Threshold Analysis and Stationary Distribution of a Stochastic Model with Relapse and Temporary Immunity

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Abstract: In this paper, a stochastic model with relapse and temporary immunity is formulated. The main purpose of this model is to investigate the stochastic properties. For two incidence rate terms, we apply the ideas of a symmetric method to obtain the results. First, by constructing suitable stochastic Lyapunov functions, we establish sufficient conditions for the extinction and persistence of this system. Then, we investigate the existence of a stationary distribution for this model by employing the theory of an integral Markov semigroup. Finally, the numerical examples are presented to illustrate the analytical findings.

Keywords: stochastic epidemic model; extinction; persistence; stationary distribution

1. Introduction

Infectious diseases have always threatened the health of human beings. In recent decades, mathematicians have tried to study the spread of infectious diseases. Many disease models have been introduced to understand and analyze epidemics [1–6]. The simplest and best-known one is the SIR (susceptible, infections, removed) model, where the total population is divided into three compartments: susceptible (S), infections (I), and removed (R). If the epidemic models are conferred on temporary immunity, then we can establish SIS (susceptible, infections, susceptible) or SIRS (susceptible, infections, removed, susceptible) models. These models have been researched widely, from the deterministic to stochastic perspective [7–16]. However, the recovery of diseases may relapse, when latent infections reactivate and revert resulting in actively infected people. These types of diseases can be modeled by the SIRI (susceptible, infections, removed, infections) model [17–20], which reads as follows

$$
\begin{cases}
\frac{dS}{dt} = \mu - \mu S - \varphi(S, I), \\
\frac{dI}{dt} = \varphi(S, I) - (\mu + \eta)I + \gamma_1 R, \\
\frac{dR}{dt} = \eta I - (\mu + \gamma_1) R,
\end{cases}
\tag{1}
$$

where μ denotes the recruitment rate of humans, and we suppose that it is equal to the natural death rate of humans. η is the recovery rate, γ_1 denotes the rate that recovered individuals are reverted to the infective state. $\varphi(S, I)$ is the transmission function, which is important in order to analyze the stochastic properties of the model. In such a model, relapse is an important feature of these types of diseases that occurs in some animals and humans, for instance, tuberculosis and herpes.

Tudor [17] and Ding [20] established SIRI models with a bilinear incidence rate. However, in real life, the types of incidence rates of the same disease may be different as the environment changes. Nevertheless, on the basis of what we have studied, few works have been done on epidemic models with different incidence rates. Thus, we propose a model with a novel type of transmission function,

$\varphi(S, I) = \sum_{i=1}^{n} p_i \varphi_i(S, I)$, where p_i $(i = 1, 2, \ldots, n)$ denotes the probability of $\varphi_i(S, I)(i = 1, 2, \ldots, n)$ occurring, and $\sum_{i=1}^{n} p_i = 1$.

Taking into account a bilinear incidence rate and the Beddington–DeAngelis incidence rate, in this regard, we assume $\varphi(S, I) = p\beta_1 SI + (1 - p)\frac{\beta_2 SI}{1 + mS + nI}$, where p represents the probability of a bilinear incidence rate occurring, the corresponding $1 - p$ is the probability of the Beddington–DeAngelis incidence rate occurring, and evidently $0 < p < 1$. m and n are nonnegative constants. $\beta_i(i = 1, 2)$ are positive constants, which represent the disease transmission coefficients. Hence, the corresponding model can be formulated as follows

$$\begin{cases} dS = \left[\mu - \mu S - p\beta_1 SI - (1 - p)\frac{\beta_2 SI}{1 + mS + nI} + \gamma_2 R\right] dt, \\ dI = \left[p\beta_1 SI + (1 - p)\frac{\beta_2 SI}{1 + mS + nI} - (\mu + \eta)I + \gamma_1 R\right] dt, \\ dR = [\eta I - (\mu + \gamma_1 + \gamma_2) R] dt, \end{cases} \tag{2}$$

where γ_2 denotes the per capita immune loss rate of removed.

Stochastic differential equations (SDEs) are ordinary differential equations (ODEs) that include random processes in their vector fields. In this paper, we establish an SDE system by introducing terms representing stochastic perturbations into the ODE system (2), which is achieved by letting $\beta_i \to \beta_i + \sigma_i B_i (i = 1, 2)$. Hence the stochastic epidemic model of system (2) is given by

$$\begin{cases} dS = \left[\mu - \mu S - p\beta_1 SI - (1 - p)\frac{\beta_2 SI}{1 + mS + nI} + \gamma_2 R\right] dt - p\sigma_1 SI dB_1 - (1 - p)\frac{\sigma_2 SI}{1 + mS + nI} dB_2, \\ dI = \left[p\beta_1 SI + (1 - p)\frac{\beta_2 SI}{1 + mS + nI} - (\mu + \eta)I + \gamma_1 R\right] dt + p\sigma_1 SI dB_1 + (1 - p)\frac{\sigma_2 SI}{1 + mS + nI} dB_2, \\ dR = [\eta I - (\mu + \gamma_1 + \gamma_2) R] dt. \end{cases} \tag{3}$$

With the following initial conditions: $S(0) > 0$, $I(0) > 0$, $R(0) > 0$ and $S(0) + I(0) + R(0) = 1$, we have

$$d(S + I + R) = [\mu - \mu(S + I + R)]dt.$$

We can also easily show that

$$S(t) + I(t) + R(t) = 1 \text{ for all } t \geqslant 0.$$

Therefore, in this paper we can also discuss the following system for $(I(t), R(t))$

$$\begin{cases} dI = [p\beta_1(1 - I - R)I + (1 - p)\frac{\beta_2(1 - I - R)I}{1 + m(1 - I - R) + nI} - (\mu + \eta)I + \gamma_1 R]dt \\ \quad + p\sigma_1(1 - I - R)I dB_1 + (1 - p)\frac{\sigma_2(1 - I - R)I}{1 + m(1 - I - R) + nI} dB_2, \\ dR = [\eta I - (\mu + \gamma_1 + \gamma_2)R]dt. \end{cases} \tag{4}$$

Precisely, the region $\Gamma_1 = \{(S, I, R) \in \mathbb{R}^3, S + I + R < 1\}$ is a positively invariant set of system (3) and the region $\Gamma_2 = \{(I, R) \in \mathbb{R}^2, I + R < 1\}$ is a positively invariant set of system (4).

The symmetric method for differential equations is a popular way to study ordinary differential equations (ODEs) and partial differential equations (PDEs). In this regard, we apply the ideas of this method shown in [21,22] to the SDE system.

Unless otherwise specified, throughout this paper, let $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\geq0}, \mathcal{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq0}$ satisfying the usual conditions (i.e., it is increasing and right continuous, while $\mathcal{F}_0$ contains all $\mathcal{P}$ -null sets). Let $\mathbb{R}^3_+ = \{x_i > 0, i = 1, 2, 3\}$, $\mathbb{R}^2_+ = \{x_i > 0, i = 1, 2\}$. Let $B(t)$ be an $\mathcal{F}_t$ -adapted Brownian motion.

The rest of this paper is organized as follows. The existence and uniqueness of the positive solution of the system (3) are provided in Section 2. In Sections 3 and 4, we explore sufficient conditions for extinction and persistence results of the epidemic. In Section 5, we use integral Markov semigroup

theory to study the existence of a unique stationary distribution for system (4). Finally, we present some numerical simulations to demonstrate our analytical results.

2. Existence and Uniqueness of the Global Positive Solution

In the following, we discuss the existence and uniqueness of the global positive solution of system (3) for any initial condition on $\mathbb{R}^+$. For further research, this is an important premise.

Theorem 1. *There is a unique and positive solution $(S(t), I(t), R(t))$ of system (3) for any initial condition on $\mathbb{R}_+$. Then the solution remains in Γ_1, almost surely.*

Proof. As the coefficients of system (3) are locally Lipschitz continuous, there exists a unique solution on $[0, \tau_e]$, where τ_e is the explosion time.

Let $n_0 \in \mathbb{N}$, such that the initial values $S(0), I(0), R(0)$ are all in the interval $(1/n_0, n_0)$. For any integer $n \geq n_0$, we consider the stop-time τ_n as

$$\tau_n = \inf \left\{ t \in [0, \tau_e) : \min\{S(t), I(t), R(t)\} \leq \frac{1}{n} \text{ or } \max\{S(t), I(t), R(t)\} \geq n \right\}.$$

Clearly, τ_n is increasing as $n \to \infty$ and less than τ_e for any $n \geq n_0$. Set $\tau_\infty = \lim_{n \to \infty} \tau_n$. We have $\tau_\infty \leq \tau_e$. If $\tau_\infty = \infty$ a.s. holds, then $\tau_e = \infty$ a.s. Otherwise, there exist $\epsilon \in (0, 1)$ and $T \in (0, \infty)$, such that $\mathbb{P}(\tau_\infty \leq T) \geq \varepsilon$. Thus, there exists an integer $n_1 \geq n_0$, such that $\mathbb{P}(\tau_n \leq T) \geq \epsilon$ for all $n \geq n_1$.

