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An Investigation of the Third Hankel Determinant Problem for Certain Subfamilies of Univalent Functions Involving the Exponential Function

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Abstract: In the current article, we consider certain subfamilies $\mathcal{S}_e^*$ and $\mathcal{C}_e$ of univalent functions associated with exponential functions which are symmetric along real axis in the region of open unit disk. For these classes our aim is to find the bounds of Hankel determinant of order three. Further, the estimate of third Hankel determinant for the family $\mathcal{S}_e^*$ in this work improve the bounds which was investigated recently. Moreover, the same bounds have been investigated for 2-fold symmetric and 3-fold symmetric functions.

Keywords: subordinations; exponential function; Hankel determinant

1. Introduction and Definitions

Let the collection of functions f that are holomorphic in $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by conditions $f(0) = f'(0) - 1 = 0$ be denoted by the symbol $\mathcal{A}$. Equivalently; if $f \in \mathcal{A}$, then the Taylor-Maclaurin series representation has the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (z \in \Delta). \tag{1}$$

Further, let we name by the notation $\mathcal{S}$ the most basic sub-collection of the set $\mathcal{A}$ that are univalent in Δ. The familiar coefficient conjecture for the function $f \in \mathcal{S}$ of the form (1) was first presented by Bieberbach [1] in 1916 and proved by de-Branges [2] in 1985. In 1916-1985, many mathematicians struggled to prove or disprove this conjecture and as result they defined several subfamilies of the set $\mathcal{S}$ of univalent functions connected with different image domains. Now we mention some of them, that is; let the notations $\mathcal{S}^*$, $\mathcal{C}$ and $\mathcal{K}$, shows the families of starlike, convex and close-to-convex functions respectively and are defined as:

$$
\begin{aligned}
\mathcal{S}^{*} &= \left\{ f \in \mathcal{S} : \frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z}, \ (z \in \Delta) \right\}, \\
\mathcal{C} &= \left\{ f \in \mathcal{S} : \frac{(zf'(z))'}{f'(z)} \prec \frac{1+z}{1-z}, \ (z \in \Delta) \right\}, \\
\mathcal{K} &= \left\{ f \in \mathcal{S} : \frac{f'(z)}{g'(z)} \prec \frac{1+z}{1-z}, \text{ for } g(z) \in \mathcal{C}, \ (z \in \Delta) \right\},
\end{aligned}
$$

where the symbol " $\prec$ " denotes the familiar subordinations between analytic functions and is define as; the function h_1 is subordinate to a function h_2, symbolically written as $h_1 \prec h_2$ or $h_1(z) \prec h_2(z)$, if we can find a function w, which is holomorphic in Δ with $w(0) = 0$ & $|w(z)| < 1$ such that $h_1(z) = h_2(w(z))$ $(z \in \Delta)$. Thus, $h_1(z) \prec h_2(z)$ implies $h_1(\Delta) \subset h_2(\Delta)$. In case of univalency of h_1 in Δ, then the following relation holds:

$$
h_1(z) \prec h_2(z) \quad (z \in \Delta) \quad \Longleftrightarrow \quad h_1(0) = h_2(0) \quad \text{and} \quad h_1(\Delta) \subset h_2(\Delta).
$$

In [3], Padmanabhan and Parvatham in 1985 defined a unified families of starlike and convex functions using familiar convolution with the function $z/(1-z)^{a}$, for all $a \in \mathbb{R}$. Later on, Shanmugam [4] generalized the idea of paper [3] and introduced the set

$$
\mathcal{S}_h^{*}(\phi) = \left\{ f \in \mathcal{A} : \frac{z(f*h)'}{(f*h)} \prec \phi(z), \ (z \in \Delta) \right\},
$$

where "$*$" stands for the familiar convolution, ϕ is a convex and h is a fixed function in $\mathcal{A}$. We obtain the families $\mathcal{S}^{*}(\phi)$ and $\mathcal{C}(\phi)$ when taking $z/(1-z)$ and $z/(1-z)^2$ instead of h in $\mathcal{S}_h^{*}(\phi)$ respectively. In 1992, Ma and Minda [5] reduced the restriction to a weaker supposition that ϕ is a function, with $Re\phi > 0$ in Δ, whose image domain is symmetric about the real axis and starlike with respect to $\phi(0) = 1$ with $\phi'(0) > 0$ and discussed some properties. The set $\mathcal{S}^{*}(\phi)$ generalizes various subfamilies of the set $\mathcal{A}$, for example:

