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Asymptotic behavior of minima and mountain pass solutions for a class of Allen-Cahn models

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In an earlier paper, the authors studied a class of Allen-Cahn models for which the solution was near 1 on a prescribed set, $T + [0, 1]^n$ where $T \subset \mathbb{Z}^n$, and near 0 on its complement. In this note, when T is finite and consists of two widely spaced subsets, T_1 and $l + T_2$ with $l \in \mathbb{Z}^n$, we study the asymptotic behavior of two special families of solutions as $l \to \infty$.

1. Introduction

In two recent papers [9], [10], the authors studied an Allen-Cahn model problem having the form

$$(1.1) \qquad -\Delta u + A_\varepsilon(x)G'(u) = 0, \quad x \in \mathbb{R}^n$$

where $G(u) = u^2(1 - u)^2$ is a double well potential, $\varepsilon > 0$, and $A_\varepsilon(x) = 1 + A(x)/\varepsilon$ with $0 \le A \in C^1(\mathbb{R}^n)$, $1-$periodic in $x_1, \cdots, x_n$, Ω is the support of $A|_{[0,1]^n}$ and has a smooth boundary, and $\overline{\Omega} \subset (0, 1)^n$. A main result of [9] is that there is an $\varepsilon_0 > 0$ such that for any finite set $T \subset \mathbb{Z}^n$ and $\varepsilon \in (0, \varepsilon_0]$, (1.1) has a solution, $U_{\varepsilon,T}$ with $0 < U_{\varepsilon,T} < 1$, $U_{\varepsilon,T}$ is near 1 on $A^T \equiv T + \overline{\Omega}$ and near 0 on $B^T \equiv (\mathbb{Z}^n \setminus T) + \overline{\Omega}$. Moreover as $\varepsilon \to 0$, $U_{\varepsilon,T} \to 1$ uniformly on A^T and $U_{\varepsilon,T} \to 0$ uniformly on B^T. When T is finite, $U_{\varepsilon,T}$ is characterized as the minimizer of a constrained variational problem associated with (1.1). Although $U_{\varepsilon,T}$ may not be unique, the set of such minimizers, $\mathcal{M}_\varepsilon(T)$, is ordered. The setting of [9] was further treated in [10] where it was shown that for each finite T, there is an $\varepsilon_1(T) > 0$ such that for $\varepsilon \in (0, \varepsilon_1(T))$, (1.1) has a solution, $V_{\varepsilon,T}$ of mountain pass type with $0 < V_{\varepsilon,T} < U_{\varepsilon,T}$.

The main goal of this note is to study the setting of when T is finite and consists of two widely separated subsets, that is, $T = T_1 \cup (l + T_2) \equiv T_l$ for $T_1, T_2 \subset \mathbb{Z}^n, l \in \mathbb{Z}^n$ and large $|l| > 0$. In particular we are interested in the asymptotic behavior as $l \to \infty$ of the minimizers, U_{ε,T_l}, and the mountain pass solutions, as well as the corresponding critical values. To describe our results, let

$$J_\varepsilon(u) = \int_{\mathbb{R}^n} \frac{1}{2}|\nabla u|^2 + A_\varepsilon(x)G(u)\,dx,$$

the functional associated with (1.1). Set

$$c_\varepsilon(T_l) = J_\varepsilon(U_{\varepsilon,T_l}).$$

A more precise characterization of c_{ε,T_l} will be given later. We will prove

Theorem 1.2. *Suppose $T \subset \mathbb{Z}^n$ is finite. Let A_ε and G be as above. Then for any $\varepsilon \in (0, \varepsilon_0]$, as $l \to \infty$,*

- 1^o *$c_\varepsilon(T_l) \to c_\varepsilon(T_1) + c_\varepsilon(T_2)$;*
- 2^o *There is a $U_{\varepsilon,T_1} \in \mathcal{M}_\varepsilon(T_1)$ such that $U_{\varepsilon,T_l} \to U_{\varepsilon,T_1}$ along a subsequence in $C^2_{loc}(\mathbb{R}^n)$;*
- 3^o *There is a $U_{\varepsilon,T_2} \in \mathcal{M}_\varepsilon(T_2)$ such that $U_{\varepsilon,T_l}(\cdot + l) \to U_{\varepsilon,T_2}$ along a subsequence in $C^2_{loc}(\mathbb{R}^n)$.*

Thus, roughly speaking, the minimizer for the T_l problem is obtained by gluing translates of the minimizers for the T_1 and T_2 problems. These results will be carried out in §2. Then in §3, we will give sharper results for the setting of [10] on mountain pass solutions. In particular for large l and small ε, it will be shown that there are two critical values of mountain pass type. One of the associated critical points of J_ε corresponds to gluing a minimum, U_{ε,T_1} of J_ε to a mountain pass solution, $V_{\varepsilon,l+T_2} = V_{\varepsilon,T_2}(\cdot - l)$, and the other to gluing a V_{ε,T_1} to a $U_{\varepsilon,l+T_2} = U_{\varepsilon,T_2}(\cdot - l)$. Moreover as $l \to \infty$, the corresponding critical values converge to the sum of $c_\varepsilon(T_1)$ and $J_\varepsilon(V_{\varepsilon,T_2})$, and the sum of $J_\varepsilon(V_{\varepsilon,T_1})$ and $c_\varepsilon(T_2)$, respectively. Some final remarks will be made in §4.

There has been a considerable amount of additional work on solutions of heteroclinic or homoclinic type of Allen-Cahn model equations. See [1]–[4], [12], [16]–[19]. The models involve forcing terms that are periodic in one or all spatial variables with the exception of [4] where there is almost periodic forcing. Aside from [16], minimization arguments are used to obtain solutions of the model equations that are in $C^2(\mathbb{R}\times\mathbb{T}^{n-1})$ or in $C^2(\mathbb{R}^2\times\mathbb{T}^{n-2})$. In the first case of $C^2(\mathbb{R}\times\mathbb{T}^{n-1})$, the solutions treated in [12], [17]–[19] are heteroclinic or homoclinic in one direction, say the x_1−direction, and are periodic in the remaining variables. Moreover the asymptotic states in the x_1 direction are spatially periodic minimizers of an associated functional. The second case of solutions in $C^2(\mathbb{R}^2 \times \mathbb{T}^{n-2})$ is studied in [1]–[4] and [17]–[19]. Here the solutions are heteroclinic in x_2 between between a pair of x_1 heteroclinics as obtained in the previous case. Solutions of mountain pass type have also been considered in [12]. Although it uses different kinds of arguments based on sub- and supersolutions and comparison arguments, [16] is the only paper aside from [9]–[10] and the current one, which treat solutions other than the heteroclinics or homoclinics in one direction mentioned above.

