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Certain Coefficient Problems for q-Starlike Functions Associated with q-Analogue of Sine Function

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Abstract: This study introduces a subclass S_{qs}^{*} of starlike functions associated with the q-analogue of the sine function defined in symmetric unit disk. This article comprises the investigation of sharp coefficient bounds, and the upper bound of the third-order Hankel determinant for this class. It also includes the findings of Zalcman and generalized Zalcman conjectures for functions of this class.

Keywords: analytic functions; q-starlike functions; univalent function; Hankel determinant; symmetric matrix; Zalcman inequalities; q-derivative operator

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1. Introduction and Preliminaries

In the study of analytic and univalent functions, coefficient problems play a vital role that helps in making many estimations about analytic functions. There is a wide range of coefficient problems that include coefficient bounds, necessary and sufficient conditions, covering results, Hankel determinants, Toeplitz determinants, and many coefficient inequalities and conjectures. The Hankel matrix was introduced by Hermann Hankel, and this matrix is a square symmetric matrix having the same entries in its skew diagonal. To relate the Hankel matrix with analytic functions, it is formed to have elements as the coefficients of certain power series of analytic functions. If an analytic function f, defined in the disk $\mathbf{E} = \{z \in \mathbb{C} : |z| < 1\}$ assumes to satisfy the conditions $f(0) = 0$ and $f'(0) = 1$, then such functions are comprised in class $\mathbf{A}$ and they will have the following form of their Taylor series

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad z \in \mathbf{E}. \tag{1}$$

The determinant of a matrix is an important number that helps in characterizing many properties of that matrix. Based on the coefficients of the series (1), the jth Hankel determinant for $f \in \mathbf{A}$ is given by

$$H_{j,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+j-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+j} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+j-1} & a_{n+j-2} & \cdots & a_{n+2j-2} \end{vmatrix}, \quad j, n \in \mathbb{N}.$$

The main goal in studying the Hankel determinant is to find the upper bound of the determinant. With certain variations in the values of j and n, the Hankel determinant $H_{j,n}(f)$ takes the following forms

$$H_{2,1}(f) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = a_1 a_3 - a_2^2, \quad H_{2,2}(f) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2 a_4 - a_3^2.$$

The determinant $H_{2,1}$ is the renowned Fekete-Szegö functional, Ref. [1] and $H_{2,2}$ is the well-known and extensively studied second Hankel determinant. Also, the third-order Hankel determinant is written as

$$\begin{aligned} H_{3,1}(f) &= \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} \\ &= a_5(a_3 - a_2^2) - a_4(a_4 - a_2 a_3) + a_3(a_2 a_4 - a_3^2). \end{aligned} \tag{2}$$

This implies that

$$|H_{3,1}(f)| \leq |a_5|\left|a_3 - a_2^2\right| + |a_4||a_4 - a_2 a_3| + |a_3||H_{2,2}(f)|. \tag{3}$$

Also, the fourth-order determinant is given by

$$\begin{aligned} H_{4,1}(f) &= \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ a_2 & a_3 & a_4 & a_5 \\ a_3 & a_4 & a_5 & a_6 \\ a_4 & a_5 & a_6 & a_7 \end{vmatrix} \\ &= a_4\Big\{a_4(a_3 a_5 - a_4^2) - a_5(a_2 a_5 - a_3 a_4) + a_6(a_4 - a_2 a_3)\Big\} \\ &\quad - a_5\Big\{a_3(a_3 a_5 - a_4^2) - a_5(a_5 - a_2 a_4) + a_6(a_4 - a_2 a_3)\Big\} \\ &\quad + a_6\Big\{a_3(a_2 a_5 - a_3 a_4) - a_4(a_5 - a_2 a_4) + a_6(a_3 - a_2^2)\Big\} \\ &\quad - a_7\Big\{a_3(a_2 a_4 - a_3^2) - a_4(a_4 - a_2 a_3) + a_5(a_3 - a_2^2)\Big\}. \end{aligned}$$

In 1966, Pommerenke [2,3] studied the Hankel determinants for univalent functions, p-valent functions, and starlike functions. In 1976, Noonan and Thomas [4] analyzed the second Hankel determinants for p-valent functions. In 1983, Noor [5] studied the Hankel determinants for close-to-convex univalent functions. Then, in 1987, Noor [6] studied the Hankel determinants for functions with bounded boundary rotations, and she studied the same for higher-order close-to-convex functions in [7]. Ehrenborg [8] studied the Hankel determinants for exponential polynomials in 2000. Following his work, in 2001, Layman [9] thoroughly studied the Hankel transform, along with its properties. In 2007, Janteng et al. [10] studied the Hankel determinants for starlike and convex functions with respect to symmetric points. Inspired by the research being performed in this area, many researchers have contributed in developing some interesting and useful results, which include some of the following recent developments.

In 2019, Mahmood et al. [11] found the third-order Hankel determinant for a family of q-starlike functions, defined by a q-Ruscheweyh derivative operator, and Srivastava et al. [12] performed the same work for close-to-convex functions associated with Lemniscate of Bernoulli. Arif et al. [13] found the bound of the third Hankel determinant for functions connected with the sine function, and Srivastava et al. [14] found the same bound of Hankel and Toeplitz determinants for q-starlike functions associated with the generalized conic domain. In 2020, Shafiq et al. [15] investigated the bound of the third Hankel determinant for q-starlike functions connected with k-Fibonacci numbers. Murugusundaramoorthy and Bulboacă [16] found the upper bound of Hankel determinants for certain analytic functions connected with the shell-shaped region. In 2021, Guney and Korfeci [17] studied the fourth-order Hankel determinant for analytic functions, which are defined by using the modified sigmoid function, Zhang and Tang [18] found the same bound for functions connected with the sine function, Srivastava et al. [19] investigated third Hankel for q-starlike functions associated with q-analogue of the exponential function, and Saliu and Noor [20] studied third Hankel for analytic functions which are defined by using the Sălăgean differential and Komatu integral operators. Recently, in 2022, Raza et al. [21] studied Hankel determinants for starlike functions connected with symmetric Booth Lemniscate, Khan et al. [22] found the bound of third-order Hankel determinants for logarithmic coefficients of starlike functions connected with Sine function, and Riaz et al. [23–25] studied the Hankel determinants for starlike and convex functions associated with the sigmoid function, lune, and cardioid domain.