Let $N(t)$ represent the population size at time t. Thus, $N(t) = S(t) + I(t) + R(t)$. For any $n \geq n_1$ and $t \in [0, \tau_n)$, we have by derivation

$$dN(t) = [\mu - \mu N(t)]dt.$$

Then, by the comparison theorem, we obtain

$$N(t) \leq 1 + [N(0) - 1]e^{-\mu t} \leq M = \max\{1, N(0)\}, \forall t \in [0, \tau_k).$$

Define a $\mathcal{C}^2$-function V_1

$$V_1(S, I, R) = (S - 1 - \log S) + (I - 1 - \log I) + (R - 1 - \log R).$$

Clearly, $V_1(t)$ is nonnegative. According to the Itô formula, one has

$$LV_1 = \left(1 - \frac{1}{S}\right)\left[\mu - \mu S - p\beta_1 SI - (1-p)\frac{\beta_2 SI}{1 + mS + nI} + \gamma_2 R\right]$$
$$+ \left(1 - \frac{1}{I}\right)\left[p\beta_1 SI + (1-p)\frac{\beta_2 SI}{1 + mS + nI} - (\mu + \eta)I + \gamma_1 R\right]$$
$$+ \left(1 - \frac{1}{R}\right)\left[\eta I - (\mu + \gamma_1 + \gamma_2)R\right]$$
$$+ \frac{1}{2}p^2\sigma_1^2 I^2 + \frac{1}{2}(1-p)^2\sigma_2^2\left(\frac{I}{1 + mS + nI}\right)^2$$
$$+ \frac{1}{2}p^2\sigma_1^2 S^2 + \frac{1}{2}(1-p)^2\sigma_2^2\left(\frac{S}{1 + mS + nI}\right)^2$$
$$= 4\mu + \eta + \gamma_1 + \gamma_2 + p\beta_1 I + (1-p)\frac{\beta_2 I}{1 + mS + nI}$$
$$+ \frac{1}{2}p^2\sigma_1^2 I^2 + \frac{1}{2}(1-p)^2\sigma_2^2\left(\frac{I}{1 + mS + nI}\right)^2$$
$$+ \frac{1}{2}p^2\sigma_1^2 S^2 + \frac{1}{2}(1-p)^2\sigma_2^2\left(\frac{S}{1 + mS + nI}\right)^2$$
$$- \left(\mu S + \mu I + \mu R + \frac{\mu}{S} + \gamma_2\frac{R}{S} + p\beta_1 S + (1-p)\frac{\beta_2 S}{1 + mS + nI}\right.$$
$$\left. + \gamma_1\frac{R}{I} + \eta\frac{I}{R}\right)$$
$$\leq 4\mu + \eta + \gamma_1 + \gamma_2 + p\beta_1 + (1-p)\beta_2 + \frac{1}{2}p^2\sigma_1^2 + \frac{1}{2}(1-p)^2\sigma_2^2 =: \Lambda.$$

Here Λ is a generic positive constant. We compute

$$dV_1 = \Lambda dt + \left(\frac{1}{S} - \frac{1}{I}\right)p\sigma_1 SI dB_1(t) + \left(\frac{1}{S} - \frac{1}{I}\right)(1-p)\frac{\sigma_2 SI}{1 + mS + nI}dB_2(t).$$

Integrating over $[0, \tau_n \wedge T]$ and taking the expectation, then

$$EV_1\left(S\left(\tau_n \wedge T\right), I\left(\tau_n \wedge T\right), R\left(\tau_n \wedge T\right)\right) \leq V_1(S(0), I(0), R(0)) + \Lambda T,$$

which leads to

$$V_1(S(0), I(0), R(0)) + \Lambda T \geq E\left[1_{\{\tau_n \wedge T\}}V_1\left(S\left(\tau_n \wedge T\right), I\left(\tau_n \wedge T\right), R\left(\tau_n \wedge T\right)\right)\right]$$
$$\geq \varepsilon\left\{(n - 1 - \log n) \wedge \left(\frac{1}{n} - 1 - \log\frac{1}{n}\right)\right\}.$$

Let $n \to \infty$, and we obtain

$$\infty > V_1(S(0), I(0), R(0)) + \Lambda T = \infty,$$

which is a contraction, then, $\tau_\infty = \infty$ a.s. This completes the proof. □

3. Extinction of the Disease

In this section, we consider the extinction of the disease of the system (4) under some sufficient assumptions.

Theorem 2. *Assume that*

$$\mathcal{R}_0 < 1, \tag{5}$$

$$\beta_1 > p\sigma_1^2, \ \beta_2 > (1-p)\sigma_2^2, \tag{6}$$

where $\mathcal{R}_0 = \dfrac{\beta_1 p + \beta_2(1-p)}{\mu + \eta + \frac{1}{2}\sigma_1^2 p^2 + \frac{1}{2}\sigma_2^2(1-p)^2 - \frac{\gamma_1 \eta}{\mu + \gamma_1 + \gamma_2}}$, *then for any initial value* $(I(0), R(0))$ *in* Γ_2, *the solution* $(I(t), R(t))$ *of system* (4) *converges to* $(0, 0)$ *a.s.*

Proof. Let $\mathcal{R}_0 < 1$ and $\theta > 1$, such that

$$-k_\theta \triangleq p\beta_1 + (1-p)\beta_2 - \left(\mu + \eta + \frac{1}{2}\sigma_1^2 p^2 + \frac{1}{2}\sigma_2^2(1-p)^2\right) + \frac{\theta\gamma_1\eta}{\mu + \gamma_1 + \gamma_2} < 0.$$

Define the stop-time

$$\tau_1 = \inf\left\{t \geq 0,\ I(t) \leq \frac{1}{\theta}\right\}.$$

We suppose that $\mathbb{E}(\tau_1) < \infty$. Actually, we assume that $I(0) \in (\frac{1}{\theta}, 1)$ (if not $\tau_1 = 0$). Hence, for all $T > 0$ and $t \leq T \wedge \tau_1$, one has

$$I(t) \geq \frac{1}{\theta}. \tag{7}$$

Now, employing Itô's formula to $\log I$, we have

$$
\begin{aligned}
d\log I &= \left[p\beta_1(1 - I - R) + (1-p)\frac{\beta_2(1-I-R)}{1 + m(1-I-R) + nI} - (\mu + \eta) + \gamma_1 \frac{R}{I} \right.\\
&\quad \left. - \frac{1}{2}p^2\sigma_1^2(1-I-R)^2 - \frac{1}{2}(1-p)^2\sigma_2^2\left(\frac{1-I-R}{1+m(1-I-R)+nI}\right)^2 \right] dt \\
&\quad + p\sigma_1(1-I-R)dB_1(t) + (1-p)\frac{\sigma_2(1-I-R)}{1+m(1-I-R)+nI}dB_2(t) \\
&\leq \left[-(\mu + \eta) + \beta_1 p - \frac{1}{2}p^2\sigma_1^2 + \beta_2(1-p) - \frac{1}{2}(1-p)^2\sigma_2^2 + \theta\gamma_1 R \right] dt \\
&\quad + p\sigma_1(1-I-R)dB_1(t) + (1-p)\frac{\sigma_2(1-I-R)}{1+m(1-I-R)+nI}dB_2(t),
\end{aligned}
\tag{8}
$$

where (6) and the monotony of the functions $x_1 \mapsto -(\mu + \gamma_1 + \gamma_2) + p\beta_1 x_1 - \frac{1}{2}p^2\sigma_1^2 x_1^2$, $x_2 \mapsto -(\mu + \gamma_1 + \gamma_2) + (1-p)\frac{\beta_2 x_2}{1 + mx_2 + nI} - \frac{1}{2}(1-p)^2\sigma_2^2\left(\frac{x_2}{1+mx_2+nI}\right)^2$ are used. Precisely, according to assumption (6), the functions x_1 and x_2 are both increasing on the interval $[0, 1]$. Combining (8) and the R-equation, one has

$$
\begin{aligned}
d\left[\log I + \frac{\theta\gamma_1}{\mu + \gamma_1 + \gamma_2}R\right] &\leq \left[-(\mu + \eta) + \beta_1 p - \frac{1}{2}p^2\sigma_1^2 + \beta_2(1-p) - \frac{1}{2}(1-p)^2\sigma_2^2 + \frac{\theta\gamma_1\eta}{\mu + \gamma_1 + \gamma_2}I \right] dt \\
&\quad + p\sigma_1(1-I-R)dB_1(t) + (1-p)\frac{\sigma_2(1-I-R)}{1+m(1-I-R)+nI}dB_2(t) \\
&\leq -k_\theta dt + p\sigma_1(1-I-R)dB_1(t) + (1-p)\frac{\sigma_2(1-I-R)}{1+m(1-I-R)+nI}dB_2(t).
\end{aligned}
$$