Using a trick involving the Maximum Principle, see [20], most of the papers mentioned above can be viewed as special cases of a more general class of quasilinear elliptic partial differential equations introduced by Moser in [15]. It's simplest semilinear form is:

$$-\Delta u + F_u(x,u) = 0 \tag{1.3}$$

where $F \in C^2(\mathbb{T}^{n+1}, \mathbb{R})$. Some papers which study (1.3) in the spirit of the research cited for (1.1) are [5]–[7], [11], [13], [15], [20].

2. The Proof of Theorem 1.2

In order to prove Theorem 1.2, some results from [9] must be recalled. In particular, the minimization characterization of $c_\varepsilon(T)$ for finite T is required as well as some decay estimates for $U_{\varepsilon,T}$. Thus let $\mathcal{W}$ denote the closure of $C_0^\infty(\mathbb{R}^n)$ functions under the norm

$$\|u\| \equiv \Big(\int_{\mathbb{R}^n} |\nabla u|^2 dx + \int_{[-1,1]^n} u^2 dx \Big)^{1/2}.$$

Let $d^* = \frac{1}{2}|\partial\Omega - \partial[0,1]^n|$ and choose any small $d \in (0, d^*)$ so that if

$$\Omega_d \equiv \{x \in \Omega \mid |x - \partial\Omega| > d\},$$

then $\partial\Omega_d$ is diffeomorphic to $\partial\Omega$. For $T \subset \mathbb{Z}^n$, set $A_T = T + \Omega_d$ and $B_T = (\mathbb{Z}^n \setminus T) + \Omega_d$. Choosing constants a and b so that $0 < b < \frac{1}{2} < a < 1$ and setting

$$\Gamma(T) = \{u \in \mathcal{W} \mid u \geq a \textit{ on } A_T \textit{ and } u \leq b \textit{ on } B_T\},$$

define

$$c_\varepsilon(T) = \inf_{u \in \Gamma(T)} J_\varepsilon(u). \tag{2.1}$$

Let χ_S denote the characteristic function of the set S. Then, as was shown in [9],

Theorem 2.2. *Let A_ε and G be as above. Then there exists an $\varepsilon_0 > 0$ such that for each $\varepsilon \in (0, \varepsilon_0]$ and each finite $T \subset \mathbb{Z}^n$,*

1^o $\mathcal{M}_\varepsilon(T) \equiv \{u \in \Gamma(T) \mid J_\varepsilon(u) = c_\varepsilon(T)\} \neq \emptyset$.
2^o *Any $U \in \mathcal{M}_\varepsilon(T)$ satisfies $0 < U < 1$ and is a classical solution of (1.1).*
3^o *$\mathcal{M}_\varepsilon(T)$ is an ordered set: $U, V \in \mathcal{M}_\varepsilon(T)$ implies $U < V$, $U > V$, or $U \equiv V$.*
4^o *If $T \subset S \subset \mathbb{Z}^n$, $U_{\varepsilon,T} \leq U_{\varepsilon,S}$ with strict inequality if $T \neq S$.*
5^o *There exist constants $C, c > 0$, independent of T and of $\varepsilon \in (0, \varepsilon_0]$, satisfying*

$$|U_{\varepsilon,T}(x) - \chi_{T+[0,1]^n}(x)| \leq C \exp(-cd(x,T)), \quad x \in \mathbb{R}^n$$

where $d(x,T) \equiv dist(x, \partial(T + [0,1]^n))$.

Now with the aid of these preliminaries, we can give the

Proof of Theorem 1.2: There is a positive integer m such that $T_i + [0,1]^n \subset [-m,m]^n$ for $i = 1,2$. For each $l \in \mathbb{Z}^n$, let $\psi_l \in C_0^\infty(\mathbb{R}^n)$ such that $\psi_l(x) = 1$ for $|x| \leq |l|/4$, $\psi_l(x) = 0$ for $|x| \geq |l|/3$, $0 \leq \psi(x) \leq 1$, and $|\nabla \psi_l(x)| \leq 20/|l|$ for any $x \in \mathbb{R}^n$. Note that for any $U_{\varepsilon,T_l} \in \mathcal{M}(T_l)$,

$$0 \leq U_{\varepsilon,T_l} = \psi_l U_{\varepsilon,T_l} + \psi_l(\cdot - l)U_{\varepsilon,T_l} + (1 - \psi_l - \psi_l(\cdot - l))U_{\varepsilon,T_l}. \tag{2.3}$$

By 5^o of Theorem 2.2 and (2.3), there exist constants $C_1, c > 0$ such that for all large $|l|$,

$$J_\varepsilon(U_{\varepsilon,T_l}) \geq J_\varepsilon(\psi_l U_{\varepsilon,T_l}) + J_\varepsilon(\psi_l(\cdot - l)U_{\varepsilon,T_l}) - C_1 \exp(-c|l|). \tag{2.4}$$

For large $|l|$, we see that

$$\psi_l U_{\varepsilon,T_l} \in \Gamma(T_1) \quad \text{and} \quad \psi_l(\cdot - l)U_{\varepsilon,T_l} \in \Gamma(l + T_2). \tag{2.5}$$

Thus (2.4)–(2.5) imply that for large $|l|$,

$$c_\varepsilon(T_l) \geq c_\varepsilon(T_1) + c_\varepsilon(T_2) - C_1 \exp(-c|l|). \tag{2.6}$$

Now to get 1^o of Theorem 1.2, take $U_{\varepsilon,T_i} \in \mathcal{M}_\varepsilon(T_i)$ for each $i = 1,2$. Then for large $|l|$,

$$u_{\varepsilon,T_l} \equiv \psi_l U_{\varepsilon,T_1} + \psi_l(\cdot - l)U_{\varepsilon,T_2}(\cdot - l) \in \Gamma(T_l).$$

