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$F(R,G)$ Cosmology through Noether Symmetry Approach

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Abstract: The $F(R,G)$ theory of gravity, where R is the Ricci scalar and G is the Gauss-Bonnet invariant, is studied in the context of existence the Noether symmetries. The Noether symmetries of the point-like Lagrangian of $F(R,G)$ gravity for the spatially flat Friedmann-Lemaitre-Robertson-Walker cosmological model is investigated. With the help of several explicit forms of the $F(R,G)$ function it is shown how the construction of a cosmological solution is carried out via the classical Noether symmetry approach that includes a functional boundary term. After choosing the form of the $F(R,G)$ function such as the case $(i): F(R,G) = f_0 R^n + g_0 G^m$ and the case $(ii): F(R,G) = f_0 R^n G^m$, where n and m are real numbers, we explicitly compute the Noether symmetries in the vacuum and the non-vacuum cases if symmetries exist. The first integrals for the obtained Noether symmetries allow to find out exact solutions for the cosmological scale factor in the cases (i) and (ii). We find several new specific cosmological scale factors in the presence of the first integrals. It is shown that the existence of the Noether symmetries with a functional boundary term is a criterion to select some suitable forms of $F(R,G)$. In the non-vacuum case, we also obtain some extra Noether symmetries admitting the equation of state parameters $w \equiv p/\rho$ such as $w = -1, -2/3, 0, 1$ etc.

Keywords: Noether symmetry approach; FLRW spacetime; action integral; variational principle; first integral; modified theories of gravity; Gauss-Bonnet cosmology

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1. Introduction

Recent observational data indicate that the current expansion of the universe is accelerating [1–8], not only expanding. Then this acceleration is explained by the existence of a dark energy, which could result from a cosmological constant Λ as the simplest candidate with the equation of state parameter $w_\Lambda = -1$, or may also be explained in the context of modified gravity models. The nature and origin of the dark energy has not been persuasively explained yet. In addition to the cosmological constant, there are different kinds of candidates for dark energy such as quintessence or phantom in the literature, and it is not even clear what type of candidates to the dark energy occur in the present universe. Therefore, there have been a number of attempts [9–15] to modify gravity to explain the origin of dark energy.

A possible modification of the standard general relativistic gravitational Lagrangian includes a wider number of curvature invariants $R, R_{ij}R^{ij}$ and $R_{ijkl}R^{ijkl}$ among others. In the so-called Gauss-Bonnet (GB) gravity theories the gravitational Lagrangian consists of a $F(R,G)$ function, where the GB invariant G is defined as $G = R^2 - 4R_{ij}R^{ij} + R_{ijkl}R^{ijkl}$. Considering the GB invariant G in dynamical equations one can recover all the curvature budget coming from the Riemann tensor. Due to of the fact that the GB invariant comes out from defining quantum fields in curved spacetimes, it should be important to take it in the context of the extended theories of gravity. It is shown in [13] that the quintessence paradigm can be recovered in the framework of $F(R,G)$ theories of gravity. The $F(R,G)$ gravity theories are generalizations $f(R)$ and $f(G)$ theory of gravities which are offered by higher order gravities, and use combinations of higher order curvature invariants constructed from the Ricci and Gauss-Bonnet scalars. In [14], some classes of $F(R,G)$ gravity have been studied with respect to the successful realization of the dark energy and of the inflationary era. We refer to readers the latest review [15] on developments of modified gravity in cosmology, emphasizing on inflation, bouncing cosmology and late-time acceleration era.

If a Lagrangian $\mathcal{L}$ for a given dynamical system admits any symmetry, this property should strongly be related with Noether symmetries that describe physical features of differential equations possessing a Lagrangian $\mathcal{L}$ in terms of first integrals admitted by them [16,17]. This can actually be seen in two ways. Firstly, one can consider a *strict Noether symmetry approach* [18–21] which yields $\pounds_{\mathbf{X}}\mathcal{L} = 0$, where $\pounds_{\mathbf{X}}$ is the Lie derivative operator along $\mathbf{X}$. On the other side, one could use the *classical Noether symmetry approach* with a functional term [22–25] which is a generalization of the strict Noether symmetry approach in the sense that the Noether symmetry equation includes a divergence of a functional boundary term. The classical Noether symmetry approach was originally established by Emmy Noether [26] and it gives a connection between a Noether symmetry and the existence of a first integral expressed in a simple form. Not only the classical Noether symmetries but also the strict ones are useful in a variety of problems arising from physics and applied mathematics. Both types of symmetries lead to the first integrals. Which type of symmetry works, i.e., gives any conserved quantity, in the first instance this is what is important. The classical Noether symmetries are directly related with the conserved quantities (first integrals) or conservation laws [17]. The strict Noether symmetry approach represents how Noether's theorem and cyclic variables are related. It is known that the conserved quantities are also related to the existence of cyclic variables into the dynamics by the strict Noether symmetry. However, it is usually required a clever choice of cyclic variables because of that the equations for the change of coordinates have not a unique solution which is also not well defined on the whole space, and thus it is not unique to find those of the cyclic variables (see References [27] for details). Furthermore, we refer to the interested readers the recent review on symmetries in differential equations [28].

The cosmological principle assume that the universe is homogeneous and isotropic in large scale structure and the geometrical model that satisfies these properties is Friedmann-Lemaitre-Robertson-Walker (FLRW) spacetime. In [19], it has been discussed the strict Noether symmetry approach for spatially flat FLRW spacetime in GB cosmology, where it was pointed out that the existence of Noether symmetries is capable of selecting suitable $F(R,G)$ models to integrate dynamics by the identification of suitable cyclic variables. After this work, the classical Noether symmetries of flat FLRW spacetime have been computed by [25], where the authors were used Noether symmetries as a geometric criterion to select the form of $F(R,G)$ function. Due to the richness of the classical Noether symmetry approach, we deduced throughout this study that it is better to use the classical Noether symmetry approach to find Noether symmetries in $F(R,G)$ gravity as in [25], rather that the approach used in [19]. If there exists any Noether symmetry with a selection of physically interesting forms of $F(R,G)$ function, then this allows us to write out the constants of motion which reduce dynamics. Furthermore, the reduced dynamics results exactly solvable cosmological model by a straightforward way. In fact, choosing an appropriate $F(R,G)$ Lagrangian, it is possible to find out conserved Noether currents which will be useful to solve dynamics. This approach is very powerful due to the fact that it allows us to find a closed system of equations, where we do not need to impose the particular form of $F(R,G)$ which is selected by the classical Noether symmetry itself. To this aim, it is possible to consider flat FLRW background metric and demonstrate that it is possible to find exact solutions via the Noether Symmetry Approach. In this study we again underline the generality of Noether's Theorem in its original form by considering the standard cosmological model.

This paper is organized as follows. In the following section, we will present an analysis of the classical Noether symmetry approach including a boundary function for the point-like $F(R,G)$ Lagrangian according to the spatially flat FLRW background. In Section 3, we will apply the classical Noether theorem to the $F(R,G)$ Lagrangian obtained in Section 2 for the flat FLRW model. In Section 4, we classify the Noether symmetries with respect to some specific forms of $F(R,G)$, and search the cosmological solutions of $F(R,G)$ gravity by considering both the vacuum and the non-vacuum cases. Finally, in Section 5, we will provide a summary of the main results obtained in the paper.

2. $F(R,G)$ Gravity

In this section we briefly present the general formalism of $F(R,G)$ gravity. The action for $F(R,G)$ gravity is given by

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2\kappa^2}F(R,G) + \mathcal{L}_m\right], \tag{1}$$

where $\kappa^2 = 8\pi G_N$, G_N is the Newton constant and $\mathcal{L}_m$ represents the matter Lagrangian. Variation of the action (1) with respect to the metric tensor g_{ij} we obtain the modified field equations

$$\begin{aligned} F_R G_{ij} &= \kappa^2 T^m_{ij} + \frac{1}{2}g_{ij}(F - RF_R) + \nabla_i\nabla_j F_R - g_{ij}\Box F_R \\ &+ F_G\Big(-2RR_{ij} + 4R_{ik}R^k_j - 2R_i^{\ klm}R_{jklm} + 4g^{kl}g^{mn}R_{ikjm}R_{ln}\Big) \\ &+ 2\left(\nabla_i\nabla_j F_G\right)R - 2g_{ij}\left(\Box F_G\right)R + 4\left(\Box F_G\right)R_{ij} - 4\left(\nabla_k\nabla_i F_G\right)R_j^{\ k} - 4\left(\nabla_k\nabla_j F_G\right)R_i^{\ k} \\ &+ 4g_{ij}\left(\nabla_k\nabla_l F_G\right)R^{kl} - 4\left(\nabla_l\nabla_n F_G\right)g^{kl}g^{mn}R_{ikjm}, \end{aligned} \tag{2}$$

where we have defined the following expressions

$$F_R \equiv \frac{\partial F(R,G)}{\partial R}, \quad F_G \equiv \frac{\partial F(R,G)}{\partial G}. \tag{3}$$

In the above field equations, ∇_i is the covariant derivative operator associate with g_{ij}, $\Box \equiv g^{ij}\nabla_i\nabla_j$ is the covariant d'Alembertian operator, and T^m_{ij} describes the ordinary matter. It is clear from the field Equation (2) that the form of $F(R,G)$ determine the dynamical behaviour of the theory.