Further,

$$\mathbb{E}\left[\log I(T \wedge \tau_1) + \frac{\theta\gamma_1}{\mu + \gamma_1 + \gamma_2}R(T \wedge \tau_1)\right] \leq \log I(0) + \frac{\theta\gamma_1}{\mu + \gamma_1 + \gamma_2}R(0) - k_\theta\mathbb{E}(T \wedge \tau_1),$$

which leads to

$$\mathbb{E}(T \wedge \tau_1) \leq \frac{\theta\gamma_1}{(\mu + \gamma_1 + \gamma_2)k_\theta} - \frac{\mathbb{E}\left[\log I(T \wedge \tau_1)\right]}{k_\theta}.$$

Then, letting $T \to \infty$ and using Fatou's lemma, we have

$$\mathbb{E}\left(\tau_1\right) \leq \frac{\theta \gamma_1}{\left(\mu + \gamma_1 + \gamma_2\right) k_\theta} + \frac{\log \theta}{k_\theta} < \infty.$$

Therefore, our claim is true. In the next step, we shall show that $I(t)$ goes by $\frac{1}{\theta^2}$ in a finite mean time. Therefore, we set

$$\tau_1' = \inf\left\{t \geq \tau_1, \ I(t) > \tfrac{1}{\theta}\right\},$$
$$\tau_2 = \inf\left\{t < \tau_1', \ I(t) < \tfrac{1}{\theta^2}\right\}.$$

Hence, for all $T > 0$ and any $t \in [\tau_1, T \wedge \tau_2]$ we obtain

$$\frac{1}{\theta^2} \leq I(t) \leq \frac{1}{\theta}. \tag{9}$$

By using (9), we can also have

$$d\left[\log I + \frac{\theta^2 \gamma_1}{\mu + \gamma_1 + \gamma_2} R\right] = \left[\beta_1 p + \beta_2(1-p) - \left(\mu + \eta + \frac{1}{2}p^2\sigma_1^2 + \frac{1}{2}(1-p)^2\sigma_2^2\right) + \frac{\theta^2 \gamma_1 \eta}{\mu + \gamma_1 + \gamma_2} I\right] dt$$
$$+ p\sigma_1(1 - I - R)dB_1(t) + (1-p)\frac{\sigma_2(1 - I - R)}{1 + m(1 - I - R) + nI}dB_2(t)$$
$$\leq -k_\theta dt + p\sigma_1(1 - I - R)dB_1(t) + (1-p)\frac{\sigma_2(1 - I - R)}{1 + m(1 - I - R) + nI}dB_2(t).$$

Integrating over $[\tau_1, T \wedge \tau_2]$, one can find

$$\mathbb{E}\left[\log I\left(T \wedge \tau_2\right)\right] \leq \log I\left(\tau_1\right) + \frac{\theta^2 \gamma_1}{\mu + \gamma_1 + \gamma_2} R\left(\tau_1\right) - k_\theta \mathbb{E}\left(T \wedge \tau_2 - \tau_1\right).$$

Thus we have

$$\mathbb{E}\left(\tau_2\right) \leq \mathbb{E}\left(\tau_1\right) + \frac{\theta^2 \gamma_1}{\left(\mu + \gamma_1 + \gamma_2\right) k_\theta} + \frac{2\log \theta}{k_\theta} < \infty.$$

By induction, we have the following definitions

$$\tau_n' = \inf\left\{t \geq \tau_{n-1}, \ I(t) > \tfrac{1}{\theta^{n-1}}\right\},$$
$$\tau_n = \inf\left\{t < \tau_n', \ I(t) < \tfrac{1}{\theta^n}\right\}.$$

We obtain

$$\mathbb{E}\left(\tau_n\right) \leq \mathbb{E}\left(\tau_{n-1}\right) + \frac{\theta^n \gamma_1}{\left(\mu + \gamma_1 + \gamma_2\right) k_\theta} + \frac{n\log \theta}{k_\theta} < \infty. \tag{10}$$

Clearly, (τ_n) is an increasing sequence. Therefore, $\tau_n \to \tau_\infty$ a.s. We denote $\Omega' = \bigcap_{n=1}^\infty \left(\tau_n < \infty\right)$, by (10) and we obtain

$$\mathbb{P}\left(\Omega'\right) = \mathbb{P}\left(\bigcap_{n=1}^\infty \left(\tau_n < \infty\right)\right) = \lim_{n \to \infty} \mathbb{P}\left(\tau_n < \infty\right) = 1. \tag{11}$$

We write Ω' in the following form

$$\Omega' = \left(\left(\tau_\infty < \infty\right) \cap \Omega'\right) \cup \left(\left(\tau_\infty = \infty\right) \cap \Omega'\right) = \left(\tau_\infty < \infty\right) \cup \left(\left(\tau_\infty = \infty\right) \cap \Omega'\right).$$

We show that $\mathbb{P}(\tau_\infty < \infty) = 0$. We assume that $\mathbb{P}(\tau_\infty < \infty) > 0$. Let $\omega \in (\tau_\infty < \infty)$. We obtain for all $n \in \mathbb{N}^*$

$$I(\tau_n(\omega)) = \frac{1}{a^n}.$$

By extending n to ∞, we obtain $I(\tau_\infty(\omega)) = 0$. Hence, $\tau_0(\omega) \leq \tau_\infty(\omega) < \infty$, where $\tau_0 = \inf\{t > 0, I(t) = 0\}$. Then

$$\mathbb{P}(\tau_0 < \infty) \geq \mathbb{P}(\tau_\infty < \infty) > 0,$$

which is a contradiction with $\mathbb{P}(\tau_0 = \infty) = 1$. Then, $\mathbb{P}(\tau_\infty < \infty) = 0$, furthermore

$$\mathbb{P}((\tau_\infty = \infty) \cap \Omega') = 1.$$

Finally, we set $\omega \in (\tau_\infty = \infty) \cap \Omega'$, $\epsilon > 0$, $t > 0$ and $n_0 = [-\log \epsilon (\log \theta)^{-1}]$. For $t > \tau_{n_0}$, there is n such that $n \geq n_0$ and $\tau_n \leq t \leq \tau_{n+1}$. Then

$$\frac{1}{\theta^{n+1}} \leq I(t) \leq \frac{1}{\theta^n} \leq \frac{1}{\theta^{n_0+1}} \leq \epsilon,$$

which suggests

$$\lim_{t \to \infty} I(t) = 0. \tag{12}$$

This indicates that $I(t)$ converges to 0 with probability 1. Using Fatou's lemma and the last equation of system (4), we obtain

$$\limsup_{t \to \infty} R(t) \leq \eta \int_0^\infty \limsup_{t \to \infty} e^{-(\mu+\gamma_1+\gamma_2)u} I(t-u) du \leq \frac{\eta}{\mu+\gamma_1+\gamma_2} \limsup_{t \to \infty} I(t),$$

which implies with (12) that $\lim_{t \to \infty} R(t) = 0$. a.s. This proof is therefore complete. $\square$

4. Persistence in the Mean

Theorem 3. *Let $(S(0), I(0), R(0)) \in \Gamma_1)$ be an initial value. Assume that*

$$\mathcal{R}_s^0 > 1$$

holds, where $\mathcal{R}_s^0 = \dfrac{p\beta_1 + \frac{(1-p)\beta_2}{1+m+n}}{\mu+\eta-\frac{\gamma_1\eta}{\mu+\gamma_1+\gamma_2}+\frac{1}{2}p^2\sigma_1^2+\frac{1}{2}(1-p)^2\sigma_2^2}$. *Then, the epidemic of (3) is permanent in the mean. Precisely, we obtain*

$$\liminf_{t \to \infty} \frac{1}{t} \int_0^t S(u) du \geq \frac{\mu}{\mu + p\beta_1 + (1-p)\beta_2 C} \quad a.s.,$$

$$\liminf_{t \to \infty} \frac{1}{t} \int_0^t I(u) du \geq \frac{\mu}{p\beta_1 + \frac{(1-p)\beta_2}{1+m+n}} \left(1 - \frac{1}{\mathcal{R}_s^0}\right) \quad a.s.,$$

$$\liminf_{t \to \infty} \frac{1}{t} \int_0^t R(u) du \geq \frac{\mu\eta}{(\mu+\gamma_1+\gamma_2)\left(p\beta_1 + \frac{(1-p)\beta_2}{1+m+n}\right)} \left(1 - \frac{1}{\mathcal{R}_s^0}\right) \quad a.s.$$

Proof. Integrating the first equation of (3) between 0 and t, we obtain

$$\mu t - (S(t) - S(0)) = \int_0^t \left(\mu S + p\beta_1 SI + \frac{(1-p)\beta_2 SI}{1+mS+nI} - \gamma_2 R\right) ds$$

$$+ \int_0^t p\sigma_1 SI dB_1(s) + \int_0^t \frac{(1-p)\sigma_2 SI}{1+mS+nI} dB_2(s)$$

$$\leq \int_0^t \left(\mu + p\beta_1 I + \frac{(1-p)\beta_2 I}{1+mS+nI}\right) S ds$$

$$+ \int_0^t p\sigma_1 SI dB_1(s) + \int_0^t \frac{(1-p)\sigma_2 SI}{1+mS+nI} dB_2(s).$$