Then, again by 5^o of Theorem 2.2, there exist constants $C_2, c > 0$ such that for large $|l|$,

$$\begin{aligned} c_\varepsilon(T_l) &\leq J_\varepsilon(u_{\varepsilon,T_l}) \\ &\leq J_\varepsilon(\psi_l U_{\varepsilon,T_1}) + J_\varepsilon(\psi_l(\cdot - l)U_{\varepsilon,T_2}(\cdot - l)) + C_2 \exp(-c|l|) \\ &= J_\varepsilon(\psi_l U_{\varepsilon,T_1}) + J_\varepsilon(\psi_l U_{\varepsilon,T_2}) + C_2 \exp(-c|l|) \\ &\leq J_\varepsilon(U_{\varepsilon,T_1}) + J_\varepsilon(U_{\varepsilon,T_2}) + 2C_2 \exp(-c|l|) \\ &= c_\varepsilon(T_1) + c_\varepsilon(T_2) + 2C_2 \exp(-c|l|). \end{aligned} \tag{2.7}$$

Combining (2.6) and (2.7), we get

$$\lim_{|l|\to\infty} c_\varepsilon(T_l) = c_\varepsilon(T_1) + c_\varepsilon(T_2). \tag{2.8}$$

To complete the proof of Theorem 1.2, note that if $U_{\varepsilon,T_l} \in \mathcal{M}_\varepsilon(T_l)$, for large $|l|$, $\psi_l U_{\varepsilon,T_l} \in \Gamma(T_1)$ and $\psi_l(\cdot - l)U_{\varepsilon,T_l} \in \Gamma(l + T_2)$. Then arguing as in (2.7), we get

$$\lim_{|l|\to\infty} J_\varepsilon(\psi_l U_{\varepsilon,T_l}) = c_\varepsilon(T_1) \tag{2.9}$$

and

$$\lim_{|l|\to\infty} J_\varepsilon(\psi_l(\cdot - l)U_{\varepsilon,T_l}) = \lim_{|i|\to\infty} J_\varepsilon(\psi_l U_{\varepsilon,T_l}(\cdot + l)) = c_\varepsilon(T_2). \tag{2.10}$$

Lastly, (2.9)–(2.10), the unform boundedness of $\{||U_{\varepsilon,T_l}||_{C^{2,\alpha}(\mathbb{R}^n)}\}$ for any fixed $\alpha \in (0,1)$, and the decay property 5^o of Theorem 2.2 yield 2^o, 3^o of Theorem 1.2.

3. Mountain pass results

In [10], for each finite $T \subset Z^n$ and each small $\varepsilon > 0$, the existence of a solution, $V_{\varepsilon,T}$, of (1.1) of mountain pass type was proved. This solution satisfies $0 < V_{\varepsilon,T} < U_{\varepsilon,T}$ where $U_{\varepsilon,T} \in \mathcal{M}_\varepsilon(T)$. In this section, we will obtain a refinement of that result which provides two mountain pass solutions when $T = T_l$ with l large. To begin, we recall some results from §3 of [10].

Let $S \subset T \subset \mathbb{Z}^n$ with T finite and $S \neq T$. Define the family of homotopies

$$\begin{aligned} \mathcal{G}_\varepsilon(S,T) \;\equiv\; &\{g \in C([0,1], W^{1,2}(\mathbb{R}^n)) \mid U_{\varepsilon,S} \leq g(\theta) \leq U_{\varepsilon,T} \\ &\quad \textit{and} \;\; g(0) = U_{\varepsilon,S},\; g(1) = U_{\varepsilon,T}\} \end{aligned}$$

and define

$$b_\varepsilon(S,T) = \inf_{g\in\mathcal{G}_\varepsilon(S,T)} \max_{\theta\in[0,1]} J_\varepsilon(g(\theta)).$$

Then by Propositions 3.1–3.2 of [10], we have

Proposition 3.1. *There are constants, $0 < \underline{\beta} < \overline{\beta} < \infty$ such that*

$$\underline{\beta} \le \liminf_{\varepsilon\to 0} \sqrt{\varepsilon} b_\varepsilon(S,T) \le \limsup_{\varepsilon\to 0} \sqrt{\varepsilon} b_\varepsilon(S,T) < \overline{\beta}.$$

With the aid of these propositions, it was proved in [10] that:

Theorem 3.2. *Let S and T be as above. Then there is an $\varepsilon_2 = \varepsilon_2(S,T) > 0$ such that for any $\varepsilon \in (0,\varepsilon_2)$, there is a solution, $V_{\varepsilon,S,T}$ of (1.1) with $U_{\varepsilon,S} < V_{\varepsilon,S,T} < U_{\varepsilon,T}$ and $J_\varepsilon(V_{\varepsilon,S,T}) = b_\varepsilon(S,T)$.*

Remark 3.3. Taking $S = T_1$ and $T = T_l$ yields an $\varepsilon_2(T_1,T_l)$ and a solution, V_{ε,T_1,T_l} of (1.1) with $U_{\varepsilon,T_1} < V_{\varepsilon,T_1,T_l} < U_{\varepsilon,T_l}$ for $\varepsilon \in (0,\varepsilon_2(T_1,T_l))$. Similarly taking $S = l+T_2$ and $T = T_l$ yields an $\varepsilon_2(l+T_2,T_l)$ and a solution, $V_{\varepsilon,(l+T_2),T_l}$ of (1.1) with $U_{\varepsilon,l+T_2} < V_{\varepsilon,T_1,T_l} < U_{\varepsilon,T_l}$ for $\varepsilon \in (0,\varepsilon_2(l+T_2,T_l))$.

