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AGGREGATION OPERATORS FROM THE ANCIENT NC AND EM POINT OF VIEW

Ana Pradera and Enric Trillas

This paper deals with the satisfaction of the well-known Non-Contradiction (NC) and Excluded-Middle (EM) principles within the framework of aggregation operators. Both principles are interpreted in a non-standard way, based on self-contradiction (as in Ancient Logic) instead of falsity (as in Modern Logic). The logical negation is represented by means of strong negation functions, and conditions are given both for those aggregation operators that satisfy NC/EM with respect to (w.r.t.) some given strong negation, as well as for those satisfying the laws w.r.t. any strong negation. The results obtained are applied to some of the most important known classes of aggregation operators.

Keywords: Non-Contradiction and Excluded-Middle principles, aggregation operators, strong negations

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1. INTRODUCTION

Information aggregation is a crucial issue in the construction of many intelligent systems, and it is used in different application domains, such as medicine, economics, engineering, statistics or decision-making processes. It is particularly useful in situations presenting some degree of uncertainty or imprecision, a feature that explains the great development that this discipline has experimented in recent years within the field of Fuzzy Logic. It is a well assorted research field, whose topics of interest range from theoretical aspects to the use of the different aggregation methods and techniques in practical situations. A large collection of distinguished classes of aggregation operators and construction methods is nowadays available, and different potential application fields have been explored (see for example [1, 4] or the recent overview on aggregation theory given in [3]).

When using aggregation techniques in practical situations, one of the first problems that one has to face up is the choice, among all the aggregation operators that are available, of the most suited one. Clearly, there is not a universal answer to this problem, since the decision is largely context-dependent. Notwithstanding, there are several criteria that may help in making this decision, such as the achievement of empirical experiments, the analysis of the expected operator's behavior (tolerant, intolerant, compensatory) or the need of some mathematical/logical properties (e. g. idempotency, symmetry, associativity, the existence of neutral or annihilator elements, etc).

This paper deals with the satisfaction of two of these mathematical properties, the well-known *Non-Contradiction* (NC) and *Excluded-Middle* (EM) laws, when interpreted in a specific way. Indeed, according to [10], each of these laws may be interpreted in at least two different ways: the standard one, based on falsity (as it is done in Modern Logic), and a non-standard one, based on self-contradiction (as in Ancient Logic). The present paper concentrates on the latter interpretation (the former one has been addressed in [7, 8]). It is organized as follows. It begins by briefly recalling the main issues on aggregation operators and negations that are needed later on, and by discussing the different interpretations of the NC and EM laws and their translation into the aggregation operators' framework (Section 2). Then Section 3 studies the satisfaction of the NC law, while Section 4 does the same for the EM law. Section 5 deals with the relationships between the two laws, and Section 6 applies all the previous results to some of the most important families of aggregation operators. Finally, the paper ends with some conclusions.

2. PRELIMINARIES

This section recalls the main concepts that conform the general framework of the paper, namely, aggregation operators and strong negations on one hand, and the interpretation of the Non-Contradiction and Excluded-Middle principles on the other.

2.1. Aggregation operators and strong negations

Although aggregation operators are defined for the general multidimensional case ([3]), in this paper we will only deal with *binary aggregation operators*, i. e., non-decreasing operators $A : [0,1]^2 \to [0,1]$ verifying the boundary conditions $A(0,0) = 0$ and $A(1,1) = 1$. Aggregation operators may be compared pointwise, that is, given two operators A_1 and A_2, it is said that A_1 is *weaker* than A_2 (or A_2 is *stronger* than A_1), and it is denoted $A_1 \le A_2$, when it is $A_1(x,y) \le A_2(x,y)$ for any $x, y \in [0,1]$. This relation is clearly a partial order (i. e., there are couples of aggregation operators which are non-comparable) and, along with the distinguished operators *Min* (minimum) and *Max* (maximum), allows to classify aggregation operators in the following four categories (see [3] for details on the different families of aggregation operators):

- *Conjunctive* operators, which are those verifying $A \le Min$. This class includes the well-known *triangular norms* (*t-norms*) as well as *copulas* (see, respectively, [5] and [6]).
- *Disjunctive* operators, verifying $Max \le A$, such as *triangular conorms (t-conorms)* and *dual copulas.*
- *Averaging* operators (or *mean* operators) which verify $Min \le A \le Max$. These operators are always idempotent (i. e., $A(x,x) = x$ holds for any $x \in [0,1]$),

and some distinguished ones in this class are those based on the arithmetic mean, such as *quasi-linear means* or *OWA operators*, as well as those based on integrals, such as *Lebesgue, Choquet* or *Sugeno* integral-based aggregations.

- Finally, the class of *hybrid* aggregation operators contains all the operators that do not belong to any of the three previous categories, i. e., operators that are not comparable with *Min* and/or are not comparable with *Max*. This class includes different aggregation operators related to t-norms and t-conorms (such as *uninorms*, *nullnorms* or *compensatory operators*) as well as *symmetric sums*.

