^i degenerates. The precise combination can be read from the corresponding vanishing vector

$$K_\alpha = \sum_{i=1}^{3} V_\alpha^i \frac{\partial}{\partial \phi^i} = v_\alpha^1 \frac{\partial}{\partial \phi^1} + v_\alpha^2 \frac{\partial}{\partial \phi^2} + \frac{\partial}{\partial \phi^3} \tag{2.69}$$

where $V_\alpha = (v_\alpha; 1)$ is the vector orthogonal to the facet. Thus, on the α-facet only one linear combination of the three angles ϕ^i degenerates. This is not enough in general to make the bivector β vanishing. On the other hand, consider the edge of $\mathcal{C}$ corresponding to the intersection of the α-th and $\alpha+1$-th facets; the vector $K_\alpha - K_{\alpha+1} = (v_\alpha - v_{\alpha+1})^1 \partial_{\phi_1} + (v_\alpha - v_{\alpha+1})^2 \partial_{\phi_2}$ also vanishes. Since the (two-dimensional) integer vectors v_α^a and $v_{\alpha+1}^a$ are not equal [7], it follows that the killing vectors ∂_{ϕ_1} and ∂_{ϕ_2} are proportional and β vanishes. Thus β vanishes precisely on the edges of the cone.

If the original $SO(6,6)$ spinor $\hat\Phi^{(0)}$ is regular, then at these points

$$\beta\,\hat\Phi^{(0)} \to 0 : \tag{2.70}$$

[7] Recall that v_α determines the toric diagram of the Calabi-Yau so no consecutive v_α can be equal.

Thus, at these degenerate points

$$\hat{\ } , \quad \hat{\ }^{(0)} : \tag{2.71}$$

Since a background is completely specified by $\hat{\ }$, $\hat{\ }_+$ and F, at the degeneration points the new background looks similar to the original one. Hence it is regular as well, as discussed from the metric point of view in [2].

3 D3 and D5 probes

The connection between gravity and field theory is provided by the study of supersymmetric D-brane probes moving on the -deformed background. We first analyse space-time filling static D-brane probes, easily extending the results of [2] to a generic Calabi-Yau background. A parallel analysis is performed for non-static probes, in particular dual giant gravitons [24], corresponding to brane probes wrapping a three-sphere in AdS_5 and spinning in the internal manifold. The case of dual giants in the -deformed $N = 4$ SYM has been analysed in [25].

In this Section we perform an analysis based on the effective Lagrangian on the world-volume of a probe moving in the deformed background. In the next Section we will discuss the same results from the point of view of T-duality and supersymmetry, using the Generalised Geometry perspective.

3.1 Static probes

The moduli space of space-time filling supersymmetric static four-branes should reproduce the mesonic moduli space of the dual gauge theory. In the undeformed background we just have a single type of static supersymmetric probe, a D3-brane which can live at every point of the internal manifold. Correspondingly, the abelian moduli space of the dual field theory is isomorphic to the Calabi-Yau cone. In the deformed background, we have two different types of static supersymmetric probes, D3-branes, and dielectric D5-branes wrapped on the (T-duality) two-torus and stabilized by a world-volume flux [2]. Supersymmetric D3-probes can only live on a set of intersecting complex lines inside the deformed Calabi-Yau, corresponding to the locus where the T^3 toric fibration degenerates to T^1. This is in agreement with the abelian moduli space of the -deformed gauge theory which indeed consists of a set of lines. In the case of rational , there exist supersymmetric D5-probes with a moduli space isomorphic to the original Calabi-Yau divided by a $Z_n \quad Z_n$ discrete symmetry. This statement is the gravity counterpart of the fact that for rational new branches are opening up in the moduli space of the gauge theory [5,6].

3.1.1 Static D3 probes

Consider a static space-time filling D3-brane probe. The dynamics is governed by the brane world-volume action

$$S_{D3} = S_{BI} + S_{CS} = -T_3 \int d^4\xi \, e^{-\phi} \sqrt{-\det G} + T_3 \int C_4 \, . \tag{3.1}$$

$G_{\alpha\beta}$ is the pull back of the space-time metric g_{MN} to the world-volume of the D3-brane

$$G_{\alpha\beta} = \frac{\partial X^M}{\partial \xi^\alpha} \frac{\partial X^N}{\partial \xi^\beta} g_{MN} \, ; \tag{3.2}$$

where $(\xi^0; \xi^1; \xi^2; \xi^3)$ are the world-volume coordinates on the brane. The ten-dimensional metric is given by

$$ds^2_{10} = r^2 dx_\mu dx^\mu + \frac{1}{r^2} ds^2_{X_6} \, . \tag{3.3}$$

By inserting in the BI and CS terms the explicit expression of the background fields (2.39)-(2.40), we see that a D3-probe feels a potential given by

$$\int d^4\xi \, V(y_i) \sim \int d^4\xi \, r^4 \left(\frac{1}{\sqrt{G}} - 1 \right) ; \tag{3.4}$$

where y_i are the coordinates on the internal space. The potential is positive definite and vanishes when $G \equiv 1$ or equivalently $h \equiv 0$. h vanishes precisely along the edges of the cone $\mathcal{C}$, where the T^3 fibration degenerates to T^1. In fact, it is easy to see from the explicit behaviour of the metric near the facets, given in equation (2.16), that h is regular and non vanishing in the interior of the cone and also in the interior of the facets. On the other hand, as follows from equation (2.16), on the edge where the adjacent facets α and $\alpha+1$ intersect, h vanishes as

$$h \sim \frac{l_\alpha(y) l_{\alpha+1}(y)}{|\langle V_\alpha, V_{\alpha+1} \rangle|^2} \, . \tag{3.5}$$

We conclude that a supersymmetric D3-probe can only move along the d edges of the symplectic cone. Recall that the topology of the deformed theory is the same as that of the original Calabi-Yau, allowing to reason in terms of fibrations. Moreover, locally, the metric near the degeneration locus is substantially identical to the original one.

We expect that a single D3-brane probes the abelian moduli space of the dual gauge theory. What we found is compatible with the results for $\mathcal{N} = 4$ SYM and the conifold discussed in Section 2.1. There we found that the abelian moduli space consists of three and four lines, respectively. These lines exactly correspond to the edges of the polyedral cone discussed in Section 2.2. From the gravity analysis we thus get the general prediction that the abelian moduli space of toric quiver gauge theories is given by a collection of d lines, where d is the number of external vertices of the toric diagram. We will verify explicitly this prediction in Section 6 with field theory methods.

3.1.2 Static D5 probes

As noticed in [2] a D5-brane wrapped on the two-torus (^{1}; ^{2}) with a world-volume flux $F = d\,{}^1 \wedge d\,{}^2 =$ is supersymmetric. It is easy to see that a similar configuration exists for all Calabi-Yau backgrounds. The supersymmetric D5-brane can live at an arbitrary point in $(y_i; {}^3)$ and can have additional moduli corresponding to Wilson lines on the two-torus. It is interesting to analyse the moduli space of such configuration, since it corresponds to a particular non abelian branch of the dual gauge theory.

Consider therefore a D5-brane wrapping the two-torus spanned by (^{1}; ^{2}) in the internal manifold. The corresponding embedding is

$$
\begin{aligned}
x &= \quad ; \quad {}^1 = {}^4; \quad {}^2 = {}^5; \\
{}^3 &= {}^3(\ ); \quad y_i = y_i(\ ) \qquad = 0;1;2;3; \qquad (3.6)
\end{aligned}
$$

where we call (^{0};:::; ^{5}) the world-volume coordinates on the brane. The world-volume action for a D5-brane is

$$
\begin{aligned}
S_{D5} &= \quad T_5 \int d^6 \ e \quad \sqrt{\quad \det(G \quad B + F)} \\
&\quad + T_5 \int C_6 + C_4 \wedge (F \quad B) + C_2 \wedge (F \quad B) \wedge (F \quad B); \qquad (3.7)
\end{aligned}
$$

where we define $F = 2 \quad {}^0 F$, with F dimensionless. We will set ${}^0 = 1$ as in the other supergravity computations.

For the six-dimensional metric we will use the expression (2.36) in symplectic coordinates

$$
\begin{aligned}
ds^2_{X_6} &= g^{ij} dy_i dy_j + g_{ij} d\,{}^i d\,{}^j \\
&= g^{ij} dy_i dy_j + G h_{ab} d\,{}^a d\,{}^b + 2G g_{a3} d\,{}^a d\,{}^3 + [g_{33} \quad (1 \quad G) h^{ab} g_{a3} g_{b3}](d\,{}^3)^2 : \qquad (3.8)
\end{aligned}
$$

Here and in the rest of this section the indices $i;j$ and $a;b$ are summed over 1;2;3 and 1;2, respectively. All the functions in the above ansatz depend on the coordinates y_i only since the angular directions are isometries of the background.

The pulled-back metric is given by

$$
\begin{pmatrix}
r^2 \quad + \frac{1}{r^2}(g^{ij} \partial \ y_i \partial \ y_j + g_{33} \partial \ {}^3 \partial \ {}^3) & G \partial \ {}^3 g_{13} & G \partial \ {}^3 g_{23} \\
G \partial \ {}^3 g_{13} & G h_{11} & G h_{12} \\
G \partial \ {}^3 g_{23} & G h_{21} & G h_{22}
\end{pmatrix} : \qquad (3.9)
$$

Similarly the pull back of the B-field has components

$$
\begin{aligned}
B_{\ 4} &= \quad h G (h^{2a} g_{a3}) \partial \ {}^3 ; \qquad (3.10) \\
B_{\ 5} &= \ h G (h^{1a} g_{a3}) \partial \ {}^3 ; \qquad (3.11) \\
B_{45} &= \ h G : \qquad (3.12)
\end{aligned}
$$

The world-volume field strength has both magnetic and electric components

$$F_{45} = \frac{1}{\gamma}; \qquad F_{\alpha 4} = \partial_\alpha A_1(\sigma); \qquad F_{\alpha 5} = \partial_\alpha A_2(\sigma): \tag{3.13}$$

The magnetic component is required by supersimmetry, while the electric components correspond to space-time fluctuations of the Wilson lines on the two-torus.

Using the above expressions the determinant in the Born–Infeld action can be written as

$$\det(G - B + F) = r^6 \frac{G}{\gamma^2}\left[\frac{1}{r^2}\left(g^{ij}\partial y_i \partial y_j + g_{33}(\partial\psi^3)^2 - 2\gamma g_a \partial\psi^3 \hat{f}^a + \gamma^2 h_{ab}\hat{f}^a\hat{f}^b\right) - r^2\right]: \tag{3.14}$$

where $\hat{f}^a = \epsilon^{ab}\partial A_b = \epsilon^{ab}F_{\cdot b}$. The overall factor of G cancels the contribution from the dilaton so that the BI action for the D5-probe takes the form [8]

$$S_{BI} = -\frac{N}{\gamma}\int d^4\sigma\, r^3 \sqrt{r^2 - \frac{1}{r^2}\left(g^{ij}\partial y_i \partial y_j + g_{33}(\partial\psi^3)^2 - 2\gamma g_a \partial\psi^3 \hat{f}^a + \gamma^2 h_{ab}\hat{f}^a\hat{f}^b\right)}: \tag{3.15}$$

The Wess-Zumino part of the action simplifies as well, since, as noticed in [2], the C_6 contribution cancels with $B_2 \wedge C_4$. The only non trivial contribution is

$$S_{WZ} = T_5 \int C_4 \wedge F_{45} = \frac{N}{\gamma}\int dt r^4: \tag{3.16}$$

The contribution to the potential vanishes for all values of the moduli y_i; ψ^3; A_a. We then obtain a six-dimensional family of supersymmetric four-branes.

We want to discuss in detail the existence and the moduli space of such configurations. First of all, due to charge quantisation, the D5-brane solutions we find exist only for rational values of $\gamma = m/n$, as discussed in details in [2[9]]. In fact, since the internal T^2 wrapped by the D5-brane supports a flux $F_{45} = 1/\gamma$, there is an induced D3-charge that has to be quantized. If we set $\gamma = m/n$, with m and n relatively prime integers, we obtain a consistent configuration by taking a D5-brane wrapped m times on the contractible T^2 [10]. This configuration can be alternatively seen as a set of n blown up D3-branes.

Our solutions should correspond to additional branches of the dual gauge theory which exist only for rational γ. These are well known for $N = 4$ SYM [5,6] and are discussed in [19] for the conifold. For a generic γ-deformed quiver gauge theory we can study the geometry of these new branches by looking at the moduli space of

[8] S_{BI} and S_{WZ} are proportional to $T_5 L^4 \alpha' Vol(T^2) = \gamma^2 N/(2Vol(X_5))$. Not to clutter formulae we will only write a factor of N.

[9] In [2] to see this they check that a configuration of $(N_{D3}; N_{D5}; N_{NS5})$ in the undeformed geometry is mapped to $(N_{D3}; N_{D5} + \gamma N_{D3}; N_{NS5})$ by the Lunin-Maldacena transformation. Hence $\gamma = m/n$ and $N_{D3} = N$ must be a multiple of n.