We seek to show that for large l these two solutions are distinct and then to study their asymptotic behavior as $|l| \to \infty$. This cannot be done directly from Theorem 3.2 since for $T = T_l$ and S as above, it gives an ε_2 which depends on l. Therefore ε_2 may go to 0 as $|l| \to \infty$. Hence sharper estimates are needed. The proof of Theorem 3.2 requires that

$$b_\varepsilon(S,T) > \max(c_\varepsilon(S), c_\varepsilon(T)).$$

Thus for our special choices of S and $T = T_l$, it suffices to show there is an $\varepsilon^* > 0$ such that

$$b_\varepsilon(S,T_l) > \max(c_\varepsilon(S), c_\varepsilon(T_l)) \tag{3.4}$$

holds for all $\varepsilon \in (0,\varepsilon^*)$ and all large l. Note that by (2.6), for large l,

$$c_\varepsilon(T_l) = \max(c_\varepsilon(S), c_\varepsilon(T_l)) \tag{3.5}$$

where $S = T_1$ or $l+T_2$. Hence to obtain (3.4), it suffices to find a constant β_ε which is independent of l for l sufficiently large such that for all $\varepsilon \in (0,\varepsilon^*)$,

$$b_\varepsilon(S,T_l) > \beta_\varepsilon > c_\varepsilon(T_l). \tag{3.6}$$

The following result is useful for that purpose. Let $\sigma > 0$ be such that $G''(s) > 0$ for $s \in [0,\sigma]$.

Proposition 3.7. *Let $\mathcal{D} \subset \mathbb{R}^n$ be an open set with a piece-wise smooth boundary and suppose $u \in W^{1,2}(\mathbb{R}^n)$ with $0 \le u \le \sigma$ on $\partial\mathcal{D}$. Set*

$$\mathcal{F}(u;\mathcal{D}) \equiv \{\varphi \in W^{1,2}(\mathbb{R}^n) \mid \varphi = u \text{ in } \mathbb{R}^n \setminus \mathcal{D}\}.$$

Then there exists a unique $w = u_{\mathcal{D}} \in \mathcal{F}(u;\mathcal{D})$ with $0 < w < \sigma$ in $\mathcal{D}$ such that

$$I_{\mathcal{D}}(w) \equiv \int_{\mathcal{D}} L_\varepsilon(w)\,dx = \inf_{\varphi \in \mathcal{F}(u;\mathcal{D})} \int_{\mathcal{D}} L_\varepsilon(\varphi)\,dx. \tag{3.8}$$

Moreover w is a solution of (1.1) in $\mathcal{D}$.

Proof: Let (u_k) be a minimizing sequence for $I_{\mathcal{D}}$. Then $(\|\nabla u_k\|_{L^2(\mathcal{D})})$ is bounded. Moreover replacing u_k by $\zeta_k = \min(\max(u_k, 0), 1)$ for which $I_{\mathcal{D}}(\zeta_k) \le I_{\mathcal{D}}(u_k)$, it can be assumed that $0 \le u_k \le 1$. Hence (u_k) is bounded in $W^{1,2}_{loc}(\mathcal{D})$ and the local weak lower semicontinuity of $I_{\mathcal{D}}$ implies that there is a $w \in \mathcal{F}(u;\mathcal{D})$ such that along a subsequence, $u_k \to w$ weakly in $W^{1,2}_{loc}(\mathcal{D})$ and (3.8) holds. To see that $0 < w < \sigma$ in $\mathcal{D}$ and therefore by standard elliptic regularity arguments in the calculus of variations, $w = u_{\mathcal{D}}$ is a solution of (1.1) in $\mathcal{D}$, we modify an argument from the proof of Theorem 3.1 of [9]. Since G is even about $1/2$, setting $q(u_k)(x) = u_k(x)$ if $u_k(x) \in [0, 1/2]$ and $q(u_k)(x) = 1 - u_k(x)$ if $u_k(x) \in [1/2, 1]$ shows $I_{\mathcal{D}}(q(u_k)) \le I_{\mathcal{D}}(u_k)$. Therefore replacing u_k by $q(u_k)$ if need be, it can be assumed that $0 \le u_k \le 1/2$ and w satisfies the same inequalities. Next set $p(u_k) = \min(u_k, \sigma)$. Then since $G(p(u_k)) \le G(u_k)$ and $\nabla p(u_k) = 0$ if $u_k > \sigma$, $I_{\mathcal{D}}(p(u_k)) \le I_{\mathcal{D}}(u_k)$ so it can be assumed that $0 \le u_k \le \sigma$ and likewise for w. To get the uniqueness, note that if w and $\hat{w}$ are minimizers,

$$\begin{aligned} 0 &= \int_{\mathcal{D}} (-\Delta(w - \hat{w}) + A_\varepsilon(G'(w) - G'(\hat{w})))(w - \hat{w})\,dx \\ &= \int_{\mathcal{D}} |\nabla w - \hat{w}|^2 + A_\varepsilon G''(z)(w - \hat{w})^2\,dx, \end{aligned}$$

where z lies between w and $\hat{w}$. Then, since $G''(s) > 0$ for $s \in [0, \sigma]$, we get $w \equiv \hat{w}$ in $\mathcal{D}$.

Now to find β_ε, let $g \in \mathcal{G}_\varepsilon(S, T_l)$. Since the argument is the same for either choice of S, let $S = T_1$. Choose a $\sigma > 0$ for which Proposition 3.7 is valid. Let $N_r(Q)$ denote an open r neighborhood of Q. By 5^o of Theorem 2.2, for all $|l| = |l(\sigma)|$ sufficiently large, $|g(\theta)(x)| \le \sigma$ for $x \in \mathbb{R}^n \setminus N_{|l|/5}(T_l)$ and each $\theta \in [0,1]$. Using the existence result of Proposition 3.7, we define

$$\hat{g}(\theta)(x) = \begin{cases} g(\theta)_{\mathbb{R}^n \setminus N_{|l|/5}(T_l)}(x) \text{ for } x \in \mathbb{R}^n \setminus N_{|l|/5}(T_l) \\ g(\theta)(x) \text{ for } x \in N_{|l|/5}(T_l). \end{cases}$$