In order to translate the Non-Contradiction and Excluded-Middle laws to the aggregation operators' field, a way for representing the logic negation is needed. The latter is usually done by means of the so-called *strong negations* ([9]), i. e., non-increasing functions $N : [0, 1] \to [0, 1]$ which are involutive, that is, verify $N(N(x)) = x$ for any $x \in [0, 1]$. Due to their definition, strong negations are continuous and strictly decreasing functions, they satisfy the boundary conditions $N(0) = 1$ and $N(1) = 0$, and they have a unique fixed point, that we will denote x_N, verifying $0 < x_N < 1$ and $N(x_N) = x_N$. Note also that, for any $x \in [0, 1]$, it is $x \leq N(x)$ if and only if $x \leq x_N$. The most commonly used strong negation is the so-called *standard negation*, defined as $N(x) = 1-x$ for all $x \in [0, 1]$. Despite its simplicity, this function plays a fundamental role in the construction of strong negations, since any strong negation may be built, by means of an automorphism, starting from the standard negation. Indeed, in [9] it was proved that a function $N : [0, 1] \to [0, 1]$ is a strong negation if and only if there exists an automorphism $\varphi : [0, 1] \to [0, 1]$ (a continuous and strictly increasing function verifying the boundary conditions $\varphi(0) = 0$ and $\varphi(1) = 1$) such that $N = \varphi^{-1} \circ (1 - Id_{[0,1]}) \circ \varphi$, i. e., $N(x) = \varphi^{-1}(1 - \varphi(x))$ for any $x \in [0, 1]$.

Recall finally that strong negations may be used to construct new aggregation operators from given ones by reversing the input scales (see e. g. [3]): if A is a binary aggregation operator and N is a strong negation, the operator $A_N : [0, 1]^2 \to [0, 1]$, defined as $A_N = N \circ A \circ (N \times N)$, is in turn a binary aggregation operator, called the N-*dual* operator of A. Thanks to the involutive nature of N, the N-dual of A_N coincides with the starting aggregation operator, i. e., it is $(A_N)_N = A$. When N is the standard negation, A_N is simply called the *dual* operator of A.

2.2. The Non-Contradiction and Excluded-Middle principles

The laws of Non-Contradiction (NC) and Excluded-Middle (EM) have been the object of many controversies along the centuries, and one can find many references concerning them. Focusing on the NC principle, it is well-known that this law, in its ancient Aristotelian formulation, can be stated as follows: for any statement p, the statements p and *not* p cannot be at the same time, i. e., $p \wedge \neg p$ is *impossible*, where the binary operation $\wedge$ represents the *and* connective and the unary operation $\neg$ models the negation. In [10] it is argued that such formulation may be interpreted in at least two different ways, depending on how the term *impossible* is understood:

- If the approach that is common in Modern Logic is adopted, the term *impossible* may be thought as *false*, and then the NC principle may be expressed, in a structure with minimum element $\mathbf{0}$, as $p \wedge \neg p = \mathbf{0}$ for any statement p.

- Another possibility, which may be considered closer to Ancient Logic, is to interpret *impossible* as *self-contradictory*, understanding that an object is self-contradictory whenever it entails its negation. In this case, the NC principle may be written as $p \wedge \neg p \vDash \neg(p \wedge \neg p)$ for any statement p, where $\vDash$ represents an entailment relation.

Similar arguments may be applied to the Excluded-Middle law. Indeed, such law may be formulated as follows: for any statement p, either p or *not* p are true, or, equivalently, $\neg(p \vee \neg p)$ is *impossible*. This allows again for two different interpretations: $\neg(p \vee \neg p) = \mathbf{0}$ (Modern Logic) and $\neg(p \vee \neg p) \vDash \neg(\neg(p \vee \neg p))$ (Ancient Logic). Note that if the negation $\neg$ is involutive (i. e., it verifies $\neg\neg p = p$ for any p), then the latter may be written as $\neg(p \vee \neg p) \vDash p \vee \neg p$.

In the setting of orthocomplemented lattices (and therefore in both quantum and classical logics) the modern and ancient interpretations coincide (both for NC and EM), since the minimum element is the only self-contradictory object. Nevertheless, [10] proves that this is not the case for more general structures, where the first approach is clearly stronger than the second one. Indeed, for the NC law case (it is similar for the EM law), if $p \wedge \neg p = \mathbf{0}$, then obviously $\mathbf{0} = p \wedge \neg p \vDash \neg(p \wedge \neg p)$ is also verified, since $\mathbf{0}$ is the minimum element, but the contrary is not always true. One of the structures where the two interpretations differ is the lattice $([0,1], \leq)$: taking for example the operations $x \wedge y = Min(x,y)$ and $\neg x = 1 - x$, it appears that these operations do not satisfy the NC law when using the first interpretation (since $Min(x, 1-x) = 0$ is only true for $x \in \{0,1\}$) but do satisfy it when using the self-contradiction view (since $Min(x, 1-x) \leq 1/2$ is true for any $x \in [0,1]$).

The lattice $([0,1], \leq)$ is precisely the framework associated to binary *aggregation operators* acting on $[0,1]$. If, in addition, *strong negations* (which, as we have just seen, are involutive functions) are used to represent the logical negation, the NC and the EM laws can be interpreted in the two following ways (where $A : [0,1]^2 \to [0,1]$ is an aggregation operator and $N : [0,1] \to [0,1]$ is a strong negation):

- Modern Logic (ML) interpretation:
  - NC(ML): $\forall\, x \in [0,1], A(x, N(x)) = 0$
  - EM(ML): $\forall\, x \in [0,1], A(x, N(x)) = 1$.
- Ancient Logic (AL) interpretation:
  - NC(AL): $\forall\, x \in [0,1], A(x, N(x)) \leq N(A(x, N(x)))$
  - EM(AL): $\forall\, x \in [0,1], N(A(x, N(x))) \leq A(x, N(x))$.

The first interpretation has been addressed by the authors in [7, 8], and the present paper deals with the second one.