[10] In the case $m = 1$ we can equivalently impose that the first Chern number for the $U(1)$ gauge bundle is integer: $\frac{1}{2\pi}\int_{T^2} F = n$, which gives $\gamma = 1/n$.

the solutions. For simplicity consider the case $N_{D5} = m = 1$. The moduli space of the brane is parameterised by $\{\phi^3; \tilde{A}_a; y_i\}$. ϕ^3 and y_i, $(i = 1;2;3)$ are four scalars deformations corresponding to transverse movements of the D5-brane in the internal geometry. Then we have two Wilson lines in the internal T^2, corresponding to the deformations of the gauge field on the brane: $e^{i\int_{\gamma_a} \mathcal{A}}$. Here $\mathcal{A} = A/(2\pi)$ such that $F = dA$, $\mathcal{F} = d\mathcal{A}$ and the integral is over the two non trivial one cycles on T^2. Notice that before T-duality the Wilson lines correspond to the position of the D3-brane on T^2. Naively the space of the deformations of the gauge field is given by the first cohomology of T^2, which is parametrized by the gauge invariants $\tilde{A}_a = \int_{\gamma_a} \mathcal{A}$, but since the holonomies, $\exp(i\tilde{A}_a)$, are the only physical observables, it is clear that they have compact range: $0 \le \tilde{A}_a \le 2\pi$.

The metric for the moduli space can be read from the DBI action, when we give a space-time dependence to all moduli. We can then interpret the electric field strengths as the space-time derivatives of the Wilson lines: $F_{\mu a} = \partial_\mu A_a = 2\pi \partial_\mu \mathcal{A}_a \simeq \partial_\mu \int_{\gamma_a} \mathcal{A} = \partial_\mu \tilde{A}_a$. By expanding (3.15) we obtain the metric on the moduli space

$$S_{D5} = \frac{N}{2\pi} \int d^4\sigma \left(g^{ij}\partial y_i \partial y_j + g_{33}(\partial\phi^3)^2 - 2\beta g_a \partial\phi^3 \hat{f}^a + \beta^2 h_{ab} \hat{f}^a \hat{f}^b \right) : \quad (3.17)$$

This metric is identical to the metric of the original Calabi-Yau when we identify

$$\partial\phi^a = \beta \hat{f}^a ; \quad \text{or} \quad \phi^a \simeq \beta\epsilon^{ab}\tilde{A}_b : \quad (3.18)$$

As discussed above, for $m = 1$ the angular variable ϕ^a associated to the Wilson lines has period $2\pi/n$. We thus see that the metric on the moduli space is just that of the original CY divided by $Z_n \times Z_n$.

Therefore the prediction from the gravity analysis is that, for every toric quiver gauge theory, at rational β, we have additional Higgs branches isomorphic to the orbifold $CY/Z_n \times Z_n$. We will give evidence for this statement in Section 6.

3.2 Dual giant gravitons

We are interested in this section in dual giant gravitons, brane probes wrapping a three-sphere in global AdS_5 and spinning in the internal manifold. Dual giants are defined in global coordinates in AdS_5.

As shown in [17], the classical phase space of a supersymmetric D3 dual giant on the undeformed Sasaki-Einstein background is isomorphic to the original Calabi-Yau, that is the abelian moduli space of the dual gauge theory. Upon geometric quantisation of the classical solutions one obtains all the mesonic BPS states of the theory[11].

In this section we will extend this discussion and study the dynamics of the dual giant gravitons in the β-deformed geometries. Since the quantisation of the classical

[11] By quantising the classical dual giant solutions we obtain states of the gauge theory on $S^3 \times R$ [24]. All these states are mapped to BPS operators via the conformal mapping to R^4.

dual giant solutions gives mesonic BPS states (corresponding to BPS operators), we expect that the classical phase space of the dual giants contains information about the mesonic moduli space of the dual gauge theory. Dual giants for the β-deformed $\mathcal{N} = 4$ SYM were already analysed in [25].

Exactly in parallel to the case of static probes, the β-deformed geometries admit BPS dual giant gravitons of two kinds. The first type of giants are present for all values of the deformation parameter β and correspond to D3-branes wrapping an S^3 in AdS_5 and spinning along the Reeb vector in the internal geometries. On the field theory side they correspond to the operators parameterising the abelian Coulomb branch of the theory. The classical phase space of the dual giants reproduces the abelian moduli space of the dual gauge theory. The other class of dual giants can exists only for rational values of the deformation parameter and consists of D5-branes wrapping the S^3 in AdS_5 and the two-torus $(\phi^1; \phi^2)$ in the internal manifold. They rotate in the angular direction orthogonal to the two-torus and have a magnetic world-volume field strength proportional to $1/\beta$. The world-volume gauge field satisfies the quantisation condition only for β rational. On the field theory side these configurations correspond to Higgs branches that are present when β is rational.

3.2.1 D3 dual giant gravitons

We want to study the dynamics of a D3-brane probe that wraps the three-sphere in AdS_5, written in global coordinates, and rotates on the internal manifold. This is still governed by the brane world-volume action (3.1) where we now take as ten-dimensional metric

$$ds^2_{10} = ds^2_{AdS_5} + ds^2_{X_5} : \tag{3.19}$$

The metric of AdS_5 is given in global coordinates

$$ds^2_{AdS_5} = -V(R)dt^2 + \frac{1}{V(R)}dR^2 + R^2(d\theta^2 + \cos^2\theta\, d\phi_1^2 + \sin^2\theta\, d\phi_2^2) \tag{3.20}$$

with $V(R) = 1 + R^2$. t is the global time in AdS_5 and the angles θ, ϕ_1 and ϕ_2 parameterise a round three-sphere. We will write the metric on X_5 as the restriction of the six-dimensional internal metric to the hypersurface with $r = 1$

$$2b^i y_i = 1 : \tag{3.21}$$

>From now on, we consider as coordinates for X_5 the angles ϕ^i and two extra angles parameterised by the y_i with the above constraint.

With this choice of coordinates the embedding $X^M(\sigma)$ corresponding to the dual giant graviton can be taken as

$$t = \tau ; \quad R = R(\tau); \quad \theta = \sigma^1; \quad \phi_1 = \sigma^2; \quad \phi_2 = \sigma^3;$$
$$\phi^i = \phi^i(\tau); \quad y_i = y_i(\tau) \qquad i = 1;\dots;3 : \tag{3.22}$$

It is then easy to see that

$$\sqrt{-\det G} = R^3 \cos\theta \sin\theta\, \Delta^{1/2}\,; \tag{3.23}$$

where we have defined (the dot represents the derivative with respect to $t = \tau$)

$$\Delta = V(R) - \frac{\dot{R}^2}{V(R)} - g^{ij}\dot{y}_i\dot{y}_j - g_{ij}\dot{\phi}^i\dot{\phi}^j\,. \tag{3.24}$$

To evaluate the WZ term we can choose the pullback of the four-form potential to be

$$C_{(4)} = R^4 \sin\theta \cos\theta\, dt \wedge d\theta \wedge d\phi_1 \wedge d\phi_2\,. \tag{3.25}$$

Substituting (3.23) and (3.25) into (3.1) we obtain the Lagrangian for the probe[12]

$$L = -N R^3 (e\sqrt{\Delta} - R)\,. \tag{3.26}$$

To find the explicit solutions for the possible motions of the D3-brane probe it is convenient to pass to the Hamiltonian formalism and solve the Hamilton equations of motion. For the dual giant graviton we are considering the canonical momenta are

$$\begin{aligned} p_R &= \frac{\partial L}{\partial \dot{R}} = e\frac{N R^3}{\sqrt{\Delta}}\frac{\dot{R}}{V}\,; \\ p_{y_i} &= \frac{\partial L}{\partial \dot{y}_i} = e\frac{N R^3}{\sqrt{\Delta}} g^{ij}\dot{y}_j\,; \\ p_{\phi_i} &= \frac{\partial L}{\partial \dot{\phi}^i} = e\frac{N R^3}{\sqrt{\Delta}} g_{ij}\dot{\phi}^j\,. \end{aligned} \tag{3.27}$$

The Hamiltonian then reads

$$\begin{aligned} H &= e\frac{N R^3}{\sqrt{\Delta}} V - N R^4 \\ &= N R^3 (\sqrt{V\Theta} - R)\,; \end{aligned} \tag{3.28}$$

where in the second line we have expressed everything in terms of the canonical momenta and we have introduced the function

$$\Theta = e^2 + \frac{1}{N^2 R^6}(V p_R^2 + g_{ij}p_{y_i}p_{y_j} + g^{ij}p_{\phi_i}p_{\phi_j})\,. \tag{3.29}$$

[12] Keeping into consideration also the factors of L, the Lagrangian for D3 dual giants is proportional to $T_3 L^4 \mathrm{Vol}(S^3) = \pi^3 N/\mathrm{Vol}(X_5)$; however we will write explicitly only the factor N in front of L.

The corresponding equations of motion are

$$\dot{R} = \frac{1+R^2}{NR^2x} p_R \; ; \tag{3.30}$$

$$\dot{p}_R = NR^3[4 \quad \frac{1}{x}(x^2 + 3e^{2} \quad + \frac{(p_R)^2}{N^2R^4})]; \tag{3.31}$$

$$\dot{y}_i = \frac{1}{NR^2x} g_{ij} p_{y_j} \; ; \tag{3.32}$$

$$\dot{p}_{y_i} = \quad \frac{NR^4}{2x} \partial_{y_i} \quad ; \tag{3.33}$$

$$\dot{\ }^i = \frac{1}{NR^2x} g^{ij} p_{\ j} \; ; \tag{3.34}$$

$$\dot{p}_{\ i} = 0 \; ; \tag{3.35}$$

where we have defined

$$x = R \sqrt{\frac{\quad}{V}} : \tag{3.36}$$

A BPS solution representing a dual giant rotating in the internal manifold is given by

$$R = \text{const}; \qquad p_R = 0 \; ; \tag{3.37}$$

$$y_i = \text{const}; \qquad p_{y_i} = 0 \; ; \tag{3.38}$$

$$\dot{\ }^i = b^i; \qquad p_{\ i} = 2NR^2 y_i \tag{3.39}$$

with y_i satisfying $\quad (y_i) = 0$.

To explicitly see it, it is convenient to introduce a set of local angular coordinates adapted to the motion of the brane probe

$$ds^2_{X_5} = g^{ij} dy_i dy_j + H (d \quad + \quad _a d \ ^a)^2 + h_{ab} d \ ^a d \ ^b \; ; \tag{3.40}$$

where $\quad$ is the angular direction in which the brane rotates, and the indices $a;b$ run from 1 to 2. As before the functions H and h_{ab} depend on the variables y_i only. In these coordinates the function $\quad$ becomes

$$= e^{\ 2} \quad + \frac{1}{N^2R^6}(V p_R^2 + g_{ij} p_{y^i} p_{y_j} + H^{\ 1} p^2 + h^{ab}(p_{\ a} \quad _a p \ )(p_{\ b} \quad _b p \ )) \; ; \tag{3.41}$$

while (3.34) and (3.35) are substituted by

$$\dot{\ } = \frac{1}{NR^2x}(H^{\ 1} p \quad h^{ab} \ _a (p_{\ b} \quad _b p \ )) \; ; \qquad \dot{p} \ = 0 \; ; \tag{3.42}$$

$$\dot{\ }^a = \frac{1}{NR^2x} h^{ab}(p_{\ b} \quad _b p \ ) \; ; \qquad \dot{p}_{\ a} = 0 : \tag{3.43}$$

Since the brane rotates in the direction $\quad$ we expect

$$\dot{y}_i = 0; \qquad \dot{\ }^a = 0; \qquad \dot{R} = 0 : \tag{3.44}$$

The first condition, together with (3.32) and (3.33), implies

$$p_{y_i} = 0 \quad \text{and} \quad \partial_{y_i} = 0 : \tag{3.45}$$

The second condition in (3.44) imposes

$$p_{\ a} = \ _a p \ : \tag{3.46}$$

And finally the third condition combined with (3.30) and (3.31) gives

$$p_R = 0 \quad \text{and} \quad x = 2 - \sqrt{4 - 3e^2} \ : \tag{3.47}$$

Observe that the condition $\partial_{y_i} = 0$ and the definitions of x and altogether imply

$$\partial_{y_i} = 0; \quad \partial_{y_i} H = 0 : \tag{3.48}$$

Up to now we have not imposed the condition that the dual giant must be BPS. This amounts to setting the Hamiltonian equal to the momentum in direction of the rotation

$$H = p \ : \tag{3.49}$$

The value of p and H on the solution are easily computed from the equations above

$$H = N R^2 [x + R^2 (x - 1)]; \tag{3.50}$$

$$p = \sqrt{H} N R^2 \sqrt{R^2 (x^2 - e^2 \ ) + x^2} \ ; \tag{3.51}$$

so that for the ratio to be equal to 1 for all values of R, one has to impose[13]

$$x = 1; \quad = 0; \quad H = 1; \tag{3.52}$$

which imply $\frac{}{} = 1$ on the BPS solutions.