In this study, we consider the spatially flat FLRW metric

$$ds^2 = -dt^2 + a(t)^2\left(dx^2 + dy^2 + dz^2\right) \tag{4}$$

where $a(t)$ is the scale factor of the Universe. Then, the Hubble parameter H is usually defined by $H \equiv \dot{a}/a$, and R and G become

$$R = 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) = 6(\dot{H} + 2H^2), \quad G = 24\frac{\dot{a}^2\ddot{a}}{a^3} = 24H^2\left(\dot{H} + H^2\right), \tag{5}$$

where the overdot denotes a derivative with respect to the time coordinate, t. For a perfect fluid matter with comoving observer $u_i = \delta_i^0$, the energy momentum tensor is $T_{ij} = (\rho + p)u_i u_j + pg_{ij}$, where ρ is the energy density and p is the isotropic pressure measured by the observer u_i. Let us assume that the matter fluid will be given under the form of a perfect fluid with the equation of state $p = w\rho$ satisfying the standard continuity equation $\dot{\rho} + 3(1+w)\rho\dot{a}/a = 0$ which yields a solution $\rho = \rho_{m0}a^{-3(1+w)}$, where ρ_{m0} is the energy density of the present universe, and w is a constant parameter. Thus, in the flat FLRW background with a perfect fluid matter, the field Equation (2) for the $F(R,G)$ gravity are given by

$$3F_R\frac{\dot{a}^2}{a^2} = \kappa^2\rho + \frac{1}{2}(RF_R + GF_G - F) - 3\dot{F}_R\frac{\dot{a}}{a} - 12\dot{F}_G\frac{\dot{a}^3}{a^3}, \tag{6}$$

$$F_R\left(\frac{2\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) = -\kappa^2 p + \frac{1}{2}(RF_R + GF_G - F) - 2\dot{F}_R\frac{\dot{a}}{a} - \ddot{F}_R - 4\frac{\dot{a}}{a}\left(\frac{\dot{a}}{a}\ddot{F}_G + \frac{2\ddot{a}}{a}\dot{F}_G\right). \tag{7}$$

In terms of the Hubble parameter H, the gravitational field Equations (6) and (7) for $F(R,G)$ gravity have the following form

$$H^2 = \frac{\kappa^2}{3}\rho_{eff}, \qquad 2\dot{H} + 3H^2 = -\kappa^2 p_{eff}, \tag{8}$$

where ρ_{eff} and p_{eff} are respectively the effective energy density and pressure of the universe, which are defined as

$$\rho_{eff} \equiv \frac{1}{F_R}\left\{\rho + \frac{1}{2\kappa^2}\left[RF_R + GF_G - F - 6H\dot{F}_R - 24H^3\dot{F}_G\right]\right\}, \tag{9}$$

$$p_{eff} \equiv \frac{1}{F_R}\left\{p + \frac{1}{2\kappa^2}\left[F - RF_R - GF_G + 4H\dot{F}_R + 2\ddot{F}_R + 16H(\dot{H} + H^2)\dot{F}_G + 8H^2\ddot{F}_G\right]\right\}. \tag{10}$$

Here we observe from (8) that $\rho_{eff} + p_{eff} = -\frac{2}{\kappa^2}\dot{H}$.

3. Noether Symmetry Approach

Recently the strict Noether symmetries of GB cosmology for the flat FLRW spacetime have been calculated, and choosing some functional form of the $F(R,G)$, the Noether symmetries related to these functional forms have been achieved [19]. Afterwards, the classical Noether symmetries have also been calculated by [25]. Both of these studies were performed in the vacuum case. In this work, after reviewing the vacuum case, we aim to generalize these studies to the non-vacuum case using the classical Noether symmetry approach described below.

The Noether symmetry generator for any point-like Lagrangian $\mathcal{L}$ is

$$\mathbf{X} = \xi(t,a,R,G)\frac{\partial}{\partial t} + \eta^1(t,a,R,G)\frac{\partial}{\partial a} + \eta^2(t,a,R,G)\frac{\partial}{\partial R} + \eta^3(t,a,R,G)\frac{\partial}{\partial G}, \tag{11}$$

if there exists a function $K(t,a,R,G)$ and the Noether symmetry condition

$$\mathbf{X}^{[1]}\mathcal{L} + \mathcal{L}(D_t\xi) = D_tK \tag{12}$$

is satisfied, where $D_t = \frac{\partial}{\partial t} + \dot{q}^i\frac{\partial}{\partial q^i}$ is the total derivative operator and $\mathbf{X}^{[1]}$ is the first prolongation of Noether symmetry generator $\mathbf{X}$, i.e.

$$\mathbf{X}^{[1]} = \mathbf{X} + \dot{\eta}^i\left(t, q^i, \dot{q}^i\right)\frac{\partial}{\partial q^i} \tag{13}$$

where $\dot{\eta}^i(t,q^k,\dot{q}^k) = D_t\eta^i - \dot{q}^iD_t\xi$, $q^i = \{a,R,G\}$ are the generalized coordinates in the three-dimensional configuration space $Q \equiv \{q^i, i = 1,2,3\}$ of the Lagrangian, whose tangent space is $TQ \equiv \{q^i, \dot{q}^i\}$. The energy functional $E_{\mathcal{L}}$ or the Hamiltonian of the Lagrangian $\mathcal{L}$ is defined by

$$E_{\mathcal{L}} = \dot{q}^i\frac{\partial\mathcal{L}}{\partial\dot{q}^i} - \mathcal{L}. \tag{14}$$

Using above definition of energy functional, the corresponding Noether flow I, which is a constant called the first integral of motion, has the expression

$$I = -\xi E_{\mathcal{L}} + \eta^i\frac{\partial\mathcal{L}}{\partial\dot{q}^i} - K, \tag{15}$$

which is a *conserved quantity*. The Noether flow (15) satisfies the conservation relation $D_tI = 0$.

It is obviously seen from a general point of view that R and G are functions of $a, \dot{a}$ and $\ddot{a}$, which yields non-canonical dynamics. The Lagrange multipliers plays a main role so as to get a canonical point-like Lagrangian [29]. Using this key future in [19], it has been accomplished that the point-like Lagrangian for $F(R,G)$ gravity becomes canonical with suitable Lagrange multipliers, where both R and G behave like effective scalar fields. We left the details for finding a canonical point-like Lagrangian by Lagrange multipliers method to the Reference [19]. For the spatially flat FLRW spacetime (4), the Lagrangian for the action of $F(R,G)$ gravity (1) has the form

$$\mathcal{L} = -6F_R a\dot{a}^2 - 6a^2\dot{a}\dot{F}_R - 8\dot{F}_G\dot{a}^3 + a^3(F - RF_R - GF_G) - 2\kappa^2\rho_{m0}a^{-3w}, \tag{16}$$

where $\dot{F}_R = F_{RR}\dot{R} + F_{RG}\dot{G}$ and $\dot{F}_G = F_{GR}\dot{R} + F_{GG}\dot{G}$. By variation of the above Lagrangian with respect to the configuration space variables a, R and G, we find respectively that

$$F_R\left(\frac{2\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) + \frac{2\dot{a}}{a}\dot{F}_R + \ddot{F}_R - \frac{4\dot{a}}{a}\left(\frac{2\ddot{a}}{a}\dot{F}_G + \frac{\dot{a}}{a}\ddot{F}_G\right) + \frac{1}{2}(RF_R + GF_G - F) = \kappa^2 p, \tag{17}$$

$$6F_{RR}\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} - \frac{R}{6}\right) - F_{GR}\left(24\frac{\dot{a}^2\ddot{a}}{a^3} - G\right) = 0, \tag{18}$$

$$6F_{GR}\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} - \frac{R}{6}\right) - F_{GG}\left(24\frac{\dot{a}^2\ddot{a}}{a^3} - G\right) = 0, \tag{19}$$

in which the Equation (17) is equivalent to the field Equation (7). Then we note that R and G coincides with the definitions of the Ricci scalar and Gauss-Bonnet invariant given by (5), respectively. Now, we calculate the energy functional $E_{\mathcal{L}}$ for the Lagrangian density (16) which has the form

$$E_{\mathcal{L}} = 2a^3\left[3F_R\frac{\dot{a}^2}{a^2} + 3\frac{\dot{a}}{a}\dot{F}_R + 12\frac{\dot{a}^3}{a^3}\dot{F}_G - \frac{1}{2}(RF_R + GF_G - F) - \kappa^2\rho\right]. \tag{20}$$

It is explicitly seen that the energy function $E_{\mathcal{L}}$ vanishes due to the $(0,0)$-field Equation (6).

