Fundamenta Informaticae 188(1) : 15–39 (2022)
Available at IOS Press through:
https://doi.org/10.3233/FI-222140

Decidability of Definability Issues in the Theory of Real Addition

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Abstract. Given a subset of $X \subseteq \mathbb{R}^n$ we can associate with every point $x \in \mathbb{R}^n$ a vector space V of maximal dimension with the property that for some ball centered at x, the subset X coincides inside the ball with a union of lines parallel to V. A point is singular if V has dimension 0.

In an earlier paper we proved that a $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$-definable relation X is $\langle \mathbb{R}, +, <, 1 \rangle$-definable if and only if the number of singular points is finite and every rational section of X is $\langle \mathbb{R}, +, <, 1 \rangle$-definable, where a rational section is a set obtained from X by fixing some component to a rational value.

Here we show that we can dispense with the hypothesis of X being $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$-definable by requiring that the components of the singular points be rational numbers. This provides a topological characterization of first-order definability in the structure $\langle \mathbb{R}, +, <, 1 \rangle$. It also allows us to deliver a self-definable criterion (in Muchnik's terminology) of $\langle \mathbb{R}, +, <, 1 \rangle$- and $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$-definability for a wide class of relations, which turns into an effective criterion provided that the corresponding theory is decidable. In particular these results apply to the class of so-called $k-$recognizable relations which are defined by finite Muller automata via the representation of the reals in a integer basis k, and allow us to prove that it is decidable whether a $k-$recognizable relation (of any arity) is $l-$recognizable for every base $l \geq 2$.

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Received July 2022; accepted November 2022.

1. Introduction

In his seminal work on Presburger Arithmetic [18], Muchnik provides a characterization of definability of a relation $X \subseteq \mathbb{Z}^n$ in $\langle \mathbb{Z}, +, < \rangle$ in terms of sections of X and local periodicity properties of X. It also shows that the characterization can be expressed as a $\langle \mathbb{Z}, +, <, X \rangle$-sentence, and thus can be decided if $\langle \mathbb{Z}, +, <, X \rangle$ is decidable. As an application Muchnik proves that it is decidable whether a $k-$recognizable relation $X \subseteq \mathbb{Z}^n$ is $\langle \mathbb{Z}, +, < \rangle$-definable. Recall that given an integer $k \geq 2$, X is $k-$recognizable if it is recognizable by some finite automaton whose inputs are the base-k encoding of integers (see [8]).

The present paper continues the line of research started in [2], which aims to extend Muchnik's results and techniques to the case of reals with addition. Consider the structure $\langle \mathbb{R}, +, <, 1 \rangle$ of the additive ordered group of reals along with the constant 1. It is well-known that the subgroup $\mathbb{Z}$ of integers is not first-order-definable in this structure. Let $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ denote the expansion of $\langle \mathbb{R}, +, <, 1 \rangle$ with the unary predicate "$x \in \mathbb{Z}$". In [2] we prove a topological characterization of $\langle \mathbb{R}, +, <, 1 \rangle$-definable relations in the family of $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$-definable relations, and use it to derive, on the one hand, that it is decidable whether or not a relation on the reals definable in $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ can be defined in $\langle \mathbb{R}, +, <, 1 \rangle$ and on the other hand that there is no intermediate structure between $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ and $\langle \mathbb{R}, +, <, 1 \rangle$ (since then, the latter result has been generalized by Walsberg [23] to a large class of $o-$minimal structures)

We recall the topological characterization of $\langle \mathbb{R}, +, <, 1 \rangle$ in $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$, [2, Theorem 6.1]. We say that the neighborhood of a point $x \in \mathbb{R}^n$ relative to a relation $X \subseteq \mathbb{R}^n$ has a *stratum* if there exists a direction such that the intersection of X with any sufficiently small neighborhood around x is the trace of a union of lines parallel to the given direction. When X is $\langle \mathbb{R}, +, <, 1 \rangle$-definable, all points have strata, except finitely many which we call singular. In [2] we give necessary and sufficient conditions for a $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$-definable relation $X \subseteq \mathbb{R}^n$ to be $\langle \mathbb{R}, +, <, 1 \rangle$-definable, namely (FSP): it has finitely many singular points and (DS): all intersections of X with arbitrary hyperplanes parallel to $n-1$ axes and having rational components on the remaining axis are $\langle \mathbb{R}, +, <, 1 \rangle$-definable. We asked whether it is possible to remove the assumption that the given relation is $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$-definable. In the present paper we prove that the answer is positive if a new assumption is added, see below. Let us first explain the structure of the proof in [2]. The necessity of the two conditions (FSP) and (DS) is easy. The difficult part was their sufficiency and it used very specific properties of the $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$-definable relations, in particular the fact that $\langle \mathbb{R}, +, <, 1 \rangle$- and $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$-definable relations are locally indistinguishible. In order to show the existence of a $\langle \mathbb{R}, +, <, 1 \rangle$-formula for X we showed two intermediate properties, (RB): for every nonsingular point x, there exists a basis of the strata subspace composed of vectors with rational components, and (FI): there are finitely many "neighborhood types", i.e., the equivalence relation $x \sim y$ on $\mathbb{R}^n$ which holds if there exists $r > 0$ such that $(x + w \in X \leftrightarrow y + w \in X$ for every $|w| < r)$ has finite index.