We can now analyse the conditions for BPS motion. Let us start with the case of the undeformed theory. In the undeformed background, is identically zero. A supersymmetric configuration can be obtained by allowing the probe to rotate along the Reeb vector. In fact the angle dual to Reeb vector is normalized to one

$$H = g(K; K) = g_{ij} b^i b^j \quad 1 \ ; \tag{3.53}$$

where we made use of equation (2.25) on the Sasaki-Einstein $r = 1$. Thus the BPS equations (3.48) and (3.52) are satisfied. This reproduces the results found in [17]: a supersymmetric dual giant must rotate along the Reeb vector and it can sit at any point in y_i. Its motion in the phase space $(q^A; p^A)$ is characterized by six free

[13] There might exist other solutions with fixed value of R. Most likely, an analysis in terms of supersymmetry transformations would reveal that these solutions are not BPS. They would correspond to truly isolated vacua in the dual field theory, that are not expected to exist in such theories.

real parameters that are the initial conditions on the Sasaki-Einstein space plus R. Altogether these parameters reconstruct a copy of the Calabi-Yau and the induced symplectic form on the phase space reduces to the natural symplectic form of the Calabi-Yau cone [17].

In the case of the deformed theory, Φ is a non trivial function of y_i and the conditions (3.48),(3.52) select a subvariety of the internal space. Since $e^{-\phi} = \sqrt{1+\hat\gamma^2 h}$ we can write the conditions for the vanishing of Φ and $\partial_{y_i}\Phi$ as

$$h = 0; \qquad \partial_{y_i} h = 0: \tag{3.54}$$

Here h is the determinant of the two-torus metric which vanishes exactly on the edges of the polyhedral cone where the torus degenerates. In addition its derivative also vanishes on the edges as equation (3.5) clearly shows. We see that the BPS condition restricts the dual giant to live on the d edges of the cone.

We still have to find the angular direction of rotation of a BPS dual giant, which is characterized by the conditions $H = 1$, $\partial_{y_i} H = 0$. We still expect our giant to rotate along the Reeb vector. We can compute the value of H for a giant rotating along the Reeb vector

$$H = g(K,K) = G + 9(1-G)(g_{33} - h^{ab} g_{a3} g_{b3}) = \frac{1 + 9\hat\gamma^2 \det g_{ij}}{1+\hat\gamma^2 h}: \tag{3.55}$$

We can easily check that along an edge where $h = \partial_{y_i} h = 0$ we have $H = 1; \partial_{y_i} H = 0$ thus solving the remaining equations of motion and BPS conditions.

Summarizing, a dual giant graviton in the beta-deformed theory is supersymmetric only when it lives on the edges of polyhedron and rotates along the Reeb vector.

Adding R to the set of initial conditions of the probe, we see that the moduli space for a dual giant can be identified with a collection of lines. We expect that the classical phase space of a single dual giant corresponds to the abelian moduli space of the dual gauge theory. Indeed what we found is consistent with the results for static probes and the field theory discussion in Section 6.

3.2.2 D5 dual giant gravitons

For $\hat\gamma$ rational another class of brane probes can be consistently embedded in the deformed geometry: D5-branes wrapping the same S^3 inside AdS_5 and the two-torus spanned by (σ^1, σ^2) in the internal manifold. The corresponding embedding is

$$\begin{aligned} &t = \sigma^0; \quad R = R(\sigma^0); \quad \theta = \sigma^1; \quad \phi_1 = \sigma^2; \quad \phi_2 = \sigma^3; \\ &\varphi^1 = \sigma^4; \quad \varphi^2 = \sigma^5; \\ &\varphi^3 = \varphi^3(\sigma^0); \quad y_i = y_i(\sigma^0) \qquad i = 1;2;3; \end{aligned} \tag{3.56}$$

where we call $(\sigma^0; \dots; \sigma^5)$ the world-volume coordinates on the brane. The discussion is completely parallel to that for a static D5-brane. The world-volume action for the dual giant is still given by (3.7) and now the pulled-back metric is given by

$$\begin{pmatrix} & 0 & 0 & 0 & G\,\dot{}^{3}g_{13} & G\,\dot{}^{3}g_{23} \\ 0 & R^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & R^2\cos^2\ \ ^1 & 0 & 0 & 0 \\ 0 & 0 & 0 & R^2\sin^2\ \ ^1 & 0 & 0 \\ G\,\dot{}^{3}g_{13} & 0 & 0 & 0 & G\,h_{11} & G\,h_{12} \\ G\,\dot{}^{3}g_{23} & 0 & 0 & 0 & G\,h_{21} & G\,h_{22} \end{pmatrix} \tag{3.57}$$

with $= V(R) \quad \frac{R^2}{V(R)} \quad g^{ij}\underline{y_i y_j} + g_{33}(\dot{}^{3})^2$. The B − eld is given by

$$B_{04} = \quad h\,G\,(h^{2a}g_{a3})\dot{}^{3}\,; \tag{3.58}$$
$$B_{05} = \quad h\,G\,(h^{1a}g_{a3})\dot{}^{3}\,; \tag{3.59}$$
$$B_{45} = \quad h\,G\,; \tag{3.60}$$

and the world-volume eld strength has both magnetic and electric components

$$F_{45} = \frac{1}{}\,; \qquad F_{04}(\ )\,; \qquad F_{05}(\ ): \tag{3.61}$$

It is a straightforward computation to verify that the BI action for the D5 probe has the same form as for the Calabi-Yau case[14]

$$S_{BI} = \frac{N}{}\int dt R^3 \sqrt{V(R) \quad \frac{R^2}{V(R)} \quad g^{ij}\underline{y_i y_j} \quad g_{33}(\dot{}^{3})^2 + 2\ g_{3a}\dot{}^{3}\hat{f}_a \quad {}^2 h_{ab}\hat{f}^a\hat{f}^b}\,; \tag{3.62}$$

where $\hat{f}^a = \ _{ab}F_{0b}$. The Wess-Zumino part of the action reduces to the Calabi one as well. This is because the only non trivial contribution is

$$S_{WZ} = T_5 \int C_4 \wedge F_{45} = \frac{N}{}\int dt R^4\,: \tag{3.63}$$

Thus the world-volume Lagrangian is

$$L = \frac{N R^3}{}(\sqrt{\ } \quad R) \tag{3.64}$$

with

$$= V(R) \quad \frac{R^2}{V(R)} \quad g^{ij}\underline{y_i y_j} \quad g_{33}(\dot{}^{3})^2 + 2\ g_{3a}\dot{}^{3}\hat{f}_a \quad {}^2 h_{ab}\hat{f}^a\hat{f}^b \tag{3.65}$$

which formally is equivalent to that of a D3 dual giant in the undeformed geometry with the replacement of $\dot{}^{a}$ with $\ ^{ab}F_{0b}$. On the undeformed Calabi-Yau a D3 dual

[14] S_{BI} and S_{WZ} are proportional to $T_5 L^4\ ^0 Vol(S^3) Vol(T^2) = \ ^4 N = Vol(X_5)$. Again we write only the factor N.

giant can live at an arbitrary point and rotates along the Reeb vector. We thus see that a class of solutions for D5 dual giants is obtained by choosing

$$F_{0a} = \frac{1}{\ }\ {}_{ab} b^b\,; \qquad \dot{\ }^3 = b^3\,: \tag{3.66}$$

We can analyse the classical phase space of the D5 dual giants. Exactly as in the case of static D5, for $= m/n$, we obtain the orbifold $CY/Z_n \ \ Z_n$. Coordinates on this space are obtained by adding R to the initial values of ^{3}, y_i and the two Wilson lines along the two-torus, and taking into account the modified periodicities of the angles. The classical phase space of the D5 dual giants is thus isomorphic to the additional Higgs branches in the moduli space of the dual gauge theory existing for rational . This is consistent with the fact that the quantisation of this classical phase space (as done for example in [17]) should reproduce the mesonic BPS operators parameterising the Higgs branch.

4 Supersymmetric D-brane probes from -transformation

In this section we analyse the existence and supersymmetry of D3 and D5 probes using generalised geometry. We show in particular that the class of dual giants found in Section 3.2 can be obtained by direct action of the -transformation on the world-volume of the D3 dual giants described in [17]. This will automatically ensure that the dual giants are supersymmetric in the -deformed background.

A simple way to do it is again using the formalism of Generalised Geometry, where a D-brane wrapping a submanifold and supporting a world-volume field strength F is described by its generalised tangent bundle $T_{(,F)}$ [22]. This can be described as a maximally isotropic subspace of $T \ \ T^\star$ [15], as follows

$$T_{(,F)} = \{X + \ \ 2 T \ \ T^\star j\ \ : X\ 2\ T \ \text{and}\ \ j = {}_X F\}\,: \tag{4.1}$$

As already mentioned, the elements of $T \ \ T^\star$ transform linearly under the action of the extended T-duality group $O(d;d)$ and so does $T_{(,F)}$. If we start from a D-brane preserving a background supersymmetry which is also preserved by the $O(d;d)$ transformation, then the D-brane obtained by 'integrating' the transformed generalised tangent bundle will be automatically supersymmetric in the transformed background.

Let us start by considering the -deformation of a static D3-brane in the undeformed toric Sasaki-Einstein background, filling the four Poincare directions and sitting at an arbitrary point of the internal Calabi-Yau cone. As it is well known, this configuration preserves all the background Poincare supersymmetries.

[15] Strictly speaking we should consider the extension of T by $T^\star$; for our class of backgrounds the two are isomorphic since B is globally defined.

If the D3-brane sits at a point where the two-torus $(\phi^1;\phi^2)$ shrinks to zero size, the generalised tangent bundle describing the new D-brane is identical to the one we started from, since the β-transformation (2.29) reduces to the identity at these points. Thus the original D3-brane is mapped to a D3-brane at the same degeneration point in the deformed background.

The situation is different when the original D3-brane sits at a point where ϕ^a are non-degenerate. Since the only coordinates playing a non-trivial role in the β-transformation are the two angles ϕ^a we can simply describe the D3-brane as a point on the two-torus $(\phi^1;\phi^2)$. Since all forms vanish when restricted to a point, the associated (two-dimensional) generalised tangent bundle (4.1) admits the basis $e^a = d\phi^a$. Acting on this basis with the β-deformation (2.29), we obtain a basis for the β-transformed generalised tangent bundle

$$\tilde{e}^a = -\gamma\,\epsilon^{ab}\frac{\partial}{\partial\phi^b} + d\phi^a\,. \tag{4.2}$$

By projecting it onto the background tangent bundle, we see that the ordinary tangent bundle of the new D-brane is spanned by ∂_{ϕ^1} and ∂_{ϕ^2}. Thus, we obtain a D5-brane wrapping $(\phi^1;\phi^2)$ in the β-deformed background. From the general definition (4.1), we also see that the D5-brane must support a world-volume gauge field $\mathcal{F} = (1/\gamma)\,d\phi^1\wedge d\phi^2$.

We can easily check this result using the supersymmetry conditions for D-branes given in terms of the (twisted) background pure-spinors [14,15]. For a D-brane wrapping the internal cycle Σ with world-volume flux $\mathcal{F}$ is

$$[\hat\Phi_-|_\Sigma\wedge e^{\mathcal{F}}]_{\text{top}-1} = 0\,;\quad [(\iota_X\hat\Phi_-)|_\Sigma\wedge e^{\mathcal{F}}]_{\text{top}} = 0\quad \forall X\in T_M \qquad (F\text{-flatness}) \tag{4.3}$$

$$[\hat\Phi_+|_\Sigma\wedge e^{\mathcal{F}}]_{\text{top}} = 0\,. \qquad (D\text{-flatness}) \tag{4.4}$$

In our case $\hat\Phi_- = e^{-\Phi}\,\gamma\,(e^{3A}\,\Omega^{(0)})$ and $\hat\Phi_+ = e^{-\Phi}\exp(-ie^{2A}J^{(0)})$. Then, we immediately see that a D3-brane is supersymmetric only where $\gamma\to 0$ (i.e. the points where the $(\phi^1;\phi^2)$ two-torus degenerates), since at the other points the F-flatness is not satisfied. On the other hand, a D5-brane wrapping the $(\phi^1;\phi^2)$ two-torus at any non-degenerate point automatically satisfies the D-flatness, since $J^{(0)}|_{T^2} = 0$, while the F-flatness imposes the condition $\mathcal{F} = (1/\gamma)\,d\phi^1\wedge d\phi^2$. We have thus recovered the result obtained from T-duality, generalising the result obtained by other means in [2] for $AdS_5\times S^5$.