1. If $\phi(z) = \frac{1+Az}{1+Bz}$ with $-1 \leq B < A \leq 1$, then $\mathcal{S}^{*}[A,B] := \mathcal{S}^{*}\left(\frac{1+Az}{1+Bz}\right)$ is the set of Janowski starlike functions, see [6]. Further, if $A = 1-2\alpha$ and $B = -1$ with $0 \leq \alpha < 1$, then we get the set $\mathcal{S}^{*}(\alpha)$ of starlike functions of order α.
2. The class $\mathcal{S}_L^{*} := \mathcal{S}^{*}(\sqrt{1+z})$ was introduced by Sokól and Stankiewicz [7], consisting of functions $f \in \mathcal{A}$ such that $zf'(z)/f(z)$ lies in the region bounded by the right-half of the lemniscate of Bernoulli given by $|w^2 - 1| < 1$.
3. For $\phi(z) = 1 + \sin z$, the class $\mathcal{S}^{*}(\phi)$ lead to the class $\mathcal{S}_{\sin}^{*}$, introduced in [8].
4. The family $\mathcal{S}_e^{*} := \mathcal{S}^{*}(e^z)$ was introduced by Mediratta et al. [9] given as:

$$
\mathcal{S}_e^{*} = \left\{ f \in \mathcal{S} : \frac{zf'(z)}{f(z)} \prec e^{z}, \ (z \in \Delta) \right\}, \tag{2}
$$

or, equivalently

$$
\mathcal{S}_e^{*} = \left\{ f \in \mathcal{S} : \left| \log \frac{zf'(z)}{f(z)} \right| < 1, \ (z \in \Delta) \right\}. \tag{3}
$$

They investigated some interesting properties and also links these classes to the familiar subfamilies of the set $\mathcal{S}$. In [9], the authors choose the function $f(z) = z + \frac{1}{4}z^2$ (Figure 1) and then sketch the following figure of the function class $\mathcal{S}_e^{*}$ by using the form (3) as:

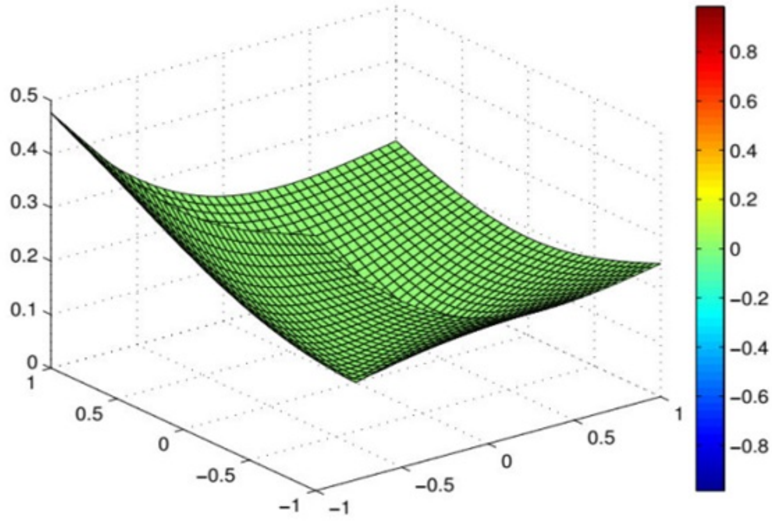

Figure 1. The figure of the function class $\mathcal{S}_1^*$ for $f(z) = z + \frac{1}{4}z^2$.

Similarly, by using Alexandar type relation in [9], we have;

$$\mathcal{C}_e = \left\{ f \in \mathcal{S} : \frac{(zf'(z))'}{f'(z)} \prec e^z, \ (z \in \Delta) \right\}. \tag{4}$$

From the above discussion, we conclude that the families $\mathcal{S}_e^*$ and $\mathcal{C}_e$ considered in this paper are symmetric about the real axis.