Let us consider the Noether symmetry conditions (12) for the point-like Lagrangian (16) to seek the dependent variables $\xi, \eta^1, \eta^2, \eta^3$ which will be solved in order that the Lagrangian (16) would admit any Noether symmetry (11). For the flat FLRW spacetime (4), the Noether symmetry conditions (12) yield 27 partial differential equations as the following

$$\begin{aligned}
&F_{GR}\xi_{,a} = 0, \quad F_{GG}\xi_{,a} = 0, \quad F_{GR}\xi_{,R} = 0, \quad F_{RR}\xi_{,R} = 0, \quad F_{GR}\xi_{,G} = 0, \quad F_{GG}\xi_{,G} = 0,\\
&F_{RR}\eta^1_{,R} = 0, \quad F_{GR}\eta^1_{,R} = 0, \quad F_{GR}\eta^1_{,G} = 0, \quad F_{GG}\eta^1_{,G} = 0, \; F_{GG}\xi_{,R} + F_{GR}\xi_{,G} = 0,\\
&F_{GR}\xi_{,R} + F_{RR}\xi_{,G} = 0, \; F_{GR}\eta^1_{,R} + F_{RR}\eta^1_{,G} = 0, \; F_{GG}\eta^1_{,R} + F_{GR}\eta^1_{,G} = 0, \; F_{GR}\eta^2_{,a} + F_{GG}\eta^3_{,a} = 0,\\
&6a\left(2F_R\eta^1_{,t} + aF_{RR}\eta^2_{,t} + aF_{GR}\eta^3_{,t}\right) + V\xi_{,a} + K_{,a} = 0, \; 6a^2F_{RR}\eta^1_{,t} + V\xi_{,R} + K_{,R} = 0,\\
&6a^2F_{GR}\eta^1_{,t} + V\xi_{,G} + K_{,G} = 0, \quad 4\left(F_{GR}\eta^2_{,t} + F_{GG}\eta^3_{,t}\right) - 3aF_R\xi_{,a} = 0,\\
&4F_{GR}\eta^1_{,t} - a\left(F_R\xi_{,R} + aF_{RR}\xi_{,a}\right) = 0, \quad 4F_{GG}\eta^1_{,t} - a\left(F_R\xi_{,G} + aF_{GR}\xi_{,a}\right) = 0,\\
&F_{GRR}\eta^2 + F_{GGR}\eta^3 + F_{GG}\eta^3_{,R} + F_{GR}\left(3\eta^1_{,a} + \eta^2_{,R} - 3\xi_{,t}\right) = 0,\\
&F_{GGR}\eta^2 + F_{GGG}\eta^3 + F_{GR}\eta^2_{,G} + F_{GG}\left(3\eta^1_{,a} + \eta^3_{,G} - 3\xi_{,t}\right) = 0,\\
&F_R\left(\frac{\eta^1}{a} + 2\eta^1_{,a} - \xi_{,t}\right) + F_{RR}\eta^2 + F_{GR}\eta^3 + a\left(F_{RR}\eta^2_{,a} + F_{GR}\eta^3_{,a}\right) = 0,\\
&F_{RR}\left(2\frac{\eta^1}{a} + \eta^1_{,a} + \eta^2_{,R} - \xi_{,t}\right) + \frac{2}{a}F_R\eta^1_{,R} + F_{RRR}\eta^2 + F_{RRG}\eta^3 + F_{GR}\eta^3_{,R} = 0,\\
&F_{GR}\left(2\frac{\eta^1}{a} + \eta^1_{,a} + \eta^3_{,G} - \xi_{,t}\right) + \frac{2}{a}F_R\eta^1_{,G} + F_{RRG}\eta^2 + F_{GGR}\eta^3 + F_{RR}\eta^2_{,G} = 0,\\
&V_{,a}\eta^1 + V_{,R}\eta^2 + V_{,G}\eta^3 + V\xi_{,t} + K_{,t} = 0,
\end{aligned} \tag{21}$$

where V is defined as $V(a,R,G) = a^3(RF_R + GF_G - F) + 2\kappa^2\rho_{m0}a^{-3w}$, which can be considered as an effective potential for the $F(R,G)$ gravity. Here , R and G act as two different scalar fields whose regimes can lead different phases of the cosmological evolution.

We note here that $\eta^1 = 0, \eta^2 = 0, \eta^3 = 0, K = const.$ and $\xi = const.$ are trivial solutions for the Noether symmetry Equation (21). This result implies that any form of $F(R,G)$ function admits the trivial Noether symmetry $\mathbf{X}_1 = \partial/\partial_t$, i.e., energy conservation, whose Noether first integral or the Hamiltonian of the system vanishes, $I = -E_{\mathcal{L}} = 0$. In the following section, we consider the form of $F(R,G)$ to find the corresponding Noether symmetries and solutions to the corresponding first integrals for each of the vacuum an the non-vacuum cases.

4. Noether Symmetries and Cosmological Solutions

Using the symmetry condition (12) to the point-like Lagrangian (16), which will fix the form of $F(R,G)$, several different cases were classified in [25] according to whether the derivative F_{RG} vanishes or not. If $F_{RG} = 0$, it means $F(R,G) = f(R) + g(G)$, which is considered as the **case (i)** below, taking $f(R) = f_0R^n$ and $g(G) = g_0G^m$. Otherwise, if $F_{RG} \neq 0$, we will take the form of $F(R,G)$ function as the **case (ii)**, i.e., $F(R,G) = f_0R^nG^m$, where n and m are real numbers.

4.1. Vacuum Case

In this case, we assume the vacuum where $L_m = 0$, i.e., $\rho_{m0} = 0$.

Case (i): $F(R,G) = f(R) + g(G)$. For this case, we choose the functional forms $f(R) = f_0R^n$ and $g(G) = g_0G^m$. Then we examine the following subcases where the powers n and m are fixed to some values, which are compatible with the Noether symmetries.

- $n = m = 1$: Then, the Noether symmetry Equation (21) imply that

$$\xi = c_1 + c_2t + c_3\frac{t^2}{2}, \; \eta^1 = \frac{a}{3}(c_2 + c_3t) + \frac{c_4t + c_5}{\sqrt{a}}, \quad \eta^2, \eta^3 \text{ arbitrary}, \tag{22}$$

$$K = -\frac{4}{3}f_0c_3a^3 - 8f_0c_4a^{\frac{3}{2}}. \tag{23}$$

This solution to Equation (21) was given in [25] by (55) together with non-trivial function (23). Thus, the Noether symmetry generators from the solution (22) together with (23) take the following forms:

$$\mathbf{X}_1 = \frac{\partial}{\partial t}, \quad \mathbf{X}_2 = \frac{1}{\sqrt{a}}\frac{\partial}{\partial a}, \quad \mathbf{X}_3 = 3t\frac{\partial}{\partial t} + a\frac{\partial}{\partial a} \quad \text{with } K = 0, \tag{24}$$

$$\mathbf{X}_4 = \frac{3t^2}{2}\frac{\partial}{\partial t} + ta\frac{\partial}{\partial a} \;\text{ with } K = -4f_0a^3; \quad \mathbf{X}_5 = \frac{t}{\sqrt{a}}\frac{\partial}{\partial a} \text{ with } K = -8f_0a^{\frac{3}{2}}, \tag{25}$$

which give the non-vanishing commutators

$$[\mathbf{X}_1, \mathbf{X}_3] = 3\mathbf{X}_1, \quad [\mathbf{X}_1, \mathbf{X}_4] = \mathbf{X}_3, \quad [\mathbf{X}_1, \mathbf{X}_5] = \mathbf{X}_2, \tag{26}$$

$$[\mathbf{X}_2, \mathbf{X}_3] = \frac{3}{2}\mathbf{X}_2, \quad [\mathbf{X}_2, \mathbf{X}_4] = \mathbf{X}_5, \quad [\mathbf{X}_3, \mathbf{X}_4] = 3\mathbf{X}_4, \quad [\mathbf{X}_3, \mathbf{X}_5] = \frac{3}{2}\mathbf{X}_5. \tag{27}$$

The first integrals of the above vector fields are the Hamiltonian, $I_1 = -E_{\mathcal{L}} = 0$, and the quantities

$$I_2 = -12f_0\sqrt{a}\dot{a}, \quad I_3 = -12a^2\dot{a}, \quad I_4 = -4f_0a^2\left(3t\dot{a} - a\right), \quad I_5 = -4f_0\sqrt{a}\left(3t\dot{a} - 2a\right). \tag{28}$$

Here we note that it is only found one Noether symmetry in Reference [19] which is $\mathbf{X}_2$ given in (24), and the remaining ones are not appeared in this reference. It follows from $E_{\mathcal{L}} = 0$ that

$\dot{a} = 0$, that is, $a(t) = a_0 = constant$, which is the Minkowski spacetime recovered in vacuum and so $I_2 = I_3 = 0$, $I_4 = 4f_0 a_0^3$ and $I_5 = 8f_0 a_0^{3/2}$ by (28).

- n arbitrary (with $n \neq 0, 1, \frac{3}{2}, \frac{7}{8}$), $m = 1$: For this case, it follows from (21) that there are *two* Noether symmetries,

$$\mathbf{X}_1 = \frac{\partial}{\partial t}, \quad \mathbf{X}_2 = 3t\frac{\partial}{\partial t} + (2n-1)a\frac{\partial}{\partial a} - 6R\frac{\partial}{\partial R}, \tag{29}$$

which gives the non-vanishing Lie algebra $[\mathbf{X}_1, \mathbf{X}_2] = 3\mathbf{X}_1$. The first integrals are $I_1 = -E_{\mathcal{L}} = 0$, that means

$$\frac{\dot{a}^2}{a^2} + (n-1)\frac{\dot{a}\dot{R}}{aR} - \frac{(n-1)}{6n}R = 0, \tag{30}$$

by using (20), and

$$I_2 = 6f_0 n a^3 R^{n-1}\left[2(n-2)\frac{\dot{a}}{a} - (n-1)(2n-1)\frac{\dot{R}}{R}\right], \tag{31}$$

for $\mathbf{X}_1$ and $\mathbf{X}_2$, respectively. Then, solving the first integral (31) in terms of a, one gets

$$a(t) = R^{\frac{(n-1)(2n-1)}{2(n-2)}}\left[a_0 + \frac{I_2}{4f_0 n(n-2)}\int R^{\frac{(n-1)(8n-7)}{2(2-n)}} dt\right]^{\frac{1}{3}}, \tag{32}$$

where a_0 is an integration constant, and $n \neq 2$. Substituting R given in (5) to the Equation (30), it follows from the integration of resulting equation with respect to t that

$$a^{-\frac{1}{n-1}}\dot{a} = a_1 R^n, \tag{33}$$

which is a constraint equation for a, and it gives

$$\frac{\dot{a}}{a} = a_1 R^2, \tag{34}$$

for $n = 2$, where a_1 is a constant of integration. Thus, the curvature scalar R given by (5) together with the relation (34) becomes

$$\dot{R} + a_1 R^3 = \frac{1}{12a_1}, \tag{35}$$

which is Abel's differential equation of first kind, and has the following solution

$$R(t) = 4a_1^2(a_1 t - a_2)\left[1 + \frac{4a_1^2(a_1 t - a_2)}{\Delta(t)}\right] + \Delta(t), \tag{36}$$

where a_2 is an integration constant, and $\Delta(t)$ is defined as

$$\Delta(t) = a_1^{2/3}\left[64a_1^4(a_1 t - a_2)^3 - 3 + 3\sqrt{2}\sqrt{3 - 64a_1^2(a_1 t - a_2)^3}\right]^{\frac{1}{3}}.$$