It is obvious that $p\beta_1 SI < p\beta_1$, $\frac{SI}{1+mS+nI} < C$, where C is a positive constant. Thus, we have

$$\mu t - (S(t) - S(0)) \le (\mu + p\beta_1 + (1-p)\beta_2 C) \int_0^t S ds$$
$$+ \int_0^t p\sigma_1 SI dB_1(s) + \int_0^t \frac{(1-p)\sigma_2 SI}{1+mS+nI} dB_2(s).$$

Multiplying both sides by $\frac{1}{t}$, we obtain

$$\frac{1}{t}\int_0^t S ds \ge \frac{1}{\mu + p\beta_1 + (1-p)\beta_2 C}\left(\mu - \frac{S(t)-S(0)}{t}\right)$$
$$- \frac{1}{\mu + p\beta_1 + (1-p)\beta_2 C}\left(\frac{1}{t}\int_0^t p\sigma_1 SI dB_1(s) + \frac{1}{t}\int_0^t \frac{(1-p)\sigma_2 SI}{1+mS+nI} dB_2(s)\right).$$

Note that $\int_0^t p\sigma_1 SI dB_1(s)$, $\int_0^t \frac{(1-p)\sigma_2 SI}{1+mS+nI} dB_2(s)$ are continuous martingales with finite quadratic variation. By employing the strong law of large numbers for local martingales, we have

$$\lim_{t\to\infty} \frac{S(t)-S(0)}{t} + \frac{1}{t}\int_0^t p\sigma_1 SI dB_1(s) + \frac{1}{t}\int_0^t (1-p)\frac{\sigma_2 SI}{1+mS+nI} dB_2(s) = 0 \ \ a.s.$$

Hence,

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t S ds \ge \frac{\mu}{\mu + p\beta_1 + (1-p)\beta_2 C} \ \ a.s.$$

Then, we define a function F on Γ_1

$$F(S, I, R) = \log I + \frac{p\beta_1 + \frac{(1-p)\beta_2}{1+m+n}}{\mu} S + B \log R, \tag{13}$$

where B is the unique solution to the equation

$$- (\mu + \gamma_1 + \gamma_2) B + 2\sqrt{\gamma_1 \eta}\sqrt{B} - \frac{\gamma_1 \eta}{\mu + \gamma_1 + \gamma_2} = 0, \tag{14}$$

given by $B = \frac{\gamma_1 \eta}{(\mu + \gamma_1 + \gamma_2)^2}$. Using $(1 - I - R)^2 \le 1$ and (8) we infer that

$$d\log I \ge \left[p\beta_1 S + (1-p)\frac{\beta_2 S}{1+mS+nI} - \left(\mu + \eta + \frac{1}{2}p^2\sigma_1^2 + \frac{1}{2}(1-p)^2\sigma_2^2\right) + \gamma_1 \frac{R}{I}\right]dt$$
$$+ p\sigma_1 S dB_1(t) + \frac{(1-p)\sigma_2 S}{1+mS+nI} dB_2(t). \tag{15}$$

Additionally, we also have

$$d\left(\frac{p\beta_1 + \frac{(1-p)\beta_2}{1+m+n}}{\mu}S\right) = \left[p\beta_1 + \frac{(1-p)\beta_2}{1+m+n} - p\beta_1 S - \frac{(1-p)\beta_2 S}{1+m+n} - \frac{p^2\beta_1^2 SI}{\mu} - \frac{p(1-p)\beta_1\beta_2 SI\frac{SI}{1+m+n}}{\mu}\right.$$
$$\left. - \frac{p(1-p)\beta_1\beta_2 SI\frac{SI}{1+mS+nI}}{\mu} - \frac{\frac{\beta_2^2(1-p)^2 SI}{(1+m+n)(1+mS+nI)}}{\mu} + \frac{\gamma_2(p\beta_1 + \frac{(1-p)\beta_2}{1+m+n})R}{\mu}\right]dt$$
$$- \frac{p\beta_1 + \frac{(1-p)\beta_2}{1+m+n}}{\mu}p\sigma_1 SI dB_1(t) - \frac{p\beta_1 + \frac{(1-p)\beta_2}{1+m+n}}{\mu}\frac{(1-p)\sigma_2 SI}{1+mS+nI} dB_2(t)$$
$$\ge \left[p\beta_1 + \frac{(1-p)\beta_2}{1+m+n} - p\beta_1 S - \frac{(1-p)\beta_2 S}{1+m+n} - \frac{(p\beta_1 + (1-p)\beta_2)^2}{\mu}I\right]dt$$
$$- \frac{p\beta_1 + \frac{(1-p)\beta_2}{1+m+n}}{\mu}p\sigma_1 SI dB_1(t) - \frac{p\beta_1 + \frac{(1-p)\beta_2}{1+m+n}}{\mu}\frac{(1-p)\sigma_2 SI}{1+mS+nI} dB_2(t). \tag{16}$$

Moreover,

$$dB \log R = \left[\eta B \frac{I}{R} - (\mu + \gamma_1 + \gamma_2)B \right] dt. \tag{17}$$

Combining (15)–(17), we have

$$
\begin{aligned}
dF(S, I, R) \geq & \left[p\beta_1 + \frac{(1-p)\beta_2}{1+m+n} - \left(\mu + \eta + \frac{1}{2}p^2\sigma_1^2 + \frac{1}{2}(1-p)^2\sigma_2^2 \right) - (\mu + \gamma_1 + \gamma_2)B \right. \\
& \left. + \left(\sqrt{\frac{\gamma_1 R}{I}} - \sqrt{\frac{B\eta I}{R}} \right)^2 + 2\sqrt{\gamma_1 \eta B} - \frac{(p\beta_1 + (1-p)\beta_2)^2}{\mu} I \right] dt \\
& + \left(1 - \frac{p\beta_1 + \frac{(1-p)\beta_2}{1+m+n}}{\mu} I \right) p\sigma_1 S dB_1(t) + \left(1 - \frac{p\beta_1 + \frac{(1-p)\beta_2}{1+m+n}}{\mu} I \right) \frac{(1-p)\sigma_2 S}{1+mS+nI} dB_2(t) \\
\geq & \left[p\beta_1 + \frac{(1-p)\beta_2}{1+m+n} - \left(\mu + \eta + \frac{1}{2}p^2\sigma_1^2 + \frac{1}{2}(1-p)^2\sigma_2^2 \right) - (\mu + \gamma_1 + \gamma_2)B \right. \\
& \left. + 2\sqrt{\gamma_1 \eta B} - \frac{(p\beta_1 + (1-p)\beta_2)^2}{\mu} I \right] dt \\
& + \left(1 - \frac{p\beta_1 + \frac{(1-p)\beta_2}{1+m+n}}{\mu} I \right) p\sigma_1 S dB_1(t) + \left(1 - \frac{p\beta_1 + \frac{(1-p)\beta_2}{1+m+n}}{\mu} I \right) \frac{(1-p)\sigma_2 S}{1+mS+nI} dB_2(t).
\end{aligned}
$$

By using (14), we have

$$
\begin{aligned}
dF(S, I, R) \geq & \left[\left(p\beta_1 + \frac{(1-p)\beta_2}{1+m+n} \right) \left(1 - \frac{1}{\mathcal{R}_s^0} \right) - \frac{(p\beta_1 + (1-p)\beta_2)^2}{\mu} I \right] dt \\
& + \left(1 - \frac{p\beta_1 + \frac{(1-p)\beta_2}{1+m+n}}{\mu} I \right) p\sigma_1 S dB_1(t) + \left(1 - \frac{p\beta_1 + \frac{(1-p)\beta_2}{1+m+n}}{\mu} I \right) \frac{(1-p)\sigma_2 S}{1+mS+nI} dB_2(t).
\end{aligned} \tag{18}
$$

Integrating the last inequality from 0 to t and multiplying by $\frac{1}{t}$, it yields that

$$
\begin{aligned}
\frac{F(S(t), I(t), R(t)) - F(S(0), I(0), R(0))}{t} \geq & \left(p\beta_1 + \frac{(1-p)\beta_2}{1+m+n} \right) \left(1 - \frac{1}{\mathcal{R}_s^0} \right) \\
& - \frac{(p\beta_1 + (1-p)\beta_2)^2}{\mu t} \int_0^t I(u) du + \frac{M_1(t)}{t} + \frac{M_2(t)}{t},
\end{aligned}
$$

where

$$M_1(t) = p\sigma_1 \int_0^t S(u) \left(1 - \frac{p\beta_1 + \frac{(1-p)\beta_2}{1+m+n}}{\mu} I(u) \right) dB_1(u),$$

and

$$M_2(t) = (1-p)\sigma_2 \int_0^t \frac{\sigma_2 S(u)}{1+mS(u)+nI(u)} \left(1 - \frac{p\beta_1 + \frac{(1-p)\beta_2}{1+m+n}}{\mu} I(u) \right) dB_2(u).$$