Note finally that the NC and EM laws, when dealing with involutive negations, are *dual* laws ([10]). In the case of aggregation operators and strong negations, this duality, which is valid for any of the two mentioned interpretations, may be stated as follows:

Proposition 1. Let A be a binary aggregation operator and let N be a strong negation. Then A satisfies NC if and only if its dual operator, A_N, satisfies EM.

3. AGGREGATION OPERATORS AND THE NON–CONTRADICTION PRINCIPLE

Note first of all that, given an aggregation operator A, the satisfaction of the NC law interpreted as in Ancient Logic, $\forall\, x \in [0,1], A(x, N(x)) \leq N(A(x, N(x)))$, may be investigated either with respect to a given strong negation N or with respect to *any* strong negation. This distinction allows for the following definition:

Definition 1. Let A be a binary aggregation operator and let N be a strong negation.

1. It is said that A *satisfies the Non-Contradiction principle with respect to N in the ancient logical sense* (NC(AL)) when $A(x, N(x)) \leq N(A(x, N(x)))$ holds for any $x \in [0,1]$.
2. It is said that A *satisfies NC(AL)* when there exists a strong negation N such that A satisfies NC(AL) w.r.t. N.
3. It is said that A *strictly satisfies NC(AL)* when it satisfies NC(AL) w.r.t. any strong negation.

The next two subsections study first the satisfaction and then the strict satisfaction of the NC(AL) principle.

3.1. Satisfaction of the Non-Contradiction principle

The satisfaction of the NC(AL) principle can be usefully characterized in two different ways, the first one in terms of the negation's fixed point and the second one involving duality:

Proposition 2. Let A be a binary aggregation operator and let N be a strong negation with fixed point x_N. Then the following conditions are equivalent:

1. A satisfies NC(AL) w.r.t. N.
2. For all $x \in [0,1]$, $A(x, N(x)) \leq x_N$ (i. e., $\sup_{x\in[0,1]} A(x, N(x)) \leq x_N$).
3. For all $x \in [0,1]$, $A(x, N(x)) \leq A_N(N(x), x)$.

P r o o f . To prove the equivalence *1-2*, it suffices to apply to Definition 1 the well-known result which states that for any strong negation N and for any $x \in [0,1]$, it is $x \leq N(x)$ if and only if $x \leq x_N$. The equivalence *1-3* is obtained by definition of the dual operator A_N, because for any $x \in [0,1]$ it is $A_N(N(x),x) = N(A(N(N(x)),N(x)))$, and, since N is involutive, this becomes $A_N(N(x),x) = N(A(x,N(x)))$. □

Let us now establish some sufficient conditions under which an aggregation operator satisfies the NC(AL) principle w.r.t. a given strong negation:

Proposition 3. Let A be a binary aggregation operator and let N be a strong negation. If any of the following conditions is verified, then A satisfies the NC(AL) principle w.r.t. N:

1. $A(x,N(x)) \leq Min(x,N(x))$ for all $x \in [0,1]$.
2. A is commutative and verifies $A \leq A_N$.
3. $A(x,N(x)) \leq B(x,N(x))$ for all $x \in [0,1]$, where $B : [0,1]^2 \to [0,1]$ is an aggregation operator satisfying NC(AL) w.r.t. N.

P r o o f .

1. If x_N is the negation's fixed point, it is clearly $Min(x,N(x)) \leq x_N$ for any $x \in [0,1]$, and then $A(x,N(x)) \leq x_N$, which means, by Proposition 2, that A satisfies NC(AL) w.r.t. N.
2. $A \leq A_N$ implies, in particular, that $A(x,N(x)) \leq A_N(x,N(x))$ for any $x \in [0,1]$. Since A is commutative (and therefore A_N is commutative), the latter is equivalent to $A(x,N(x)) \leq A_N(N(x),x)$ for any $x \in [0,1]$, and then, according to Proposition 2, A satisfies NC(AL) w.r.t. N.
3. With the given conditions, it is $\sup x \in [0,1] A(x,N(x)) \leq \sup_{x \in [0,1]} B(x,N(x)) \leq x_N$, i. e., A satisfies NC(AL) w.r.t. N. □

The above conditions allow to find both averaging, hybrid and conjunctive aggregation operators satisfying NC(AL). Examples in the two first categories will be given in Section 6 Regarding conjunctive operators, note that the first condition in Proposition 3 shows that *any* conjunctive aggregation operator (and hence any triangular norm and any copula) satisfies NC(AL) regardless of the strong negation that is chosen, that is:

Corollary 1. Conjunctive aggregation operators strictly satisfy the NC(AL) principle.

On the other hand, next Proposition provides some necessary conditions that any aggregation operator must necessarily fulfill in order to satisfy the NC(AL) principle w.r.t. a given strong negation:

Proposition 4. Let A be a binary aggregation operator and let N be a strong negation with fixed point x_N. If A satisfies the NC(AL) principle w.r.t. N, then:

1. For all $x \in [0,1]$, $A(x, N(x)) \leq Max(x, N(x))$.
2. $\sup_{x\in[0,1]} A(x, N(x)) \neq 1$ (in particular, $A(0,1) \neq 1$, $A(1,0) \neq 1$).
3. $x_N \geq Max(A(0,1), A(1,0))$.
4. If A has an annihilator $a \in [0,1]$, then $x_N \geq a$ (and therefore $a \neq 1$).
5. If A has a neutral element $e \in [0,1]$, then $x_N \leq e$ (and therefore $e \neq 0$).