Now, we intend to find the upper bound of the third-order Hankel determinant for a subclass of starlike univalent functions, denoted by S^*_{qs}, which is defined below. Also, certain coefficient inequalities named Zalcman and generalized Zalcman inequalities are also part of our investigations. Before introducing the class S^*_{qs}, we need to know about some preliminary concepts, which are stated as follows.

A function f is called univalent in $\mathbf{E}$ if there exists a one-to-one correspondence between $\mathbf{E}$ and $f(\mathbf{E})$. That is, for $z_1, z_2 \in \mathbf{E}$, if $f(z_1) = f(z_2)$ leads to $z_1 = z_2$. The class $\mathbf{S}$ consists of functions that are not only analytic, but univalent as well in $\mathbf{E}$, and the conditions $f(0) = 0, f'(0) = 1$ normalize these functions. That means, $\mathbf{S} = \{f \in A : f$ is univalent in $\mathbf{E}\}$. Starlikeness is a very important geometric property. To define a starlike domain, we join every point of the set with a fixed point through a straight line, and if all such straight lines lie entirely in that domain, then that domain is called starlike with respect to that fixed point to say, w_0. Geometrically speaking, if every point of the domain is visible from that fixed point w_0, then the domain will be starlike or star-shaped with respect to w_0. The function that maps $\mathbf{E}$ onto a domain that is starlike and whose fixed point is the origin is called the starlike function. Furthermore, all those functions of the class $\mathbf{S}$ that satisfy the condition $\Re\left(\frac{zf'(z)}{f(z)}\right) > 0,\ z \in \mathbf{E}$ form the class $\mathbf{S}^*$ of starlike univalent functions. The next important class is the class $\mathbf{P}$ whose elements are all those functions p, which is analytic, and the condition that normalizes these functions is $p(0) = 1$, such that $\Re p(z) > 0,\ z \in \mathbf{E}$. That is,

$$\mathbf{P} = \{p : p(0) = 1 \text{ and } \Re p(z) > 0,\ z \in \mathbf{E}\}$$

and having Taylor series expansion of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n. \tag{4}$$

Let w be an analytic function in $\mathbf{E}$, and it is called the Schwarz function, if $w(0) = 0$, such that $|w(z)| < 1$ for $z \in \mathbf{E}$. Let $f(z)$ and $g(z)$ be analytic functions in $\mathbf{E}$, and if a Schwarz function w exists in $\mathbf{E}$ such that

$$f(z) = g(w(z)),\ z \in \mathbf{E}, \tag{5}$$

then f is said to subordinate g and is denoted by $f \prec g$. If the function g is univalent in $\mathbf{E}$ and $f(0) = g(0)$, then $f(\mathbf{E}) \subset g(\mathbf{E})$. For more details, see [26–28]. By using the relation (5), one can write the formation of classes $\mathbf{P}$ and $\mathbf{S}^*$ as follows.

$$\mathbf{P} = \left\{p : p(0) = 1 \text{ and } p(z) \prec \frac{1+z}{1-z},\ z \in \mathbf{E}\right\},$$

$$\mathbf{S}^* = \left\{f \in \mathbf{S} : \frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z},\ z \in \mathbf{E}\right\}.$$

Quantum calculus, also known as q-calculus, is just the same as classical calculus, but with a major difference, in that unlike the classical form, we do not use limits in q-calculus. We define derivatives as differences and antiderivatives as sums. By definition, the q-derivative of a function f, which is particularly complex-valued and is defined in the domain $\mathbf{D}$, is given as follows

$$(D_q f)(z) = \begin{cases} \frac{f(z)-f(qz)}{(1-q)z}, & z \neq 0, \\ f'(0), & z = 0, \end{cases} \tag{6}$$

where $0 < q < 1$. Also, we see that

$$\lim_{q \to 1^-} (D_q f)(z) = \lim_{q \to 1^-} \frac{f(z) - f(qz)}{(1-q)z} = f'(z),$$

provided that the function f is differentiable in domain $\mathbf{D}$. The Maclaurin's series representation of the function $D_q f$ given in (1) is given by

$$D_q f(z) = \sum_{n=0}^{\infty} [n]_q a_n z^{n-1},$$

where

$$[n]_q = \begin{cases} \frac{1-q^n}{1-q}, & \text{if } n \in \mathbb{C}, \\ \sum_{n=0}^{n-1} q^n, & \text{if } n \in \mathbb{N}. \end{cases} \tag{7}$$

For more details, see [29,30]. To take a brief overview of the applications of q-calculus, we observe mainly that the q-derivative operator D_q is an important tool that is used to define and to thoroughly investigate the numerous subclasses of analytic functions. Similarly, using this derivative operator, a q-extension of the class of starlike functions was firstly given in [31]. However, the development that provided a strong foundation for the application of the q-calculus in the context of Geometric Function Theory was presented by Srivastava, and he achieved this by introducing the basic (or q-) hypergeometric functions; for details, see [32]. To access the recent work on q-derivative, we refer to the following recent developments.