Due to the uniqueness result of Proposition 3.7, $\hat{g} \in \mathcal{G}_\varepsilon(T_1, T_l)$, and because of its definition, $J_\varepsilon(\hat{g}(\theta)) \le J_\varepsilon(g(\theta))$. Choose a function $\phi_l \in C^\infty(\mathbb{R}^n; [0,1])$ such that $\phi_l(x) = 1$ for $x \in N_{|l|/2}(l+T_2)$, $\phi_l(x) = 0$ for $x \notin N_{3|l|/4}(l+T_2)$ and $|\nabla \phi_l| \le 10/|l|$. Then define $\tilde{g}(\theta) \equiv \hat{g}(\theta)\phi_l$. We see from the decay property 5^o of Theorem 2.2 that there exist constants $D, d > 0$ such that for any $x \in \mathbb{R}^n \setminus N_{|l|/5}(T_l)$, $\hat{g}(\theta)(x) \le D\exp(-d|l|)$. Since

$$-\Delta \hat{g}(\theta) + A_\varepsilon(x) G'(\hat{g}(\theta)) = 0 \ \text{ in } \ \mathbb{R}^n \setminus N_{|l|/5}(T_l),$$

by standard local elliptic estimates [14], there exist constants $C', c' > 0$ such that for any $x \in N_{3|l|/4}(l+T_2) \setminus N_{|l|/2}(l+T_2)$, $|\nabla \hat{g}(\theta)(x)| \le C'\exp(-c'|l|)$. Thus, there are constants, $c, C > 0$, independent of large $|l|$ and $g \in \mathcal{G}_\varepsilon(T_1, T_l)$ such that

$$J_\varepsilon(g(\theta)) \ge \int_{N_{3|l|/4}(l+T_2)} L_\varepsilon(\hat{g}(\theta))\, dx \ge J_\varepsilon(\tilde{g}(\theta)) - C\exp(-c|l|). \tag{3.9}$$

Now we define $h \in C([0,1], W^{1,2}(\mathbb{R}^n))$ by

$$h(\theta)(x) = \begin{cases} 3\theta \min\{\tilde{g}(0)(x), U_{\varepsilon, l+T_2}(x)\} \text{ for } \theta \in [0, 1/3] \\ \min\{\tilde{g}(3\theta - 1)(x), U_{\varepsilon, l+T_2}(x)\} \text{ for } \theta \in [1/3, 2/3] \\ (3\theta - 2)U_{\varepsilon, l+T_2}(x) + (3 - 3\theta)h(2/3)(x) \text{ for } \theta \in (2/3, 1]. \end{cases}$$

Hence $h \in \mathcal{G}_\varepsilon(\emptyset, l+T_2)$. Since $U_{\varepsilon, l+T_2}(x) = U_{\varepsilon, T_2}(x - l)$ and $\{J_\varepsilon(U_{\varepsilon, T_2})\}$ is uniformly bounded for small $\varepsilon > 0$, taking $|l|$ large shows there is a constant, $C_1 > 0$, independent of small $\varepsilon > 0$ and large $|l| > 0$, such that

$$\max_{\theta \in [0,1] \setminus [1/3, 2/3]} J_\varepsilon(h(\theta)) \le C_1, \tag{3.10}$$

and

$$\max_{\theta \in [1/3, 2/3]} J_\varepsilon(h(\theta)) \le \max_{\theta \in [0,1]} J_\varepsilon(\tilde{g}(\theta)) + C_1. \tag{3.11}$$

Consequently by the form of h, (3.9)–(3.11), and Proposition 3.1, there is a constant $C_2 > 0$, independent of small ε and large l such that

$$\begin{aligned}\max_{\theta\in[0,1]} J_\varepsilon(\tilde{g}(\theta)) &\geq \max_{\theta\in[0,1]} J_\varepsilon(h(\theta)) - C_2 \geq b_\varepsilon(\emptyset, l+T_2) - C_2\\ &= b_\varepsilon(\emptyset, T_2) - C_2 \geq \underline{\beta}/\sqrt{\varepsilon} - C_2.\end{aligned}$$

Since $\{c_\varepsilon(T_l) \mid l \in \mathbb{Z}^n\}$ is bounded, taking $\beta_\varepsilon = \underline{\beta}/\sqrt{\varepsilon} - C_2 - 1$, we conclude that for large $l > 0$,

$$b_\varepsilon(T_1, T_l) = \inf_{g\in\mathcal{G}_\varepsilon(T_1,T_l)} \max_{\theta\in[0,1]} J_\varepsilon(g(\theta)) \geq \beta_\varepsilon - 1 > c_\varepsilon(T_l). \tag{3.12}$$

As a consequence of the above observations, we have:

Corollary 3.13. *There is an $r_0 > 0$ and $\varepsilon^* = \varepsilon^*(T_1, T_2) > 0$ such that for $|l| \geq r_0, \varepsilon \in (0, \varepsilon^*)$, and $S = T_1$ or $S = l + T_2$, $b_\varepsilon(S, T_l)$ is a critical value of J_ε defined on $\mathcal{G}_\varepsilon(S, T_l)$.*

Next the asymptotic behavior as $l \to \infty$ of $b_\varepsilon(S, T_l)$ and the corresponding critical points of J_ε will be studied.

Theorem 3.14. *Let A_ε and G be as above. Let $U_{\varepsilon,T_1}, U_{\varepsilon,T_2}$ be respectively the largest members of $\mathcal{M}_\varepsilon(T_1), \mathcal{M}_\varepsilon(T_2)$ and U_{ε,T_l} be the smallest member of $\mathcal{M}_\varepsilon(T_l)$. Then there is an $\varepsilon_2 = \varepsilon_2(T_1, T_2) \in (0, \varepsilon^*)$ such that for any $\varepsilon \in (0, \varepsilon_2)$, as $l \to \infty$,*

- 1^o $b_\varepsilon(T_1, T_l) \to c_\varepsilon(T_1) + b_\varepsilon(T_2)$,
- 2^o *$V_{\varepsilon,T_1,T_l} \to U_{\varepsilon,T_1}$ and there is a solution, $V_{\varepsilon,2}$, of (1.1) with $J_\varepsilon(V_{\varepsilon,2}) = b_\varepsilon(T_2)$ such that $V_{\varepsilon,T_1,T_l}(\cdot - l) \to V_{\varepsilon,2}$, convergence being along a subsequence in C^2_{loc}.*
- 3^o $b_\varepsilon(l + T_2, T_l) \to b_\varepsilon(T_1) + c_\varepsilon(T_2)$.
- 4^o *There is a solution, $V_{\varepsilon,1}$, of (1.1) with $J_\varepsilon(V_{\varepsilon,1}) = b_\varepsilon(T_1)$ such that $V_{\varepsilon,l+T_2,T_l} \to V_{\varepsilon,1}$ and $V_{\varepsilon,T_l}(\cdot - l) \to U_{\varepsilon,T_2}$, convergence being along a subsequence in C^2_{loc}.*

In particular, by 1^o–4^o, $V_{\varepsilon,T_1,T_l} \neq V_{\varepsilon,l+T_2,T_l}$ for large l.

