LONGITUDINAL STABILITY OF A RACETRACK MICROTRON

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ABSTRACT

Both a linear matrix theory and a Hamiltonian description are given to describe the longitudinal oscillations of electrons in a Racetrack Microtron (RTM). The stable phase domain is determined and an optimum synchronous phase deduced. This theory is used as a basis for more accurate computer calculations for two RTM's being built at the Eindhoven University of Technology. The asymptotically synchronous phase has been calculated and the stable phase/energy injection region determined. The effect of slow deviations in the cavity voltage amplitude on the area of these stability regions has been investigated. We conclude that deviations of up to 7% have very little effect on the acceleration process.

INTRODUCTION

We set up and compare three methods for examining longitudinal momentum and phase oscillations in a RTM. First, a linear matrix theory is given [1]. Although this theory does yield a stability domain for the synchronous phase φ_s, no expression for the allowed oscillation amplitudes can be found. Therefore, secondly, a description using the Hamilton formalism is given. This description can be refined such that its results closely match those from the matrix theory. Additionally, a maximum phase-oscillation amplitude can be derived. The above theories assume the relativistic velocity β to be 1, i.e. the electrons travel at the speed of light. So, thirdly, for examining the effect of $\beta \neq 1$, computer simulations have been carried through. These have also been used for examining the effect of slow and small deviations in the cavity voltage amplitude.

MATRIX THEORY

The behaviour of a particle with small deviations of momentum (δP) and phase $(\delta\varphi)$ with respect to a reference particle of momentum P_0 and synchronous phase φ_s is inspected. The momentum deviation causes a change of phase deviation due to the different radius of curvature in the bending magnets (magnetic field B_0). Since we use $\beta = 1$, we can neglect the drift region of the RTM. For the change of phase deviation per revolution, we obtain

$$(\delta\varphi)_{\text{final}} = (\delta\varphi)_{\text{initial}} + \frac{4\pi^2}{eB_0\lambda_{\text{RF}}}(\delta P)_{\text{initial}} \tag{1}$$

with λ_{RF} the RF-cavity wavelength. Likewise, a phase deviation $\delta\varphi$ causes a change of momentum deviation as the particle is accelerated by a different cavity voltage. For a single acceleration (in first order approximation):

$$(\delta P)_{\text{final}} = (\delta P)_{\text{initial}} - \frac{eV_{\text{cav}}\sin(\varphi_s)}{c}(\delta\varphi)_{\text{initial}} \tag{2}$$

with V_{cav} the cavity voltage amplitude. Eqs. (1) and (2) can be joined in a single matrix $\boldsymbol{A}$, describing the transfer in $(\delta\varphi, \delta P)$-space caused by a single acceleration and a subsequent revolution over 2π:

$$\boldsymbol{A} = \begin{pmatrix} 1 - 2\pi h\tan(\varphi_s) & \frac{4\pi^2}{eB_0\lambda_{\text{RF}}} \\ -\frac{eV_{\text{cav}}}{c}\sin(\varphi_s) & 1 \end{pmatrix}. \tag{3}$$

Here, use is made of the isochronism condition, which can be written as $2\pi V_{\text{cav}}\cos(\varphi_s) = hcB_0\lambda_{\text{RF}}$, with h the path length difference between successive orbits, expressed in units of λ_{RF}. Stability is found for $|\text{Tr}(\boldsymbol{A})| < 2$, yielding

$$0 < \tan(\varphi_s) < \frac{2}{h\pi}. \tag{4}$$

For $h = 1$, we find $\varphi_{s,\max} = 32.5°$; for $h = 2$, $\varphi_{s,\max} = 17.7°$. The oscillation frequency ν_s follows from the matrix trace as well

$$2\cos(2\pi\nu_s) = 2 - 2\pi h\tan(\varphi_s) \quad \Rightarrow \quad \nu_s \approx \sqrt{\frac{h\tan(\varphi_s)}{2\pi}}. \tag{5}$$

It has been shown by Kapitza [1] that the oscillation amplitudes are smallest for a matrix trace equal zero, resulting in $\varphi_{s,\text{opt}} = 17.7°$ at h=1 and $\varphi_{s,\text{opt}} = 9.0°$ at h=2. In order to be able to find precise expressions for the oscillation amplitudes, we now turn to a Hamiltonian description.