Mahmood et al. [33] studied the coefficient problems of q-starlike functions associated with conic domains. Mahmood et al. [34] studied the geometric properties of certain analytic functions that are defined by using the q-integral operator. Raza et al. [35] studied the q-analogue of differential subordinations by considering the Janowski functions and Lemniscate of Bernoulli. Zainab et al. [36] studied q-starlike functions defined by the q-version of the Ruscheweyh differential operator. Riaz et al. [37] studied the q-starlike functions of negative order, and Saliu et al. [38] studied q-symmetric starlike functions of Janowski type. Moreover, to access the fractional version of certain derivatives like the Caputo fractional derivative, and the conformable fractional derivative, see [39,40] and the references therein. Motivated by this referred work, we now introduce the class S^*_{qs} of q-starlike functions associated with the q-analogue of the function $1 + \sin(qz)$, as follows.

Definition 1. *A function $f \in \mathbf{S}$ is said to be in the class S^*_{qs}, if it satisfies the following condition*

$$\frac{zD_q f(z)}{f(z)} \prec 1 + \sin(qz), \quad z \in \mathbf{E}. \tag{8}$$

That is,

$$S^*_{qs} = \left\{ f \in \mathbf{S} : \frac{zD_q f(z)}{f(z)} \prec 1 + \sin(qz), \quad z \in \mathbf{E} \right\}. \tag{9}$$

The class S_{qs}^* generalizes the class S_s^* of starlike functions associated with the function $1+\sin(z)$ and $\lim\limits_{q\longrightarrow 1^-} S_{qs}^* \equiv S_s^*$. The class S_s^* was introduced and studied by Cho et al. [41].

Now, we proceed to discuss certain coefficient problems for the class S_{qs}^*, for which we need the following lemmas.

Lemma 1 ([42,43]). *If $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \in \mathbf{P}$, then*

$$2c_2 = c_1^2 + x(4 - c_1^2)$$

for some x ($|x| \leq 1$) and

$$4c_3 = c_1^3 + 2(4 - c_1^2)c_1 x - (4 - c_1^2)c_1 x^2 + 2(4 - c_1^2)\left(1 - |x|^2\right)z$$

for some z ($|z| \leq 1$).

Lemma 2. *Let the function $p \in \mathbf{P}$ be given by (4). Then, $\left|c_3 - 2Bc_1c_2 + Dc_1^3\right| \leq 2$ if $0 \leq B \leq 1$ and $B(2B-1) \leq D \leq B$.*

Lemma 3 ([43]). *Let the function $p \in \mathbf{P}$ be given by (4). Then,*

$$|c_n| \leq 2 \qquad (n \in \mathbf{N}). \tag{10}$$

and the inequality is sharp. Also,

$$|c_n - \mu c_k c_{n-k}| \leq 2,\ n > k,\ \mu \in [0,1]. \tag{11}$$

Lemma 4 ([44]). *Let the function $p \in \mathbf{P}$ be given by (4), $0 < a < 1$, $0 < \alpha < 1$ and*

$$8a(1-a)\{(\alpha\beta - 2\lambda)^2 + (\alpha(a+\alpha) - \beta)^2\} + \alpha(1-\alpha)(\beta - 2a\alpha)^2 \leq 4\alpha^2 a(1-\alpha)^2(1-a).$$

Then,

$$\left|\lambda c_1^4 + ac_2^2 + 2\alpha c_1 c_3 - \frac{3}{2}\beta c_1^2 c_2 - c_4\right| \leq 2. \tag{12}$$

Lemma 5 ([45]). *Let $\overline{\mathbf{E}} := \{z \in \mathbb{C} : |z| \leq 1\}$, and for real numbers A, B, C, let*

$$Y(A,B,C) := \max\left\{|A + Bx + Cx^2| + 1 - |x|^2 : x \in \overline{\mathbf{E}}\right\}. \tag{13}$$

If $AC \geq 0$, then

$$Y(A,B,C) = \begin{cases} |A| + |B| + |C|, & |B| \geq 2(1 - |C|), \\ 1 + |A| + \dfrac{B^2}{4(1 - |C|)}, & |B| < 2(1 - |C|). \end{cases}$$

By making use of the above lemmas, we contribute to the study of coefficient problems by investigating the following for the class S_{qs}^*.

1. First, four coefficient bounds $|a_n|$, $n = 2,3,4,5$.
2. The Zalcman inequality $\left|a_n^2 - a_{2n-1}\right| \leq (n-1)^2$ for $n = 2$.
3. The generalized Zalcman inequality $|a_n a_m - a_{n+m-1}| \leq (n-1)(m-1)$ for certain values of m and n.
4. The upper bounds of the second Hankel $|H_{2,2}(f)|$ and the third Hankel determinant $|H_{3,1}(f)|$.

2. Main Results

Theorem 1. *If $f \in S^*_{qs}$ has the series form as given in (1), then*

$$|a_n| \leq \frac{1}{\sum_{j=0}^{n-2} q^j}, \qquad 0 < q < 1, \quad n = 2,3,4 \tag{14}$$

and

$$|a_5| \leq \frac{1}{\sum_{j=0}^{3} q^j}, \quad 0 < q \leq 0.8651682397.$$

The result is sharp.