The first integral (31) for $n = 2$ yields $I_2 = -36f_0 a^3\dot{R}$, and then the Equation (35) gives rise to the scale factor as

$$a(t) = \left[\frac{a_1 I_2}{3f_0(12a_1^2 R(t)^3 - 1)}\right]^{\frac{1}{3}}. \tag{37}$$

- $n = \frac{3}{2}, m = 1$: This case admits extra Noether symmetries as pointed out in Reference [30]. The existence of the extra Noether symmetries put even further first integrals which raise the possibility to find an exact solution. The Noether symmetries obtained from (21) are $\mathbf{X}_1$ and

$$\mathbf{X}_2 = \frac{1}{a}\frac{\partial}{\partial a} - \frac{2R}{a^2}\frac{\partial}{\partial R}, \quad \mathbf{X}_3 = 3t\frac{\partial}{\partial t} + 2a\frac{\partial}{\partial a} - 6R\frac{\partial}{\partial R}, \quad \mathbf{X}_4 = t\mathbf{X}_2 \quad \text{with } K = -9f_0 a\sqrt{R}, \tag{38}$$

with the non-vanishing Lie brackets

$$[\mathbf{X}_1, \mathbf{X}_3] = 3\mathbf{X}_1, \quad [\mathbf{X}_1, \mathbf{X}_4] = \mathbf{X}_2, \quad [\mathbf{X}_2, \mathbf{X}_3] = 4\mathbf{X}_2, \quad [\mathbf{X}_3, \mathbf{X}_4] = -\mathbf{X}_4. \tag{39}$$

The corresponding Noether constants are $I_1 = -E_{\mathcal{L}} = 0$, which give

$$\frac{\dot{a}^2}{a^2} + \frac{\dot{a}\dot{R}}{2aR} - \frac{R}{18} = 0, \tag{40}$$

and

$$I_2 = -9f_0 a\sqrt{R}\left(\frac{\dot{a}}{a} + \frac{\dot{R}}{2R}\right), \quad I_3 = -9f_0 a^3\sqrt{R}\left(\frac{\dot{a}}{a} + \frac{\dot{R}}{R}\right), \quad I_4 = I_2 t + 9f_0 a\sqrt{R}. \tag{41}$$

Using above first integrals, we find the scale factor and the Ricci scalar as follows:

$$a(t) = \frac{1}{6\bar{I}_2}\sqrt{(\bar{I}_2 t - \bar{I}_4)^4 + 18\bar{I}_2\bar{I}_3}, \quad R(t) = \frac{36\bar{I}_2^2(\bar{I}_2 t - \bar{I}_4)^2}{(\bar{I}_2 t - \bar{I}_4)^4 + 18\bar{I}_2\bar{I}_3}, \tag{42}$$

where it is defined $\bar{I}_2 = -9f_0 I_2, \bar{I}_3 = -9f_0 I_3$ and $\bar{I}_4 = -9f_0 I_4$.

- $n = \frac{7}{8}$, $m = 1$: In addition to $\mathbf{X}_1$, this case includes extra *two* Noether symmetries [30]

$$\mathbf{X}_2 = 4t\frac{\partial}{\partial t} + a\frac{\partial}{\partial a} - 8R\frac{\partial}{\partial R}, \quad \mathbf{X}_3 = 2t^2\frac{\partial}{\partial t} + ta\frac{\partial}{\partial a} - 8tR\frac{\partial}{\partial R} \quad \text{with } K = -\frac{21}{4}f_0 a^3 R^{-\frac{1}{8}}. \tag{43}$$

Then the non-zero Lie brackets are

$$[\mathbf{X}_1, \mathbf{X}_2] = 4\mathbf{X}_1, \quad [\mathbf{X}_1, \mathbf{X}_3] = \mathbf{X}_2, \quad [\mathbf{X}_2, \mathbf{X}_3] = 4\mathbf{X}_3. \tag{44}$$

Thus the first integrals of this case are $I_1 = -E_{\mathcal{L}} = 0$, which yield

$$\frac{\dot{a}^2}{a^2} - \frac{\dot{a}\dot{R}}{8aR} + \frac{R}{42} = 0, \tag{45}$$

and

$$I_2 = \frac{21}{4}f_0 R^{-\frac{1}{8}}\left(-3a^2\dot{a} + a^3\frac{\dot{R}}{8R}\right), \quad I_3 = I_2 t + \frac{21}{4}f_0 a^3 R^{-\frac{1}{8}}. \tag{46}$$

Substituting the Ricci scalar R given in (5) to the Equation (45), and integrating the resulting equation, one gets

$$a^8\dot{a} = a_0 R^{\frac{7}{8}}, \tag{47}$$

where a_0 is a constant of integration. Defining $\bar{I}_2 = -4I_2/(21f_0)$ and $\bar{I}_3 = -4I_3/(21f_0)$, the first integrals (46) become

$$\bar{I}_2 = \left(a^3 R^{-\frac{1}{8}}\right)^{\cdot}, \quad \bar{I}_3 = \bar{I}_2 t - a^3 R^{-\frac{1}{8}}, \tag{48}$$

which give

$$a(t) = \left[R^{\frac{1}{8}}(\bar{I}_2 t - \bar{I}_3)\right]^{\frac{1}{3}}. \tag{49}$$

Putting the latter form of scale factor into (47), after integration for R, one finds

$$R(t) = \frac{(\bar{I}_2 t - \bar{I}_3)^4}{\left[\frac{2a_0}{\bar{I}_2} + R_0(\bar{I}_2 t - \bar{I}_3)^6\right]^2}, \tag{50}$$

then the scale factor becomes

$$a(t) = \sqrt{\bar{I}_2 t - \bar{I}_3}\left[\frac{2a_0}{\bar{I}_2} + R_0(\bar{I}_2 t - \bar{I}_3)^6\right]^{-\frac{1}{12}}, \tag{51}$$

where R_0 is an integration constant.

- $n = \frac{1}{2}$, $m = \frac{1}{4}$: Here there exist *two* Noether symmetries,

$$\mathbf{X}_1 = \frac{\partial}{\partial t}, \qquad \mathbf{X}_2 = t\frac{\partial}{\partial t} - 2R\frac{\partial}{\partial R} - 4G\frac{\partial}{\partial G}, \tag{52}$$

with the non-vanishing Lie algebra $[\mathbf{X}_1, \mathbf{X}_2] = \mathbf{X}_1$. Then the first integrals related with these Noether symmetries are $I_1 = -E_{\mathcal{L}} = 0$, which yield

$$\frac{\dot{a}^2}{a^2} - \frac{\dot{a}\dot{R}}{2aR} + \frac{R}{6} + \frac{g_0}{4f_0}\sqrt{R}G^{\frac{1}{4}}\left(1 - \frac{6\dot{a}^3\dot{G}}{a^3G^2}\right) = 0, \tag{53}$$

and

$$I_2 = -6\left(\frac{f_0}{2}R^{-\frac{1}{2}}a^2\dot{a} + g_0 G^{-\frac{3}{4}}\dot{a}^3\right). \tag{54}$$

The Noether symmetries (52) have also been obtained in [25] with the symmetry vector (41). In order to determine the invariant functions of the Noether symmetry $\mathbf{X}_2$ given in (52), after solving the Lagrange system [28]

$$\frac{dt}{t} = \frac{dR}{-2R} = \frac{dG}{-4G}, \tag{55}$$

one find the solutions for $R(t)$ and $G(t)$ as

$$R(t) = \frac{R_0}{t^2}, \qquad G(t) = \frac{G_0}{t^4}. \tag{56}$$

Here we get a power-law solution $a(t) = a_0 t^2$, where the Equations (53) and (54) yield

$$g_0 = -\frac{4f_0 G^{\frac{3}{4}}}{6\sqrt{R_0}}\frac{(R_0+36)}{(G_0+192)}, \qquad I_2 = \frac{6f_0 a_0^3}{\sqrt{R_0}}\left[\frac{16(R_0+36)}{3(G_0+192)} - 1\right]. \tag{57}$$

For the obtained R and G in (56), if we take into account the definitions of R and G given by (5), then we get the values of constants as $R_0 = 36$ and $G_0 = 192$. Thus, the relation (57) becomes $g_0 = -(4/3)^{1/4} f_0$ and $I_2 = 0$.