HAMILTONIAN DESCRIPTION

We here give an outline of the Hamiltonian theory. A more detailed description will be published in the future.
The general expression for the Hamiltonian $\mathcal{H}$ describing the motion of a relativistic particle in a curvi-linear coordinate frame, neglecting the vertical component, is given by

$$\mathcal{H} = \sqrt{E_0^2 + c^2(p_x - eA_x)^2 + c^2\left(\frac{P_s}{1 + x/\rho} - eA_s\right)^2}. \tag{6}$$

Here, E_0 is the electron rest mass, p_x and P_s are the electron momentum in the x-direction (rectangularly to the orbit) and in the s-direction (along the orbit). The local radius of curvature is ρ and A_x and A_s represent the time-dependent vector potential in the transverse and longitudinal directions respectively. We want to make a second order expansion of this Hamiltonian. Since p_x is already of first order, and since A_x is of second order in the spatial coordinates (as far the magnetic field is concerned), we set $A_x = 0$, also neglecting electric fringe fields. We split the remaining longitudinal vector potential term in a fast and slowly varying part, corresponding to the electric and magnetic fields respectively. Then, we can neglect the change of the slow part with time for calculating the electric field in the cavity, i.e.

$$E_s = -\frac{\partial}{\partial t}A_s = -\frac{\partial}{\partial t}(A_{s,\text{fast}} + A_{s,\text{slow}}) \approx -\frac{\partial}{\partial t}A_{s,\text{fast}}. \tag{7}$$

Next, we transform the Hamiltonian to small variables and expand it up to second degree in the variables. We also introduce a new time τ, according to $\tau = \Omega(t)dt$ with $\Omega(t)$ the angular revolution frequency through the RTM, which decreases at increasing energy $W_0(t)$. We also remove first-order terms from the Hamiltonian by applying suitable transformations and obtain

$$\widetilde{\mathcal{H}} = \tfrac{1}{2}\left(\frac{1}{cB_0}\right)p_x^2 + \tfrac{1}{2}\left(\frac{1}{cB_0\gamma^2}\right)\bar{p}_s^2 - x\bar{p}_s + \tfrac{1}{2}cB_0Q_x x^2 + \qquad (8)$$
$$+\bar{s}\,\frac{dP_{s,0}}{d\tau} - \frac{W_0}{B_0c^2}\cos\left(\frac{\omega_{RF}}{\Omega}\tau + \varphi_s\right)\int^{\bar{s}+\int^{\tau} P_0/(eB_0)d\tau'} E_s(s')ds'.$$

Here, γ is the usual relativistic energy and ω_{RF} the angular RF frequency. The quantity Q_x is the radial betatron number, $P_{s,0}$ is the reference momentum, and the variables x, $\bar{s}$, p_x and $\bar{p}_s$ are small, unscaled deviations of position and momentum.

Note how we have split the time- and position dependent part of the RF electric field. The integral over the position dependent part can also be split up in two terms: a reference part, depending on time only, plus a small term describing the difference in accelerating voltage felt by a deviating particle. We neglect the former term as it does not contribute to the equations of motion, and make a Fourier expansion of the remaining small part (which is periodic in the scaled time τ). Next, we average the obtained Hamiltonian over one revolution and we find that, of the entire Fourier expansion, only one term does *not* disappear, viz. the frequency component corresponding to the RF-frequency.

After all these manipulations, a smoothed Hamiltonian results, which has, however, still got a coupling term between the x and s direction. We can remove this coupling by applying a generating function. Additionally, we scale the variables and finally obtain a Hamiltonian which is just the sum of two harmonic oscillators, given by

$$\widehat{\mathcal{H}} = \tfrac{1}{2}\hat{p}_x^2 + \tfrac{1}{2}\frac{2\gamma^2}{Q_x}\hat{x}^2 - \tfrac{1}{2}\frac{\gamma^2}{Q_x}\hat{p}_s^2 - \tfrac{1}{2}\frac{h\tan(\varphi_s)}{2\pi\gamma^2}\hat{s}^2. \qquad (9)$$

From this Hamiltonian we can isolate the "longitudinal" part (also setting $Q_x = 1$). The minus-signs are irrelevant, merely denoting a reversed direction of time, and can be forgotten. We also scale our variables such that the γ-terms disappear and obtain (omitting bars for convenience)

$$\mathcal{H}_S = \tfrac{1}{2}p_s^2 + \tfrac{1}{2}\frac{h\tan(\varphi_s)}{2\pi}s^2 \qquad (10)$$

with an oscillation frequency

$$\nu_s = \sqrt{\frac{h\tan(\varphi_s)}{2\pi}} \qquad (11)$$

which clearly agrees with the result from matrix theory. In order to be able to get more information from our Hamiltonian, we can make two adjustments in the procedure described above [2]. First, we have replaced the position-dependent acceleration by a smooth one (i.e. at all positions in the orbit). This can be corrected by including the δ-shaped acceleration in the potential term of the Hamiltonian. Next, we have made a second-order expansion of the difference in accelerating voltage seen by a deviating electron with respect to the reference electron. If we include the unapproximated expression for the accelerating voltage, we get a non-symmetric and non-harmonic potential well, from which a maximum oscillation amplitude can be derived. We discuss both adjustments below.