Proof. If $f \in S^*_{qs}$, then from (5) and (8)

$$\frac{zD_q f(z)}{f(z)} = 1 + \sin(qw(z)), \;\; z \in \mathbf{E}, \tag{15}$$

where $w(z) = \frac{p(z)-1}{1+p(z)}$. If p follows the form of (4), then

$$w(z) = \frac{c_1 z + c_2 z^2 + c_3 z^3 + \cdots}{2 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots}.$$

Using this, one can have

$$\begin{aligned} 1 + \sin(qw(z)) &= 1 + \sin\left(q\left(\frac{c_1 z + c_2 z^2 + c_3 z^3 + \ldots}{2 + c_1 z + c_2 z^2 + c_3 z^3 + \ldots}\right)\right) \\ &= 1 + q\frac{c_1}{2}z + q\left(\frac{c_2}{2} - \frac{c_1^2}{4}\right)z^2 + \left(\frac{-q^3 c_1^3}{48} + q\left(-\frac{c_1 c_2}{4} + \frac{c_3}{2} - \frac{c_1 c_2}{4} + \frac{c_1^3}{8}\right)\right)z^3 \\ &\quad + \left(q\left(-\frac{c_2^2}{4} + \frac{c_1^2 c_2}{8} - \frac{c_1 c_3}{2} + \frac{c_4}{2} + \frac{c_1^2 c_2}{4} - \frac{c_1^4}{8}\right) - \frac{q^3}{8}\left(\frac{c_1^2 c_2}{8} - \frac{c_1^4}{8}\right)\right)z^4 + \cdots. \end{aligned} \tag{16}$$

Now, from (1) and (6), we consider

$$\begin{aligned} \frac{zD_q f(z)}{f(z)} &= \frac{z\left\{\left(z + a_2 z^2 + a_3 z^3 + \ldots\right) - \left(qz + qa_2 z^2 + qa_3 z^3 + \cdots\right)\right\}}{z(1-q)(z + a_2 z^2 + a_3 z^3 + \ldots)} \\ &= 1 + a_2 q z + \left\{q(q+1)a_3 - qa_2^2\right\}z^2 + \left\{q\left(\sum_{j=0}^{2} q^j\right)a_4 - q(2+q)a_2 a_3 + qa_2^3\right\}z^3 \\ &\quad + q\left\{a_5 q^3 + (-a_2 a_4 + a_5)q^2 + (-a_3^2 + a_3 a_2^2 - a_2 a_4 + a_5)q \right. \\ &\quad \left. + 3a_3 a_2^2 - 2a_2 a_4 - a_2^4 - a_3^2 + a_5\right\}z^4 + \cdots \end{aligned} \tag{17}$$

Thus, by using the above series, (15) takes the form

$$
\begin{aligned}
& 1+a_2qz+\left\{q(q+1)a_3-qa_2^2\right\}z^2+\left\{q\left(\sum_{j=0}^{2}q^j\right)a_4-q(2+q)a_2a_3+qa_2^3\right\}z^3+ \\
& +q\left\{a_5q^3+(-a_2a_4+a_5)q^2+(-a_3^2+a_3a_2^2-a_2a_4+a_5)q+3a_3a_2^2\right. \\
& \left.-2a_2a_4-a_2^4-a_3^2+a_5\right\}z^4+\cdots \\
= \; & 1+q\frac{c_1}{2}z+q\left(\frac{c_2}{2}-\frac{c_1^2}{4}\right)z^2+\left(\frac{-q^3c_1^3}{48}+q\left(-\frac{c_1c_2}{4}+\frac{c_3}{2}-\frac{c_1c_2}{4}+\frac{c_1^3}{8}\right)\right)z^3+ \\
& \left(q\left(-\frac{c_2^2}{4}+\frac{c_1^2c_2}{8}-\frac{c_1c_3}{4}+\frac{c_4}{2}+\frac{c_1^2c_2}{4}-\frac{c_1c_3}{4}-\frac{c_1^4}{8}\right)-\frac{q^3}{8}\left(\frac{c_1^2c_2}{8}-\frac{c_1^4}{8}\right)\right)z^4 \\
& +\cdots. \qquad (18)
\end{aligned}
$$

The comparison of coefficients of z, z^2, z^3, and z^4, together with precise computation, yields the following:

$$a_2 = \frac{c_1}{2}, \qquad (19)$$

$$a_3 = \frac{c_2}{2(q+1)}, \qquad (20)$$

$$a_4 = -\frac{q^2c_1^3}{48\left(\sum_{j=0}^{2}q^j\right)}+\frac{c_3}{2\left(\sum_{j=0}^{2}q^j\right)}-\frac{q}{4(q+1)\left(\sum_{j=0}^{2}q^j\right)}c_1c_2, \qquad (21)$$

and

$$
\begin{aligned}
a_5 = \; & \frac{-1}{2(\sum_{j=0}^{3}q^j)}\left[-\frac{2q^4+2q^3+q^2}{48\left(\sum_{j=0}^{2}q^j\right)}c_1^4+\frac{qc_2^2}{2(q+1)}+\frac{q(q+1)c_1c_3}{2\left(\sum_{j=0}^{2}q^j\right)}\right. \\
& \left.-\frac{q^2(1-q-q^2)c_1^2c_2}{8\left(\sum_{j=0}^{2}q^j\right)}-c_4\right]. \qquad (22)
\end{aligned}
$$

Applying (10) to (19) and (20), we obtain $|a_2| \leq 1$ and $|a_3| \leq \frac{1}{q+1}$. Now consider,

$$|a_4| = \frac{1}{2\left(\sum_{j=0}^{2}q^j\right)}\left|c_3-\frac{q}{2(q+1)}c_1c_2-\frac{q^2c_1^3}{24}\right|.$$