- $n = 1$, $m = \frac{1}{2}$: In this case, there are also *two* Noether symmetries

$$\mathbf{X}_1 = \frac{\partial}{\partial t}, \qquad \mathbf{X}_2 = 3t\frac{\partial}{\partial t} + a\frac{\partial}{\partial a} - 12G\frac{\partial}{\partial G}, \tag{58}$$

which give rise to the first integrals $I_1 = -E_{\mathcal{L}} = 0$, which can be written by using (20) as follows

$$\frac{\dot{a}^2}{a^2} + \frac{g_0}{f_0}\sqrt{G}\left(\frac{1}{12} - \frac{\dot{a}^3\dot{G}}{a^3G^2}\right) = 0, \tag{59}$$

and

$$I_2 = 6\left[-2f_0 a^2\dot{a} + \frac{g_0}{\sqrt{G}}a\dot{a}^2\left(\frac{\dot{G}}{G} - 4\frac{\dot{a}}{a}\right)\right]. \tag{60}$$

Here we have to point out that the Noether symmetries (58) are of the form (41) in [25]. By solving the Lagrange system for $\mathbf{X}_2$

$$\frac{dt}{3t} = \frac{da}{a} = \frac{dG}{-12G}, \tag{61}$$

the invariant functions can be obtained as

$$a(t) = a_0 t^{\frac{1}{3}}, \qquad G(t) = \frac{G_0}{t^4}. \tag{62}$$

Then, substituting these into the Equations (59) and (60), we can find the constraint relations

$$g_0 = -\frac{12 f_0 \sqrt{|G_0|}}{9G_0 + 16}, \qquad I_2 = 4 f_0 a_0^3 \left[\frac{32}{3(9G_0+16)} - 1\right]. \tag{63}$$

Here the definition of G in terms of $a(t)$ by (5) gives rise to the value $G_0 = -16/27$, which means $g_0 = -\sqrt{3} f_0/2$ and $I_2 = 0$ after substituting G_0 into (63).

Case (ii): $F(R,G) = f_0 f(R) g(G)$. Here we will consider the functional forms $f(R) = R^n$ and $g(G) = G^m$. These types of functional forms are appeared in some references such as [9,19,25,31,32].

- n, m arbitrary: This theory admits the following Noether symmetries

$$\mathbf{X}_1 = \frac{\partial}{\partial t}, \qquad \mathbf{X}_2 = 3t\frac{\partial}{\partial t} + (4m+2n-1)a\frac{\partial}{\partial a} - 6R\frac{\partial}{\partial R} - 12G\frac{\partial}{\partial G}, \tag{64}$$

and the corresponding first integrals are

$$\frac{\dot{a}^2}{a^2} + \frac{\dot{a}}{a}\left[(n-1)\frac{\dot{R}}{R} + m\frac{\dot{G}}{G}\right] + \frac{12mR}{G}\frac{\dot{a}^3}{a^3}\left[\frac{\dot{R}}{R} + \frac{(m-1)}{n}\frac{\dot{G}}{G}\right] - (n+m-1)\frac{R}{6n} = 0, \tag{65}$$

$$\begin{aligned} I_2 =& 6 f_0 R^n G^m a^3 \Big\{ 2(2m+n-2)\frac{\dot{a}}{a}\left(\frac{n}{R} + \frac{4m\dot{a}^2}{Ga^2}\right) \\ & - (4m+2n-1)\left(\frac{n}{R}\left[(n-1)\frac{\dot{R}}{R} + m\frac{\dot{G}}{G}\right] + \frac{4m\dot{a}^2}{Ga^2}\left[n\frac{\dot{R}}{R} + (m-1)\frac{\dot{G}}{G}\right]\right)\Big\}. \end{aligned} \tag{66}$$

These are very general statements and one can find any solution choosing the arbitrary powers n and m. The invariant functions for the vector field $\mathbf{X}_2$ can be determined by solving the associated Lagrange system

$$\frac{dt}{3t} = \frac{da}{(4m+2n-1)a} = \frac{dR}{-6R} = \frac{dG}{-12G}, \tag{67}$$

which yields

$$a(t) = a_0 t^{\frac{4m+2n-1}{3}}, \quad R(t) = \frac{R_0}{t^2}, \quad G(t) = \frac{G_0}{t^4}. \tag{68}$$

Now one can find the constants R_0 and G_0 in terms of powers of $a(t)$ as $R_0 = 2(4m+2n-1)(8m+4n-5)/3$ and $G_0 = 16(4m+2n-1)^3(2m+n-2)/27$ by considering (5). Thus, using the obtained quantities by (68) in (65) and (66), we find the constraints

$$(10m+2n-1)(4m+2n-1)(8m+4n-5) = 0, \qquad I_2 = 6(2m+n) f_0 a_0^3 R_0^n G_0^m. \tag{69}$$

- $m = 1-n$: This case is considered in the reference [19] as a simplest non-trivial case with the selection of $n = 2$. In general, the solution of Noether symmetry equations (21) becomes

$$\xi = c_1 + c_2 t, \quad \eta^1 = c_2(3-2n)\frac{a}{3}, \quad \eta^2 = \eta^2(t,a,R,G), \quad \eta^3 = \frac{G}{R}(-2c_2 R + \eta^2), \; K = c_3, \tag{70}$$

where c_i's ($i = 1,2,3$) are constant parameters, and η^2 is an arbitrary function of t, a, R and G. This arbitrariness means that there are infinitely many Noether symmetries and it gives us to decide a selection of consistent solution for the scale factor a. Therefore, we choose $\eta^2 = -2c_2 R$ to get a consistent power-law solution for the scale factor a, using the associated Lagrange system. It has to be mentioned here that this type of selection is not necessary for non power-law solutions. We proceed considering $\eta^2 = -2c_2R$ at (70), which yields that there are *two* Noether symmetries

$$\mathbf{X}_1 = \frac{\partial}{\partial t}, \qquad \mathbf{X}_2 = 3t\frac{\partial}{\partial t} + (3-2n)a\frac{\partial}{\partial a} - 6R\frac{\partial}{\partial R} - 12G\frac{\partial}{\partial G}. \tag{71}$$

The first integrals of the above vector fields are

$$\frac{\dot{a}}{a} + (n-1)\left(1 - \frac{4R\dot{a}^2}{Ga^2}\right)\left(\frac{\dot{R}}{R} - \frac{\dot{G}}{G}\right) = 0, \tag{72}$$

and

$$I_2 = -6f_0 n\left(\frac{R}{G}\right)^{n-1} a^2\dot{a}\left\{4n - 3 + 24(n-1)\frac{R\dot{a}^2}{Ga^2}\right\}. \tag{73}$$

By choosing the variable $\zeta = \frac{R}{G}$, the first integrals (72) and (73) take the form

$$\frac{\dot{a}}{a} + (n-1)\left(\frac{1}{\zeta} - 4\frac{\dot{a}^2}{a^2}\right)\dot{\zeta} = 0, \quad 6f_0 n\zeta^{n-1}a^2\dot{a}\left(4n - 3 + 24(n-1)\zeta\frac{\dot{a}^2}{a^2}\right) + I_2 = 0. \tag{74}$$

For the selection of $n = 2$, it is seen that the first equation of (74) is similar to the Equation (38) of the Reference [19]. After solving the associated Lagrange system for the vector field $\mathbf{X}_2$ given in (71), we have

$$a(t) = a_0 t^{\frac{3-2n}{3}}, \quad R(t) = \frac{R_0}{t^2}, \quad G(t) = \frac{G_0}{t^4}, \tag{75}$$

Using the definitions of R and G in (5), the constants R_0 and G_0 are found as $R_0 = (16n^2 - 36n + 18)/3$ and $G_0 = 16n(2n-3)^3/27$.

As a simple selection for the component η^2, we choose $\eta^2 = 0$ in (70). Then there are again *two* Noether symmetries

$$\mathbf{X}_1 = \frac{\partial}{\partial t}, \qquad \mathbf{X}_2 = 3t\frac{\partial}{\partial t} + (3-2n)a\frac{\partial}{\partial a} - 6G\frac{\partial}{\partial G}. \tag{76}$$

The Noether constants for these vector fields are $I_1 = -E_{\mathcal{L}} = 0$, which yield the same relation with (72), and

$$I_2 = 6f_0 n a^3\left(\frac{R}{G}\right)^{n-1}\left\{(2n-3)\left[\frac{2\dot{a}}{a} + (n-1)\left(1 - \frac{4R\dot{a}^2}{Ga^2}\right)\left(\frac{\dot{R}}{R} - \frac{\dot{G}}{G}\right)\right]\right.$$
$$\left. + 2(1-n)\frac{\dot{a}}{a}\left(3 - \frac{4R\dot{a}^2}{Ga^2}\right)\right\}, \tag{77}$$

which becomes

$$I_2 = 6f_0 n\left(\frac{R}{G}\right)^{n-1} a^2\dot{a}\left\{2n - 3 + 2(1-n)\left(3 - \frac{4R\dot{a}^2}{Ga^2}\right)\right\}, \tag{78}$$

by using (72). It is easily seen that the Noether symmetry $\mathbf{X}_2$ in (76) does not have a consistent solution for a power-law form of the scale factor a. The reason of this inconsistency is follows from analysing of the associated Lagrange system for $\mathbf{X}_2$ in such a way that it gives the scale factor $a(t)$ as in (75), but $G(t) = G_0 t^{-2}$ which contradicts the form of $G(t) \sim t^{-4}$ from the definition (5).

4.2. Non-Vacuum Case

In this section, we assume that the matter has a constant equation of state (EoS) parameter $w \equiv p/\rho$ with the perfect fluid matter. We mention that Equations (9) and (10) imply that the contribution of the $F(R,G)$ gravity can formally be included in the effective energy density and pressure of the universe. For the GR with $F(R,G)=R$, $\rho_{eff}=\rho$ and $p_{eff}=p$, and so the Equations (9) and (10) are the FLRW equations.