Remark 3.15. By 1^o–4^o, $V_{\varepsilon,T_1,T_l} \neq V_{\varepsilon,l+T_2,T_l}$ for large l, i.e. we have two distinct solutions of (1.1) of mountain pass type.

Proof of Theorem 3.14: We will prove 1^o–2^o. The remaining items are proved in the same way. Set $p(\theta) = \theta U_{\varepsilon,T_l} + (1-\theta)U_{\varepsilon,T_1}$ for $\theta \in [0, 1]$ so

$p \in \mathcal{G}_\varepsilon(T_1, T_l)$. Then there is a constant $M = M(\varepsilon)$ which is independent of large l such that

$$J_\varepsilon(p(\theta)) \le M(\varepsilon) \tag{3.16}$$

for $\theta \in [0,1]$. Hence by (3.16),

$$b_\varepsilon(T_1, T_l) \le M(\varepsilon). \tag{3.17}$$

Now we argue somewhat as in the proof of (3.6). Choose any σ for which Proposition 3.7 is valid. For any $g_l \in \mathcal{G}_\varepsilon(T_1, T_l)$, we see that if $|l| > 0$ is large, $|g_l(\theta)(x)| \le \sigma$ for $x \in \mathbb{R}^n \setminus N_{|l|/10}(T_l)$ and each $\theta \in [0,1]$. Using the existence result of Proposition 3.7, define

$$\hat{g}_l(\theta)(x) = \begin{cases} g_l(\theta)_{\mathbb{R}^n \setminus N_{|l|/10}(T_l)}(x) \text{ for } x \in \mathbb{R}^n \setminus N_{|l|/10}(T_l) \\ g_l(\theta)(x) \text{ for } x \in N_{|l|/10}(T_l). \end{cases}$$

The uniqueness result of Proposition 3.7 implies that $\hat{g}_l \in \mathcal{G}_\varepsilon(T_1, T_l)$. Choose a function $\psi_l \in C^\infty(\mathbb{R}^n; [0,1])$ such that $\psi_l(x) = 1$ for $x \in N_{|l|/8}(T_l)$, $\psi_l(x) = 0$ for $x \notin N_{|l|/4}(T_l)$ and $|\nabla \psi_l| \le 10/|l|$ and define $\tilde{g}_l(\theta) \equiv \hat{g}_l(\theta)\psi_l$. As in (3.9), we find constants, $c, C > 0$, independent of large $|l| > 0$ such that

$$J_\varepsilon(g_l(\theta)) \ge J_\varepsilon(\hat{g}_l(\theta)) \ge J_\varepsilon(\tilde{g}_l(\theta)) - C\exp(-c|l|). \tag{3.18}$$

With χ_S denoting the characteristic function of S as in §2, define

$$\tilde{g}_{l,1}(\theta) \equiv \tilde{g}_l(\theta)\chi_{N_{|l|/2}(T_1)}, \quad \tilde{g}_{l,2}(\theta) \equiv \tilde{g}_l(\theta)\chi_{N_{|l|/2}(l+T_2)},$$

Then, by (3.18),

$$J_\varepsilon(g_l(\theta)) \ge J_\varepsilon(\tilde{g}_{l,1}(\theta)) + J_\varepsilon(\tilde{g}_{l,2}(\theta)) - C\exp(-c|l|). \tag{3.19}$$

Next define h_l as follows:

$$h_l(\theta)(x) = \begin{cases} 3\theta \min\{\tilde{g}_{l,2}(0)(x), U_{\varepsilon,l+T_2}(x)\} \text{ for } \theta \in [0,1/3] \\ \min\{\tilde{g}_{l,2}(3\theta-1)(x), U_{\varepsilon,l+T_2}(x)\} \text{ for } \theta \in [1/3,2/3] \\ (3\theta-2)U_{\varepsilon,l+T_2}(x) + (3-3\theta)h_l(2/3)(x) \text{ for } \theta \in (2/3,1]. \end{cases}$$

Since $h_l \in \mathcal{G}_\varepsilon(\emptyset, l+T_2)$, Proposition 3.1 shows that

$$\max_{\theta \in [0,1]} J_\varepsilon(h_l(\theta)) \ge b_\varepsilon(\emptyset, l+T_2) = b_\varepsilon(\emptyset, T_2) \ge \underline{\beta}/\sqrt{\varepsilon}. \tag{3.20}$$

Theorem 1.2 and the form of h_l show that for some constant $C_1 > 0$, independent of small $\varepsilon > 0$ and large $|l| > 0$,

$$\max_{\theta\in[0,1]\setminus(1/3,2/3)} J_\varepsilon(h_l)(\theta)) \le C_1$$

and this implies that for small $\varepsilon > 0$ and large l,

$$\max_{\theta\in[0,1]} J_\varepsilon(h_l(\theta)) = \max_{\theta\in[1/3,2/3]} J_\varepsilon(h_l)(\theta)). \tag{3.21}$$

By Theorem 1.2 and the decay property 5^o of Theorem 2.2, for any $x \in \mathbb{R}^n$, $\tilde{g}_{l,2}(3\theta - 1)(x) \le U_{\varepsilon,T_l}(x)$ and

$$\lim_{|l|\to\infty} \|U_{\varepsilon,T_l} - U_{\varepsilon,T_2}(\cdot - l)\|_{C^1(\mathrm{supp}(\tilde{g}_{l,2}))} = 0.$$

Therefore

$$\lim_{|l|\to\infty} \max_{\theta\in[1/3,2/3]} J_\varepsilon(h_l(\theta)) = \lim_{|l|\to\infty} \max_{\theta\in[0,1]} J_\varepsilon(\tilde{g}_{l,2}(\theta)). \tag{3.22}$$

Thus, by (3.20)–(3.22),

$$\lim_{|l|\to\infty} \max_{\theta\in[0,1]} J_\varepsilon(\tilde{g}_{l,2}(\theta)) \ge b_\varepsilon(\emptyset, T_2). \tag{3.23}$$