Proof. All the proofs are based on the characterizations given in Proposition 2:

1. If there was some $x_0 \in [0,1]$ such that $A(x_0, N(x_0)) > Max(x_0, N(x_0))$, since $Max(x, N(x)) \geq x_N$ for any $x \in [0,1]$, this would imply $A(x_0, N(x_0)) > x_N$, which means that A would not satisfy NC(AL) w.r.t. N.
2. If it was $\sup_{x\in[0,1]} A(x, N(x)) = 1$, this would imply $1 \leq x_N$, i. e., $x_N = 1$, and it is always $x_N \neq 1$ for any strong negation N.
3. Obvious since $\sup_{x\in[0,1]} A(x, N(x)) \geq Max(A(0,1), A(1,0))$.
4. Obvious since $\sup_{x\in[0,1]} A(x, N(x)) \geq A(a, N(a)) = a$.
5. Obvious since $\sup_{x\in[0,1]} A(x, N(x)) \geq A(e, N(e)) = N(e)$ and this implies $x_N \geq N(e)$, which is equivalent to $x_N \leq e$. □

This Proposition implies that aggregation operators verifying $A(0,1) = 1$ or $A(1,0) = 1$ will never satisfy the NC(AL) principle (note that this class includes, in particular, any aggregation operator having $a = 1$ as annihilator element or $e = 0$ as neutral element). This allows to easily find examples of both averaging and hybrid operators not satisfying NC(AL) (see Section 6), as well as to include inside this category the whole class of disjunctive aggregation operators (since these operators obviously verify $A(0,1) = A(1,0) = 1$):

Corollary 2. Disjunctive aggregation operators do never satisfy the NC(AL) principle.

In summary, the following results regarding the satisfaction of the NC(AL) principle have been obtained:

1. Conjunctive aggregation operators do always (strictly) satisfy NC(AL).
2. Disjunctive aggregation operators do never satisfy NC(AL).
3. The classes of averaging and hybrid aggregation operators include operators satisfying NC(AL) as well as operators not satisfying NC(AL).

Note finally that aggregation operators satisfying the NC(AL) principle may be characterized in the following way:

Proposition 5. Let A be a binary aggregation operator and let N be a strong negation with fixed point x_N. A satisfies NC(AL) w.r.t. N if and only if $A \leq A^{nc}_{(N)}$, where for any $x, y \in [0,1]$ it is:

$$A^{nc}_{(N)}(x,y) = \begin{cases} 0, & \text{if } x = y = 0 \\ x_N, & \text{if } y \leq N(x), (x,y) \neq (0,0) \\ 1, & \text{otherwise.} \end{cases}$$

Proof. If A is an aggregation operator satisfying NC(AL) w.r.t. N, the inequality $A(x, N(x)) \leq x_N$ for any $x \in [0,1]$ implies, by monotonicity, $A(x,y) \leq x_N$ for any $y \leq N(x)$. The converse is obvious. □

3.2. Strict satisfaction of the Non-Contradiction principle

With regards to the strict satisfaction of the NC(AL) principle, the previous section has shown that conjunctive operators do always fulfill this property, disjunctive operators do never fulfill it, and that there are both averaging and hybrid operators that do not fulfill it (those that do not even satisfy NC(AL)). Therefore, it remains to known which averaging/hybrid operators, among those satisfying NC(AL), do also strictly satisfy the law. The following necessary conditions help in answering this question:

Proposition 6. Let A be a binary aggregation operator. If A strictly satisfies the NC(AL) principle, then:

1. $A(0,1) = A(1,0) = 0$.
2. $A(x,y) \leq Min(x,y)$ for any $(x,y) \in [0,1[^2$.

Proof.

1. Since it is $1 = N(0)$ and $0 = N(1)$ for any strong negation N, if A strictly satisfies NC(AL) it must be $A(0,1) \leq x_N$ and $A(1,0) \leq x_N$ for any $x_N \in]0,1[$, i. e., $A(0,1) = A(1,0) = 0$.
2. If $x = 0$ or $y = 0$, the inequality is obvious since by monotonicity of A it is $A(0,z) \leq A(0,1)$ and $A(z,0) \leq A(1,0)$ for any $z \in [0,1]$, and then, using *1.*, it is $A(0,z) = A(z,0) = 0$ for any $z \in [0,1]$. Otherwise, let us suppose that there exist $a, b \in]0,1[^2$ such that $A(a,b) > Min(a,b)$. Clearly, it is possible to find a strong negation N such that $b = N(a)$ and $x_N < A(a,b)$. Then it is $A(a, N(a)) = A(a,b) > x_N$, i. e., A does not satisfy NC(AL) w.r.t. N, which is a contradiction. □

An immediate consequence of the above result is that any aggregation operator strictly satisfying NC(AL) has necessarily $a = 0$ as annihilator element and may only have $e = 1$ as neutral element. Moreover, operators strictly satisfying NC(AL) can be characterized as follows:

Proposition 7. Let A be a binary aggregation operator. A strictly satisfies NC(AL) if and only if $A \leq A^{nc}$, where for any $x, y \in [0,1]$ it is:

$$A^{nc}(x,y) = \begin{cases} Min(x,y), & \text{if } (x,y) \in [0,1[^2 \quad \text{or } (x,y) \in \{(0,1),(1,0)\} \\ 1, & \text{otherwise.} \end{cases}$$

Proof. If A strictly satisfies NC(AL), the result is easily obtained using Proposition 6 and the definition of aggregation operator. The converse is obvious since it is always $Min(x, N(x)) \leq x_N$ for any strong negation N with fixed point x_N. □