Assuming the values $B=\frac{q}{4(q+1)}$ and $D=-\frac{q^2}{24}$, which satisfy $B(2B-1) \leq D \leq B$ for the application of Lemma 2, we get

$$|a_4| \leq \frac{1}{q^2+q+1} = \frac{1}{\sum_{j=0}^{2}q^j}.$$

Now, from (22), consider,

$$\begin{aligned}
|a_5| &= \left| \frac{1}{\sum_{j=0}^{3} q^j} \left[\frac{2q^4+2q^3+q^2}{96\left(\sum_{j=0}^{2} q^j\right)} c_1^4 - \frac{qc_2^2}{4(q+1)} - \frac{q(q+1)c_1c_3}{4\left(\sum_{j=0}^{2} q^j\right)} \right.\right. \\
&\quad \left.\left. - \frac{(-q^5-2q^4+q^2)c_1^2c_2}{8q(q+1)\left(\sum_{j=0}^{2} q^j\right)} + \frac{c_4}{2} \right] \right| \\
&= \left| -\frac{1}{2\left(\sum_{j=0}^{3} q^j\right)} \right| \left| -\frac{2q^4+2q^3+q^2}{48\left(\sum_{j=0}^{2} q^j\right)} c_1^4 + \frac{qc_2^2}{2(q+1)} \right. \\
&\quad \left. + \frac{q(q+1)c_1c_3}{2\left(\sum_{j=0}^{2} q^j\right)} - \frac{(q^5+2q^4-q^2)c_1^2c_2}{4q(q+1)\left(\sum_{j=0}^{2} q^j\right)} - c_4 \right| \\
&= \frac{1}{2\left(\sum_{j=0}^{3} q^j\right)} \left| \lambda c_1^4 + ac_2^2 + 2\alpha c_1c_3 - \frac{3}{2}\beta c_1^2c_2 - c_4 \right|,
\end{aligned}$$

where

$$\begin{aligned}
\lambda &:= -\frac{2q^4+2q^3+q^2}{48\left(\sum_{j=0}^{2} q^j\right)}, \qquad a := \frac{q}{2(q+1)}, \\
\alpha &:= \frac{q(q+1)}{4\left(\sum_{j=0}^{2} q^j\right)}, \quad \beta := \frac{q^2(1-q-q^2)}{12\left(\sum_{j=0}^{2} q^j\right)}.
\end{aligned}$$

We see that $0 < a < 1, 0 < \alpha < 1$. Now,

$$\begin{aligned}
8a(1-a)\{(\alpha\beta-2\lambda)^2+(\alpha(a+\alpha)-\beta)^2\}+\alpha(1-\alpha)(\beta-2a\alpha)^2-4\alpha^2a(1-\alpha)^2(1-a) &= \\
\frac{\begin{array}{c}(53q^{11}+318q^{10}+589q^9+2128q^8+3210q^7+3304q^6\\ +1867q^5-452q^4-2091q^3-2118q^2-1152q-288)q^3\end{array}}{2304(q+1)^2\left(\sum_{j=0}^{2} q^j\right)^4} &= \Psi(q)
\end{aligned}$$

A calculation shows that the equation $\Psi(q) \leq 0$ when $0 < q \leq 0.8651682397$.
Hence, $\Psi(q) \leq 0$. Now, by using Lemma 4, we can have

$$|a_5| \leq \frac{1}{\sum_{j=0}^{3} q^j}.$$

For sharpness, consider the function $f_n : \boldsymbol{E} \longrightarrow \mathbb{C}$, defined by

$$\frac{zD_qf_n(z)}{f_n(z)} = 1 + \sin(qz^n), \quad n = 2,3,4,5. \tag{23}$$

Then, it is easy to see that the function $f_n \in S_{qs}^*$. A simple calculations shows that

$$\begin{aligned}
f_2(z) &= z + \frac{q}{[2]_q - 1} z^2 + \cdots, \\
f_3(z) &= z + \frac{q}{[3]_q - 1} z^3 + \frac{q^2}{-[3]_q + [5]_q[3]_q + 1 - [5]_q} z^5 + \cdots, \\
f_4(z) &= z + \frac{q}{[4]_q - 1} z^4 + \cdots, \\
f_5(z) &= z + \frac{q}{[5]_q - 1} z^5 + \cdots.
\end{aligned}$$

Hence, by using (7), the result is sharp. □

Considering $q \longrightarrow 1^-$ in the above result, we obtain the following improved result than the one proven in [46].

Corollary 1. *If $f \in S_s^*$ has the series form as given in (1), then,*

$$|a_n| \leq \frac{1}{n-1}, n = 2,3,4.$$

3. Zalcman and Generalized Zalcman Conjecture

In 1960, Zalcman proposed a remarkable conjecture for univalent functions whose generalized version were given by Ma [47] in 1999. These conjectures are still open but have been proven for certain subclasses of univalent functions. Zalcman's conjecture states that every $f \in \mathbf{S}$ having the form of (1) satisfies the following sharp inequality.

$$\left|a_n^2 - a_{2n-1}\right| \leq (n-1)^2, \quad n \geq 2. \tag{24}$$

Whereas, the generalized Zalcman conjecture states that the Taylor series coefficients from (1) of univalent functions $f \in \mathbf{S}$ satisfy the following inequality.

$$|a_n a_m - a_{n+m-1}| \leq (n-1)(m-1), \quad \forall\, m,n \in \mathbb{N},\ n \geq 2,\ m \geq 2. \tag{25}$$

We intend to find these inequalities for the considered class S_{qs}^* for certain values of n and m. For $n = 2$, the inequality (24) takes the form $\left|a_2^2 - a_3\right| \leq 1$.