Therefore

$$
\begin{aligned}
\frac{1}{t} \int_0^t I(u) du \geq & \frac{\mu(p\beta_1 + \frac{(1-p)\beta_2}{1+m+n})}{(p\beta_1 + (1-p)\beta_2)^2} \left(1 - \frac{1}{\mathcal{R}_s^0} \right) - \frac{\mu}{(p\beta_1 + (1-p)\beta_2)^2} \frac{F(S(t), I(t), R(t))}{t} \\
& + \frac{\mu}{(p\beta_1 + (1-p)\beta_2)^2} \frac{F(S(0), I(0), R(0))}{t} \\
& + \frac{\mu}{(p\beta_1 + (1-p)\beta_2)^2} \frac{M_1 t}{t} + \frac{\mu}{(p\beta_1 + (1-p)\beta_2)^2} \frac{M_2 t}{t}.
\end{aligned}
$$

Clearly, $M_1(t), M_2(t)$ are continuous martingales and $M_1(0) = 0$, $M_2(0) = 0$ with their quadratic variation

$$[M_1](t) = p^2 \sigma_1{}^2 \int_0^t \left[S^2(u) \left(1 - \frac{p\beta_1 + \frac{(1-p)\beta_2}{1+m+n}}{\mu} I(u) \right) \right]^2 du < p^2 \sigma_1{}^2 \left(1 + \frac{p\beta_1 + \frac{(1-p)\beta_2}{1+m+n}}{\mu} \right)^2 t,$$

$$[M_2](t) = (1-p)^2 \sigma_2{}^2 \int_0^t \left(1 - \frac{p\beta_1 + \frac{(1-p)\beta_2}{1+m+n}}{\mu} I(u) \right)^2 \left(\frac{S(u)}{1+mS(u)+nI(u)} \right)^2 du$$

$$< (1-p)^2 \sigma_2{}^2 \left(1 + \frac{p\beta_1 + \frac{(1-p)\beta_2}{1+m+n}}{\mu} \right)^2 t.$$

Hence, by the strong law of large numbers for local martingales, we have

$$\lim_{t \to \infty} \frac{1}{t} \frac{M_1(t)}{t} = 0 \ a.s.,$$

$$\lim_{t \to \infty} \frac{1}{t} \frac{M_2(t)}{t} = 0 \ a.s.$$

Consequently, we have

$$\liminf_{t \to \infty} \frac{1}{t} \int_0^t I(u) du \geq \frac{\mu \left(p\beta_1 + \frac{(1-p)\beta_2}{1+m+n} \right)}{(p\beta_1 + (1-p)\beta_2)^2} \left(1 - \frac{1}{\mathcal{R}_s^0} \right) - \frac{\mu}{(p\beta_1 + (1-p)\beta_2)^2} \limsup_{t \to \infty} \frac{F(S(t), I(t), R(t))}{t} \quad a.s. \quad (19)$$

Clearly,

$$\limsup_{t \to \infty} \frac{F(S(t), I(t), R(t))}{t} \leq \limsup_{t \to \infty} \frac{p\beta_1 + \frac{(1-p)\beta_2}{1+m+n}}{\mu t} = 0.$$

Combining (19), we find the desired result.

$$\liminf_{t \to \infty} \frac{1}{t} \int_0^t I(u) du \geq \frac{\mu \left[p\beta_1 + \frac{(1-p)\beta_2}{1+m+n} \right]}{(p\beta_1 + (1-p)\beta_2)^2} \left(1 - \frac{1}{\mathcal{R}_s^0} \right) \quad a.s. \quad (20)$$

Integrating the last equation of (3) and multiplying by $\frac{1}{t}$, we obtain

$$\frac{\mu + \gamma_1 + \gamma_2}{t} \int_0^t R(s) ds = \frac{\eta}{t} \int_0^t I(s) ds + \frac{R(0) - R(t)}{t}.$$

As $\lim\limits_{t \to \infty} \frac{R(0) - R(t)}{t} = 0$, using (20), we deduce that

$$\liminf_{t \to \infty} \frac{1}{t} \int_0^t R(s) ds \geq \frac{\eta \mu \left(p\beta_1 + \frac{(1-p)\beta_2}{1+m+n} \right)}{(\mu + \gamma_1 + \gamma_2)(p\beta_1 + (1-p)\beta_2)^2} \left(1 - \frac{1}{\mathcal{R}_s^0} \right).$$

Hence, we complete the proof. $\square$

5. Existence of a Stationary Distribution

In Section 4, we obtain that the persistence in the mean of the system starts from any initial condition of the invariant region Γ_1 on certain conditions. In this section, to better understand the asymptotic behavior of the diseases shown in the SDE system (4), in the case of $\mathcal{R}_s^0 > 1$, we present

that there exists a stationary distribution of the process $(I(t), R(t))$. The diffusion matrix of system (4) is shown as

$$\sum(i,r) = \begin{pmatrix} \left(p\sigma_1(1-i-r)i + (1-p)\frac{\sigma_2(1-i-r)i}{1+m(1-i-r)+ni}\right)^2 & 0 \\ 0 & 0 \end{pmatrix}.$$

As the diffusion matrix is degenerate and dissatisfies the uniform ellipticity condition, we can not directly prove the asymptotic stability using Khasminskii's theorem [23]. To handle this situation, we apply the integral Markov semigroup theory, presented in [14,24,25].

Now, we consider the space $(\Gamma_2, \mathcal{B}(\Gamma_2), m)$, where $\mathcal{B}(\Gamma_2)$ is the σ-algebra of Borel subsets of Γ_2 and m is the Lebesgue measure on $(\Gamma_2, \mathcal{B}(\Gamma_2))$. Throughout this section, we denote the transition probability function for the diffusion process $(I(t), R(t))$, i.e.,

$$P(t, i_0, r_0, B) = \mathbb{P}((I(t), R(t)) \in B | I(0) = i_0, R(0) = r_0).$$

In Theorem 4, we show the absolute continuity of the transition function of the degenerate diffusion processes, as described by the Stratonovich equation, using the Hörmander condition, given in [26]. Let $\mathcal{K}(t, i, r, i_0, r_0)$ be the density of $P(t, i_0, r_0, .)$. For any $t \geq 0$, the operator $P(t)$ can be defined as follows

$$P(t)g(i,r) = \int_{\Gamma_2} \mathcal{K}(t, i, r, i_0, r_0)g(i,r)m(di, dr), \quad g \in \mathcal{D}.$$

Hence, according to system (4), we define the integral Markov semigroup $\{P(t)\}_{t \geq 0}$.

Theorem 4. *The transition probability function $P(t, i_0, r_0, .)$ of the solution $(I(t), R(t))$ of system (4), has a density $\mathcal{K}(t, i, r, i_0, r_0) \in \mathcal{C}^\infty(\mathbb{R}, \Gamma_2, \Gamma_2)$.*

Proof. Denote $a(x)$ and $b(x)$ two vectors fields defined on $\mathbb{R}^n$, then the Lie bracket $[a, b](x)$ is presented as

$$[a, b]_j(x) = \sum_{k=1}^{n} \left[a_k \frac{\partial b_j}{\partial x_k}(x) - b_k \frac{\partial a_j}{\partial x_k}(x) \right],$$

$j = 1, \ldots, n$. We write the Itô system (4) as the Stratonovitch SDE

$$\begin{pmatrix} dI \\ dR \end{pmatrix} = \begin{pmatrix} a_1(I,R) \\ a_2(I,R) \end{pmatrix} dt + \begin{pmatrix} p\sigma_1(1-I-R)I + (1-p)\frac{\sigma_2(1-I-R)I}{1+m(1-I-R)+nI} \\ 0 \end{pmatrix} \circ dB(t), \quad (21)$$

where

$$a_1(I,R) = p\beta_1(1-I-R)I + (1-p)\frac{\beta_2(1-I-R)I}{1+m(1-I-R)+nI} - (\mu+\eta)I + \gamma_1 R,$$
$$a_2(I,R) = \eta I - (\mu + \gamma_1 + \gamma_2)R.$$

Let $a(I,R) = (a_1(I,R), a_2(I,R))'$ and $b(I,R) = (p\sigma_1(1-I-R)I + (1-p)\frac{\sigma_2(1-I-R)I}{1+m(1-I-R)+nI}, 0)'$. By direct calculation, we obtain $\det([a,b](I,R), b(I,R)) = \eta(p\sigma_1(1-I-R)I + (1-p)\frac{\sigma_2(1-I-R)I}{1+m(1-I-R)+nI})^2 > 0$ for any $[I,R] \in \Gamma_2$. Accordingly, the vectors $[a,b](I,R)$ and $b(I,R)$ span the space $\Gamma 2$. By employing the Hörmander theorem, the probability function $P(t, i_0, r_0, .)$ has a density $\mathcal{K}(t, i, r, i_0, r_0) \in \mathcal{C}^\infty(\mathbb{R}, \Gamma_2, \Gamma_2)$. $\square$