Note that for each $\theta \in [0,1]$, $\tilde{g}_l^1(\theta) \in \Gamma(T_1)$ so

$$J_\varepsilon(\tilde{g}_l^1(\theta)) \ge c_\varepsilon(T_1). \tag{3.24}$$

Thus, combining (3.24) with (3.19) and (3.23), gives

$$\liminf_{l\to\infty} b_\varepsilon(T_1, T_l) \ge c_\varepsilon(T_1) + b_\varepsilon(T_2). \tag{3.25}$$

To get an upper bound for $b_\varepsilon(T_1, T_l)$, a gluing argument will be used. Let p be as in (3.16). Note that by the 5^o of Theorem 2.2 and the fact that U_{ε,T_l} and U_{ε,T_1} are solutions of (1.1), there are constants, $C, c > 0$ such that

$$|p(x)| + |\nabla p(x)| \le C\exp(-cd(x, T_l)). \tag{3.26}$$

Let $\delta > 0$ and choose $h \in \mathcal{G}_\varepsilon(\emptyset, l + T_2)$ such that

$$\max_{\theta\in[0,1]} J_\varepsilon(h(\theta)) \le b_\varepsilon(T_2) + \delta. \tag{3.27}$$

Roughly speaking, we would like to glue p restricted to a neighborhood of T_1 to h restricted to a neighborhood of $l+T_2$ and use the resulting function to get the upper bound for $b_\varepsilon(T_1,T_l)$. However there are some technical problems in doing so because h need not have good enough decay properties and $p(0)=U_{\varepsilon,T_1}\neq h(0)$. To get around these difficulties, using Proposition 3.7, set

$$\hat{h}(\theta)(x)=\begin{cases} h_{\mathbb{R}^n\setminus N_{|l|/4}(l+T_2)}(\theta)(x) \text{ for } x\in\mathbb{R}^n\setminus N_{|l|/4}(l+T_2)\\ h(\theta)(x) \text{ for } x\in N_{|l|/4}(l+T_2)\end{cases}$$

Then due to the properties of $h_{\mathbb{R}^n\setminus N_{|l|/4}(l+T_2)}$, by (3.27),

$$\max_{\theta\in[0,1]} J_\varepsilon(\hat{h}(\theta))\leq b_\varepsilon(T_2)+\delta. \tag{3.28}$$

Set $h^*(\theta)(x)=\max(\hat{h}(\theta)(x),U_{\varepsilon,T_1}(x))$ so $h^*(0)=U_{\varepsilon,T_1}$ and for l large, by (3.28),

$$\max_{\theta\in[0,1]}\int_{N_{|l|/2}(l+T_2)} L_\varepsilon(h^*(\theta))\leq b_\varepsilon(T_2)+2\delta. \tag{3.29}$$

Choose $r\in(0,|l|)$ with r large enough so that $g|_{\partial N_r(T_1)}<\sigma$ where σ is as in Proposition 3.7. Define $f\in\mathcal{G}_\varepsilon(T_1,T_l)$ by

$$f(\theta)(x)=\begin{cases} p(\theta)(x) \text{ for } x\in N_{2r}(T_1)\\ q(\theta)(x) \text{ for } x\in\mathbb{R}^n\setminus N_{2r}(T_1)\cup N_{|l|/2}(l+T_2)\\ h^*(\theta)(x) \text{ for } x\in N_{|l|/2}(l+T_2).\end{cases}$$

where $q(\theta)$, as given by Proposition 3.7, extends the function whose restriction to $N_{2r}(T_1)$ is $p(\theta)$ and whose restriction to $N_{|l|/2}(l+T_2)$ is $h^*(\theta)$. Then as earlier, for some constants, $C^*,c^*>0$,

$$J_\varepsilon(f(\theta))\leq\int_{N_{2r}(T_1)} L_\varepsilon(p(\theta))\,dx+\int_{N_{\frac{|l|}{2}}(l+T_2)} L_\varepsilon(h^*(\theta))\,dx+C^*\exp(-c^*r) \tag{3.30}$$

Observe that for fixed r, as $|l|\to\infty$,

$$\int_{N_r(T_1)} L_\varepsilon(p(\theta))\,dx\to\int_{N_r(T_1)} L_\varepsilon(U_{\varepsilon,T_1})\,dx\leq c_\varepsilon(T_1) \tag{3.31}$$

uniformly in $\theta \in [0,1]$. Therefore by (3.29) and (3.31), for large l,

$$J_\varepsilon(f(\theta)) \le \int_{N_{2r}(T_1)} L_\varepsilon(U_{\varepsilon,T_1})\,dx + b_\varepsilon(T_2) + C^* \exp(-c^* r) + 2\delta, \tag{3.32}$$

so by (3.31)–(3.32),

$$\limsup_{l\to\infty} b_\varepsilon(T_1, T_l) \le c_\varepsilon(T_1) + b_\varepsilon(T_2) + C^* \exp(-c^* r) + 2\delta. \tag{3.33}$$

Letting $r \to \infty$, and then $\delta \to 0$, and combining the result with (3.25) yields 1^o of Theorem 3.14.

To prove 2^o of the Theorem, note first that by Theorem 1.2, as $l \to \infty$, $U_{\varepsilon,T_l} \to U \in \mathcal{M}_\varepsilon(T_1)$ in $C^2_{loc}(\mathbb{R}^n)$, where $U \ge U_{\varepsilon,T_1}$. The choice of U_{ε,T_1} as the largest member of $\mathcal{M}_\varepsilon(T_1)$ implies $U = U_{\varepsilon,T_1}$. Since $U_{\varepsilon,T_1} < V_{\varepsilon,T_1,T_l} < U_{\varepsilon,T_l}$ and V_{ε,T_1,T_l} is a solution of (1.1), the first assertion of 2^o follows. The second requires more work. The uniform bounds for V_{ε,T_1,T_l} in $C^{2,\alpha}(\mathbb{R}^n)$, Theorem 1.2, and the choice of U_{ε,T_2} imply there is a solution, V, of (1.1) such that $V_{\varepsilon,T_1,T_l}(\cdot - l) \to V$ in $C^{2,\alpha}_{loc}(\mathbb{R}^n)$ along a subsequence as $l \to \infty$. Therefore with $1 << r < |l|/2$, estimating as earlier,

$$\left| J_\varepsilon(V_{\varepsilon,T_1,T_l}) - \int_{N_r(T_1)} L_\varepsilon(V_{\varepsilon,T_1,T_l})\,dx - \int_{N_r(T_2)} L_\varepsilon(V_{\varepsilon,T_1,T_l}(\cdot - l))\,dx \right| \le C_3 \exp(-r). \tag{3.34}$$