Theorem 2. *If $f \in S_{qs}^*$ has the series form as given in (1), Then,*

$$\left|a_3 - a_2^2\right| \leq \frac{1}{q+1}. \tag{26}$$

The above inequality is sharp, which can be obtained with the function f_3, given in (23).

Proof. From (19) and (20), consider

$$\left|a_3 - a_2^2\right| = \frac{1}{2(q+1)}\left|c_2 - \frac{(q+1)c_1^2}{2}\right| = \frac{1}{2(q+1)}\left|c_2 - vc_1^2\right|,$$

where $v = (q+1)/2$. Since $q \in (0,1)$, therefore, $0 < v < 1$. Now, by using (11) for $n = 2$ and $k = 1$, we obtain (26). □

Upon letting $q \longrightarrow 1^-$, the above result reduces to the following, proven in [46].

Corollary 2. *If $f \in S_s^*$ has the series form as given in (1), then,*

$$\left|a_3 - a_2^2\right| \leq \frac{1}{2}.$$

For $n = 3,\ m = 2$, the inequality (25) reduces to $|a_2 a_3 - a_4| \leq 2$. We discuss it as follows:

Theorem 3. *If $f \in S_{qs}^*$ has the series form as given in (1), then,*

$$|a_4 - a_3 a_2| \leq \frac{1}{q^2+q+1}, \quad q \in \left(0, \frac{\sqrt{3}}{2}\right). \tag{27}$$

The result is sharp for the function f_4, given in (23).

Proof. From (19), (20), and (21), consider

$$\begin{aligned} a_4 - a_3 a_2 &= -\frac{q^2 c_1^3}{48\left(\sum_{j=0}^{2} q^j\right)} + \frac{c_3}{2\left(\sum_{j=0}^{2} q^j\right)} - \frac{q}{4(q+1)\left(\sum_{j=0}^{2} q^j\right)} c_1 c_2 - \frac{c_1 c_2}{4(q+1)}, \\ &= -\frac{q^2 c_1^3}{48\left(\sum_{j=0}^{2} q^j\right)} + \frac{c_3}{2\left(\sum_{j=0}^{2} q^j\right)} - \frac{(q^2+2q+1)}{4(q+1)\left(\sum_{j=0}^{2} q^j\right)} c_1 c_2. \end{aligned}$$

Taking the modulus, we get

$$\begin{aligned} |a_4 - a_3 a_2| &= \left| -\frac{q^2 c_1^3}{48\left(\sum_{j=0}^{2} q^j\right)} + \frac{c_3}{2\left(\sum_{j=0}^{2} q^j\right)} - \frac{(q^2+2q+1)}{4(q+1)\left(\sum_{j=0}^{2} q^j\right)} c_1 c_2 \right| \\ &= \left| \frac{1}{2\left(\sum_{j=0}^{2} q^j\right)} \right| \left| c_3 - \frac{(q^2+2q+1)}{2(q+1)} c_1 c_2 - \frac{q^2 c_1^3}{24} \right|. \end{aligned}$$

Assuming the values $B = \frac{(q^2+2q+1)}{2(q+1)}$ and $D = -\frac{q^2}{24}$, we see that

$$B(2B-1) - D = \frac{1}{24}\left(4q^2 - 3\right) < 0, \quad \text{for } q \in \left(0, \frac{\sqrt{3}}{2}\right)$$

which shows that $B(2B-1) \leq D \leq B$. Thus, the application of Lemma 2 gives that

$$|a_4 - a_3 a_2| \leq \frac{1}{q^2+q+1}.$$

□

Now, the following results investigate the inequality (24) for $m = n = 3$.

Theorem 4. *If $f \in S_{qs}^*$ has the series form as given in (1), then,*

$$\left|a_3^2 - a_5\right| \leq \frac{1}{\sum_{j=0}^{3} q^j}, \qquad q \in (0, 0.3898584501).$$

The result is sharp for the function f_5, given in (23).

Proof. From (20) and (22), consider

$$\begin{aligned} \left|a_3^2 - a_5\right| &= \left| \left(\frac{c_2}{2(q+1)}\right)^2 - \left(\frac{1}{\sum_{j=0}^{3} q^j}\right)\left(\frac{2q^4+2q^3+q^2}{96\left(\sum_{j=0}^{2} q^j\right)} c_1^4 - \right.\right. \\ &\quad \left.\left. - \frac{q c_2^2}{4(q+1)} - \frac{q(q+1)c_1 c_3}{4\left(\sum_{j=0}^{2} q^j\right)} - \frac{(q^5+2q^4+q^2)c_1^2 c_2}{8q(q+1)\left(\sum_{j=0}^{2} q^j\right)} + \frac{c_4}{2}\right) \right| \\ &= \frac{1}{2\left(\sum_{j=0}^{3} q^j\right)} \left| \left(-\frac{q^2(1+2q+2q^2)}{48\left(\sum_{j=0}^{2} q^j\right)} c_1^4 + \frac{q^2+q+1}{2(q+1)} c_2^2 \right.\right. \\ &\quad \left.\left. + \frac{q(q+1)}{2\left(\sum_{j=0}^{2} q^j\right)} c_1 c_3 + \frac{q^2(q^2+q-1)}{8\left(\sum_{j=0}^{2} q^j\right)} c_1^2 c_2 - c_4 \right) \right| \\ &= \frac{1}{2\left(\sum_{j=0}^{3} q^j\right)} \left| \lambda c_1^4 + a c_2^2 + 2\alpha c_1 c_3 - \frac{3}{2}\beta c_1^2 c_2 - c_4 \right|, \end{aligned}$$

where

$$\lambda \; : \; = -\frac{q^2(1+2q+2q^2)}{48\left(\sum_{j=0}^{2} q^j\right)}, \qquad a := \frac{q^2+q+1}{2(q+1)},$$
$$\alpha \; : \; = \frac{q(q+1)}{4\left(\sum_{j=0}^{2} q^j\right)}, \qquad \beta := -\frac{q^2(q^2+q-1)}{8\left(\sum_{j=0}^{2} q^j\right)}.$$