Now we use the approach given in [24,25] to verify that $\mathcal{K}$ is positive. We fix a function $\phi \in C([0, T], R)$. Consider the system of differential equations below

$$\begin{cases} \dot{I}_\phi(t) = a_1(I_\phi(t), R_\phi(t)) + (p\sigma_1(1-I_\phi(t)-R_\phi(t))I_\phi(t) + (1-p)\frac{\sigma_2(1-I_\phi(t)-R_\phi(t))I_\phi(t)}{1+m(1-I_\phi(t)-R_\phi(t))+nI_\phi(t)}\varphi(t), \\ \dot{R}_\phi(t) = a_2(I_\phi(t), R_\phi(t)), \end{cases} \quad (22)$$

with the initial condition $I_\phi(0) = i_0$, $R_\phi(0) = r_0$ and let $D_{i_0,r_0,\phi}$ be the Frechét derivative of the function $h \to \begin{pmatrix} I_{\phi+h}(T) \\ R_{\phi+h}(T) \end{pmatrix}$ from $\mathcal{C}([0,T],R)$ to $\mathbb{R}^2$. The derivative $D_{i_0,r_0,\phi}$ could be found as follows. Let $E(t) = a'(I_\phi(t), R_\phi(t)) + \phi(t)b'(I_\phi(t), R_\phi(t))$, where a' and b' are the Jacobian of $a(I,R)$ and $b(I,R)$, respectively. Denote $Q(t,t_0)$, for $0 \le t_0 \le t \le T$, as the function satisfying $Q(t_0,t_0) = I$, $\frac{\partial Q}{\partial t}(t,t_0) = E(t)Q(t,t_0)$. Then

$$D_{i_0,r_0,\phi}h = \int_0^T Q(T,s)b(I_\phi(s), R_\phi(s))h(s)ds.$$

If, for some $\phi \in \mathcal{C}([0,T],R)$, the derivative $D_{i_0,r_0,\phi}$ has rank 2, then $\mathcal{K}(t,i,r,i_0,r_0) > 0$ for $i = I_\phi(T)$ and $r = R_\phi(T)$.

Theorem 5. *For each $(i_0,r_0) \in \Gamma_2$ and $\phi \in \mathcal{C}([0,T],\mathbb{R})$, the derivative $D_{i_0,r_0,\phi}$ has rank 2.*

Proof. Let $\epsilon \in (0,T)$ and $h = 1_{[T-\epsilon,T]}$. By a first-order Taylor series approximation, we obtain $Q(T,s) = 1 - E(T)(T-s) + o(T-s)$. Then $D_{i_0,r_0,\phi}h = \epsilon e - \frac{1}{2}\epsilon^2 E(T)e + o(\epsilon^2)$ where $e = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. By computation we have

$$E(T)e = \begin{pmatrix} p(\beta_1 + \phi\sigma_1)\varphi_\phi(t) + (1-p)(\beta_2 + \phi\sigma_2)\left(\frac{\varphi_\phi(t)}{1+m\rho_\phi(t))+nI_\phi(t)} - \frac{(n-m)\rho_\phi(t)I_\phi(t)}{(1+m\rho_\phi(t))+nI_\phi(t))^2}\right) - (\mu+\eta) \\ \eta \end{pmatrix}.$$

where $\varphi_\phi(t) = 1 - 2I_\phi(t) - R_\phi(t)$ and $\rho_\phi(t) = 1 - I_\phi(t) - R_\phi(t)$. Therefore, e and $E(t)e$ are linearly independent and the derivative $D_{i_0,r_0,\phi}$ has rank 2. $\square$

Now we verify the positivity of $\mathcal{K}$. Hence, we investigate the controllability of the region Γ_2.

Theorem 6. *For each $(I_0,R_0) \in \Gamma_2$ and for almost every $(I_1,R_1) \in \Gamma_2$, there exists $T > 0$, such that $\mathcal{K}(T,I_1,R_1,I_0,R_0) > 0$.*

Proof. Fixed (i_0,r_0), $(i_1,r_1) \in \Gamma_2$. Without a loss of generality, we assume that $r_1 < r_0$. Now we show that there is a control function ϕ and $T > 0$ such that $I_\phi(0) = i_0$, $R_\phi(0) = r_0$, $I_\phi(T) = i_1$, $R_\phi(T) = r_1$. We first consider the following ODEs

$$\begin{cases} \dot{X}_1 = -(\mu + \gamma_1 + \gamma_2)X_1, \\ \dot{Y}_1 = \eta - (\mu + \gamma_1 + \gamma_2 + \eta)Y_1, \\ X_1(0) = Y_1(0) = r_0, \end{cases} \tag{23}$$

with positive solutions X_1 and Y_1. We define the function $R_1(t)$, as follows

$$R_1(t) = X_1(t) + \frac{i_0}{1-r_0}(Y_1(t) - X_1(t)).$$

Thus, we have

$$R_1(0) = r_0, \quad \dot{R}_1(0) = \eta i_0 - (\mu + \gamma_1 + \gamma_2). \tag{24}$$

Furthermore

$$\dot{R}_1(t) = -(\mu + \gamma_1 + \gamma_2)R_1(t) + \frac{\lambda i_0}{1-r_0}(1 - Y_1(t)),$$

$$= \eta - (\mu + \gamma_1 + \gamma_2 + \eta)R_1(t) + \frac{\eta(1 - i_0 - r_0)}{1-r_0}(X_1(t) - 1).$$

As $X_1(t), Y_1(t) \in (0,1)$ for all $t \geq 0$. Then

$$-(\mu + \gamma_1 + \gamma_2)R_1(t) < \dot{R}_1(t) < \eta - (\mu + \gamma_1 + \gamma_2 + \eta)R_1(t). \tag{25}$$

Similarly, we consider X_2 and Y_2 the solutions of the ODEs

$$\begin{cases} \dot{X}_2 = -(\mu + \gamma_1 + \gamma_2)X_2, \\ \dot{Y}_2 = \eta - (\mu + \gamma_1 + \gamma_2 + \eta)Y_2, \\ X_2(0) = Y_2(0) = r_1. \end{cases} \tag{26}$$

Then, the function

$$Z(t) = X_2(t) + \frac{i_1}{1 - r_1}(Y_2(t) - X_2(t)),$$

verifies the following

$$Z(0) = r_1, \ \dot{Z}(0) = \eta i_1 - (\mu + \gamma_1 + \gamma_2)r_1. \tag{27}$$

Now, we choose a differentiable function, $R_3(t) = Z(t - T)$, where T is a sufficiently large number, such that

$$X_1(t) \leq R_3(t) \leq Y_1(t), \ \ \forall t \in [T - \varepsilon, \varepsilon], \tag{28}$$

where ε is a small enough positive number. Hence, from (27) and (28) we get that for all $t \in [T - \varepsilon, T]$, $R_3(t)$ verifies

$$-(\mu + \gamma_1 + \gamma_2)R_3(t) < \dot{R}_3(t) < \eta - (\mu + \gamma_1 + \gamma_2 + \eta)R_3(t), \tag{29}$$

$$R_2(T) = r_1, \ \dot{R}_2(T) = \eta i_1 - (\mu + \gamma_1 + \gamma_2)r_1. \tag{30}$$

Finally, we define a function $R_2(t)$ on $[\varepsilon, T - \varepsilon]$ such that $X_1(t) \leq R_2(t) \leq Y_1(t)$, then we have

$$-(\mu + \gamma_1 + \gamma_2)R_2(t) < \dot{R}_2(t) < \eta - (\mu + \gamma_1 + \gamma_2 + \eta)R_2(t). \tag{31}$$

In addition, we choose $R_2(t)$ appropriate to define the $\mathcal{C}^1$-function R_ϕ as follows

$$R_\phi(t) = \begin{cases} R_1(t), 0 \leq t \leq \varepsilon, \\ R_2(t), \varepsilon \leq t \leq T - \varepsilon, \\ R_3(t), T - \varepsilon \leq t \leq T. \end{cases}$$

Hence the function R_ϕ, constructed above, possesses the properties below

$$\begin{aligned} & R_\phi(0) = r_0, \ R_\phi(t) = r_1, \\ & \dot{R}_\phi(0) = \eta i_0 - (\mu + \gamma_1 + \gamma_2)r_0, \ \dot{R}_\phi(T) = \eta i_1 - (\mu + \gamma_1 + \gamma_2)r_1, \\ & -(\mu + \gamma_1 + \gamma_2)R_\phi(t) < \dot{R}_\phi(t) < \eta - (\mu + \gamma_1 + \gamma_2 + \eta)R_\phi(t). \end{aligned} \tag{32}$$

Now, we define the function $I_\phi(t)$ on the interval $[0, T]$ by

$$I_\phi(t) = \frac{1}{\eta}(\dot{R}_\phi(t) + (\mu + \gamma_1 + \gamma_2)R_\phi(t)).$$