Remark 1. Note that the above result says, in other words, that aggregation operators strictly satisfying NC(AL) are always -and only- of the form:

$$A(x,y) = \begin{cases} A_1(x,y), & \text{if } (x,y) \in [0,1[^2 \quad \text{or} (x,y) \in \{(0,1),(1,0)\} \\ A_2(x,y), & \text{otherwise} \end{cases}$$

where A_1 is a non-decreasing binary operator such that $A_1 \leq Min$ and A_2 is a non-decreasing binary operator such that $A_2(1,1) = 1$. Then, the following cases are possible:

- If A_2 is taken such that $A_2 \leq Min$, then A is a conjunctive aggregation operator.
- If $A_1 = Min$ and $A_2 \geq Min$, then A is an averaging aggregation operator.
- Otherwise, A is a hybrid aggregation operator.

4. AGGREGATION OPERATORS AND THE EXCLUDED–MIDDLE PRINCIPLE

Similarly to Non-Contradiction, the Excluded-Middle principle, when applied to aggregation operators according to the Ancient Logic interpretation, allows to establish the following definitions:

Definition 2. Let A be a binary aggregation operator and let N be a strong negation.

1. It is said that A *satisfies the Excluded-Middle principle with respect to N in the ancient logical sense* (EM(AL)) when $A(x, N(x)) \geq N(A(x, N(x)))$ holds for any $x \in [0,1]$.
2. It is said that A *satisfies EM(AL)* when there exists a strong negation N such that A satisfies EM(AL) w.r.t. N.
3. It is said that A *strictly satisfies EM(AL)* when it satisfies EM(AL) w.r.t. any strong negation.

As it was pointed out in the preliminaries section (Proposition 1), the laws NC(AL) and EM(AL) are dual laws, and this duality allows to translate all the results obtained for the NC(AL) law to the case of its dual law EM(AL). Therefore, the results corresponding to the EM(AL) law are presented in the sequel without proofs.

4.1. Satisfaction of the Excluded-Middle principle

Proposition 8. Let A be a binary aggregation operator and let N be a strong negation with fixed point x_N. Then the following conditions are equivalent:

1. A satisfies EM(AL) w.r.t. N
2. For all $x \in [0,1]$, $x_N \leq A(x, N(x))$ (i. e., $x_N \leq \inf_{x\in[0,1]} A(x, N(x))$)
3. For all $x \in [0,1]$, $A(x, N(x)) \geq A_N(N(x), x)$.

Proposition 9. Let A be a binary aggregation operator and let N be a strong negation. If any of the following conditions is verified, then A satisfies the EM(AL) principle w.r.t. N:

1. $A(x, N(x)) \geq Max(x, N(x))$ for all $x \in [0,1]$.
2. A is commutative and verifies $A \geq A_N$.
3. $A(x, N(x)) \geq B(x, N(x))$ for all $x \in [0,1]$, where B is an aggregation operator satisfying the EM(AL) principle w.r.t. N.

Corollary 3. Disjunctive aggregation operators strictly satisfy the EM(AL) principle.

Proposition 10. Let A be a binary aggregation operator and let N be a strong negation with fixed point x_N. If A satisfies the EM(AL) principle w.r.t. N, then:

1. For all $x \in [0,1]$, $A(x, N(x)) \geq Min(x, N(x))$
2. $\inf_{x\in[0,1]} A(x, N(x)) \neq 0$ (in particular, $A(0,1) \neq 0$, $A(1,0) \neq 0$)
3. $x_N \leq Min(A(0,1), A(1,0))$
4. If A has an annihilator $a \in [0,1]$, then $x_N \leq a$ (and therefore $a \neq 0$)
5. If A has a neutral element $e \in [0,1]$, then $x_N \geq e$ (and therefore $e \neq 1$)

Corollary 4. Conjunctive aggregation operators do never satisfy the EM(AL) principle.

4.2. Strict satisfaction of the Excluded-Middle principle

Proposition 11. Let A be a binary aggregation operator. If A strictly satisfies the EM(AL) principle, then:

1. $A(0,1) = A(1,0) = 1$.

2. $A(x,y) \geq Max(x,y)$ for any $(x,y) \in]0,1]^2$.

Proposition 12. Let A be a binary aggregation operator. A strictly satisfies EM(AL) if and only if $A \geq A^{em}$, where for any $x, y \in [0,1]$ it is:

$$A^{em}(x,y) = \begin{cases} Max(x,y), & \text{if } (x,y) \in]0,1]^2 \quad \text{or } (x,y) \in \{(0,1),(1,0)\} \\ 0, & \text{otherwise.} \end{cases}$$

5. RELATIONSHIPS BETWEEN NON–CONTRADICTION AND EXCLUDED–MIDDLE

It is interesting to remark that the duality relationship between the NC(AL) and EM(AL) principles is not exclusive, i. e., it is possible to find aggregation operators satisfying both laws:

Proposition 13. Let A be a binary aggregation operator and let N be a strong negation with fixed point x_N. Then A satisfies both NC(AL) and EM(AL) w.r.t. N if and only if $A(x, N(x)) = N(A(x, N(x)))$ for all $x \in [0,1]$, or, equivalently, $A(x, N(x)) = x_N$ for all $x \in [0,1]$.