Case (i): $F(R,G)=f(R)+g(G)$.

For this case, we again choose $f(R)=f_0R^n$, $g(G)=g_0G^m$, and determine the Noether symmetries in the presence of matter.

- $n=m=1$: This gives the usual GR theory. For some value of the constant EoS parameter, we would like to give the Noether symmetries in the following. First of all, for $w=-1$ (the cosmological constant), the present value of the energy density becomes $\rho_{m0}=4f_0/(3\kappa^2\alpha^2)$, and there exist *five* Noether symmetries

$$\begin{aligned}
&\mathbf{X}_1=\frac{\partial}{\partial t},\quad \mathbf{X}_2=\frac{e^{\frac{t}{\alpha}}}{\sqrt{a}}\frac{\partial}{\partial a}\quad \text{with } K=-\frac{8f_0}{\alpha}a^{\frac{3}{2}}e^{\frac{t}{\alpha}},\quad \mathbf{X}_3=\frac{e^{-\frac{t}{\alpha}}}{\sqrt{a}}\frac{\partial}{\partial a}\quad \text{with } K=\frac{8f_0}{\alpha}a^{\frac{3}{2}}e^{-\frac{t}{\alpha}},\\
&\mathbf{X}_4=e^{\frac{2t}{\alpha}}\frac{\partial}{\partial t}+\frac{2}{3\alpha}e^{\frac{2t}{\alpha}}a\frac{\partial}{\partial a}\qquad \text{with } K=-\frac{16f_0}{3\alpha^2}a^3e^{\frac{2t}{\alpha}},\\
&\mathbf{X}_5=e^{-\frac{2t}{\alpha}}\frac{\partial}{\partial t}-\frac{2}{3\alpha}e^{-\frac{2t}{\alpha}}a\frac{\partial}{\partial a}\quad \text{with}\quad K=-\frac{16f_0}{3\alpha^2}a^3e^{-\frac{2t}{\alpha}},
\end{aligned}\tag{79}$$

with the non-vanishing commutators

$$\begin{aligned}
&[\mathbf{X}_1,\mathbf{X}_2]=\frac{1}{\alpha}\mathbf{X}_2,\quad [\mathbf{X}_1,\mathbf{X}_3]=-\frac{1}{\alpha}\mathbf{X}_3,\quad [\mathbf{X}_1,\mathbf{X}_4]=\frac{2}{\alpha}\mathbf{X}_4,\\
&[\mathbf{X}_1,\mathbf{X}_5]=-\frac{2}{\alpha}\mathbf{X}_5,\quad [\mathbf{X}_2,\mathbf{X}_5]=-\frac{2}{\alpha}\mathbf{X}_3,\quad [\mathbf{X}_3,\mathbf{X}_4]=\frac{2}{\alpha}\mathbf{X}_2,\quad [\mathbf{X}_4,\mathbf{X}_5]=-\frac{4}{\alpha}\mathbf{X}_1,
\end{aligned}\tag{80}$$

where α is a constant. Then the first integrals are $I_1=-E_{\mathcal{L}}=0$, that gives $\kappa^2\rho_{m0}=3f_0\dot{a}^2/a^2$, and the quantities

$$\begin{aligned}
&\bar{I}_2=e^{\frac{t}{\alpha}}\sqrt{a}\left(-3\dot{a}+\frac{2}{\alpha}a\right),\quad \bar{I}_3=-e^{-\frac{t}{\alpha}}\sqrt{a}\left(3\dot{a}+\frac{2}{\alpha}a\right),\\
&\bar{I}_4=\frac{2}{3\alpha}e^{\frac{2t}{\alpha}}\left(-3a^2\dot{a}+\frac{2}{\alpha}a^3\right),\quad \bar{I}_5=-\frac{2}{3\alpha}e^{-\frac{2t}{\alpha}}\left(3a^2\dot{a}+\frac{2}{\alpha}a^3\right),
\end{aligned}\tag{81}$$

where we have defined $I_2=4f_0\bar{I}_2$, $I_3=4f_0\bar{I}_3$, $I_4=4f_0\bar{I}_4$ and $I_5=4f_0\bar{I}_5$. After solving these first integrals for a, we find that the Noether constants become $\bar{I}_3=0$, $\bar{I}_5=0$, $\bar{I}_4=\bar{I}_2^2/6$, and the scale factor is

$$a(t)=a_0\exp\left(-\frac{2t}{3\alpha}\right),\tag{82}$$

where $a_0=(\alpha/4)^{2/3}$. This is the well-known de Sitter solution.

In the case of $w=-1/2$, we also find *five* Noether symmetries

$$\begin{aligned}
&\mathbf{X}_1=\frac{\partial}{\partial t},\quad \mathbf{X}_2=\frac{1}{\sqrt{a}}\frac{\partial}{\partial a}\quad \text{with } K=-48f_0t,\quad \mathbf{X}_3=\frac{t}{\sqrt{a}}\frac{\partial}{\partial a}\quad \text{with } K=-8f_0a^{\frac{3}{2}}-24f_0t^2,\\
&\mathbf{X}_4=t\frac{\partial}{\partial t}+\left(\frac{a}{3}+\frac{3t^2}{\sqrt{a}}\right)\frac{\partial}{\partial a}\qquad \text{with}\quad K=-48f_0ta^{\frac{3}{2}}-48f_0t^3,\\
&\mathbf{X}_5=\frac{t^2}{2}\frac{\partial}{\partial t}+t\left(\frac{a}{3}+\frac{t^2}{\sqrt{a}}\right)\frac{\partial}{\partial a}\quad \text{with}\quad K=-24f_0t^2a^{\frac{3}{2}}-4f_0a^3-12f_0t^4.
\end{aligned}\tag{83}$$

Thus the non-vanishing Lie brackets of the above vector fields are

$$
\begin{aligned}
&[\mathbf{X}_1, \mathbf{X}_3] = \mathbf{X}_2, \qquad [\mathbf{X}_1, \mathbf{X}_4] = \mathbf{X}_1 + 6\mathbf{X}_3, \quad [\mathbf{X}_1, \mathbf{X}_5] = \mathbf{X}_4, \\
&[\mathbf{X}_2, \mathbf{X}_4] = \frac{1}{2}\mathbf{X}_2, \quad [\mathbf{X}_2, \mathbf{X}_5] = \frac{1}{2}\mathbf{X}_3, \quad [\mathbf{X}_3, \mathbf{X}_4] = -\frac{1}{2}\mathbf{X}_3, \quad [\mathbf{X}_4, \mathbf{X}_5] = \mathbf{X}_5. \qquad (84)
\end{aligned}
$$

Under the change of the Noether constants $I_2 \to 12f_0 I_2, I_3 \to 12f_0 I_3, I_4 \to 12f_0 I_4, I_5 \to 12f_0 I_5$ for the Noether symmetries (83), the first integrals for $\mathbf{X}_1, ..., \mathbf{X}_5$ become

$$
\begin{aligned}
&3f_0 \frac{\dot{a}^2}{\sqrt{a}} = \kappa^2 \rho_{m0}, \\
&I_2 = -\sqrt{a}\dot{a} + 4t, \quad I_3 = -t\sqrt{a}\dot{a} + \frac{2}{3}a^{\frac{3}{2}} + 2t^2, \\
&I_4 = \frac{a^2}{3}\dot{a} - 3t^2\sqrt{a}\dot{a} + 4ta^{\frac{3}{2}} + 4t^3, \quad I_5 = -\frac{t}{3}a^2\dot{a} - t^3\sqrt{a}\dot{a} + 2t^2 a^{\frac{3}{2}} + \frac{a^3}{9} + t^4, \qquad (85)
\end{aligned}
$$

Taking into account these first integrals, we find that

$$
a(t) = a_0 \left(4t - I_2\right)^{\frac{4}{3}}, \qquad (86)
$$

$$
\rho_{m0} = \frac{16 f_0}{\kappa^2}, \quad I_3 = \frac{I_2^2}{8}, \quad I_4 = \frac{I_2^3}{16}, \quad I_5 = \frac{I_3^2}{4}, \qquad (87)
$$

where $a_0 = (3/16)^{2/3}$.