First, we examine the effect of a δ-shaped acceleration. We do this by multiplying the potential term in the Hamiltonian by $\delta(\tau)$:

$$\mathcal{H}_S = \tfrac{1}{2}p_s^2 + \tfrac{1}{2}\frac{h\tan(\varphi_s)}{2\pi}\delta(\tau)s^2 \qquad (12)$$

We transform this Hamiltonian to action and angle variables (J, ϕ) by way of a generating function and examine the appropriate resonance. We also average the Hamiltonian over one revolution and get

$$\mathcal{H}_S = J\left\{\left(\frac{h\tan(\varphi_s)}{2\pi} - \frac{1}{4}\right) + \frac{h\tan(\varphi_s)}{2\pi}\cos(2\phi)\right\}. \qquad (13)$$

As the resonance is excited when the angular part of the Hamiltonian becomes zero (i.e. when the action part goes to infinity), we can find the maximum value of φ_s by demanding the occurence of the resonance, yielding

$$2\,\frac{h\tan(\varphi_{s,max})}{2\pi} = \frac{1}{4} \quad\Rightarrow\quad \tan(\varphi_{s\,max}) = \frac{\pi}{4h}. \qquad (14)$$

For $h = 1$ we find $\varphi_{s,max} = 38.1°$, which has to be compared to the value of 32.5° that resulted from the matrix theory. The difference is caused by the averaging procedure above: we have neglected oscillating terms. It is possible to apply a canonical transformation on the Hamiltonian that transforms these oscillating terms to higher order. If this is done, the resonance turns out to occur at 33.2°, which is evidently much closer to the value obtained by matrix theory. A second canonical transformation will better the results even more.

A second improvement on the basic Hamiltonian (10) can be obtained by extending the potential function. For obtaining the simple harmonic oscillator, we have made a second order expansion of the cavity voltage sine wave near the synchronous phase. Assuming small radial deviations, we can as yet still replace the oscillator by the true potential term. In the potential obtained this way, a small well is present and thus an expression for the maximum momentum spread allowed can be found. Reversing the applied scalings, we find

$$\left(\frac{p_s}{P_0}\right)_{max} = \frac{\Omega}{\omega_{RF}}\sqrt{\frac{2h}{\pi}(\tan(\varphi_s) - \varphi_s)}. \qquad (15)$$

It has been shown that the Hamiltonian theory is a flexible model, very able to describe various aspects of longitudinal focusing. The model outlined above is only a first approach, yet gives rather good results. One should combine the results of both corrections discussed in order to get an idea of the major properties of RF focusing in a RTM.

COMPUTER SIMULATIONS

Computer simulations have been carried out [3], which include the effect of $\beta \neq 1$. The longitudinal motion is described by two coupled difference equations:

$$\begin{cases} T_i &= T_{i-1} + eV_{cav}\cos(\varphi_{i-1}) \\ \varphi_i &= \varphi_{i-1} + \dfrac{2\pi}{\beta\lambda_{RF}}(2\pi\rho_i + 2L) \end{cases} \qquad 1 \leq i \leq N. \qquad (16)$$

Here, T_i, φ_i and ρ_i represent the kinetic energy, particle phase and bending radius in the (assumed) homogeneous dipole magnets respectively, all in the i'the orbit. The quantity L is the length of the drift region in between the magnets; for a microtron, we set $L = 0$. Finally, N denotes the number of turns from injection to extraction.

A stable phase/energy domain, the "acceptance" and "emittance," can be calculated, taking into account the so-called asymptotically synchronous phase φ_{ASP}, being the synchronous phase for $\beta \to 1$. These calculations were done for our 5 to 25 MeV / 1.3 GHz RTM. Its parameters are given elsewhere in these proceedings [4]. Figure 1 shows an example of results for $\varphi_{ASP} = 15°$. The central region of the acceptance (left figure) has an energy spread of about 4% (200 keV) and a phase spread of 25° (53 ps). The righthand figure shows the matching emittance with a linear structure in the central region. The energy spread of the extracted beam is about 0.9% (230 keV).