Let us now pass to the description of the action of the β-transformation on the D3 dual giant gravitons. D3 dual giants in the undeformed background have been found and discussed in [17]. In any toric Sasaki-Einstein background, they wrap a static S^3 of arbitrary radius at the center of AdS_5, sit at any point described by the y_i coordinates (constrained by the condition $2b^i y_i = 1$) and run along the angular coordinates as follows

$$t = \tau\,;\qquad \phi^i = b^i\,\tau + \text{const}\,. \tag{4.5}$$

As for the case above, if a D3 dual giant sits at a point in the y_i coordinates where the two-torus described by $(\phi^1;\phi^2)$ degenerates, its β-transformation is trivial and gives again a D3 described by the same embedding (4.5). These are nothing but the D3-brane dual giants described in Subsection 3.2.1, which are thus supersymmetric.

In order to study the β-transformation of D3 dual giants sitting at non-degeneration points, we can restrict our attention on the time t and the three angles ϕ^i. From (4.1) we see that a basis for the generalised tangent bundle of these D3 dual giants is given by the tangent vectors and a basis of one forms vanishing along the trajectory

$$e^0 = \frac{\partial}{\partial\sigma} = \frac{\partial}{\partial t} + b^i\frac{\partial}{\partial\phi^i} \quad ; \quad e^3 = dt - g_{ij}b^j d\phi^i \quad ; \quad e^\alpha = c_{(\alpha)i}d\phi^i \; ; \qquad (4.6)$$

where $\alpha = 1,2$, $i,j = 1,2,3$ and $c_{(\alpha)i}$ are such that $c_{(\alpha)i}b^i = 0$. By β-transforming it

$$\tilde{e}^0 = \frac{\partial}{\partial t} + b^i\frac{\partial}{\partial\phi^i} \quad ; \quad \tilde{e}^3 = -\gamma\,\epsilon^{ab}g_{aj}b^j\frac{\partial}{\partial\phi^b} + dt - g_{ij}b^j d\phi^i \; ;$$
$$\tilde{e}^\alpha = -\gamma\,\epsilon^{ab}c_{(\alpha)b}\frac{\partial}{\partial\phi^a} + c_{(\alpha)i}d\phi^i \; . \qquad (4.7)$$

Projecting this basis to the background tangent bundle we obtain a basis for the tangent bundle to the β-transformed brane, which is thus a D5-brane described by the embedding

$$(\sigma;\sigma^a) \mapsto (t = \sigma \; ; \; \phi^3 = b^3\sigma + \text{const} \; ; \; \phi^a = \sigma^a) \; . \qquad (4.8)$$

As above, from the 'twisting' of the basis (4.7) we see that the D5-brane must support a non-trivial world-volume field strength, which can be easily calculated to be

$$F = \frac{1}{\gamma}\epsilon_{ab}b^b d\sigma\wedge d\sigma^a + d\sigma^1\wedge d\sigma^2 = \frac{1}{2\gamma}\epsilon_{ab}\left(\gamma b^a d\sigma + d\sigma^a\right)\wedge\left(\gamma b^b d\sigma + d\sigma^b\right) . \qquad (4.9)$$

We have thus recovered the D5 dual giants described in Subsection 3.2.2. Again, they are automatically supersymmetric by $O(2;2)$ symmetry. As already discussed in Section 3.1, the gauge field must be quantised, giving the condition $\gamma = m/n$ rational.

In Sections 3.1 and 3.2.2 we showed that the moduli space of D5-brane probes (static or dual giants) is given by $CY/\mathbb{Z}_n \times \mathbb{Z}_n$. Here we will briefly show that the same result can be obtained as the β-deformation of the moduli space of a probe D3 in the undeformed geometry.

For simplicity, consider a static D3-brane in an undeformed Sasaki-Einstein background (the analysis of dual giants is completely analogous). As explained in [15], the infinitesimal deformations of a D-brane wrapping a cycle Σ with field strength F are described by sections of the generalised normal bundle: $N_{(\Sigma,F)} = E|_\Sigma / T_{(\Sigma,F)} \simeq T^{*}_{(\Sigma,F)}$. In the case of the static D3-brane, focusing again on the $(\phi^1;\phi^2)$ directions, a basis for the sections of $N_{(\Sigma,F)}$ is given by the following representatives

$$e_a = \frac{\partial}{\partial\phi^a} \; ; \qquad (4.10)$$

which clearly generate the motion of the D3-brane in the $(\phi^1;\phi^2)$ directions. We can now apply the β-transformation (2.29) to obtain representatives of the corresponding sections of the generalised normal bundle to the D5-brane in the β-deformed background. The are given by

$$\tilde{e}_a = \frac{1}{\gamma}\epsilon_{ba}d\phi^b\,. \tag{4.11}$$

The displacement

$$\phi^a \to \phi^a + c^a \tag{4.12}$$

of the D3-brane in the Sasaki-Einstein background is generated by the generalized normal vector $c^a e_a$. The β-transformation maps it into $c^a\tilde{e}_a$, which corresponds, as discussed in [15], to a shift $\delta A = c^a\tilde{e}_a$ of the gauge field on the D5-brane in the β-deformed background. In components this reads

$$A_a \to A_a + \frac{1}{\gamma}\epsilon_{ab}c^b = A_a + n\,\epsilon_{ab}c^b \tag{4.13}$$

Thus, in particular, a periodic shift $\Delta_a\phi^b = 2\pi\delta_a^b$ of the D3-brane corresponds to a shift

$$\Delta_a\int_{\gamma_b} A = 2\pi n\,\epsilon_{ba} \tag{4.14}$$

of the Wilson line on the D5-brane. As before the Wilson lines are defined by $\int_{\gamma_a}\tilde{A}$, with $\tilde{A} = A/2\pi$, have period 2π and parameterise a two-torus $\tilde{T}^2$.

This result have a natural interpretation taking into account that the β-deformation maps n D3-branes to a single D5-brane. From this point of view, the angular positions ϕ^a in the undeformed background actually corresponds to the average $\langle\phi^a\rangle = \sum_{r=1}^{n}\phi^a_{(r)}/n$ of the angular positions $\phi^a_{(r)}, r = 1,\ldots,n$, of the n D3-branes, while the Wilson lines on the D5-brane in the β-deformed background are associated to the sums $\sum_{r=1}^{n}\phi^a_{(r)}$ (the trace of the corresponding $n\times n$ matrix in the complete non-abelian description of the n D3-branes) by the β-deformation. A constant periodic shift $\Delta_a\langle\phi^b\rangle = 2\pi\delta_a^b$ of the average D3-brane position then produces the shift (4.14) of the D5-brane Wilson lines. From (4.14), we see that going once around a 1-cycle in T^2_{SE} corresponds to going n-times around a 1-cycle in $\tilde{T}^2$

$$\tilde{T}^2 \simeq T^2_{SE}/(\mathbb{Z}_n\times\mathbb{Z}_n)\,. \tag{4.15}$$

We can conclude that the moduli space of the static D5-branes in the β-deformed background corresponds to the quotient $CY/(\mathbb{Z}_n\times\mathbb{Z}_n)$ of the CY cone of the undeformed theory. The same arguments presented above can be applied to the case of D5 dual giants in the β-deformed background and lead to the expected conclusion that their moduli space again corresponds to $CY/(\mathbb{Z}_n\times\mathbb{Z}_n)$.

However, until now we have given only a one-to-one map between the coordinates on the moduli space and the coordinates on $CY/(\mathbb{Z}_n \times \mathbb{Z}_n)$. To complete the identification we still have to compute the metric on the moduli space and see that it coincides with the metric of $CY/(\mathbb{Z}_n \times \mathbb{Z}_n)$.

Consider the moduli space of a static supersymmetric D5-brane described above. Its tangent vectors correspond to the fluctuations in the internal space that preserve the supersymmetry condition and can thus be seen as massless chiral fields in an effective four-dimensional description. The Kahler metric for these chiral fields can be in principle obtained by looking at their kinetic term obtained by expanding the DBI+CS action for the D5-brane. This is exactly the metric we are interested in.

We can apply the results of [15,16] to identify the Kahler structure of the moduli space. To find the correct holomorphic parametrization of the D5 massless fluctuations we can use once again the action of the β-deformation. The fluctuation of a general D-brane are given by the sections of the generalised normal bundle $N_{(\Sigma,\mathcal{F})}$ [15]. For a D3-brane in a Sasaki-Einstein background, the moduli space corresponds to the CY cone M itself, $N_{(\Sigma,\mathcal{F})} \simeq T_M$ and the associated complex structure is nothing but the complex structure of the CY. Now, a basis for the holomorphic tangent space to the moduli space is given by the following sections of the generalised normal bundle

$$e_i = \frac{\partial}{\partial z^i}\ ; \tag{4.16}$$

where z^i are the holomorphic coordinates on the CY. A basis for the holomorphic deformations for the corresponding D5-brane in the β-deformed background can be obtained simply by taking the β-transformation of the basis (4.16)

$$\tilde{e}_i = O_{LM} \cdot e_i . \tag{4.17}$$

We can now use the general formula for the Kahler metric given in [15,16], which was in fact obtained by expanding the DBI+CS D-brane action. In the basis (4.17) it is given by

$$\begin{aligned}
G_{i\bar{j}} &= -i\int_\Sigma [\tilde{e}_i \cdot \tilde{e}_{\bar{j}} \cdot \mathrm{Im}(\tilde{e}^{2A-\phi}\hat{\Phi}_+)]|_\Sigma \wedge e^{\mathcal{F}} = \\
&= -i\int_\Sigma e^{2A} e^{-\beta\,\iota_{e_i}\iota_{e_{\bar{j}}}}\, \mathrm{Im}\left[\exp(-ie^{2A}J_{(0)})\right]|_\Sigma \wedge e^{\mathcal{F}} = \\
&= i\int_\Sigma J^{(0)}_{i\bar{j}}\,\mathcal{F} = -i(2\pi)^2 n J^{(0)}_{i\bar{j}}\ ;
\end{aligned} \tag{4.18}$$

where $J^{(0)}$ is Kahler form on the CY cone. We thus see that we obtain (locally) exactly the CY metric, up to an overall factor which comes from the fact that the D5-brane with n units of $\mathcal{F}$ flux corresponds to n D3-branes in the undeformed SE background. From the coordinate identification discussed above, we can conclude that the Kahler moduli space for the D5-brane is indeed $CY/(\mathbb{Z}_n \times \mathbb{Z}_n)$.

5 Comments on giant gravitons

There exist other BPS string configurations. Of particular interest are the giant gravitons, configurations of D3-brane wrapping 3 cycles in the internal space. It would be quite interesting to perform a complete analysis of the spectrum of giant gravitons on the β-deformed background. As shown in [26–31], in the undeformed case, the quantisation of the classical supersymmetric giant graviton solutions gives a complete information about the spectrum and the partition function of BPS mesonic operators in the field theory.

In the Calabi-Yau case, giant gravitons can be parameterised by Euclidean D3-branes living inside the internal six-manifold [26,32]. We restrict to the minimal giant gravitons without world-volume flux, which parametrize all the bosonic BPS states. The argument given in [26] suggests that the same parameterisation can be used in all solutions with AdS_5 factor. The supersymmetric conditions for Euclidean D-branes on a generalised geometry background have been derived in [33] and shown to be identical to the conditions for the internal part of space-filling branes discussed in [14,15][16], that we have already written in (4.3) and (4.4). So they can be easily applied to an Euclidean D3-brane, given the form of the pure spinors discussed in Section 2.3.

The F-flatness condition (4.3) for Euclidean D3-brane wrapping Σ with $F = 0$ reduces to

$$\Omega_{(0)}|_{\Sigma} = 0 \, , \tag{5.1}$$

where we recall that $\Omega_{(0)}$ is the holomorphic (3,0) on the original CY geometry. The condition (5.1) exactly requires that the 4-cycle wrapped by the Euclidean D3-brane must be holomorphic with respect to the CY complex structure. Consider for example four-cycles in β-deformed toric vacua defined by the embedding $w_3 = g(z^1,z^2,\bar{z}^1,\bar{z}^2)$, where $z^{1,2},\bar{z}^{1,2}$ are chosen as coordinates on the cycle. Then the F-flatness (5.1) becomes

$$dz^1 \wedge dz^2 \wedge dg = 0 \quad , \quad \bar{\partial} g = 0 \, , \tag{5.2}$$

which indeed requires that the embedding is holomorphic with respect to the old variables. Of course, other supersymmetric embeddings might exist which are not parameterised by $z^{1,2}$.