We see that $0 < a < 1, 0 < \alpha < 1$. Now,

$$\begin{aligned} 8a(1-a)\{(\alpha\beta - 2\lambda)^2 + (\alpha(a+\alpha)-\beta)^2\} + \alpha(1-\alpha)(\beta - 2a\alpha)^2 - 4\alpha^2 a(1-\alpha)^2(1-a) &= \\ \frac{\begin{array}{c} q^2(-770q^3 - 4011q^4 - 8430q^5 - 11127q^6 - 10282q^7 - 6180q^8 \\ -1810q^9 + 845q^{10} + 72 + 1340q^{11} + 835q^{12} + 296q^{13} + 50q^{14} + 348q + 416q^2) \end{array}}{2304(q+1)^2\left(\sum_{j=0}^{2} q^j\right)^4} &= \Psi_1(q) \end{aligned}$$

A calculation shows that the equation $\Psi_1(q) \le 0$ when $0 < q \le 0.3898584501$. Hence, $\Psi_1(q) \le 0$. Now by using Lemma 4, we can have

$$\left|a_3^2 - a_5\right| \le \frac{1}{\sum_{j=0}^{3} q^j}$$

and $\frac{1}{\sum_{j=0}^{3} q^j}$, for $q \in (0, 0.3898584501)$. This shows that the inequality (24) is satisfied for $n = 3$. □

The following result investigates the inequality (25) for $n = 4,\ m = 2$.

Theorem 5. *If the function $f \in S_{qs}^*$ has the form (1), then,*

$$|a_4 a_2 - a_5| \le \frac{1}{\sum_{j=0}^{3} q^j}, \quad q \in (0, 0.1889972572].$$

Proof. From (19), (21), and (22), consider

$$\begin{aligned} |a_5 - a_2 a_4| &= \frac{1}{2\left(\sum_{j=0}^{3} q^j\right)}\left|\frac{-q^2(q+2)}{48}c_1^4 + \frac{qc_2^2}{2(q+1)} + \frac{(q+1)c_1c_3}{2} + \frac{q(q-2)c_1^2c_2}{8} - c_4\right| \\ &= \frac{1}{2\left(\sum_{j=0}^{3} q^j\right)}\left|\lambda c_1^4 + ac_2^2 + 2\alpha c_1 c_3 - \frac{3}{2}\beta c_1^2 c_2 - c_4\right|, \end{aligned}$$

where

$$\lambda \; : \; = \frac{-q^2(q+2)}{48}, \qquad a := \frac{q}{2(q+1)},$$
$$\alpha \; : \; = \frac{(q+1)}{4}, \qquad \beta := \frac{q(2-q)}{12}.$$

We see that $0 < a < 1, 0 < \alpha < 1$. Now,

$$\begin{aligned} 8a(1-a)\{(\alpha\beta - 2\lambda)^2 + (\alpha(a+\alpha)-\beta)^2\} + \alpha(1-\alpha)(\beta - 2a\alpha)^2 - 4\alpha^2 a(1-\alpha)^2(1-a) &= \\ \frac{q(40q^5 - 146q^4 + 82q^6 + 751q^3 + 17q^7 + 1930q^2 + 276q - 126)}{2304(q+1)^2} &= \Psi_2(q) \end{aligned}$$

A calculation shows that the equation $\Psi_2(q) \leq 0$ when $0 < q \leq 0.1889972572$. For the application of (12), we have

$$|a_5 - a_2 a_4| \leq \frac{1}{\sum_{j=0}^{3} q^j},$$

which completes the proof. □

4. Hankel Determinants

Theorem 6. *If $f \in S^*_{qs}$ has the series form as given in (1), then,*

$$|H_{2,2}(f)| \leq \frac{1}{(q+1)^2}. \tag{28}$$

The result is sharp for the function f_3 given in (23).

Proof. From (19), (20), and (21), consider

$$H_{2,2}(f) = a_2 a_4 - a_3^2 = -\frac{q^2 c_1^4}{96\left(\sum_{j=0}^{2} q^j\right)} - \frac{q}{8(q+1)\left(\sum_{j=0}^{2} q^j\right)} c_1^2 c_2 + \frac{c_1 c_3}{4\left(\sum_{j=0}^{2} q^j\right)} - \frac{c_2^2}{4(q+1)^2}.$$

Since the class S^*_{qs} and the functional $H_{2,2}$ are rotationally invariant, we can consider that $c := c_1 \in [0,2]$. Then, using Lemma 1 and after simplification, we obtain

$$H_{2,2}(f) = \frac{1}{8\left(\sum_{j=0}^{2} q^j\right)} \left(\begin{array}{c} -\frac{q^2\left(q^2+2q+7\right)c^4}{12(q+1)^2} + \frac{q(4-c^2)(1-q)xc^2}{2(q+1)^2} \\ -\frac{\left(qc^2+4\left(\sum_{j=0}^{2} q^j\right)\right)(4-c^2)x^2}{2(q+1)^2} + c(4-c^2)\left(1-|x|^2\right)z \end{array} \right)$$

where x and z are such that $|x| \leq 1$, $|z| \leq 1$.