Proof. Immediate from Definitions 1 and 2 and the fact that strong negations have a unique fixed point. □

In particular, the next result provides a sufficient condition for finding aggregation operators in this situation:

Proposition 14. Let A a binary aggregation operator and let N be a strong negation. If A is commutative and N-self-dual (i. e., $A = A_N$) then it satisfies NC(AL) and EM(AL) w.r.t. N.

Proof. It suffices to apply the sufficient conditions given in Propositions 3 and 9. □

On the other hand, the necessary conditions for NC(AL) and EM(AL) given, respectively, in Propositions 4 and 10, allow to establish the following results:

Proposition 15. Let A be a binary aggregation operator and let N be a strong negation with fixed point x_N. If A satisfies NC(AL) and EM(AL) w.r.t. N, then:

1. For all $x \in [0,1]$, $Min(x, N(x)) \leq A(x, N(x)) \leq Max(x, N(x))$.
2. $A(0,1) = A(1,0) = x_N$.
3. If A has an annihilator $a \in [0,1]$, then $a = x_N$ (and therefore $a \neq 0, 1$).
4. If A has a neutral element $e \in [0,1]$, then $e = x_N$ (and therefore $e \neq 0, 1$).

A first immediate consequence of the above results is that operators satisfying the two laws may only be found among averaging or hybrid operators:

Corollary 5. Neither conjunctive nor disjunctive aggregation operators can satisfy both NC(AL) and EM(AL).

Note also that Propositions 6 and 11 show that it is not possible to find aggregation operators *strictly* satisfying the two laws:

Proposition 16. No aggregation operator can strictly satisfy both the NC(AL) and EM(AL) principles.

Moreover, operators strictly satisfying one of the laws do not satisfy the other:

Proposition 17. Let A be a binary aggregation operator. If A strictly satisfies NC(AL), then it does not satisfy EM(AL). If A strictly satisfies EM(AL), then it does not satisfy NC(AL).

Proof. If A strictly satisfies NC(AL), then, according to Proposition 6, it is $A(0,1) = A(1,0) = 0$, but Proposition 10 states that this is incompatible with A satisfying EM(AL) w.r.t. some strong negation N. The second result is dual. □

Observe finally that there are aggregation operators that do not satisfy neither NC(AL) nor EM(AL). Such operators may only be found among either averaging or hybrid operators (since conjunctive/disjunctive operators do always satisfy NC(AL) and EM(AL), respectively). In particular, Propositions 4 and 10 show that aggregation operators such that either $[A(0,1) = 0, A(1,0) = 1]$ or $[A(0,1) = 1, A(1,0) = 0]$ do not satisfy neither NC(AL) nor EM(AL). Examples of operators of this kind are the projections of the first and last coordinates, given, respectively, by $P_F(x,y) = x$ and $P_L(x,y) = y$ for all $x, y \in [0,1]$.

6. EXAMPLES

This section applies all the results obtained in the previous ones in order to find examples of aggregation operators satisfying NC(AL), EM(AL) or both of them (see [3] for details and references regarding the different families of aggregation operators).

6.1. Conjunctive aggregation operators

Conjunctive aggregation operators do always strictly satisfy the NC(AL) principle and do never satisfy EM(AL) (see Corollaries 1 and 4).

6.2. Averaging aggregation operators

The most interesting types of operators included in this class are the following:

1. Operators strictly satisfying NC(AL).
2. Operators satisfying NC(AL) and not satisfying EM(AL).
3. Operators satisfying both NC(AL) and EM(AL).
4. Operators satisfying EM(AL) and not satisfying NC(AL).
5. Operators strictly satisfying EM(AL).

In the following we provide examples of averaging operators of each of these types, excluding types 4 and 5, since such operators may be easily obtained, by means of duality, from the ones belonging to classes 2 and 1, respectively.

Strict NC(AL)

Remark 1 provides a general characterization of aggregation operators strictly satisfying NC(AL), that may be particularized to obtain averaging operators by choosing $A_1 = \mathit{Min}$ and $A_2 \geq \mathit{Min}$. Therefore, there is a wide family of averaging operators strictly satisfying NC(AL), but note nevertheless that this characterization excludes, by construction, the most important known classes of averaging operators, such as quasi-linear means or OWA operators.

NC(AL) and not EM(AL)

According to Proposition 17, operators strictly satisfying NC(AL) do never satisfy EM(AL), and therefore the ones given above are examples of averaging operators satisfying NC(AL) and not satisfying EM(AL). Nevertheless, there are also averaging operators that do not strictly satisfy NC(AL), but satisfy NC(AL) and do not satisfy EM(AL). Such operators may be found, for instance, in the well-known class of *quasi-arithmetic means* (see e. g. [3]), i. e., operators of the form $M_f(x,y) = f^{-1}\left(\frac{f(x)+f(y)}{2}\right)$ where $f : [0,1] \to [-\infty,+\infty]$ is a continuous strictly monotone function. Indeed, it is clear that such operators do not strictly satisfy NC(AL) (see Proposition 7). In addition:

Proposition 18. Let M_f be a quasi-arithmetic mean such that f is increasing, concave and verifies $f(0) = -\infty$. Then M_f does not satisfy EM(AL), and, if N is a twice differentiable concave strong negation and f is also twice differentiable, then M_f satisfies NC(AL) w.r.t. N.