On the other hand, the general D-flatness condition is (4.4) in the β-deformed toric-vacua, for the above four-cycles with $F = 0$, becomes

$$(J \wedge J \wedge J)|_{\Sigma} \sim dx^1 \wedge dx^2 \wedge dg \wedge d\bar{g} = 0 \quad , \quad \mathrm{Im}\,(\partial_1 g\, \partial_2 \bar{g}) = 0 \, . \tag{5.3}$$

Interestingly, all the supersymmetric conditions can be written in terms of the original complex coordinates of the Calabi-Yau. This is in agreement with field theory, where the moduli space for the deformed theory remains a complex manifold and

[16] Indeed, the results of this section can be equally used to identify and study flavor D7-branes on this general class of β-deformed backgrounds (see [34,35] for work in this direction).

the original complex structure of the moduli space can be still used to characterize it. We can easily find many solutions of the F and D – flatness conditions. For example, all monomials of definite charge $w_3 = e^{n_1 z_1} e^{n_2 z_2}$ solve the constraints. At first sight, we are left with more solutions than expected from the spectrum of BPS states of the deformed theory. However a more careful analysis of the giant graviton characterization as Euclidean D3-branes, of their global properties, of their world-volume flux and, in general, of the quantisation procedure should be performed before extracting correct results. We leave this interesting analysis for future work.

6 The gauge theory

In this Section we discuss the moduli space for a β-deformed quiver gauge theory. Rather than giving general proofs for all toric quiver theories we examine various examples and we give some general arguments.

6.1 Non abelian BPS conditions

In order to understand the full mesonic moduli space of the gauge theory we need to study general non-abelian solutions of the F term equations.

Before attacking the general construction, we consider $\mathcal{N} = 4$ SYM and the conifold. In the $\mathcal{N} = 4$ SYM case, we form mesons out of the three adjoint fields (ϕ_i). The non-abelian BPS conditions for these mesonic fields are given in equation (2.9) and can be considered as equations for three $N \times N$ matrices. In the conifold case, we can define four composite mesonic fields which transform in the adjoint representation of one of the two gauge groups

$$x = (A_1 B_1) \; ; \; y = (A_2 B_2) \; ; \; z = (A_1 B_2) \; ; \; w = (A_2 B_1) \tag{6.1}$$

and consider the four mesons $x; y; z; w$ as $N \times N$ matrices. We could use the second gauge group without changing the results. With a simple computation using the F-term conditions (2.10) we derive the following matrix commutation equations

$$\begin{aligned} xz &= b^{-1} zx \\ xw &= bwx \\ yz &= bzy \\ yw &= b^{-1} wy \\ xy &= yx \\ zw &= wz \end{aligned} \tag{6.2}$$

and the matrix equation

$$xy = bwz \tag{6.3}$$

which is just the conifold equation. Here and in the following $b = e^{2\pi i\beta}$. For $\beta = 0$ these conditions simplify. All the mesons commute and the $N \times N$ matrices $x; y; z; w$

can be simultaneous diagonalized. The eigenvalues are required to satisfy the conifold equation (6.3) and therefore the moduli space is given by the symmetrized product of N copies of the conifold, as expected.

An interesting observation is that, for the $\mathcal{N}=4$ SYM and (6.2) for the conifold, the F-term conditions for $\beta \neq 0$ can be obtained by using the non commutative product defined in (2.5)

$$f * g \equiv e^{i\pi\beta(Q^f \wedge Q^g)} fg\,. \tag{6.4}$$

The charges of mesons for $\mathcal{N}=4$ and the conifold are shown in Figure 2.

The BPS conditions for the Calabi-Yau case, which require that every pair of mesonic fields f and g commute, are replaced in the β-deformed theory by a non commutative version

$$[f,g]=0 \quad \to \quad [f,g]_\beta \equiv f*g - g*f = 0\,. \tag{6.5}$$

It is an easy exercise, using the assignment of charges shown in Figure 2, to show that these modified commutation relations reproduce equations (2.9) and (6.2).

This simple structure extends to a generic toric gauge theory. The algebraic equations of the Calabi-Yau give a set of matrix equations for mesons. In the undeformed theory, all mesons commute, while in the β-deformed theory the original commutation properties are replaced by their non commutative version. In order to fully appreciate these statements we need to understand the structure of the mesonic chiral ring for toric theories [36–42].

6.1.1 The mesonic chiral ring

We briefly review the structure of the mesonic chiral ring for quiver gauge theories. The reader is referred to [36–42] for an exhaustive discussion. The reader who wants to avoid technical details can directly jump to the next Sections, where most of the examples are self-explaining.

From the algebraic-geometric point of view the data of a conical toric Calabi-Yau are encoded in a rational polyedral cone C in $\mathbb{Z}^3$ defined by a set of vectors V_α $\alpha=1,\dots,d$. For a CY cone, using an $SL(3,\mathbb{Z})$ transformation, it is always possible to carry these vectors in the form $V_\alpha=(x_\alpha,y_\alpha,1)$. In this way the toric diagram can be drawn in the x,y plane (see for example Figure 2). The CY equations can be reconstructed from this set of combinatorial data using the dual cone C^*. This is defined in equation (2.14) and it was already used to write the metric as a T^3 fibration. The two cones are related as follow. The geometric generators for the cone C^*, which are vectors aligned along the edges of C^*, are the perpendicular vectors to the facets of C.

To give an algebraic-geometric description of the CY, we need to consider the cone C^* as a semi-group and to find its generators over the integer numbers. The primitive vectors pointing along the edges generate the cone over the real numbers but we generically need to add other vectors to obtain a basis over the integers. Denote by W_j with $j = 1;\ldots;k$ a set of generators of C over the integers. To every vector W_j it is possible to associate a coordinate x_j in some ambient space. k vectors in Z^3 are clearly linearly dependent for $k > 3$, and the additive relations satisfied by the generators W_j translate into a set of multiplicative relations among the coordinates x_j. These are the algebraic equations defining the six-dimensional CY cone.

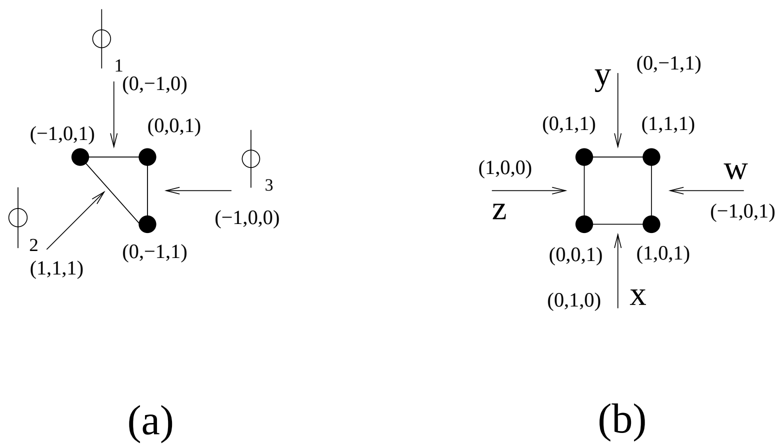

Figure 2: The toric diagram C and the generators of the dual cone C with the associated mesonic fields for: (a) $N = 4$, (b) conifold. The $U(1)^3$ charges of the mesons are explicitly indicated; the first two entries of the charge vectors give the $U(1)^2$ global charge used to define the non commutative product.

All the relations between points in the dual cone become relations among mesons in the field theory. In fact, using toric geometry and dimer technology, it is possible to show that there exists a one to one correspondence between the integer points inside C and the mesonic operators in the dual field theory, modulo F-term constraints [37,40]. To every integer point m_j in C we indeed associate a meson M_{m_j} in the gauge theory with $U(1)^3$ charge m_j. In particular, the mesons are uniquely determined by their charge under $U(1)^3$. The first two coordinates

$$Q^{m_j} = (m_j^1; m_j^2) \tag{6.6}$$

of the vector m_j are the charges of the meson under the two flavour $U(1)$ symmetries. Since the cone C is generated as a semi-group by the vectors W_j the generic meson will be obtained as a product of basic mesons M_{W_j}, and we can restrict to these generators for all our purposes. The multiplicative relations satisfied by the coordinates x_j become a set of multiplicative relations among the mesonic operators M_{W_j} inside the chiral ring of the gauge theory. It is possible to prove that these relations are a consequence of the F-term constraints of the gauge theory. The abelian version of this set of relations is just the set of algebraic equations defining the CY variety as embedded in C^k. The examples of $N = 4$ SYM and the conifold are shown in Figure 2. In the case of $N = 4$, the three mesons ϕ_j correspond to independent charge vectors and we obtain the variety C^3. In the case of the conifold, the four mesons $x; y; z; w$ correspond to four vectors with one linear relation and we obtain the description of the conifold as a quadric $xy = zw$ in C^4.

We need now to understand the non abelian structure of the BPS conditions. Mesons correspond to closed loops in the quiver and, as shown in [36,38], for any meson there is an F-term equivalent meson that passes for a given gauge group. We can therefore assume that all meson loops have a base point at a specific gauge group and consider them as $N \times N$ matrices M. In the undeformed theory, the F-term equations imply that all mesons commute and can be simultaneously diagonalized. The additional F-term constraints require that the mesons, and therefore all their eigenvalues, satisfy the algebraic equations defining the Calabi-Yau. This gives a moduli space which is the N-fold symmetrized product of the Calabi-Yau. This has been explicitly verified in [43] for the case of the quiver theories [44] corresponding to the L^{pqr} manifolds. In the β-deformed theory the commutation relations among mesons are replaced by β-deformed commutators

$$M_{m_1} M_{m_2} = e^{2i\pi\beta(Q^{m_1} \wedge Q^{m_2})} M_{m_2} M_{m_1} = b^{(Q^{m_1} \wedge Q^{m_2})} M_{m_2} M_{m_1} : \qquad (6.7)$$

The prescription (6.7) will be our short-cut for computing the relevant quantities we will be interested in. This fact becomes computationally relevant in the generic toric case. As we will show in an explicit example in the Appendix B this procedure is equivalent to using the β-deformed superpotential defined in (2.8) and deriving the constraints for the mesonic fields from the F-term relations.

Finally the mesons still satisfy a certain number of algebraic equations

$$f(M) = 0 \qquad (6.8)$$

which are isomorphic to the defining equations of the original Calabi-Yau.

6.2 Abelian moduli space

In this section, we give evidence from the gauge theory side that the abelian moduli space of the β-deformed theories is a set of lines. There are exactly d such lines, where d is the number of vertices in the toric diagram. In fact, the lines correspond to the geometric generators of the dual cone of the undeformed geometry, or, in other words, the edges of the polyedron C where the T^3 fibration degenerates to T^1. Internal generators of C as a semi-group do not correspond to additional lines in the moduli space. These statements are the field theory counterpart of the fact that the D3 probes can move only along the edges of the symplectic cone.

We explained in the previous section how to obtain a set of modified commutation relations among mesonic fields. In the abelian case the mesons reduce to commuting c-numbers. >From the relations (6.7) with non a trivial b factor, we obtain the constraint

$$M_{m_1} M_{m_2} = 0 : \qquad (6.9)$$

Adding the algebraic constraints (6.8) defining the CY, we obtain the full set of constraints for the abelian mesonic moduli space.

We now solve the constraints in a selected set of examples, which are general enough to exemplify the result. We analyse $\mathcal{N}=4$, the conifold, the Suspended Pinch Point (SPP) singularity and a more sophisticated example, PdP_4, which covers the case where the generators of C as a semi-group are more than the geometric generators.

6.2.1 The case of $\mathbb{C}^3$

The $\mathcal{N}=4$ theory is simple and was already discussed in Section 2.1. The three lines correspond to the geometric generators of the dual cone as in Figure 2.

6.2.2 The conifold

The abelian mesonic moduli space of the conifold theory was already discussed in Section 2.1 using elementary fields. From the equations (6.2) we obtain the same result: four lines corresponding to the external generators of the dual cone as shown in Figure 2.