For $w = 0$ (the dust), the dynamical system admits the following *five* Noether symmetries

$$
\begin{aligned}
&\mathbf{X}_1 = \frac{\partial}{\partial t}, \quad \mathbf{X}_2 = \frac{1}{\sqrt{a}}\frac{\partial}{\partial a}, \quad \mathbf{X}_3 = \frac{t}{\sqrt{a}}\frac{\partial}{\partial a} \quad \text{with } K = -8f_0 a^{\frac{3}{2}}, \\
&\mathbf{X}_4 = t\frac{\partial}{\partial t} + \frac{a}{3}\frac{\partial}{\partial a} \qquad \text{with } K = -\kappa^2 \rho_{m0} t, \qquad (88) \\
&\mathbf{X}_5 = \frac{t^2}{2}\frac{\partial}{\partial t} + \frac{ta}{3}\frac{\partial}{\partial a} \quad \text{with} \quad K = -\frac{4f_0}{3}a^3 - \kappa^2 \rho_{m0} t^2,
\end{aligned}
$$

and then the non-vanishing commutators are

$$
\begin{aligned}
&[\mathbf{X}_1, \mathbf{X}_3] = \mathbf{X}_2, \quad [\mathbf{X}_1, \mathbf{X}_4] = \mathbf{X}_1, \quad [\mathbf{X}_1, \mathbf{X}_5] = \mathbf{X}_4, \\
&[\mathbf{X}_2, \mathbf{X}_4] = \frac{1}{2}\mathbf{X}_2, \quad [\mathbf{X}_2, \mathbf{X}_5] = \frac{1}{2}\mathbf{X}_3, \quad [\mathbf{X}_3, \mathbf{X}_4] = -\frac{1}{2}\mathbf{X}_3, \quad [\mathbf{X}_4, \mathbf{X}_5] = \mathbf{X}_5. \qquad (89)
\end{aligned}
$$

The corresponding first integrals of the Noether symmetries (88) are

$$
I_1 = -E_{\mathcal{L}} = 0, \quad I_2 = -12f_0\sqrt{a}\dot{a}, \quad I_3 = I_2 t + 8f_0 a^{\frac{3}{2}}, \qquad (90)
$$

$$
I_4 = -4f_0 a^2 \dot{a} + 2\kappa^2 \rho_{m0} t, \qquad I_5 = 4f_0\left(-ta^2\dot{a} + \frac{1}{3}a^3\right) + \kappa^2 \rho_{m0} t^2. \qquad (91)
$$

Using the above first integrals one can find the scale factor and the constraints on Noether constants as follows

$$
a(t) = a_0 \left(I_3 - I_2 t\right)^{\frac{2}{3}}, \qquad (92)
$$

$$
\rho_{m0} = \frac{I_2^2}{48 f_0 \kappa^2}, \quad I_4 = \frac{I_2 I_3}{24 f_0}, \quad I_5 = \frac{I_3^2}{48 f_0}, \qquad (93)
$$

where $a_0 = (8f_0)^{-2/3}$.

Finally, for $w = 1$ (stiff matter), we find *three* Noether symmetries

$$\mathbf{X}_1 = \frac{\partial}{\partial t}, \quad \mathbf{X}_2 = 3t\frac{\partial}{\partial t} + a\frac{\partial}{\partial a}, \quad \mathbf{X}_3 = \frac{3}{2}t^2\frac{\partial}{\partial t} + ta\frac{\partial}{\partial a} \quad \text{with } K = -4f_0a^3, \tag{94}$$

which yields the non-vanishing Lie algebra: $[\mathbf{X}_1, \mathbf{X}_2] = 3\mathbf{X}_1$, $[\mathbf{X}_1, \mathbf{X}_3] = \mathbf{X}_2$, $[\mathbf{X}_2, \mathbf{X}_3] = 3\mathbf{X}_3$. The Noether constants for $\mathbf{X}_1, \mathbf{X}_2$ and $\mathbf{X}_3$ are

$$I_1 = 0 \quad \Rightarrow 3f_0a^4\dot{a}^2 = \kappa^2\rho_{m0}, \quad I_2 = -12f_0a^2\dot{a}, \quad I_3 = I_2t + 4f_0a^3, \tag{95}$$

having the solution

$$a(t) = a_0\left(I_3 - I_2t\right)^{\frac{1}{3}}, \qquad \rho_{m0} = \frac{I_2^2}{48f_0\kappa^2}, \tag{96}$$

where $a_0 = (4f_0)^{-1/3}$.

- n arbitrary (with $n \neq 0, \frac{3}{2}, \frac{7}{8}$), $m = 1$: In this case we have the same Noether symmetries $\mathbf{X}_1, \mathbf{X}_2$ given by (29) in the vacuum case. For this case we are led to the constant EoS parameter w as

$$w = \frac{1}{2n-1}. \tag{97}$$

Using this EoS parameter, the first integral for $\mathbf{X}_1$ gives

$$\frac{\dot{a}^2}{a^2} + (n-1)\frac{\dot{a}\dot{R}}{aR} - \frac{(n-1)}{6n}R = \frac{\kappa^2\rho_{m0}}{3f_0n}a^{-\frac{6n}{2n-1}}R^{1-n}. \tag{98}$$

The scale factor for this case has the same form with (32), which is not a power-law form, and the Equations (31) and (98) are the constraint equations to be considered. It is interesting to see from (97) that $n = 0$ if $w = -1$ (the cosmological constant) which is excluded in this case, $n = 1$ if $w = 1$ (the stiff matter), and $n = 2$ if $w = 1/3$ (the relativistic matter), etc. Therefore, this case includes some important values of the EoS parameter.

This model admits power-law solution of the form $a(t) = a_0t^{(2n-1)/3}$, and the Ricci scalar and the GB invariant become $R(t) = R_0t^{-2}$ and $G(t) = G_0t^{-4}$, where the constants R_0 and G_0 follow from (5) as $R_0 = 2(2n-1)(4n-5)/3$ and $G_0 = 16(n-2)(2n-1)^3/27$. Meanwhile, the constraint relations (31) and (98) for this power-law scale factor give

$$\rho_{m0} = \frac{f_0}{3\kappa^2}(5-4n)(2n-1)^2R_0^{n-1}a_0^{\frac{6n}{2n-1}}, \qquad I_2 = 4n(4n-5)(2n-1)f_0a_0^3R_0^{n-1}, \tag{99}$$

where $n \neq \frac{1}{2}, \frac{5}{4}$ due to $\rho_{m0} \neq 0$. The power-law solution of this case works for $n = 2$, i.e., $w = 1/3$, and it gives negative energy density as $\rho_{m0} = -54f_0a_0^4/\kappa^2$.

- $n = \frac{3}{2}, m = 1$: We will firstly consider the case $w = -2/3$ which requires that $\rho_{m0} = \alpha/2\kappa^2$, α is a constant. For this case, there are *three* Noether symmetries $\mathbf{X}_1, \mathbf{X}_2$ with $K = -2\alpha t$, and $\mathbf{X}_3 = t\mathbf{X}_2$ with $K = -9f_0a\sqrt{R} - \alpha t^2$, where $\mathbf{X}_2$ is the same as given in (38). Thus the constants of motion for the vector fields $\mathbf{X}_1, \mathbf{X}_2$ and $\mathbf{X}_3$ are, respectively,

$$\frac{\dot{a}^2}{a^2} + \frac{\dot{a}\dot{R}}{2aR} - \frac{R}{18} = \frac{\alpha}{9f_0a\sqrt{R}}, \tag{100}$$

and

$$I_2 = -9f_0a\sqrt{R}\left(\frac{\dot{a}}{a} + \frac{\dot{R}}{2R}\right) + 2\alpha t, \qquad I_3 = I_2t - \alpha t^2 + 9f_0a\sqrt{R}. \tag{101}$$

Using above Noether constants, the scale factor for the case $w = -2/3$ yields

$$a(t) = \frac{1}{9f_0\sqrt{R}}(I_3 - I_2 t + \alpha t^2). \tag{102}$$

For $w = 0$, the Noether symmetries are identical to vector fields given in (38), but $\mathbf{X}_3$ has a non-zero function $K = -6\kappa^2\rho_{m0}$. Redefining the Noether constants such as $I_2 = -9f_0\bar{I}_2$, $I_3 = -9f_0\bar{I}_3$ and $I_4 = -9f_0\bar{I}_4$, after some algebra, we find the scale factor

$$a(t) = \frac{1}{6\bar{I}_2}\sqrt{(\bar{I}_2 t - \bar{I}_4)^4 + \frac{16}{f_0}\kappa^2\rho_{m0}(\bar{I}_2 t - \bar{I}_4) + 18\bar{I}_2\bar{I}_3 + \frac{12}{f_0}\kappa^2\rho_{m0}\bar{I}_4}, \tag{103}$$

and the Ricci scalar

$$R(t) = \frac{36\bar{I}_2^2(\bar{I}_2 t - \bar{I}_4)^2}{a(t)^2}. \tag{104}$$

- $n = \frac{7}{8}, m = 1$: If $w = 4/3$, there are *three* Noether symmetries, in which $\mathbf{X}_2$ and $\mathbf{X}_3$ are the same as (43). The first integral $I_1 = -E_{\mathcal{L}} = 0$ due to $\mathbf{X}_1$ becomes

$$\frac{\dot{a}^2}{a^2} - \frac{\dot{a}\dot{R}}{8aR} + \frac{R}{42} = \frac{8}{21f_0}\kappa^2\rho_{m0}a^{-7}R^{\frac{1}{8}}. \tag{105}$$

The scale factor for this case has the same form as (49), but now it is difficult to gain the explicit form of $a(t)$ using (105).

For the dust matter ($w = 0$), there are *three* Noether symmetries which are the same form as (43), but the function K is non-trivial such that $K = -8\kappa^2\rho_{m0}t$ for $\mathbf{X}_2$ and $K = -\frac{21}{4}f_0 a^3 R^{-1/8} - 4\kappa^2\rho_{m0}t^2$ for $\mathbf{X}_3$. Thus the first integrals for $\mathbf{X}_1, \mathbf{X}_2$ and $\mathbf{X}_3$ are given by, respectively,

$$I_1 = -E_{\mathcal{L}} = 0 \quad \Leftrightarrow \quad \frac{\dot{a}^2}{a^2} - \frac{\dot{a}\dot{R}}{8aR} + \frac{R}{42} = \frac{8\kappa^2\rho_{m0}R^{\frac{1}{8}}}{21f_0a^3}, \tag{106}$$

and

$$I_2 = \frac{21}{4}f_0R^{-\frac{1}{8}}\left(-3a^2\dot{a} + a^3\frac{\dot{R}}{8R}\right) + 8\kappa^2\rho_{m0}t, \quad I_3 = I_2 t + \frac{21}{4}f_0a^3R^{-\frac{1}{8}} + 4\kappa^2\rho_{m0}t^2. \tag{107}$$

After redefining $I_2 = -\frac{21}{4}f_0\bar{I}_2$ and $I_3 = -\frac{21}{4}f_0\bar{I}_3$, the second relation in (107) implies the scale factor

$$a(t) = R^{\frac{1}{24}}\left(\alpha t^2 + \bar{I}_2 t - \bar{I}_3\right)^{\frac{1}{3}}, \tag{108}$$

where $\alpha \equiv \frac{16\kappa^2\rho_{m0}}{21f_0}$.