Thus, we can deduce from (32) that $0 < I_\phi(t) + R_\phi(t) < 1$ for all $t \in [0, T]$. After that, we choose the control function $\phi(t)$ as follows

$$\phi(t) = \frac{\dot{I}_\phi(t) - a_1(I_\phi(t), R_\phi(t))}{[(p\sigma_1(1 - I_\phi(t) - R_\phi(t)) + (1 - p)\frac{\sigma_2(1 - I_\phi(t) - R_\phi(t))}{1 + m(1 - I_\phi(t) - R_\phi(t)) + nI_\phi(t)}]I_\phi(t)}.$$

Therefore, with the above determinate continuous function ϕ, $(I_\phi(t), R_\phi(t))$ is the solution to system (22). Furthermore, from (32) and the chosen of $I_\phi(t)$, we find that this solution satisfies

$$(I_\phi(0), R_\phi(0)) = (i_0, r_0), \ (I_\phi(T), R_\phi(T)) = (i_1, r_1).$$

This completes the proof. $\square$

Theorem 7. *If $\mathcal{R}_s^0 > 1$, then the semigroup $\{P(t)\}_{t \geq 0}$ is asymptotically stable. This indicates that there exists a unique density $\mathcal{K}^*(i, r)$, such that*

$$\int \int_{\Gamma_2} |\mathcal{K}(t, i, r, i_0, r_0) - \mathcal{K}^*(i, r)| di dr = 0 \ \text{ for all } (i_0, r_0) \in \Gamma_2.$$

Proof. First, we construct a positive $\mathcal{C}^2$-function $\mathbf{V}_2$ and a closed set $\Delta \subset \Gamma_2$ such that

$$\sup_{(I,R) \in \Gamma_2 \backslash \Delta} \mathcal{L}\mathbf{V}_2(I, R) < 0,$$

where $\mathcal{L}$ is the differential operator related to the system (4) and $\Delta = (\alpha_1, \alpha_2) \times (\alpha_3, \alpha_4) \subset \Gamma_2$ in which α, α_1, α_2, α_3, and α_4 are positive constants to be chosen later. Then the semigroup $\{P(t)\}_{t \geq 0}$ is not sweeping from the set Δ. It is clear that the function $f_{u,v}(x) = u \log x - vx$, $u, v > 0$ has its maximum at $\frac{u}{v}$. Using the definition of $F(S, I, R)$ in (13), we can easily deduce that

$$F(I, R) + a \log R = \log I + \frac{p\beta_1 + \frac{(1-p)\beta_2}{1+m+n}}{\mu}(1 - I - R) + B \log R + a \log \ R$$

$$= f_{1, \frac{p\beta_1 + \frac{(1-p)\beta_2}{1+m+n}}{\mu}}(I) + f_{B+a, \frac{p\beta_1 + \frac{(1-p)\beta_2}{1+m+n}}{\mu}}(R) + \frac{p\beta_1 + \frac{(1-p)\beta_2}{1+m+n}}{\mu}$$

$$\leq f_{1, \frac{p\beta_1 + \frac{(1-p)\beta_2}{1+m+n}}{\mu}}\left(\frac{\mu}{p\beta_1 + \frac{(1-p)\beta_2}{1+m+n}}\right) + f_{B+a, \frac{p\beta_1 + \frac{(1-p)\beta_2}{1+m+n}}{\mu}}\left(\frac{\mu(B+a)}{p\beta_1 + \frac{(1-p)\beta_2}{1+m+n}}\right) + \frac{p\beta_1 + \frac{(1-p)\beta_2}{1+m+n}}{\mu}.$$

Hence,

$$\mathbf{V}_2(I, R) = f_{1, \frac{p\beta_1 + \frac{(1-p)\beta_2}{1+m+n}}{\mu}}\left(\frac{\mu}{p\beta_1 + \frac{(1-p)\beta_2}{1+m+n}}\right) + f_{B+a, \frac{p\beta_1 + \frac{(1-p)\beta_2}{1+m+n}}{\mu}}\left(\frac{\mu(B+a)}{p\beta_1 + \frac{(1-p)\beta_2}{1+m+n}}\right) + \frac{p\beta_1 + \frac{(1-p)\beta_2}{1+m+n}}{\mu} + 1$$

$$- F(I, R) - a \log R > 0.$$

Combining (18) and the differential operator $\mathcal{L}$, we have

$$L\mathbf{V}_2(I, R) \leq -\left(p\beta_1 + \frac{(1-p)\beta_2}{1+m+n}\right)\left(1 - \frac{1}{\mathcal{R}_s^0}\right) + \frac{(p\beta_1 + (1-p)\beta_2)^2}{\mu}I + a(\mu + \gamma_1 + \gamma_2) - \frac{a\eta I}{R}. \quad (33)$$

Let $(I, R) \in \Gamma_2 \backslash \Delta$. We consider the following four cases:

- If $I < a_1$. In this case, we choose a_1 and a to be sufficiently small, such that

$$L\mathbf{V}_2(I, R) \leq -\left(p\beta_1 + \frac{(1-p)\beta_2}{1+m+n}\right)\left(1 - \frac{1}{\mathcal{R}_s^0}\right) + \frac{(p\beta_1 + (1-p)\beta_2)^2}{\mu}a_1 + a(\mu + \gamma_1 + \gamma_2) < 0.$$

- If $a_1 \leq I < 1$ and $R < a_3$, we choose a_3 to be sufficiently small, such that

$$L\mathbf{V}_2(I, R) \leq \frac{(p\beta_1 + (1-p)\beta_2)^2}{\mu}a_1 + a(\mu + \gamma_1 + \gamma_2) - \frac{a\eta a_1}{a_3} < 0.$$

- If $a_2 \leq I < 1 - a_3$ and $a_3 \leq R < 1 - a_2$, we choose $a_2 < 1$ to be sufficiently close to 1, such that

$$L\mathbf{V}_2(I, R) \leq \frac{(p\beta_1 + (1-p)\beta_2)^2}{\mu}a_1 + a(\mu + \gamma_1 + \gamma_2) - \frac{a\eta a_2}{1 - a_2} < 0.$$

- If $a_1 \leq I < 1 - a_4$ and $a_4 \leq R < 1 - a_1$, we choose $a_4 < 1$ to be sufficiently close to 1 and a to be sufficiently small, such that

$$L\mathbf{V}_2(I, R) \leq -\left(p\beta_1 + \frac{(1-p)\beta_2}{1+m+n}\right)\left(1 - \frac{1}{\mathcal{R}_s^0}\right) + \frac{(p\beta_1 + (1-p)\beta_2)^2}{\mu}a_1(1 - a_4) + a(\mu + \gamma_1 + \gamma_2) < 0.$$

Then, with the above appropriate choice on the five parameters, we conclude that

$$\mathcal{L}\mathbf{V}_2(I, R) < 0, \ \forall (I, R) \in \Gamma_2 \setminus \Delta,$$

as intended. We complete the proof of this theorem. $\square$

6. Numerical Simulations

We simulate the system (3) for various parameter sets to verify the main theoretical results raised in this paper. For simplicity, we choose the following parameters $\eta = 0.1$, $p = 0.3$.

Example 1. *Figure 1 presents a single path of the solution $(S(t), I(t), R(t))$ to the stochastic system (3) with initial condition $(0.7, 0.2, 0.1)$ for the parameters $\beta_1 = \beta_2 = 0.3$, $\sigma_1 = \sigma_2 = 0.5$, $\gamma_1 = 0.04$, $\gamma_2 = 0.01$, $m = 0.25$, $n = 0.3$, where $\mathcal{R}_0 = 0.8415 < 1$. Then the disease of system (4) will die out.*

Example 2. *Figure 2 presents the solution $(S(t), I(t), R(t))$ to system (3) with an initial conditions $(0.7, 0.2, 0.1)$ for the parameters $\mu = 0.3$, $\beta_1 = \beta_2 = 0.6$, $\sigma_1 = \sigma_2 = 0.2$, $\gamma_1 = 0.2$, $\gamma_2 = 0.1$, $m = 0.15$, $n = 0.2$, where $\mathcal{R}_s^0 = 1.2983 > 1$. Then from Theorem 4 the disease of system (3) is permanent in mean, and admits a stationary distribution.*