6.2.3 SPP

The gauge theory obtained as the near horizon limit of a stack of D3-branes at the tip of the conical singularity

$$xy^2 = wz \tag{6.10}$$

is called the SPP gauge theory [45]. The toric diagram and the quiver of this theory are given in Figure 3. Its superpotential is

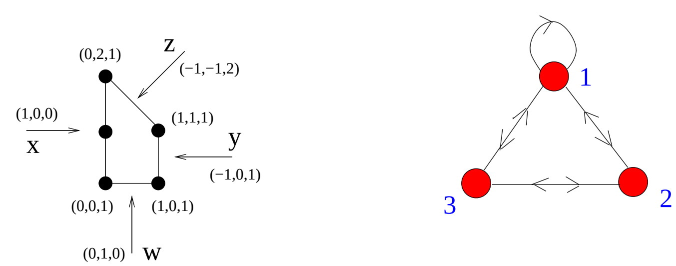

Figure 3: The toric diagram and the quiver of the SPP singularity

$$W = X_{21}X_{12}X_{23}X_{32} + X_{13}X_{31}X_{11} - X_{32}X_{23}X_{31}X_{13} - X_{12}X_{21}X_{11} \tag{6.11}$$

The generators of the mesonic chiral ring are

$$\begin{aligned} w &= X_{13}X_{32}X_{21}\,; \quad x = X_{11}\,; \\ z &= X_{12}X_{23}X_{31}\,; \quad y = X_{12}X_{21}\,. \end{aligned} \tag{6.12}$$

These mesons correspond to the generators of the dual cone in Figure 3. Their flavour charges can be read from the dual toric diagram

$$Q_x = (1;0)\,,\quad Q_z = (-1;-1)\,,\quad Q_y = (-1;0)\,,\quad Q_w = (0;1): \tag{6.13}$$

Using the deformed commutation rule for mesons (6.7) we obtain the following relations

$$\begin{aligned} &xw = bwx\,;\quad zx = bxz\,;\quad wz = bzw\,;\\ &wy = byw\,;\quad yz = bzy: \end{aligned} \tag{6.14}$$

In the abelian case they reduce to

$$\begin{aligned} &xw = 0\,;\quad zx = 0\,;\quad wz = 0\,;\\ &wy = 0\,;\quad yz = 0\,;\quad xy^2 \sim wz\,; \end{aligned} \tag{6.15}$$

where the last equation is the additional F-term constraint giving the original CY manifold. The presence of the symbol "$\sim$" is due to the fact that the original CY equation is deformed by an unimportant power of the deformation parameter b, which can always be reabsorbed by rescaling the variables. The solutions to these equations are

$$\begin{aligned} &(x = 0;\; y = 0;\; z = 0) \to \{w\}\,;\\ &(x = 0;\; y = 0;\; w = 0) \to \{z\}\,;\\ &(x = 0;\; z = 0;\; w = 0) \to \{y\}\,;\\ &(w = 0;\; y = 0;\; z = 0) \to \{x\}\,; \end{aligned} \tag{6.16}$$

corresponding to the four complex lines associated to the four generators of the dual cone.

6.2.4 PdP_4

This is probably the simplest example with internal generators: the perpendicular to the toric diagram are enough to generate the dual cone on the real numbers but other internal vectors are needed to generate the cone on the integer numbers. The discussion in Section 3.2 suggests that the moduli space seen by the dual giant gravitons and hence the abelian mesonic moduli space of the gauge theory are exhausted by the external generators. We will see evidence of this fact.

The PdP_4 gauge theory, [46], is the theory obtained as the near horizon limit of a stack of D3-branes at the tip of the non complete intersection singularity defined by the set of equations

$$\begin{aligned} &z_1 z_3 = z_2 t\,,\; z_2 z_4 = z_3 t\,,\; z_3 z_5 = z_4 t\\ &z_2 z_5 = t^2\,,\; z_1 z_4 = t^2: \end{aligned} \tag{6.17}$$

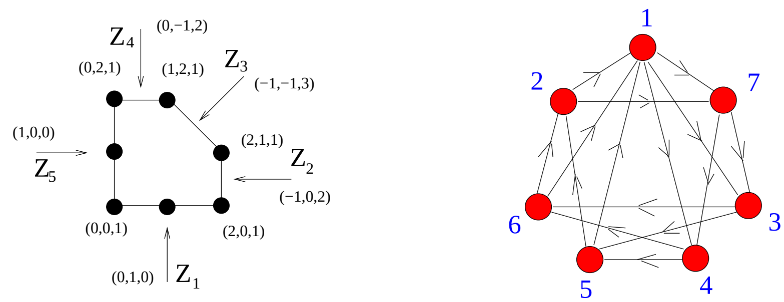

Figure 4: The toric diagram and the quiver of the PdP_4 singularity

The toric diagram and the quiver of the theory are given in Figure 4. The superpotential of the theory is

$$
\begin{aligned}
W = {} & X_{61}X_{17}X_{74}X_{46} + X_{21}X_{13}X_{35}X_{52} + X_{27}X_{73}X_{36}X_{62} + X_{14}X_{45}X_{51} \\
& - X_{51}X_{17}X_{73}X_{35} - X_{21}X_{14}X_{46}X_{62} - X_{27}X_{74}X_{45}X_{52} - X_{13}X_{36}X_{61}. \qquad (6.18)
\end{aligned}
$$

The generators of the mesonic chiral ring are

$$
\begin{aligned}
& z_1 = X_{51}X_{13}X_{35}; \quad z_2 = X_{51}X_{17}X_{74}X_{45}; \quad z_3 = X_{21}X_{17}X_{74}X_{45}X_{52}; \\
& z_4 = X_{14}X_{45}X_{52}X_{21}; \quad z_5 = X_{14}X_{46}X_{61}; \quad t = X_{13}X_{36}X_{61}. \qquad (6.19)
\end{aligned}
$$

From the toric diagram we can easily read the charges of the mesonic generators

$$
Q_{z_1} = (0,1); \quad Q_{z_2} = (-1,0); \quad Q_{z_3} = (-1,-1); \quad Q_{z_4} = (0,-1); \quad Q_{z_5} = (1,0). \qquad (6.20)
$$

To generate the cone on the integers we need to add the internal generator $t = (0,0,1)$ with flavour charges $Q_t = (0,0)$. The generators satisfy the equations (6.17) for the PdP_4 singularity modified just by some irrelevant proportional factors given by powers of β. We must add the relations obtained from the mesonic β-deformed commutation rule (6.7)

$$
\begin{aligned}
& z_1 z_2 = \beta z_2 z_1; \quad z_1 z_3 = \beta z_3 z_1; \quad z_5 z_1 = \beta z_1 z_5; \quad z_2 z_3 = \beta z_3 z_2 \\
& z_2 z_4 = \beta z_4 z_2; \quad z_3 z_4 = \beta z_4 z_3; \quad z_3 z_5 = \beta z_5 z_3; \quad z_4 z_5 = \beta z_5 z_4; \qquad (6.21)
\end{aligned}
$$

that in the abelian case reduce to

$$
\begin{aligned}
& z_1 z_2 = 0; \quad z_1 z_3 = 0; \quad z_5 z_1 = 0; \quad z_2 z_3 = 0; \\
& z_2 z_4 = 0; \quad z_3 z_4 = 0; \quad z_3 z_5 = 0; \quad z_4 z_5 = 0. \qquad (6.22)
\end{aligned}
$$

The solutions to the set of equations (6.17) and (6.22) are

$$
\begin{aligned}
& (z_2 = 0; \; z_3 = 0; \; z_4 = 0; \; z_5 = 0; \; t = 0) \to \{z_1\}; \\
& (z_1 = 0; \; z_3 = 0; \; z_4 = 0; \; z_5 = 0; \; t = 0) \to \{z_2\}; \\
& (z_1 = 0; \; z_2 = 0; \; z_4 = 0; \; z_5 = 0; \; t = 0) \to \{z_3\}; \\
& (z_1 = 0; \; z_2 = 0; \; z_3 = 0; \; z_5 = 0; \; t = 0) \to \{z_4\}; \\
& (z_1 = 0; \; z_2 = 0; \; z_3 = 0; \; z_4 = 0; \; t = 0) \to \{z_5\}; \qquad (6.23)
\end{aligned}
$$

corresponding to the five external generators. We observe in particular that the complex line corresponding to the internal generators t is not a solution.

6.3 Non abelian moduli space and rational β

The F-term equations

$$M_{m_1} M_{m_2} = e^{2\pi i \beta (Q^{m_1}\wedge Q^{m_2})} M_{m_2} M_{m_1} \tag{6.24}$$

give a non commutative 't Hooft-Weyl algebra for the $N \times N$ matrices M_I. By diagonalizing the matrix $\beta_{m_1 m_2} = \beta(Q^{m_1}\wedge Q^{m_2})$ we can reduce the problem to various copies of the algebra for a non commutative torus

$$M_1 M_2 = e^{2\pi i \beta} M_2 M_1 \tag{6.25}$$

whose representations are well known.

For generic β, corresponding to irrational values of β, the 't Hooft-Weyl algebra has no non trivial finite dimensional representations: we can only find solutions where all the matrices are diagonal, and in particular equation (6.25) implies $M_1 M_2 = M_2 M_1 = 0$. The problem is thus reduced to the abelian one and the moduli space is obtained by symmetrizing N copies of the abelian moduli space, which consists of d lines. This is the remaining of the original Coulomb branch of the undeformed theory.

For rational $\beta = m/n$, instead, new branches are opening up in the moduli space [5,6]. In fact, for rational β, we can have finite dimensional representations of the 't Hooft-Weyl algebra which are given by $n \times n$ matrices $(O^I)_{ij}$. The explicit form of the matrices $(O^I)_{ij}$ can be found in [47] but it is not of particular relevance for us. For gauge groups $SU(N)$ with $N = nM$ we can have vacua where the mesons have the form

$$(M_I)_{\alpha\beta} = \mathrm{Diag}(M_a) \otimes (O^I)_{ij}; \quad a = 1,\ldots,M; \quad i,j = 1,\ldots,n; \quad \alpha,\beta = 1\ldots N: \tag{6.26}$$

The M variables M_a are further constrained by the algebraic equations (6.8) and are due to identifications by the action of the gauge group. A convenient way of parameterising the moduli space is to look at the algebraic constraints satisfied by the elements of the centre of the non-commutative algebra [5].

We will give arguments showing that the centre of the algebra of mesonic operators is the algebraic variety $CY/Z_n \times Z_n$. Here CY means the original undeformed variety, and the two Z_n factors are abelian discrete sub-groups of the two flavours symmetries. This statement is the field theory counterpart of the fact that the moduli space of D5 dual giant gravitons is the original Calabi-Yau divided by $Z_n \times Z_n$.

The generic vacuum (6.26) corresponds to M D5 dual giants moving on the geometry. The resulting branch of the moduli space is the M-fold symmetrized product of the original Calabi-Yau divided by $Z_n \times Z_n$. Each D5 dual giant should be considered as a fully non-abelian solution of the dual gauge theory carrying n color indices so that the total number of colors is $N = nM$. We can obtain a different perspective on this branch of our gauge theory by considering it as the world-volume theory of D3-branes sitting at a discrete torsion $Z_n \times Z_n$ orbifold of the original singularity [48]. In this picture, the D5 dual giants correspond to the physical branes surviving the orbifold projection. This perspective has been discussed in details in the literature for $N = 4$ SYM [5] and it can be easily extended to generic toric singularities.

6.3.1 The case of C^3

The case of the β-deformation of $N = 4$ gauge theory is simple and well known [5].

The generators of the algebra of mesonic operators are the three elementary fields Φ_1, Φ_2, Φ_3. Equation (2.9) implies that it possible to write the generic element of the algebra in the ordered form

$$\mathcal{O}_{k_1,k_2,k_3} = \Phi_1^{k_1}\Phi_2^{k_2}\Phi_3^{k_3} \tag{6.27}$$

The centre of the algebra is given by the subset of operators in (6.27) such that:

$$\begin{aligned}
\mathcal{O}_{k_1,k_2,k_3}\Phi_1 &= b^{k_3-k_2}\Phi_1\mathcal{O}_{k_1,k_2,k_3} = \Phi_1\mathcal{O}_{k_1,k_2,k_3};\\
\mathcal{O}_{k_1,k_2,k_3}\Phi_2 &= b^{k_1-k_3}\Phi_2\mathcal{O}_{k_1,k_2,k_3} = \Phi_2\mathcal{O}_{k_1,k_2,k_3};\\
\mathcal{O}_{k_1,k_2,k_3}\Phi_3 &= b^{k_2-k_1}\Phi_3\mathcal{O}_{k_1,k_2,k_3} = \Phi_3\mathcal{O}_{k_1,k_2,k_3}.
\end{aligned} \tag{6.28}$$

Since $b^n = 1$, the center of the algebra is given by the set of $\mathcal{O}_{k_1,k_2,k_3}$ such that $k_1 = k_2 = k_3 \bmod n$.

The generators of the center of the algebra are: $\mathcal{O}_{n,0,0}$; $\mathcal{O}_{0,n,0}$; $\mathcal{O}_{0,0,n}$; $\mathcal{O}_{1,1,1}$. We call them $x;y;w;z$ respectively. They satisfy the equation

$$xyw = z^n \tag{6.29}$$

which defines the variety $C^3/Z_n \times Z_n$. To see this, take C^3 with coordinate $Z^1;Z^2;Z^3$, and consider the action of the group $Z_n \times Z_n$ on C^3

$$(Z^1;Z^2;Z^3) \to (Z^1\omega^{-1};Z^2\eta;Z^3\omega\eta^{-1}) \tag{6.30}$$

with $\omega^n = \eta^n = 1$. The basic invariant monomials under this action are $x = (Z^1)^n$; $y = (Z^2)^n$; $w = (Z^3)^n$; $z = Z^1Z^2Z^3$ and they clearly satisfy the equation (6.29).

This fact can be represented in a diagrammatic way as in Figure 5. This representation of the rational value β-deformation is valid for every toric CY singularity.