- $n = \frac{1}{2}$, $m = \frac{1}{4}$: In addition to $\mathbf{X}_1 = \partial/\partial t$, the condition for existing extra Noether symmetry is that the EoS parameter should be $w = 0$. Thus, an additional Noether symmetry is obtained as follows

$$\mathbf{X}_2 = t\frac{\partial}{\partial t} - 2R\frac{\partial}{\partial R} - 4G\frac{\partial}{\partial G} \quad \text{with} \quad K = -2\kappa^2\rho_{m0}t. \tag{109}$$

Then the Noether constants for these vector fields yield

$$H^2 - \frac{\dot{R}}{2R}H + \frac{R}{6} + \frac{g_0}{4f_0}\sqrt{R}G^{\frac{1}{4}}\left(1 - 6H^3\frac{\dot{G}}{G^2}\right) = \frac{2\kappa^2\rho_{m0}\sqrt{R}}{3f_0a^3}, \tag{110}$$

and

$$I_2 = -3a^3\left(f_0 R^{-\frac{1}{2}} H + 2g_0 G^{-\frac{3}{4}} H^3\right) + 2\kappa^2 \rho_{m0} t, \tag{111}$$

which can be written as

$$H^3 + \frac{f_0}{2g_0}\left(\frac{G^3}{R^2}\right)^{\frac{1}{4}} H - \frac{(2\kappa^2\rho_{m0}t - I_2)}{6g_0 a^3} G^{\frac{3}{4}} = 0. \tag{112}$$

This is a cubic equation for H.

- $n = 1$, $m = \frac{1}{2}$: For $w = 0$, it is found the Noether symmetries $\mathbf{X}_1$ and $\mathbf{X}_2$ which are the same as (58), but $\mathbf{X}_2$ has the non-trivial function $K = -6\kappa^2\rho_{m0}t$. Then the Noether constants for $\mathbf{X}_1$ and $\mathbf{X}_2$ take the following forms

$$H^2 + \frac{g_0}{f_0}\sqrt{G}\left(\frac{1}{12} - H^3\frac{\dot{G}}{G^2}\right) = \frac{\kappa^2\rho_{m0}}{3a^3}, \tag{113}$$

and

$$I_2 = 6a^3\left[-2f_0 H + \frac{g_0 H^2}{\sqrt{G}}\left(\frac{\dot{G}}{G} - 4H\right) + \frac{\kappa^2\rho_{m0}t}{a^3}\right]. \tag{114}$$

For $w = 1$, there are *two* Noether symmetries that are the same as (58), where $K = 0$ for both of symmetries. Therefore, the first integral for $\mathbf{X}_2$ is the same as (60), and the first integral for $\mathbf{X}_1$ becomes

$$H^2 + \frac{g_0}{f_0}\sqrt{G}\left(\frac{1}{12} - H^3\frac{\dot{G}}{G^2}\right) = \frac{\kappa^2\rho_{m0}}{3a^6}. \tag{115}$$

Case (ii): $F(R,G) = f_0 f(R) g(G)$. The functional forms $f(R) = R^n$ and $g(G) = G^m$ are also assumed in this section.

- n, m arbitrary: For this case, there exist *two* Noether symmetries which are the same as (64), and the EoS parameter becomes

$$w = \frac{1}{4m + 2n - 1}. \tag{116}$$

The first integral for $\mathbf{X}_2$ is the same as (66), and it has the following form

$$H^2 + \left[(n-1)\frac{\dot{R}}{R} + m\frac{\dot{G}}{G}\right] H + \frac{12mR}{G}\left[\frac{\dot{R}}{R} + \frac{(m-1)}{n}\frac{\dot{G}}{G}\right] H^3 - (n+m-1)\frac{R}{6n} = \frac{\kappa^2\rho_{m0} a^{\frac{6(2n+m)}{4m+2n-1}}}{3f_0 n R^{n-1} G^m}, \tag{117}$$

for $\mathbf{X}_1$. Note that the Equation (116) includes important EoS parameters, for example $w = -1$ if $n = -2m$; $w = -1/3$ if $n = -(2m+1)$; $w = 1/3$ if $n = 2(1-m)$ and $w = 1$ if $n = 2, m = -1/2$. In the case of dust matter ($w = 0$), we have *two* Noether symmetries given by (64), but where the function K for $\mathbf{X}_2$ is $K = -6\kappa^2\rho_{m0}t$. Therefore, the first integrals for $\mathbf{X}_1$ and $\mathbf{X}_2$ are, respectively,

$$H^2 + \left[(n-1)\frac{\dot{R}}{R} + m\frac{\dot{G}}{G}\right] H + \frac{12mR}{G}\left[\frac{\dot{R}}{R} + \frac{(m-1)}{n}\frac{\dot{G}}{G}\right] H^3 - (n+m-1)\frac{R}{6n} = \frac{\kappa^2\rho_{m0}R^{1-n}}{3f_0 n a^3 G^m}, \tag{118}$$

$$\begin{aligned} I_2 = 6f_0 R^n G^m a^3 \Big\{ & 2(n+2m-2)H\left(\frac{n}{R} + \frac{4mH^2}{G}\right) \\ & -(4m+2n-1)\left(\frac{n}{R}\left[(n-1)\frac{\dot{R}}{R} + m\frac{\dot{G}}{G}\right] + \frac{4mH^2}{G}\left[n\frac{\dot{R}}{R} + (m-1)\frac{\dot{G}}{G}\right]\right)\Big\} + 6\kappa^2\rho_{m0}t. \end{aligned} \tag{119}$$

- $m = 1 - n$: In this case, the EoS parameter takes the form $w = \frac{1}{3-2n}$, and this model admits *two* Noether symmetries, which are the same as (71). The first integral due to $\mathbf{X}_1$ yields

$$H^2 + (n-1)H\left(1 - \frac{4R}{G}H^2\right)\left(\frac{\dot{R}}{R} - \frac{\dot{G}}{G}\right) = \frac{\kappa^2 \rho_{m0}}{3f_0 n}\left(\frac{R}{G}\right)^{n-1} a^{\frac{6(n-1)}{3-2n}}, \tag{120}$$

and the first integral for $\mathbf{X}_2$ becomes

$$I_2 = 6f_0 n\left(\frac{R}{G}\right)^{n-1} a^3 H\left\{2n - 3 + 2(1-n)\left(3 - \frac{4R}{G}H^2\right) + \frac{\kappa^2\rho_{m0}}{3f_0 n}\left(\frac{R}{G}\right)^{1-n}\frac{a^{\frac{6(n-1)}{3-2n}}}{H^2}\right\}, \tag{121}$$

using (120).

5. Conclusions

In this work, we have considered both the vacuum and the non-vacuum theories of $F(R,G)$ gravity admitting Noether symmetries. First of all, we have obtained the dynamical field equations for those of gravity theories, which also come from the Lagrangian of $F(R,G)$ gravity in the background of spatially flat FLRW spacetime such that it gives rise to the dynamical field equations varying with respect to the configuration space variables. Afterwards we have used the point-like $F(R,G)$ Lagrangian (16) to write out the Noether symmetry equations, and solve them to get the Noether symmetries in both the vacuum and the non-vacuum cases. It has been appeared very rich cosmological structures from the Noether symmetries for the several functional form of the $F(R,G)$ functions in each of the cases.

The main results of this study can be summarized in the following. First of all, we can verify that all the $F(R,G)$ models studied here admit trivial first integral, namely $E_{\mathcal{L}} = 0$, as they should. Secondly, it is obtained the previous results choosing the $F(R,G)$ function, for example, the case (i) in the vacuum recovers the results of [30] on the Noether symmetries for $n = \frac{3}{2}, \frac{7}{8}$. Using the first integrals directly, we found the analytical solutions (42) and (51) for $n = \frac{3}{2}$ and $n = \frac{7}{8}$, respectively. These cases are also generalized to the non-vacuum and it is found analytical solutions (103) for $n = \frac{3}{2}$ related with the EoS parameter $w = -2/3$, and (108) for $n = \frac{7}{8}$ with the EoS parameter $w = 4/3$. For other values of n, the scale factor $a(t)$ is analytically calculated by (32) in the vacuum section of this study. In each of the cases (i) and (ii) for the vacuum and the non-vacuum, we found the first integrals of Noether symmetries which can be used to provide analytical solutions. As it is pointed out in [25], we also note that the classical Noether symmetry approach with a boundary term K constrains the $F(R,G)$ gravity as a selection criterion that can distinguish the $F(R,G)$ models to utilize the existence of non-trivial Noether symmetries. In this study, we found the maximum number of symmetries as five at the non-vacuum case, but it is four at the vacuum case [28].

This work not only plays complementary role to the previous two studies [19,25], but also includes the the non-vacuum case and it is explicitly found some scale factors in the vacuum case. It might be interesting to perform an analysis of the cosmological parameters for the obtained cosmological models in both of the cases. This will be an argument of future work.

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