For given parameters $q, n \in \mathbb{N} = \{1, 2, \ldots\}$, the Hankel determinant $H_{q,n}(f)$ was defined by Pommerenke [10,11] for a function $f \in \mathcal{S}$ of the form (1) as follows:

$$H_{q,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}. \tag{5}$$

The concept of Hankel determinant is very useful in the theory of singularities [12] and in the study of power series with integral coefficients. For deep insight, the reader is invited to read [13–15]. Specifically, the absolute sharp bound of the functional $H_{2,2}(f) = a_2a_4 - a_3^2$ for each of the sets $\mathcal{S}^*$ and $\mathcal{C}$ were proved by Janteng et al. [16,17] while the exact estimate of this determinant for the family of close-to-convex functions is still unknown (see, [18]). On the other side for the set of Bazilevič functions, the sharp estimate of $|H_{2,2}(f)|$ was given by Krishna et al. [19]. Recently, Srivastava and his coauthors [20] found the estimate of second Hankel determinant for bi-univalent functions involving symmetric q-derivative operator while in [21], the authors discussed Hankel and Toeplitz determinants for subfamilies of q-starlike functions connected with a general form of conic domain. For more literature see [22–29]. The determinant with entries from (1)

$$H_{3,1}(f) = \begin{vmatrix} 1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}$$

is known as Hankel determinant of order three and the estimation of this determinant $|H_{3,1}(f)|$ is very hard as compared to derive the bound of $|H_{2,2}(f)|$. The very first paper on $H_{3,1}(f)$ visible in 2010 by

Babalola [30] in which he got the upper bound of $H_{3,1}(f)$ for the families of $\mathcal{S}^*$ and $\mathcal{C}$. Later on, many authors published their work regarding $|H_{3,1}(f)|$ for different sub-collections of univalent functions, see [8,31–36]. In 2017, Zaprawa [37] upgraded the results of Babalola [30] by giving

$$|H_{3,1}(f)| \leq \begin{cases} 1, & \text{for} \quad f \in \mathcal{S}^*, \\ \frac{49}{540}, & \text{for} \quad f \in \mathcal{C}, \end{cases}$$

and claimed that these bounds are still not best possible. Further for the sharpness, he examined the subfamilies of $\mathcal{S}^*$ and $\mathcal{C}$ consisting of functions with m-fold symmetry and obtained the sharp bounds. Moreover this determinant was further improved by Kwon et al. [38] and proved $|H_{3,1}(f)| \leq 8/9$ for $f \in \mathcal{S}^*$, yet not best possible. The authors in [39–41] contributed in similar direction by generalizing different classes of univalent functions with respect to symmetric points. In 2018, Kowalczyk et al. [42] and Lecko et al. [43] got the sharp inequalities

$$|H_{3,1}(f)| \leq 4/135, \quad \text{and} \quad |H_{3,1}(f)| \leq 1/9,$$

for the recognizable sets $\mathcal{K}$ and $\mathcal{S}^*(1/2)$ respectively, where the symbol $\mathcal{S}^*(1/2)$ indicates to the family of starlike functions of order $1/2$. Also we would like to cite the work done by Mahmood et al. [44] in which they studied third Hankel determinant for a subset of starlike functions in q-analogue. Additionally Zhang et al. [45] studied this determinant for the set $\mathcal{S}_e^*$ and obtained the bound $|H_{3,1}(f)| \leq 0.565$.

In the present article, our aim is to investigate the estimate of $|H_{3,1}(f)|$ for both the above defined classes $\mathcal{S}_e^*$ and $\mathcal{C}_e$. Moreover, we also study this problem for m-fold symmetric starlike and convex functions associated with exponential function.

2. A Set of Lemmas

Let $\mathcal{P}$ denote the family of all functions p that are analytic in $\mathbb{D}$ with $\Re(p(z)) > 0$ and has the following series representation

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \,(z \in \Delta)\,. \tag{6}$$

Lemma 1. *If $p \in \mathcal{P}$ and has the form , then*

$$\begin{aligned} |c_n| &\leq 2 \ \textit{for} \ n \geq 1, & (7)\\ |c_{n+k} - \mu c_n c_k| &< 2, \textit{ for } 0 \leq \mu \leq 1, & (8)\\ |c_m c_n - c_k c_l| &\leq 4 \ \textit{for} \ m+n = k+l, & (9)\\ \left|c_{n+2k} - \mu c_n c_k^2\right| &\leq 2(1+2\mu); \ \textit{for} \ \mu \in \mathbb{R}, & (10)\\ \left|c_2 - \frac{c_1^2}{2}\right| &\leq 2 - \frac{|c_1|^2}{2}, & (11) \end{aligned}$$

and for complex number λ, we have

$$\left|c_2 - \lambda c_1^2\right| \leq 2\max\{1, |2\lambda - 1|\}\,. \tag{12}$$

For the inequalities (7), (11), (8), (10), (9) see [46] and (12) is given in [47].