(1) Suppose first that $c = 2$. Then, $|H_{2,2}(f)| = \frac{q^2\left(q^2+2q+7\right)}{6(q+1)^2\left(\sum_{j=0}^{2} q^j\right)} < \frac{1}{(q+1)^2}$.

(2) When $c = 0$,

$$|H_{2,2}(f)| \leq \frac{|x|}{(q+1)^2} \leq \frac{1}{(q+1)^2}.$$

(3) Next, assume that $c \in (0,2)$ and the application of the triangle inequality gives

$$|H_{2,2}(f)| \leq \frac{c(4-c^2)}{8\left(\sum_{j=0}^{2} q^j\right)} \Psi(A,B,C),$$

where

$$\Psi(A,B,C) := |A + Bx + Cx^2| + 1 - |x|^2, \quad |x| \leq 1,$$

with

$$A := -\frac{q^2\left(q^2+2q+7\right)c^3}{12(q+1)^2(4-c^2)}, \quad B := \frac{q(1-q)c}{2(q+1)^2}, \quad C := -\frac{\left(qc^2+4\left(\sum_{j=0}^{2} q^j\right)\right)}{2c(q+1)^2}.$$

Clearly,

$$AC = \frac{q^2\left(qc^2+4\left(\sum_{j=0}^{2} q^j\right)\right)\left(q^2+2q+7\right)c^2}{24(q+1)^4(4-c^2)} > 0, \quad c \in (0,2), \quad q \in (0,1).$$

We now show that $|B| \geq 2(1-|C|)$. For this, we define the function

$$\varphi(c) = 2(q+1)^2 c(|B| - 2(1-|C|)) = q(3-q)c^2 - 4(q^2+2q+1)c + 8\left(\sum_{j=0}^{2} q^j\right).$$

Now,

$$\begin{aligned} \varphi'(c) &= 2q(3-q)c - 4(q^2+2q+1), \\ \varphi''(c) &= 2q(3-q) > 0, \quad q \in (0,1). \end{aligned}$$

This shows that the function φ' is increasing and

$$\max \varphi'(c) = \varphi'(2) = -8q^2 + 4q - 4 < 0, \quad q \in (0,1).$$

This implies that

$$\varphi'(c) < 0,\ c \in (0,2), \quad q \in (0,1).$$

Hence, φ is a decreasing function and

$$\min \varphi(c) = \varphi(2) = 4q(1-q) > 0, \quad q \in (0,1).$$

We conclude that $(|B| - 2(1-|C|)) > 0$. Then, by Lemma 5,

$$|H_{2,2}(f)| \leq \frac{c(4-c^2)}{8\left(\sum_{j=0}^{2} q^j\right)}(|A| + |B| + |C|) = g(c), \tag{29}$$

where

$$g(c) := -\frac{1}{96(q+1)^2\left(\sum_{j=0}^{2} q^j\right)}\left(\begin{array}{c} (-7\,q^2 - 2\,q^3 + 6\,q(1-q) + 6\,q - q^4)c^4 \\ +\,(-24\,q(1-q) + 24\,q^2 + 24)c^2 \;-96(1+\,q+\,q^2) \end{array}\right).$$

Now,

$$g'(c) = \frac{1}{96(q+1)^2\left(\sum_{j=0}^{2} q^j\right)}\left(Lc^3 + Mc\right),$$

where

$$\begin{aligned} L &= -\left(-28\,q^2 - 8\,q^3 + 24\,q(1-q) + 24\,q - 4\,q^4\right) < 0, \quad q \in (0,1), \\ M &= -\left(48 - 48\,q(1-q) + 48\,q^2\right) < 0, \quad q \in (0,1). \end{aligned}$$

We conclude that g is a decreasing function and

$$g(c) \leq g(0) = \frac{1}{(q+1)^2}, \quad q \in (0,1).$$

Hence,

$$|H_{2,2}(f)| \leq \frac{1}{(q+1)^2}.$$

The result is sharp, which can be obtained with the function f_3, given in (23). □

Upon letting $q \longrightarrow 1^-$, the above result reduces to the following, proven in [46].

Corollary 3. *If $f \in S_s^*$ has the series form as given in (1), then,*

$$\left|a_2 a_4 - a_3^2\right| \leq \frac{1}{4}.$$

Theorem 7. *If $f \in S^*_{qs}$ has the series form as given in (1), then,*

$$|H_{3,1}(f)| \leq \frac{1}{\left(\sum_{j=0}^{3} q^j\right)(q+1)} + \frac{1}{\left(\sum_{j=0}^{2} q^j\right)^2} + \frac{1}{(q+1)^3}, \quad q \in (0, 0.86). \tag{30}$$

The proof follows easily by using the inequalities of Theorem 1, Theorem 6, Theorem 3, and Theorem 2 in (3).

5. Conclusions

This work has introduced a new class S^*_{qs} of q-starlike functions linked with the q-analogue of the sine function through subordination relation. This class generalizes the class S^*_s of starlike functions. The investigations of certain coefficient inequalities like sharp coefficient bounds, the upper bound of the third-order Hankel determinant, Zalcman inequalities, and generalized Zalcman inequalities for the class S^*_{qs} have been included in this work. It also improves the coefficient bounds and the upper bound of the third-order Hankel determinant for the class S^*_s. Moreover, for future work, the defined class S^*_{qs} can be further investigated for finding the upper bounds of higher-order Hankel and Toeplitz determinants. Meanwhile, the obtained coefficient bounds may be used for studying many other unaddressed coefficient problems for this class.

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