Example 3. *Figure 3 presents the results of SDE (3) with initial conditions $(0.7, 0.2, 0.1)$ for $\gamma_1 = 0, 0.5, 1$, and the other parameters are the same as in Example 2. Then we have $1 < \mathcal{R}_s^0(0) = 1.1932 < \mathcal{R}_s^0(0.5) = 1.3794 < \mathcal{R}_s^0(1) = 1.4437$. Therefore, a relapse can increase the incidence of the disease.*
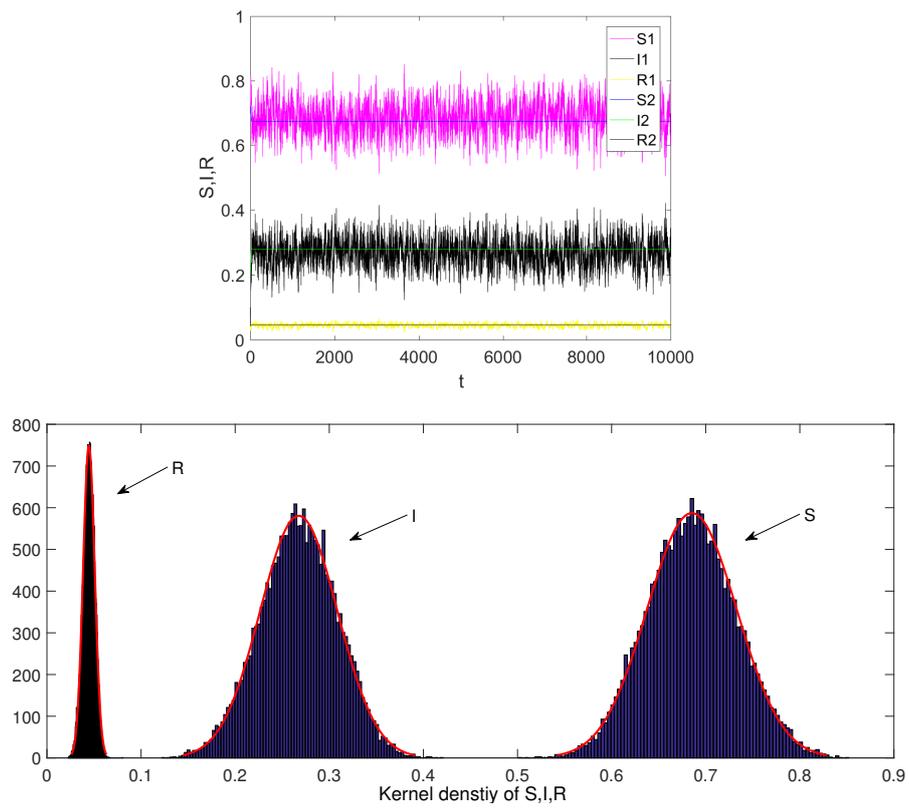
Figure 1. Paths of the solution and the kernel density functions to the system (3) using the data of Example 2.
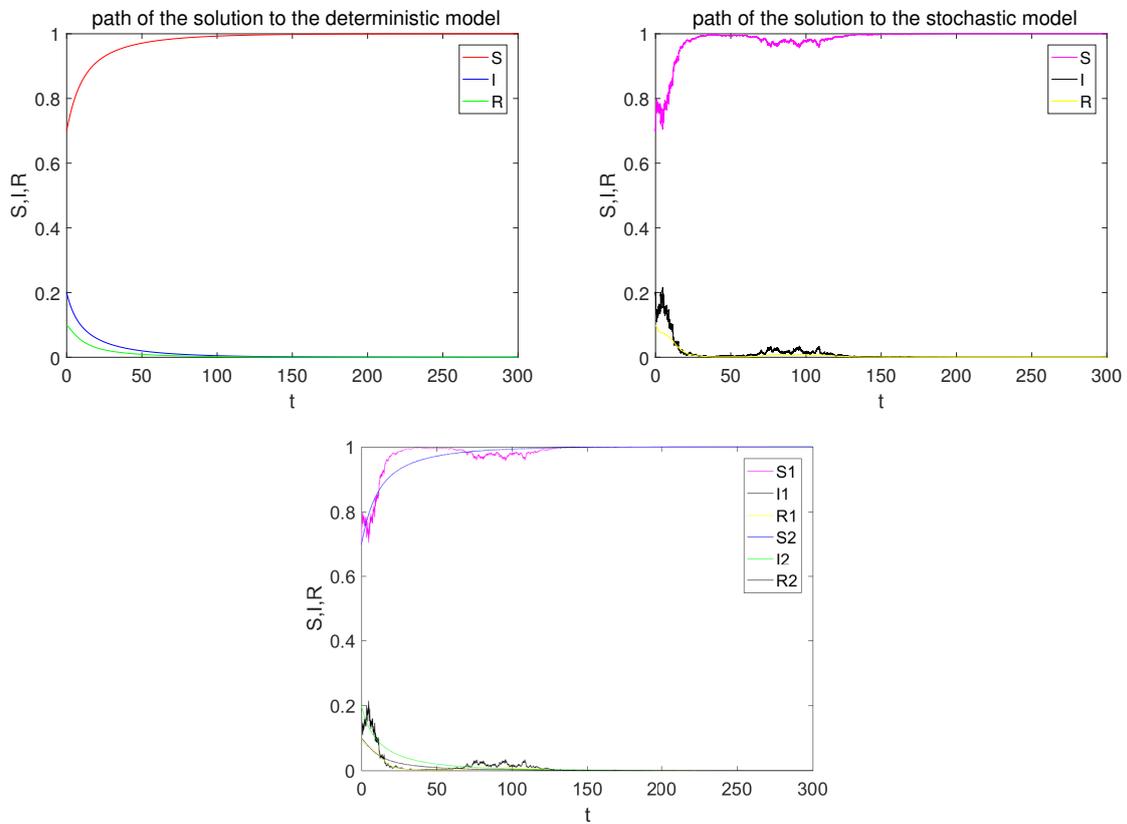

Figure 2. Paths of the solution to the system (3) using the data of Example (1).
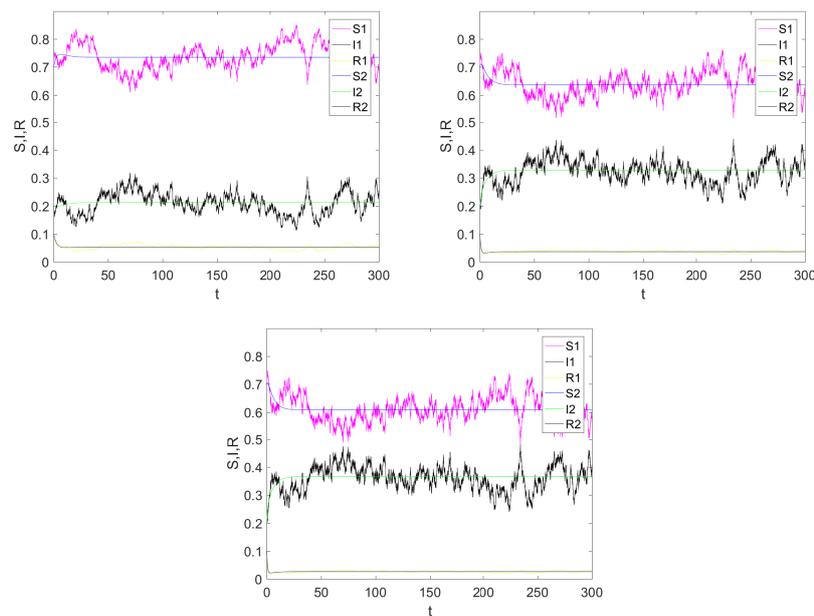
Figure 3. Paths of the solution to the system (3) using the data of Example 3.

7. Conclusions

In this paper, we focus on the principle properties of a stochastic epidemic model with relapse and temporary immunity. The highlight of this paper is that, considering that the incidence rate of diseases may change as the environment changes, we put forward a new version of the transmission function $\varphi(S, I) = \sum_{i=1}^{n} p_i \varphi_i(S, I)$, $\sum_{i=1}^{n} p_i = 1$. We assume $\varphi(S, I) = p\beta_1 SI + (1-p)\frac{\beta_2 SI}{1 + mS + nI}$ $(0 < p < 1)$, and this transmission function indicates that, in different environments, the disease has two different incidence rates, which are the bilinear incidence rate and the Beddington–DeAngelis incidence rate. Moreover, if the incidence rate of this disease is only the bilinear incidence rate, then $p = 1$. In contrast, if $p = 0$, this implies that the disease has only one incidence, which is the Beddington–DeAngelis incidence rate. Therefore, our assumptions are both mathematically and biologically reasonable. The extinction result shows that the disease will die out almost surely when $\mathcal{R}_0 < 1$. Then we obtain that the disease persists in the mean when $\mathcal{R}_s^0 > 1$, and there exists a stationary distribution for the stochastic model in the meantime (see Figure 2). We also mention that a relapse can increase the risk of diseases, as $\mathcal{R}_0$ increases with the relapse rate γ_1 (see Figure 3).

There are also many related problems to be solved with further investigation. For example:

- The long-time behavior of the system (3) when $\mathcal{R}_s^0 = 1$.
- If there exists a threshold $\mathcal{R}^*$, when $\mathcal{R}_s^0 < \mathcal{R}^* < \mathcal{R}_0$, for the behavior of the system (3).

In order to make this model more reasonable, we can also investigate delay stochastic differential equation models, control stochastic differential equation models, and impulsive stochastic differential equation models of system (2) for further work. The approaches are shown in [27–41].

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