3. Improved Bound of $|H_{3,1}(f)|$ for the Set $\mathcal{S}_e^*$

Theorem 1. *If f belongs to $\mathcal{S}_e^*$, then*

$$|H_{3,1}(f)| \leq 0.50047781.$$

Proof. Let $f \in \mathcal{S}_e^*$. Then we can write (2), in terms of Schwarz function as

$$\frac{zf'(z)}{f(z)} = e^{w(z)}.$$

If $h \in \mathcal{P}$, then it can be written in form of Schwarz function as

$$h(z) = \frac{1+w(z)}{1-w(z)} = 1 + c_1 z + c_2 z^2 + \cdots.$$

From above, we can get

$$w(z) = \frac{h(z)-1}{h(z)+1} = \frac{c_1 z + c_2 z^2 + c_3 z^3 + \cdots}{2 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots}.$$

$$\begin{aligned} \frac{zf'(z)}{f(z)} &= 1 + a_2 z + \left(2a_3 - a_2^2\right) z^2 + \left(3a_4 - 3a_2 a_3 + a_2^3\right) z^3 \\ &+ \left(4a_5 - 2a_3^2 - 4a_2 a_4 + 4a_2^2 a_3 - a_2^4\right) z^4 = 1 + p_1 z + p_2 z^2 + \cdots. \end{aligned} \tag{13}$$

and from the series expansion of w along with some calculations, we have

$$e^{w(z)} = 1 + w(z) + \frac{(w(z))^2}{2!} + \frac{(w(z))^3}{3!} + \frac{(w(z))^4}{4!} + \frac{(w(z))^5}{5!} + \cdots.$$

After some computations and rearranging, it yields

$$\begin{aligned} e^{w(z)} &= 1 + \frac{1}{2}c_1 z + \left(\frac{c_2}{2} - \frac{c_1^2}{8}\right) z^2 + \left(\frac{c_1^3}{48} + \frac{c_3}{2} - \frac{c_1 c_2}{4}\right) z^3 \\ &+ \left(\frac{1}{384}c_1^4 + \frac{1}{2}c_4 - \frac{1}{8}c_2^2 + \frac{1}{16}c_1^2 c_2 - \frac{1}{4}c_1 c_3\right) z^4 + \cdots. \end{aligned} \tag{14}$$

Comparing (13) and (14), we have

$$a_2 = \frac{c_1}{2}, \tag{15}$$

$$a_3 = \frac{1}{4}\left(c_2 + \frac{c_1^2}{4}\right), \tag{16}$$

$$a_4 = \frac{1}{6}\left(c_3 + \frac{c_1 c_2}{4} - \frac{c_1^3}{48}\right), \tag{17}$$

$$a_5 = \frac{1}{4}\left(\frac{c_1^4}{288} + \frac{c_4}{2} + \frac{c_1 c_3}{12} - \frac{c_1^2 c_2}{24}\right). \tag{18}$$

From (5), the Third Hankel determinant can be written as

$$H_{3,1}(f) = -a_2^2 a_5 + 2a_2 a_3 a_4 - a_3^3 + a_3 a_5 - a_4^2.$$

Using (15), (16), (17) and (18), we get

$$H_{3,1}(f) = \frac{35}{27648}c_1^4 c_2 + \frac{53}{6912}c_1^3 c_3 + \frac{c_2 c_4}{32} + \frac{19}{576}c_1 c_2 c_3 - \frac{211}{331776}c_1^6 - \frac{c_2^3}{64} - \frac{3}{128}c_1^2 c_4 - \frac{13}{2304}c_1^2 c_2^2 - \frac{c_3^2}{36}.$$

After rearranging, it yields

$$\begin{aligned} H_{3,1}(f) &= \frac{211}{165888}c_1^4\left(c_2-\frac{c_1^2}{2}\right)+\frac{3}{64}c_4\left(c_2-\frac{c_1^2}{2}\right)-\frac{c_1c_3}{96}\left(c_2-\frac{c_1^2}{2}\right)+\frac{1}{165888}c_1^3(c_3-c_1c_2) \\ &+\frac{407}{165888}c_1^2\left(c_1c_3-c_2^2\right)-\frac{c_3}{36}(c_3-c_1c_2)-\frac{c_2}{64}(c_4-c_1c_3)-\frac{529}{165888}c_1^2c_2^2-\frac{c_2^3}{64}. \end{aligned}$$