Hence letting l and then $r \to \infty$, (3.34) and 1^o of this theorem give

$$c_\varepsilon(T_1) + b_\varepsilon(T_2) = J_\varepsilon(U_{\varepsilon,T_1}) + J_\varepsilon(V_{\varepsilon,2}) \tag{3.35}$$

so

$$b_\varepsilon(T_2) = J_\varepsilon(V_{\varepsilon,2})$$

and Theorem 3.14 is proved.

4. Some concluding remarks

Remark 4.1. In §2–§3, we have shown that there are solutions of (1.1) when $T = T_l$ corresponding to gluing minima for T_1 and $l + T_2$ and gluing minima for T_1 to mountain pass solutions for $l + T_2$ (as well as the other way around). It is therefore natural to ask whether one can find additional solutions of (1.1) by gluing a mountain pass solution for T_1 to one for $l + T_2$.

We believe this to be the case. Indeed such solutions were obtained in a related situation in [8] although the question of asymptotic behavior there also remains open.

Remark 4.2. If $T_1, T_2, \cdots, T_k$ are subsets of $\mathbb{Z}^n$, one can use the arguments of this note to find solutions of (1.1) corresponding to minima for $T_1, l_2 + T_2, \cdots, l_k + T_k$ provided that the sets $T_1, l_2 + T_2, \cdots, l_k + T_k$ are widely spaced. Similarly one can glue one mountain pass solution to $k-1$ widely separated minima. We expect that there are higher order analogues of these results in the spirit of Remark 4.1.

Remark 4.3. If in Theorem 1.2, T_1 or T_2 and hence T_l is an infinite set, $c_\varepsilon(T_l)$ is infinite. Therefore 1^o of Theorem 1.2 is not meaningful. However most of 2^o–3^o of the theorem can be preserved if the distance between T_1 and $l + T_2$, $dist(T_1, l + T_2) \to \infty$ as $l \to \infty$ for some unbounded set of $l's$. To be more precise, recall that from Theorem 1.1 of [9], if $S \subset \mathbb{Z}^n$ is infinite, there is still a solution, $\mathcal{U}_{\varepsilon,S}$, of (1.1) in $\Gamma(S)$. Moreover $\mathcal{U}_{\varepsilon,T}$ is a minimal solution, i.e. for all smooth φ having compact support,

$$\int_{\mathbb{R}^n} (L_\varepsilon(\mathcal{U}_{\varepsilon,S} + \varphi) - L_\varepsilon(\mathcal{U}_{\varepsilon,S}))\, dx \geq 0. \tag{4.4}$$

As was the case in Theorem 2.2, $\mathcal{U}_{\varepsilon,S}$ need not be unique. Returning to the current setting, suppose that $S = T_l = T_1 \cup (l + T_2)$ and assume

(l^*) $\mathcal{I} \equiv \{l \in \mathbb{Z}^n \mid dist(T_1, l + T_2) > 0\}$ is unbounded.

As a simple example, suppose that T_1 and T_2 are infinite subsets of $\mathbb{R}^{n-1} \times \{0\}$. Then we can take $\mathcal{I} = \{l_n e_n \mid l_n \in \mathbb{N}\}$.

Now we have:

Theorem 4.5. *Suppose that* (l^*) *holds. Then for each* $\varepsilon \in (0, \varepsilon_0)$,

(1^o) *there is a minimal solution,* $\mathcal{U}_{\varepsilon,T_1}$ *of (1.1) with* $\mathcal{U}_{\varepsilon,T_1} \in \Gamma(T_1)$ *such that along a sequence of* $(l_p) \subset \mathcal{I}$ *with* $l_p \to \infty$ *as* $p \to \infty$, $\mathcal{U}_{\varepsilon,T_{l_p}} \to \mathcal{U}_{\varepsilon,T_1}$ *in* $C^2_{loc}(\mathbb{R}^n)$;

(2^o) *there is a minimal solution,* $\mathcal{U}_{\varepsilon,T_2}$ *of (1.1) with* $\mathcal{U}_{\varepsilon,T_2} \in \Gamma(T_2)$ *such that along a sequence of* $(m_p) \subset \mathcal{I}$ *with* $m_p \to \infty$ *as* $p \to \infty$, $\mathcal{U}_{\varepsilon,T_{m_p}}(\cdot + m_p) \to \mathcal{U}_{\varepsilon,T_2}$ *in* $C^2_{loc}(\mathbb{R}^n)$.

Proof: Since $\|\mathcal{U}_{\varepsilon,T_l}\|_{L^\infty(\mathbb{R}^n)} \leq 1$ for all $l \in \mathcal{I}$, using the local $W^{k,p}$ and Schauder estimates, as e.g. in [9], shows for any $\alpha \in (0,1)$, there is a constant, $K = K(\alpha)$ such that

$$\|\mathcal{U}_{\varepsilon,T_l}\|_{C^{2,\alpha}(\mathbb{R}^n)} \leq K(\alpha) \tag{4.6}$$

independently of $l \in \mathcal{I}$. Therefore the Arzela-Ascoli Theorem and (1.1) imply the existence of the solution, $\mathcal{U}_{\varepsilon,T_1}$, as a limit of $\mathcal{U}_{\varepsilon,T_{l_p}}$. That $\mathcal{U}_{\varepsilon,T_{l_p}} \in \Gamma(T_{l_p})$ for all $p \in \mathcal{I}$ implies $\mathcal{U}_{\varepsilon,T_1} \in \Gamma(T_1)$. Moreover, as the C^2_{loc} limit of minimal solutions of (1.1), (4.4) shows $\mathcal{U}_{\varepsilon,T_1}$ is minimal. A similar argument establishes 2^o.

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