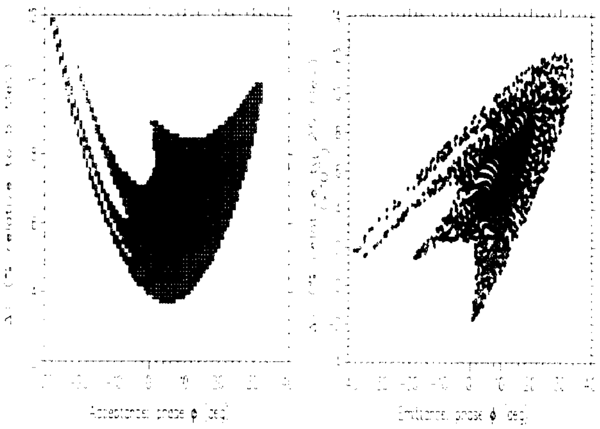

Figure 1: *The acceptance and emittance at a phase of 15°.*

the phase spread 15° (32 ps). Figure 2 shows the area inside the separatrix at injection as a function of φ_{ASP}. Up to 20° the area rapidly increases. Between 20 and 25° the graph has some peculiar peaks, probably arising from the existence of a lot of "spaghetti" around the central stable region, i.e. accidentally stable points. From 25° onwards the area decreases again, eventually reaching the zero-line at 33° (not shown).
An important assumption contained in the preceeding results is a stable cavity voltage amplitude. In practice, the cavity voltage will show small and slow changes. The term "slow" indicates that the cavity voltage amplitude should not significantly change whilst one specific bunch of electrons is being accelerated. In our RTM, a bunch of electrons stays about 85 ns in the machine. Therefore, the cavity voltage amplitude is

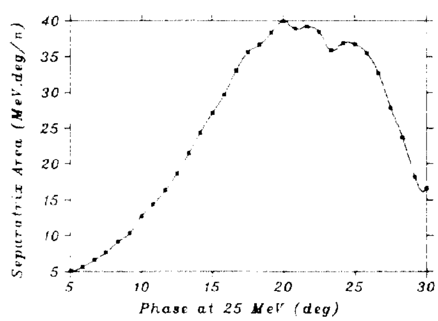

Figure 2: *Separatrix area as a function of synchronous phase.*

allowed to swing with a frequency much smaller than 10 MHz. In that case, one can test the effect of such changes by assuming a stable cavity voltage amplitude for each bunch, differing a few percent from the central value.
As an example, figure 3 shows the separatrix area at $\varphi_{ASP} = 15°$ as a function of the deviation in cavity voltage amplitude. The graph clearly shows that a rather sharply bounded region exists in which the separatrix area remains large, strange peaks once more being caused by shaghetti. For this region, the deviation of extraction energy as a function of cavity voltage amplitude is shown in figure 4. It can be concluded that large deviations of accelerating amplitude are allowed (7%), without considerable change of extraction energy (0.7%). Apparently, the acceleration process is very able to self-correct deviations in cavity voltage amplitude as to stabilise the extraction energy.

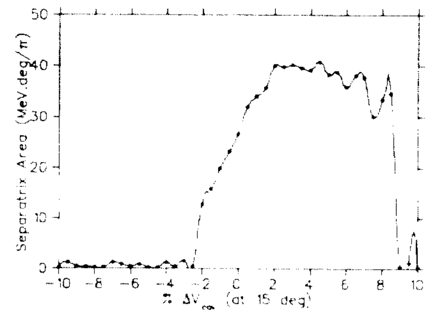

Figure 3: *Separatrix area as function of deviations in cavity voltage amplitude.*

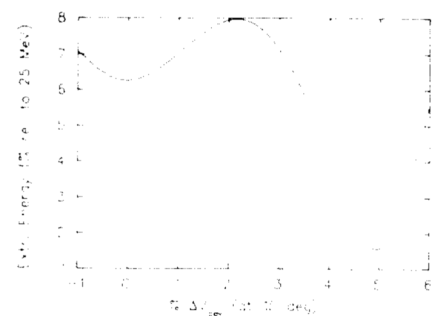

Figure 4: *Extraction energy as a function of deviations in cavity voltage amplitude.*

CONCLUSIONS

Both a matrix- and a Hamiltonian description of synchrotron oscillations in a Racetrack Microtron have been given. Additionally, computer calculations have been performed.

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