Using triangle inequality along with (7), (11), (8) and (9), provide us

$$\begin{aligned} |H_{3,1}(f)| &\leq \frac{211}{165888}|c_1|^4\left(2-\frac{|c_1|^2}{2}\right)+\frac{3}{32}\left(2-\frac{|c_1|^2}{2}\right)+\frac{|c_1|}{48}\left(2-\frac{|c_1|^2}{2}\right)+\frac{1}{82944}|c_1|^3 \\ &+\frac{407}{41472}|c_1|^2+\frac{1}{9}+\frac{1}{16}+\frac{529}{41472}|c_1|^2+\frac{1}{8}. \end{aligned}$$

If we substitute $|c_1| = x \in [0,2]$, we obtain a function of variable x. Therefore, we can write

$$\begin{aligned} |H_{3,1}(f)| &\leq \frac{211}{165888}x^4\left(2-\frac{x^2}{2}\right)+\frac{3}{32}\left(2-\frac{x^2}{2}\right)+\frac{x}{48}\left(2-\frac{x^2}{2}\right)+\frac{1}{82944}x^3 \\ &+\frac{407}{41472}x^2+\frac{1}{9}+\frac{1}{16}+\frac{529}{41472}x^2+\frac{1}{8}. \end{aligned}$$

The above function attains its maximum value at $x = 0.64036035$, which is

$$|H_{3,1}(f)| \leq 0.50047781.$$

Thus, the proof is completed. □

4. Bound of $|H_{3,1}(f)|$ for the Set $\mathcal{C}_e$

Theorem 2. *Let f has the form (1) and belongs to $\mathcal{C}_e$. Then*

$$|a_2| \leq \frac{1}{2}, \tag{19}$$

$$|a_3| \leq \frac{1}{4}, \tag{20}$$

$$|a_4| \leq \frac{17}{144}, \tag{21}$$

$$|a_5| \leq \frac{7}{96}. \tag{22}$$

The first three inequalities are sharp.

Proof. If $f \in \mathcal{C}_e$, then we can write (4), in form of Schwarz function as

$$1+\frac{zf''(z)}{f'(z)} = e^{w(z)}.$$

From (1), we can write

$$\begin{aligned} 1+\frac{zf''(z)}{f'(z)} &= 1+2a_2z+\left(6a_3-4a_2^2\right)z^2+\left(12a_4-18a_2a_3+8a_2^3\right)z^3 \\ &+\left(20a_5-18a_3^2-32a_2a_4+48a_2^2a_3-16a_2^4\right)z^4+\cdots. \end{aligned} \tag{23}$$

By comparing (23) and (14), we get

$$a_2 = \frac{c_1}{4}, \tag{24}$$

$$a_3 = \frac{1}{12}\left(c_2 + \frac{c_1^2}{4}\right), \tag{25}$$

$$a_4 = \frac{1}{24}\left(\frac{c_1c_2}{4} + c_3 - \frac{c_1^3}{48}\right), \tag{26}$$

$$a_5 = \frac{1}{20}\left(\frac{c_1^4}{288} + \frac{c_4}{2} + \frac{c_1c_3}{12} - \frac{c_1^2c_2}{24}\right). \tag{27}$$

Implementing (7), in (24) and (25), we have

$$|a_2| \leq \tfrac{1}{2} \quad \text{and} \quad |a_3| \leq \tfrac{1}{4}.$$

Reshuffling (26), we have

$$|a_4| = \frac{1}{24}\left|\frac{5}{24}c_1c_2 + \frac{c_1}{24}\left(c_2 - \frac{c_1^2}{2}\right) + c_3\right|.$$

Application of triangle inequality and (7) and (11) leads us to

$$|a_4| \leq \frac{1}{24}\left\{\frac{5}{12}|c_1| + \frac{|c_1|}{24}\left(2 - \frac{|c_1|^2}{2}\right) + 2\right\}.$$

If we insert $|c_1| = x \in [0,2]$, then we get

$$|a_4| \leq \frac{1}{24}\left\{\frac{5}{12}x + \frac{x}{24}\left(2 - \frac{x^2}{2}\right) + 2\right\}.$$

The overhead function has a maximum value at $x = 2$, thus

$$|a_4| \leq \frac{17}{144}.$$

Reordering (27), we have

$$|a_5| = \frac{1}{20}\left|\frac{1}{2}\left(c_4 - \frac{c_1^2c_2}{48}\right) - \frac{c_1^2}{96}\left(c_2 - \frac{c_1^2}{3}\right) + \frac{c_1}{12}\left(c_3 - \frac{c_1c_2}{4}\right)\right|.$$