6.4 Conifold

The case of the conifold is a bit more intricate and can be a useful example for the generic CY toric cone. The generators of the mesonic algebra $x;y;z;w$ satisfy the

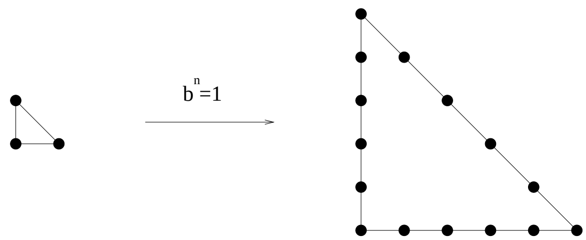

Figure 5: $\mathbb{C}^3 \to \mathbb{C}^3/\mathbb{Z}_n \times \mathbb{Z}_n$ in the toric picture, $b^5 = 1$.

equations (6.2). It follows that we can write the generic monomial element of the algebra in the ordered form

$$_{k_1,k_2,k_3,k_4} = x^{k_1} y^{k_2} w^{k_3} z^{k_4}\,. \tag{6.31}$$

The centre of the algebra is given by the subset of the operators (6.31) that satisfy the equations

$$\begin{aligned}
{}_{k_1,k_2,k_3,k_4}\, x &= b^{k_4-k_3}\, x\, {}_{k_1,k_2,k_3,k_4} = x\, {}_{k_1,k_2,k_3,k_4}\,;\\
{}_{k_1,k_2,k_3,k_4}\, y &= b^{k_3-k_4}\, y\, {}_{k_1,k_2,k_3,k_4} = y\, {}_{k_1,k_2,k_3,k_4}\,;\\
{}_{k_1,k_2,k_3,k_4}\, w &= b^{k_1-k_2}\, w\, {}_{k_1,k_2,k_3,k_4} = w\, {}_{k_1,k_2,k_3,k_4}\,;\\
{}_{k_1,k_2,k_3,k_4}\, z &= b^{k_2-k_1}\, z\, {}_{k_1,k_2,k_3,k_4} = z\, {}_{k_1,k_2,k_3,k_4}\,.
\end{aligned} \tag{6.32}$$

Because $b^n = 1$, the elements of the centre of the algebra are the subset of the operators of the form (6.31) such that $k_1 = k_2$, $k_3 = k_4$, mod n.

The centre is generated by ${}_{n,0,0,0}$, ${}_{0,n,0,0}$, ${}_{0,0,n,0}$, ${}_{0,0,0,n}$, ${}_{1,1,0,0}$, ${}_{0,0,1,1}$; we call them respectively A, B, C, D, E, G. The F-term relation

$$xy = bwz \tag{6.33}$$

then implies that E and G are not independent: $E = bG$. Moreover the generators of the centre of the algebra satisfy the equations

$$AB = CD = E^n\,. \tag{6.34}$$

As in the previous example, it is easy to see that these are the equations of the $\mathbb{Z}_n \times \mathbb{Z}_n$ orbifold of the conifold. Take indeed the coordinates x, y, w, z defining the conifold as a quadric embedded in $\mathbb{C}^4$. The action of $\mathbb{Z}_n \times \mathbb{Z}_n$ is

$$x, y, w, z \to x, y^{-1}, w^{-1}, z\,, \tag{6.35}$$

where ${}^n = {}^n = 1$. The basic invariants of this action are A, B, C, D, E, G, and they are subject to the constraint (6.33). Hence the equations (6.34) define the variety $\mathcal{C}(T^{1,1})/\mathbb{Z}_n \times \mathbb{Z}_n$.

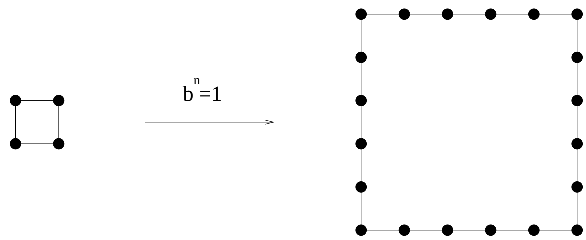

Figure 6: $C(T^{1,1}) \to C(T^{1,1})/\mathbb{Z}_n \times \mathbb{Z}_n$ in the toric picture, $b^5 = 1$

6.5 The general case

Now we want to analyse the generic case and show that the centre of the mesonic algebra for the rational β-deformed ($b^n = 1$) gauge theory is the $\mathbb{Z}_n \times \mathbb{Z}_n$ quotient of the undeformed CY.

For a generic toric quiver gauge theory we take a set of basic mesons M_{W_j} (we will call them simply x_j from now on) corresponding to the generators W_j of the cone C. These are the generators of the mesonic chiral ring of the given gauge theory. Because they satisfy the relations (6.24) it is always possible to write the generic monomial element of the mesonic algebra generated by x_j in the ordered form

$$\mathcal{O}_{p_1,\ldots,p_k} = x_1^{p_1} x_2^{p_2} \ldots x_k^{p_k} \,. \tag{6.36}$$

We are interested in the operators that form the centre of the algebra, or, in other words, that commute with all the elements of the algebra. To find them it is enough to find all the operators that commute with all the generators of the algebra, namely $x_1,\ldots,x_k$. The generic operator (6.36) has charge $Q_{p_1,\ldots,p_k}$ under the two flavour $U(1)$ symmetries, and the generators x_j have charges Q_j. They satisfy the following relations

$$\mathcal{O}_{p_1,\ldots,p_k}\, x_j = x_j\, \mathcal{O}_{p_1,\ldots,p_k}\, b^{Q_{p_1,\ldots,p_k} \wedge Q_j} \,. \tag{6.37}$$

This implies that the centre of the algebra is formed by the set of $\mathcal{O}_{p_1,\ldots,p_k}$ such that

$$Q_{p_1,\ldots,p_k} \wedge Q_j = 0 \mod n \,, \; j = 1,\ldots,k \,. \tag{6.38}$$

At this point it is important to realize that the Q_j contain the two dimensional vectors perpendicular to the edges of the two dimensional toric diagram. The fact that the toric diagram is convex implies that the Q_j span the T^2 flavour torus. In particular the operator $\mathcal{O}_{p_1,\ldots,p_k}$ must commute (modulo n) with the operators with charges $(1,0)$ and $(0,1)$. The first condition gives all the operators in the algebra that are invariant under the $\mathbb{Z}_n$ in the second $U(1)$, while the second gives all the operators invariant under the $\mathbb{Z}_n$ contained in the first $U(1)$. All together the set of operators in the centre of the algebra consists of all operators $\mathcal{O}_{p_1,\ldots,p_k}$ invariant under the $\mathbb{Z}_n \times \mathbb{Z}_n$ discrete subgroup of the T^2.

The monomials made with the free $x_1,\ldots,x_k$ coordinates of $\mathbb{C}^k$ that are invariant under $\mathbb{Z}_n \times \mathbb{Z}_n$, form, by definition, the quotient variety $\mathbb{C}^k/\mathbb{Z}_n \times \mathbb{Z}_n$. The toric variety

V is defined starting from a ring over $\mathbb{C}^k$ with relations given by a set of polynomials $\{q_1, \dots, q_l\}$ defined by the toric diagram

$$\mathbb{C}[V] = \frac{\mathbb{C}[x_1, \dots, x_k]}{\{q_1, \dots, q_l\}} . \tag{6.39}$$

Indeed the elements of the centre of the algebra are the monomials made with the x_j, subject to the relations $\{q_1, \dots, q_l\}$, invariant under $\mathbb{Z}_n \times \mathbb{Z}_n$. This fact allows us to conclude that the centre of the algebra in the case $b^n = 1$ is the quotient of the original CY

$$V_b = \frac{CY}{\mathbb{Z}_n \times \mathbb{Z}_n} . \tag{6.40}$$

The -deformed $\mathcal{N} = 4$ gauge theory and the -deformed conifold gauge theory are special cases of this result. In the appendix we will discuss a more sophisticated example, which includes SPP as a particular case.

7 Conclusions

In this paper we discussed general properties of the -deformation of toric quiver gauge theories and of their gravitational duals, which have a very simple characterization in terms of generalised complex geometry.

We analysed the moduli space of vacua of the -deformed theory using D-branes probes and field theory analysis. An important class of supersymmetric probes, the giant gravitons, has still to be analysed. It would be interesting to study the classical configurations of giant gravitons in the -deformed background and their quantisation. This should give information about the spectrum of BPS operators and, as it happens in the undeformed theory, it should help in computing partition functions for the chiral ring of the gauge theory [27–31, 40–42].

On the gravity side, we clarified the geometrical structure of the supersymmetric -deformed background. The description in terms of pure spinors is remarkably simple. It would be interesting to see whether this description can be extended to the analysis of other marginal deformations of superconformal theories. In particular $\mathcal{N} = 4$ SYM and other quiver gauge theories admit deformations that breaks the $U(1)^3$ symmetry whose supergravity dual is still elusive. It would be interesting to extend our methods to the search of these missing solutions.

Acknowledgments

D.F. would like to thank Loriano Bonora, Umut Gursoy, Alberto Mariotti, Marco Pirrone for valuable discussions. D.F. would like to thank the Laboratoire de Physique Theorique de l'Ecole Normale Superieure and Universites Paris VI et VII, Jussieu for the kind hospitality during part of this work. A.Z. would like to thank the Laboratoire de Physique Theorique de l'Universites Paris VI et VII, Jussieu, the Galileo Galilei Institute in Florence and the Newton Institute in Cambridge for hospitality and support during part of this work. D.F. is supported in part by INFN and the Marie Curie fellowship under the programme EUROTHEPHY-2007-1. A.Z. is supported in part by INFN and MIUR under contract 2005-024045-004 and 2005-023102 and by the European Community's Human Potential Program MRTN-CT-2004-005104. A.B and M.P. are supported in part by the RTN contract MRTN-CT-2004-512194 and by ANR grant BLAN05-0079-01. R.M. is supported in part by RTN contract MRTN-CT-2004-005104 and by ANR grant BLAN06-3-137168. L.M. is supported by the DFG cluster of excellence "Origin and Structure of the Universe" and would like to acknowledge the Galileo Galilei Institute for Theoretical Physics for hospitality and the INFN for partial support.

A β-deformed $\mathcal{N}=4$ Super Yang-Mills

For the β-deformation of $\mathcal{N}=4$ SYM it is possible to use the pure spinor formalism to determine the precise relation between the parameter γ entering the supergravity background and the β parameter deforming the superpotential of the dual gauge theory. Even if the computation does not apply to the β-deformation of a generic toric Calabi-Yau, we report it here since it provides a nice application of the formalism of Generalised Complex Geometry.

The computation is based on the observation that for a generic deformation of $\mathcal{N}=4$ SYM it possible to relate the integrable pure spinor of the gravity solution ($\hat{\Phi}_-$ for us) and the superpotential of the dual gauge theory [11,15]. More precisely it possible to write the superpotential for a single D-brane probe, with a world-volume flux F and wrapping a cycle Σ in the internal manifold, in terms of the closed pure spinor [15]. Since $e^{3A}\hat{\Phi}_-$ is closed, one can locally write $e^{3A}\hat{\Phi}_- = d\mathcal{C}$ and the superpotential can be written as

$$W = \int_\Sigma \mathcal{C}|_\Sigma \wedge e^F \, . \tag{A.1}$$

Notice that (A.1) has precisely the form of the CS term in the standard D-brane action, where $\mathcal{C}$ plays the role of the twisted RR-potentials $C \wedge e^B$. A non-abelian generalisation of such CS term for multiple D-branes was obtained by Myers in [49], using an argument essentially based on T-duality. Since the pure spinor $\hat{\Phi}_-$ transforms precisely as the RR-field strengths under T-duality, the same argument can be applied in our case, and the resulting non-abelian superpotential has exactly the same form of Myers' non-abelian CS term, with $C \wedge e^B$ substituted by $\mathcal{C}$.

For the background obtained by β-deforming $AdS_5 \times S^5$, using the standard flat complex coordinates on the internal warped C^3, we have

$$e^{3A}\hat{\Phi}_- = \gamma(z^1 z^2 dz^3 + \text{cyclic}) + dz^1 \wedge dz^2 \wedge dz^3 \; ; \tag{A.2}$$

and thus

$$= \quad z^1 z^2 z^3 + \frac{1}{3!}\,{}_{ijk} z^i dz^j \wedge dz^k : \tag{A.3}$$

Then, from the above argument and Myers' non-abelian CS action we get the following non-abelian superpotential for a stack of D3-branes (in units $^0 = 1$)

$$\begin{aligned} W \quad = \quad & \mathrm{Str}[e^{2i} \quad]_{(0)} \\ & \mathrm{Tr}[(1 + i \quad)_{1\ 2\ 3} \quad (1 \quad i \quad)_{1\ 3\ 2}] ; \end{aligned} \tag{A.4}$$

where $_i$ is the non-abelian scalar eld describing the D3-brane uctuations, which is canonically associated to $z^i/(2\quad ^0)$. Comparing with (2.2), since we need $\quad 1$ to trust the supergravity approximation, we conclude that

$$= \quad : \tag{A.5}$$

B Some explicit eld theory examples

In this appendix we illustrate few points of the eld theory analysis. Using the SPP example, we show how the non commutative product acts on the undeformed superpotential and motivate formula (2.8). We also discuss the non abelian branches of the theories $L^{p,q,q}$ for rational $\quad$.