P r o o f . The property $f(0) = -\infty$ implies that $a = 0$ is an annihilator element for M_f, and this, according to Proposition 10, entails that M_f does not satisfy EM(AL). On the other hand, we need to prove that M_f, under the given conditions, satisfies NC(AL) w.r.t. N, i. e., that $M_f(x, N(x)) \leq x_N$ holds for any $x \in [0,1]$, or, equivalently, $f(x) + f(N(x)) \leq 2f(x_N)$. If we denote $g(x) = f(x) + f(N(x))$, its first derivative is $g'(x) = f'(x) + f'(N(x)).N'(x)$, which, since $N'(x_N) = -1$, has an extreme point in $x = x_N$. The second derivative of g is given by $g''(x) = f''(x) + f'(N(x))N''(x) + f''(N(x))N'(x)^2$, and it is $g'' \leq 0$, since $f' \geq 0$ (f increasing) and $f'', N'' \leq 0$ (because f and N are concave). This means that g is concave, which entails that its extreme point x_N is a maximum, and then $f(x) + f(N(x)) = g(x) \leq g(x_N) = 2f(x_N)$. □

A concrete example of quasi-arithmetic mean fulfilling the above conditions is the well-known geometric mean, $G(x, y) = \sqrt{xy}$, since it is generated by $f(x) = log(x)$.

NC(AL) and EM(AL)

Proposition 14 shows that, given a strong negation N, any commutative N-self-dual operator satisfies both NC(AL) and EM(AL) w.r.t. N. In the case of averaging operators, this result may be applied, for example, to a wide sub-class of quasi-arithmetic means:

Proposition 19. Let $\varphi : [0,1] \to [0,1]$ be an automorphism and let $N_\varphi = \varphi^{-1} \circ (1 - Id) \circ \varphi$ be the strong negation generated by φ. Then the quasi-arithmetic mean M_φ, defined as $M_\varphi(x, y) = \varphi^{-1}\left(\frac{\varphi(x)+\varphi(y)}{2}\right)$ for any $x, y \in [0,1]$, satisfies NC(AL) and EM(AL) w.r.t. N_φ.

P r o o f . It suffices to apply Proposition 14, since M_φ is clearly commutative and N_φ-self-dual. □

Examples of quasi-arithmetic means with the above properties are the so-called *root power operators*, obtained when choosing $\varphi(x) = x^\alpha$, $\alpha > 0$, as generating function.

6.3. Hybrid aggregation operators

The class of hybrid aggregation operators contains operators in the same categories as averaging operators:

Strict NC(AL)

Aggregation operators built as in Remark 1 but choosing $A_2 \nleq Min$ and either $A_1 \neq Min$ or $A_2 \ngeq Min$ are clearly hybrid operators strictly satisfying NC(AL). Regarding the best known families of hybrid operators, note that Proposition 6 entails that neither *uninorms* (because they have neutral element $e \neq 1$) nor *nullnorms* (because they have annihilator element $a \neq 0$) strictly satisfy NC(AL) (the definition of these two classes of aggregation operators will be recalled below).

NC(AL) AND NOT EM(AL)

Of course, hybrid operators strictly satisfying NC(AL) are examples of operators satisfying NC(AL) and not satisfying EM(AL) (Proposition 17). More operators of this class may be found among *uninorms*. Uninorms ([11]) are commutative and associative aggregation operators $U : [0,1]^2 \to [0,1]$ possessing a neutral element $e \in]0,1[$. They behave as t-norms in $[0,e]^2$, as t-conorms in $[e,1]^2$ and otherwise verify $Min(x,y) \leq U(x,y) \leq Max(x,y)$. They may be classified into two different classes: those with annihilator element $a = 0$, known as *conjunctive uninorms* and those with annihilator $a = 1$, known as *disjunctive uninorms*. An important class of conjunctive uninorms (see e. g. [3]) is the one given below (where T is a t-norm, S is a t-conorm and $e \in]0,1[$):

$$U^c_{e,T,S}(x,y) = \begin{cases} e \cdot T(\frac{x}{e}, \frac{y}{e}), & \text{if } Max(x,y) \leq e \\ e + (1-e) \cdot S(\frac{x-e}{1-e}, \frac{y-e}{1-e}), & \text{if } Min(x,y) \geq e \\ Min(x,y), & \text{otherwise.} \end{cases}$$

Proposition 20. Let $U^c_{e,T,S}$ be the conjunctive uninorm given above and let N be a strong negation with fixed point x_N. Then:

1. $U^c_{e,T,S}$ does not satisfy EM(AL).
2. $U^c_{e,T,S}$ satisfies NC(AL) w.r.t. N if and only if $x_N \leq e$.

Proof. $U^c_{e,T,S}$ does not satisfy EM(AL) because of its zero annihilator (Proposition 10), and Proposition 4 shows that $x_N \leq e$ is a necessary condition for $U^c_{e,T,S}$ satisfying NC(AL) w.r.t. N. It is not difficult to check that this condition is also sufficient, since, as it is $Min(x, N(x)) \leq x_N \leq e$, the situation $Min(x, N(x)) > e$ never holds, and in the two remaining situations it is clearly $U^c_{e,T,S}(x, N(x)) \leq Min(x, N(x)) \leq x_N$ (since $T \leq Min$). □

NC(AL) AND EM(AL)

Proposition 14 may be used again in order to find hybrid operators satisfying both NC(AL) and EM(AL) w.r.t. some strong negation N, since there exist hybrid operators which are commutative and N-self-dual. For example (see [3]), any operator of the form

$$A(x,y) = \frac{B(x,y)}{B(x,y) + B(1-x, 1-y)}$$

with convention $\frac{0}{0} = 0.5$, where B is a commutative aggregation operator, is a commutative aggregation operator which is self-dual w.r.t. the standard negation $N(x) = 1 - x$, and, therefore, satisfies both NC(AL) and EM(AL) w.r.t. N.