By using triangle inequality along with (7), and (8), we get

$$|a_5| \leq \frac{7}{96}.$$

Equalities are obtain if we take

$$f(z) = \int_0^z e^{J(t)}dt = z + \frac{1}{2}z^2 + \frac{1}{4}z^3 + \frac{17}{144}z^4 + \frac{19}{360}z^5 + \cdots \tag{28}$$

where

$$J(t) = \int_0^t \frac{e^x - 1}{x}dx.$$

□

Theorem 3. *If f is of the form (1) belongs to $\mathcal{C}_e$, then*

$$\left|a_3-\gamma a_2^2\right| \leq \frac{1}{6}\max\left\{1, \frac{3}{2}\left|\gamma-1\right|\right\}, \tag{29}$$

where γ is a complex number.

Proof. From (24) and (25), we get

$$\left|a_3-\gamma a_2^2\right| = \left|\frac{c_2}{12}+\frac{c_1^2}{48}-\frac{\gamma}{16}c_1^2\right|.$$

By reshuffling it, provides

$$\left|a_3-\gamma a_2^2\right| = \frac{1}{12}\left|\left(c_2-\frac{1}{2}\left(\frac{3\gamma-1}{2}\right)c_1^2\right)\right|.$$

Application of (12), leads us to

$$\left|a_3-\gamma a_2^2\right| \leq \max\left\{\frac{1}{6}, \frac{1}{12}\left|3\gamma-3\right|\right\}.$$

□

Substituting $\gamma = 1$, we obtain the following inequality.

Corollary 1. *If $f \in \mathcal{C}_e$ and has the series represntaion (1), then*

$$\left|a_3-a_2^2\right| \leq \frac{1}{6}. \tag{30}$$

Theorem 4. *If f has the form (1) belongs to $\mathcal{C}_e$, then*

$$\left|a_2a_3-a_4\right| \leq \frac{31}{288}. \tag{31}$$

Proof. Using (24), (25) and (26), we have

$$\left|a_2a_3-a_4\right| = \left|\frac{c_1c_2}{96}+\frac{7}{1152}c_1^3-\frac{c_3}{24}\right|.$$

By rearranging it, gives

$$\left|a_2a_3-a_4\right| = \left|-\frac{1}{48}\left(c_3-\frac{c_1c_2}{2}\right)-\frac{1}{48}\left(c_3-\frac{7}{24}c_1^3\right)\right|.$$

By applying triangle inequality plus (8) and (10), we get

$$\left|a_2a_3-a_4\right| \leq \left\{\frac{1}{24}+\frac{19}{288}\right\} = \frac{31}{288}.$$

□

Theorem 5. *Let $f \in \mathcal{C}_e$ be of the form (1). Then*

$$\left|a_2a_4-a_3^2\right| \leq \frac{3}{64}. \tag{32}$$

Proof. From (24), (25) and (26), we have

$$\left|a_2a_4-a_3^2\right|=\left|\frac{c_1c_3}{96}-\frac{c_1^4}{1536}-\frac{c_1^2c_2}{1152}-\frac{c_2^2}{144}\right|.$$

By reordering it, yields

$$\left|a_2a_4-a_3^2\right|=\left|\frac{c_1}{576}\left(c_3-\frac{c_1c_2}{2}\right)+\frac{c_1}{576}\left(c_3-\frac{3}{8}c_1^3\right)+\frac{1}{144}\left(c_1c_3-c_2^2\right)\right|.$$

Application of triangle inequality plus (7), (11), (10) and (9), we obtain

$$\left|a_2a_4-a_3^2\right|\leq\frac{4}{576}+\frac{7}{576}+\frac{4}{144}=\frac{3}{64}.$$

□

Theorem 6. *If $f\in\mathcal{C}_e$ and has the form* (1), *then*

$$|H_{3,1}(f)|\leq 0.0234598.$$

Proof. Using (5), the Hankel determinant of order three can be formed as;

$$H_{3,1}(f)=-a_2^2a_5+2a_2a_3a_4-a_3^3+a_3a_5-a_4^2.$$

Using (24), (25), (26) and (27), gives us

$$H_{3,1}(f)=\frac{7}{5760}c_1c_2c_3-\frac{c_3^2}{576}-\frac{c_2^3}{1728}-\frac{173}{6635520}c_1^6+\frac{23}{276480}c_1^4c_2+\frac{c_2c_4}{480}-\frac{13}{46980}c_1^2c_2^2-\frac{c_1^2c_4}{960}+\frac{23}{69120}c_1^3c_3.$$