B.1 Action of the non commutative product

To obtain the -deformed gauge theory we pass from the simple product between elds to the star product:

$$X_i X_j \;! \;\; X_i \quad X_j \quad e^{i \quad (Q^i \wedge Q^j)} X_i X_j \tag{B.1}$$

where X_i are the elementary elds in the quiver.

The star product is non commutative but associative and the product of a string of n elds takes the form:

$$X_{a_1} \quad ::: \quad X_{a_n} \quad b^{1/2(\sum_{i<j} Q_{a_i} \wedge Q_{a_j})} X_{a_1} ::: X_{a_n} \tag{B.2}$$

Let us consider two generic mesonic elds with base point in the same gauge group: $M = X_{a_1} ::: X_{a_m}$, $N = X_{b_1} ::: X_{b_n}$. In the undeformed theory they commute $M N = N M$, but when we turn on the -deformation this relation becomes: $\tilde{M} \quad \tilde{N} = \tilde{N} \quad \tilde{M}$, for the quantities $\tilde{M} = X_{a_1} \quad ::: \quad X_{a_m}$, $\tilde{N} = X_{b_1} \quad ::: \quad X_{b_n}$. This gives, using (B.2):

$$\tilde{M}\tilde{N} = b^{(Q_M \wedge Q_N)} \tilde{N}\tilde{M} \tag{B.3}$$

where we dened the charges of the composite elds: $Q_M = Q_{a_1} + ::: + Q_{a_m}$, $Q_N = Q_{b_1} + ::: + Q_{b_n}$. Note that relation (B.3) also holds in the same form for mesons M and N, since they are proportional to $\tilde{M}$ and $\tilde{N}$ respectively, thanks again to (B.2). We obtain therefore our general method (6.7) for computing commutation relations for mesons.

We would like now to understand the structure of the superpotential W for the β-deformed theory, obtained by replacing the standard product with the star product in (B.1). First of all, since W is a trace of mesons, consistency requires the star product to be invariant under cyclic permutations of the fields. This happens because of the conservation of charge [17]: the two $U(1)$ flavour charges of each meson are zero.

Then we want to show that W can always be put into the form (2.8) by rescaling fields. Consider a generic toric gauge theory with G gauge groups, E elementary fields and V monomials in the superpotential. We have the relation [18]:

$$G - E + V = 0 \tag{B.4}$$

The superpotential W of the undeformed theory is a sum of V monomials $m_I; n_J$ made with traces of products of elementary fields. Every elementary field appears in the superpotential W once with the positive sign and once with the negative sign,

$$W = \sum_{I=1}^{V/2} c_I^+ m_I - \sum_{J=1}^{V/2} c_J^- n_J \tag{B.5}$$

After β-deformation the coefficients c_I^+, c_J^- are replaced by generic complex numbers.

Rescaling the elementary chiral fields produces a rescaling also of the coefficients c_I^+, c_J^-, but note that the quantity

$$\frac{\prod_I c_I^+}{\prod_J c_J^-} = \text{const} \tag{B.6}$$

remains constant since every chiral field contributes just once in the numerator and just once in the denominator. In the undeformed theory this constant is 1, while in the β-deformed case its value can be written as $b^{V/2}$, for some rational β.

Consider the action of the E dimensional group of chiral fields rescalings over the V dimensional space of coefficients c_I^+, c_J^- in the superpotential. The subgroup that leaves invariant a generic point (with all coefficients different from zero) is the group of global symmetries of the superpotential. It is known that toric theories have $G+1$ global symmetries[18], therefore the dimension of a generic orbit is $E - (G+1) = V - 1$, thanks to (B.4). This shows that (B.6) is the only algebraic constraint under field rescalings, and hence it is always possible to put the superpotential in the form:

[17] This is the analog of the cyclic invariance of the factor $\exp\left(\frac{i}{2}\theta_{ij}\sum_{0<\alpha<\beta<n} k_\alpha^i k_\beta^j\right)$ in the n point vertex interaction of the perturbative expansion of space-time non-commutative quantum field theories, due to the conservation of momenta at each vertex.

[18] These are the 2 flavour non anomalous symmetries plus $G-1$ baryonic symmetries (anomalous and non anomalous).

$$W = \sum_I m_I \quad b \quad \sum_J n_J \tag{B.7}$$

Let us explain in more detail a particular case, SPP.

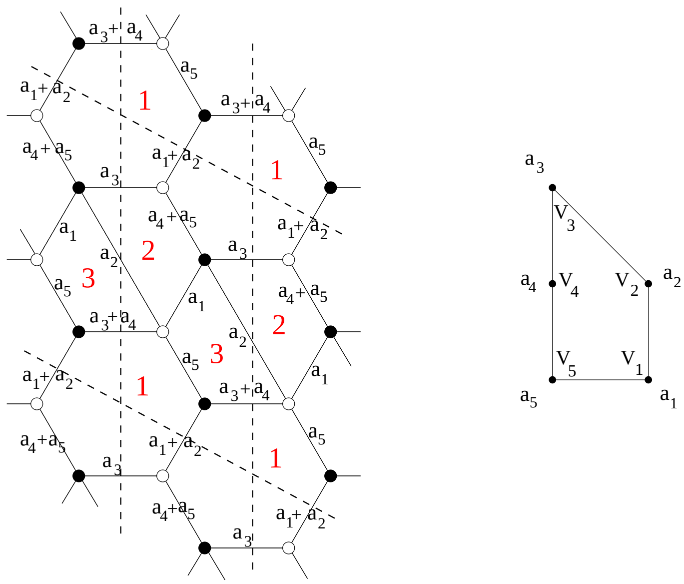

Figure 7: Dimer configuration and toric diagram for the SPP singularity.

All the information of a toric quiver gauge theory is encoded in a dimer graph [18] (see Figure 7). The idea is very simple: you draw a graph on T^2 such that it contains all the information of the gauge theory: every link is a field, every node a superpotential term, and every face is a gauge group. There exist efficient algorithms to compute the distribution of charges a_i for the various $U(1)$ global symmetries of the gauge theory [50]. The charges for every fields in the SPP gauge theory are given in Figure 7. For the two global flavour symmetries we are interested in, the trial charges are such that $\sum_i a_i = 0$ (conservation of flavour charges at every node). We can thus write the charges of the mesonic fields in terms of the trial charges:

$$\begin{aligned} x &= X_{11} \to a_1 + a_2 \;, y = X_{12}X_{21} \to a_3 + a_4 + a_5 \\ w &= X_{13}X_{32}X_{21} \to a_2 + 2a_3 + a_4 \;, z = X_{12}X_{23}X_{31} \to a_1 + a_4 + 2a_5 \end{aligned} \tag{B.8}$$

Using the values of the mesonic charges given in (6.13) one can now compute the charges a_i for the elementary fields. These will be a set of rational numbers. We can now use these charges to pass from the simple product to the star product (B.1) in every term in the superpotential. This procedure will generate a phase factor in front of every term in the superpotential. The interesting quantity is the invariant constant in (B.6):

$$\frac{\prod_I C_I^+}{\prod_J C_J} = e^{2i} = b^{1} \tag{B.9}$$

The actual value of this constant implies that we can rescale the elementary fields in such a way that the superpotential assumes the form:

$$W = X_{21}X_{12}X_{23}X_{32} + X_{13}X_{31}X_{11} \quad b^{1=2}(X_{32}X_{23}X_{31}X_{13} + X_{12}X_{21}X_{11}) \tag{B.10}$$

Using the F-term equations from the -deformed superpotential (B.10) one can reproduce the commutation rules among mesons (6.14) given in the main text plus the -deformed version of the CY singularity: $wz = bxy^2$.

B.2 $L^{p;q;q}$

In this Section we give another example of the moduli space for rational . $L^{p;q;q}$ with q p are an infinite class of Sasaki-Einstein spaces. For some values of $p;q$ these spaces are very well known. Indeed $L^{1;1;1} = C(T^{1;1})$, and $L^{1;2;2} = SPP$. The real cone over $L^{p;q;q}$ is a toric Calabi-Yau cone that can be globally described as an equation in C^4:

$$C(L^{p;q;q}) \; ! \; x^p y^q = wz \tag{B.11}$$

All the algebraic geometric information regarding these singularities can be encoded in a toric diagram, see Figure 8. The variety is a complete intersection in C^4. Indeed

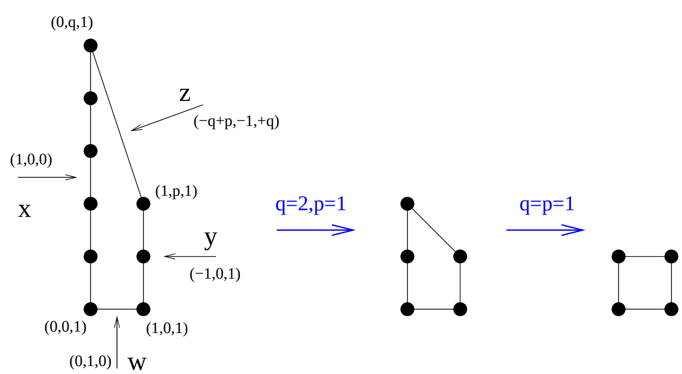

Figure 8: The toric diagrams of the $C(L^{p;q;q})$ singularity and their two well known special cases: SPP, $C(T^{1;1})$.

to each generator of the dual cone we can assign a coordinate like in Figure 8. These coordinates are in one to one correspondence with the mesonic field in the field theory generating the chiral ring, and the first two coordinates of the vectors are their charges under the two $U(1)$ flavour symmetries. The generators of the mesonic algebra are $x;y;w;z$ and thanks to their commutation relations

$$\begin{aligned} xy &= yx, \; xw = bwx, \; xz = b^{1}zx \\ yw &= b^{1}wy, \; yz = bzy, \; wz = b^{q\;p}zw \end{aligned} \tag{B.12}$$

we can write the generic monomial element of the algebra in the ordered form:

$$ {}_{k_1,k_2,k_3,k_4} = x^{k_1} y^{k_2} w^{k_3} z^{k_4} \qquad \text{(B.13)}$$

The center of the algebra is given by the subset of the operators (B.13) that satisfy the equations:

$$\begin{aligned}
{}_{k_1,k_2,k_3,k_4}\, x &= b^{k_4 \;\; k_3}\, x\, {}_{k_1,k_2,k_3,k_4} = x\, {}_{k_1,k_2,k_3,k_4} \\
{}_{k_1,k_2,k_3,k_4}\, y &= b^{k_3 \;\; k_4}\, y\, {}_{k_1,k_2,k_3,k_4} = y\, {}_{k_1,k_2,k_3,k_4} \\
{}_{k_1,k_2,k_3,k_4}\, w &= b^{k_1 \;\; k_2 \;\; (q \;\; p)k_4}\, w\, {}_{k_1,k_2,k_3,k_4} = w\, {}_{k_1,k_2,k_3,k_4} \\
{}_{k_1,k_2,k_3,k_4}\, z &= b^{k_2 \;\; k_1 + (q \;\; p)k_3}\, z\, {}_{k_1,k_2,k_3,k_4} = z\, {}_{k_1,k_2,k_3,k_4}
\end{aligned} \qquad \text{(B.14)}$$

Because $b^n = 1$ the elements of the center of the algebra are the subset of the operators of the form (B.13) such that $k_3 = k_4$, $k_1 = k_2 + (q \;\; p)k_4$, $k_1 = k_2 + (q \;\; p)k_3$ mod n. The generators of this algebra are ${}_{n,0,0,0}$; ${}_{0,n,0,0}$; ${}_{0,0,n,0}$; ${}_{0,0,0,n}$; ${}_{1,1,0,0}$; ${}_{q \; p,0,1,1}$; we call them respectively A, B, C, D, E, G. Using the F-term relation $x^p y^q = wz$ we see that G depends on the other generators through: $G = E^q$. Moreover the relations among generators are:

$$A^p B^q = CD; \qquad E^n = AB. \qquad \text{(B.15)}$$

In the special case of $q = p = 1$ these equations reduce to those for the quotient of the conifold. It is easy to see that equations (B.15) define the $Z_n \;\; Z_n$ orbifold of the $C(L^{p,q,q})$. Take the coordinates x, y, w, z realizing $C(L^{p,q,q})$ as a quadric embedded in C^4. The action of $Z_n \;\; Z_n$ is:

$$x, y, w, z \;!\;\; x \;\; , y \;\; {}^{1} , w \;\; , z \;\; {}^{q+p \;\;\; 1} \qquad \text{(B.16)}$$

where ${}^n = {}^n = 1$. The independent invariants of this action are A, B, C, D, E, and they are subject to the constraints (B.15). Hence the equations (B.15) define the variety $C(L^{p,q,q})/Z_n \;\; Z_n$.

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