Other examples of operators in this class may be found among the so-called *nullnorms* ([2]), i. e., aggregation operators $V : [0,1]^2 \to [0,1]$ which are associative, commutative and possess an element $a \in]0,1[$ such that for any $x \in [0,1]$, it is

$$V(x,0) = x \text{ if } x \leq a, \qquad V(x,1) = x \text{ if } x \geq a.$$

Thanks to monotonicity, the element a acts as an annihilator for V, which behaves as a t-conorm in $[0,a]^2$, as a t-norm in $[a,1]^2$ and verifies $V(x,y) = a$ otherwise. Any nullnorm (see e. g. [3]) has the following structure (where T is a t-norm, S is a t-conorm and $a \in]0,1[$):

$$V_{a,T,S}(x,y) = \begin{cases} a \cdot S(\frac{x}{a}, \frac{y}{a}), & \text{if } Max(x,y) \leq a \\ a + (1-a) \cdot T(\frac{x-a}{1-a}, \frac{y-a}{1-a}), & \text{if } Min(x,y) \geq a \\ a, & \text{otherwise.} \end{cases}$$

Regarding the satisfaction of the NC(AL) and EM(AL) laws, the following result on nullnorms may be established:

Proposition 21. Let $V_{a,T,S}$ be a nullnorm and let N be a strong negation with fixed point x_N. Then:

1. $V_{a,T,S}$ satisfies NC(AL) w.r.t. N if and only if $a \leq x_N$.
2. $V_{a,T,S}$ satisfies EM(AL) w.r.t. N if and only if $a \geq x_N$.
3. $V_{a,T,S}$ satisfies NC(AL) and EM(AL) w.r.t. N if and only if $a = x_N$.

Proof. Let us prove the first statement (the proof of the second one is similar, and the third one directly follows from the two previous ones). The necessity of the condition $a \leq x_N$ comes from Proposition 4 and the fact that a is an annihilator for $V_{a,T,S}$. In order to prove that such condition is also sufficient for the satisfaction of NC(AL) (i. e., $V_{a,T,S}(x, N(x)) \leq x_N$ holds for any $x \in [0,1]$), note that it is always $Max(x, N(x)) \geq x_N$, and then $a \leq x_N$ means that the situation $Max(x, N(x)) \leq a$ cannot hold. In the case $Min(x,y) \geq a$, it is clearly $V_{a,T,S}(x, N(x)) \leq Min(x, N(x)) \leq x_N$ (since $T \leq Min$), and otherwise it is $V_{a,T,S}(x, N(x)) = a \leq x_N$. □

6.4. Disjunctive aggregation operators

Disjunctive aggregation operators do always strictly satisfy the EM(AL) principle and never satisfy NC(AL) (see Corollaries 4 and 2).

7. CONCLUSIONS

This paper has studied the satisfaction of the Non-Contradiction and Excluded-Middle principles within the framework of aggregation operators. The two laws have been analyzed from a non-standard point of view, based, as in Ancient Logic, on self-contradiction. Different characterizations, as well as necessary and sufficient conditions, have been found and applied to some examples.

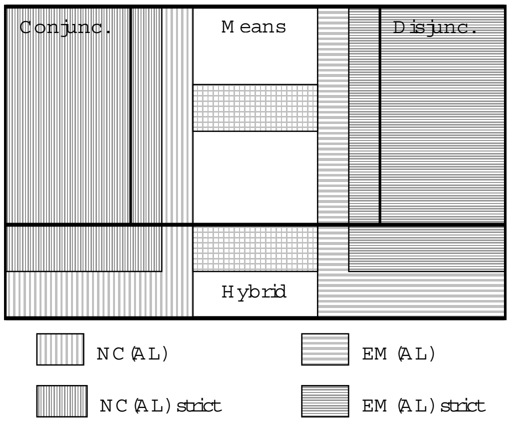

Fig. 1. Aggregation operators and the NC/EM principles in Ancient Logic.

Figure 1, which summarizes the obtained results, shows the impact that the study that has been performed has on the available classification of aggregation operators into four general classes (conjunctive, disjunctive, averaging – mean – and hybrid operators). Two main conclusions can be underlined:

- The study of the satisfaction of the NC(AL)/EM(AL) laws does not provide any new knowledge neither for conjunctive operators nor for disjunctive operators. Indeed – as it could have been intuitively expected – the first ones do always strictly satisfy the NC(AL) law, whereas the second ones do always strictly satisfy the EM(AL) law.
- On the other hand, the laws NC(AL)/EM(AL) allow for new and substantial distinctions among averaging and hybrid aggregation operators. Indeed, this paper has shown that operators in these two classes may be further classified into the following six sub-categories:
  - Operators strictly satisfying NC(AL).
  - Operators satisfying NC(AL) and not satisfying EM(AL).
  - Operators satisfying EM(AL) and not satisfying NC(AL).
  - Operators satisfying both NC(AL) and EM(AL).
  - Operators not satisfying neither NC(AL) nor EM(AL).
  - Operators strictly satisfying EM(AL).

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Ana Pradera, Departamento de Informática, Estadística y Telemática, Universidad Rey Juan Carlos. 28933 Móstoles. Madrid. Spain.
e-mail: ana.pradera@urjc.es

Enric Trillas, Departamento de Inteligencia Artificial, Universidad Politécnica de Madrid. 28660 Boadilla del Monte. Madrid. Spain.
e-mail: etrillas@fi.upm.es