Now, rearranging it provides

$$\begin{aligned}H_{3,1}(f)\;=\;&\frac{173}{3317760}c_1^4\left(c_2-\frac{c_1^2}{2}\right)-\frac{103}{1658880}c_1^2c_2\left(c_2-\frac{c_1^2}{2}\right)+\frac{c_4}{480}\left(c_2-\frac{c_1^2}{2}\right)\\&+\frac{11}{17280}c_1c_2\left(c_3-\frac{365}{1056}c_1c_2\right)+\frac{c_2}{1728}\left(c_1c_3-c_2^2\right)-\frac{c_3}{576}\left(c_3-\frac{23}{120}c_1^3\right).\end{aligned}$$

Application of triangle inequality plus (7), (11), (8), (10) and (9), leads us to

$$|H_{3,1}(f)|\leq\frac{173}{3317760}|c_1|^4\left(2-\frac{|c_1|^2}{2}\right)+\frac{103}{829440}|c_1|^2\left(2-\frac{|c_1|^2}{2}\right)+\frac{1}{4320}|c_1|+\frac{83}{8640}+\frac{1}{216}.$$

Now, replacing $|c_1|=x\in[0,2]$, then, we can write

$$|H_{3,1}(f)|\leq\frac{173}{3317760}x^4\left(2-\frac{x^2}{2}\right)+\frac{103}{829440}x^2\left(2-\frac{x^2}{2}\right)+\frac{1}{240}\left(2-\frac{x^2}{2}\right)+\frac{11}{4320}x+\frac{41}{2880}.$$

The above function gets its maximum at $x=0.7024858$, Therefore, we have

$$|H_{3,1}(f)|\leq 0.02345979.$$

Thus the proof is completed. □

5. Bounds of $|H_{3,1}(f)|$ for 2-Fold and 3-Fold Functions

Let $m \in \mathbb{N} = \{1, 2, \ldots\}$. If a rotation $\triangle$ about the origin through an angle $2\pi/m$ carries $\triangle$ on itself, then such a domain $\triangle$ is called m-fold symmetric. An analytic function f is m-fold symmetric in Δ, if

$$f\left(e^{2\pi i/m}z\right) = e^{2\pi i/m} f(z),\ (z \in \Delta).$$

By $\mathcal{S}^{(m)}$, we define the set of m-fold univalent functions having the following Taylor series form

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1},\ \ (z \in \Delta). \tag{33}$$

The sub-families $\mathcal{S}_e^{*(m)}$ and $\mathcal{C}_e^{(m)}$ of $\mathcal{S}^{(m)}$ are the sets of m-fold symmetric starlike and convex functions respectively associated with exponential functions. More intuitively, an analytic function f of the form (33), belongs to the families $\mathcal{S}_e^{*(m)}$ and $\mathcal{C}_e^{(m)}$, if and only if

$$\frac{zf'(z)}{f(z)} = \exp\left(\frac{p(z)-1}{p(z)+1}\right),\ p \in \mathcal{P}^{(m)}, \tag{34}$$

$$1 + \frac{zf''(z)}{f'(z)} = \exp\left(\frac{p(z)-1}{p(z)+1}\right),\ p \in \mathcal{P}^{(m)}. \tag{35}$$

where the set $\mathcal{P}^{(m)}$ is defined by

$$\mathcal{P}^{(m)} = \left\{ p \in \mathcal{P} : p(z) = 1 + \sum_{k=1}^{\infty} c_{mk} z^{mk},\ \ (z \in \Delta) \right\}. \tag{36}$$

Here we prove some theorems related to 2-fold and 3-fold symmetric functions.

Theorem 7. *If $f \in \mathcal{S}_e^{*(2)}$ and has the form (33), then*

$$|H_{3,1}(f)| \leq \frac{1}{8}.$$

Proof. Let $f \in \mathcal{S}_e^{*(2)}$. Then, there exists a function $p \in \mathcal{P}^{(2)}$, such that

$$\frac{zf'(z)}{f(z)} = \exp\left(\frac{p(z)-1}{p(z)+1}\right).$$

Using the series form (33) and (36), when $m = 2$ in the above relation, we can get

$$a_3 = \frac{c_2}{4}, \tag{37}$$

$$a_5 = \frac{c_4}{8}. \tag{38}$$

Now,

$$H_3(f) = a_3 a_5 - a_3^3.$$

Utilizing (37) and (38), we get

$$H_{3,1}(f) = -\frac{c_2^3}{64} + \frac{c_2 c_4}{32}.$$

By rearranging, it yields

$$H_{3,1}(f) = \frac{c_2}{32}\left(c_4 - \frac{c_2^2}{2}\right).$$

Using triangle inequality long with (8) and (7), gives us

$$|H_{3,1}(f)| \leq \frac{1}{8}.$$

Hence, the proof is done. □

Theorem 8. *If $f \in \mathcal{S}_e^{*(3)}$ and has the series form (33), then*

$$|H_{3,1}(f)| \leq \frac{1}{9}.$$

This result is sharp for the function

$$f(z) = \exp\left(\int_0^z \frac{e^{x^3}}{x} dx\right) = z + \frac{1}{3}z^4 + \frac{5}{36}z^7 + \cdots \tag{39}$$

Proof. As, $f \in \mathcal{S}_e^{*(3)}$, therefore there exists a function $p \in \mathcal{P}^{(3)}$, such that

$$\frac{zf'(z)}{f(z)} = \exp\left(\frac{p(z)-1}{p(z)+1}\right).$$

Utilizing the series form (33) and (36), when $m = 3$ in the above relation, we can obtain

$$a_4 = \frac{c_3}{6}.$$

Then,

$$H_{3,1}(f) = -a_4^2 = -\frac{c_3^2}{36}.$$

Utilizing (7) and triangle inequality, we have

$$|H_{3,1}(f)| \leq \frac{1}{9}.$$

Thus the proof is ended. □

Theorem 9. *Let $f \in \mathcal{C}_e^{(2)}$ and has the form given in (33). Then*

$$|H_{3,1}(f)| \leq \frac{1}{120}.$$

Proof. As, $f \in \mathcal{C}_e^{(2)}$, then there exists a function $p \in \mathcal{P}^{(2)}$, such that

$$1 + \frac{zf''(z)}{f'(z)} = \exp\left(\frac{p(z)-1}{p(z)+1}\right).$$

Utilizing the series form (33) and (36), when $m = 2$ in the above relation, we can obtain

$$a_3 = \frac{c_2}{12}, \tag{40}$$

$$a_5 = \frac{c_4}{40}. \tag{41}$$

$$H_{3,1}(f) = a_3 a_5 - a_3^3.$$

Using (40) and (41), we have

$$H_{3,1}(f) = -\frac{c_2^3}{1728} + \frac{c_2 c_4}{480}.$$

Now, reordering the above equation, we obtain

$$H_3(f) = \frac{c_2}{480}\left(c_4 - \frac{5}{18}c_2^2\right).$$

Application of (7), (8) and triangle inequality, leads us to

$$|H_{3,1}(f)| \leq \frac{1}{120}.$$

Thus, the required result is completed. □

Theorem 10. *If $f \in \mathcal{C}_e^{(3)}$ and has the form given in (33), then*

$$|H_{3,1}(f)| \leq \frac{1}{144}. \tag{42}$$

This result is sharp for the function

$$f(z) = \int_0^z e^{I(t)}dt = z + \frac{1}{12}z^4 + \frac{5}{252}z^7 + \cdots \tag{43}$$

where

$$I(t) = \int_0^t \frac{e^{x^3} - 1}{x}dx.$$

Proof. Let, $f \in \mathcal{C}_e^{(3)}$. Then there exists a function $p \in \mathcal{P}^{(3)}$, such that

$$1 + \frac{zf''(z)}{f'(z)} = \exp\left(\frac{p(z) - 1}{p(z) + 1}\right).$$

Utilizing the series form (33) and (36), when $m = 3$ in the above relation, we can obtain

$$a_4 = \frac{c_3}{24}.$$

Then,

$$H_{3,1}(f) = -\frac{c_3^2}{576}.$$

Implementing (7) and triangle inequality, we have

$$|H_{3,1}(f)| \leq \frac{1}{144}.$$

Hence, the proof is done. □

6. Conclusions

In this article, we studied Hankel determinant $H_{3,1}(f)$ for the families $\mathcal{S}_e^*$ and $\mathcal{C}_e$ whose image domain are symmetric about the real axis. Furthermore, we improve the bound of third Hankel determinant for the family $\mathcal{S}_e^*$. These bounds are also discussed for 2-fold symmetric and 3-fold symmetric functions.

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