When passing from the characterization of $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$-definable relations to the characterization of general ones the topological characterization uses the same intermediate properties but they are much more delicate to establish and an extra condition (RSP) is required: all singular points of X have rational components. Moreover we show that this characterization is effective under natural conditions. Indeed, if every nonempty $\langle \mathbb{R}, +, <, 1, X \rangle$-definable relation contains a point with rational components, then the $\langle \mathbb{R}, +, <, 1\rangle$-definability of X is expressible in the structure $\langle \mathbb{R}, +, <, 1, X\rangle$ itself. The crucial point is the notion of *quasi-singular* points generalizing that of singular points. We were forced to consider this new notion because the $\langle \mathbb{R}, +, <, 1, X\rangle$-predicate which defines singular points in $\langle \mathbb{R}, +, <, \mathbb{Z}\rangle$ no longer defines them in general structures. In so doing we can turn the criterion for $\langle \mathbb{R}, +, <, 1\rangle$-definability into an effective criterion provided that the theory of $\langle \mathbb{R}, +, <, 1, X\rangle$ is decidable. More precisely we show that for every decidable expansion $\mathcal{M}$ of $\langle \mathbb{R}, +, <, 1\rangle$ such that every nonempty $\mathcal{M}$-definable relation contains a point with rational components, one can decide whether or not a given $\mathcal{M}$-definable relation is $\langle \mathbb{R}, +, <, 1\rangle$-definable.

We extend the result of $\langle \mathbb{R}, +, <, 1\rangle$-definability of a general relation to that of $\langle \mathbb{R}, +, <, \mathbb{Z}\rangle$-definability. Every relation on the reals can be uniquely decomposed into some relations on the integers and some relations on the unit hypercubes ([10], see also [13]). This decomposition yields a simple characterization of the $\langle \mathbb{R}, +, <, \mathbb{Z}\rangle$-definable relations, which is expressible in $\langle \mathbb{R}, +, <, \mathbb{Z}, X\rangle$ provided that all nonempty $\langle \mathbb{R}, +, <, \mathbb{Z}, X\rangle$-definable relations contain a point with rational components. Combining the result on $\langle \mathbb{R}, +, <, 1\rangle$-definability for the reals and Muchnik's result on $\langle \mathbb{Z}, +, <, 1\rangle$-definable integer relations we show that for every decidable expansion $\mathcal{N}$ of $\langle \mathbb{R}, +, <, \mathbb{Z}\rangle$ such that every nonempty $\mathcal{N}$-definable relation contains a point with rational components, one can decide whether or not a given $\mathcal{N}$-definable relation is $\langle \mathbb{R}, +, <, \mathbb{Z}\rangle$-definable.

We also study a particularly significant case. The notion of k-recognizability for relations on integers can be extended to the case of relations on reals, by considering Muller automata which read infinite words encoding reals written in base k, see [9, Definition 1]. The class of $k-$recognizable relations coincides with the class of relations definable in some expansion of $\langle \mathbb{R}, +, <, \mathbb{Z}\rangle$ of the form $\langle \mathbb{R}, \mathbb{Z}, +, <, X_k\rangle$ where X_k is a base dependent ternary predicate [9, section 3]. This expansion satisfies the above required condition since it has a decidable theory and every nonempty $k-$recognizable relation contains a point with rational components. The $\langle \mathbb{R}, +, <, \mathbb{Z}\rangle$-definable relations define a subclass which has a very specific relevance since it coincides with the class of relations which are $k-$recognizable for every $k \geq 2$ [4, 5, 6]. A consequence of our result is that given a $k-$recognizable relation it can be decided if it is $\ell-$recognizable for all bases $\ell \geq 2$. This falls into the more general issue of finding effective characterizations of subclasses of $k-$recognizable relations. A previous result of this type was proved by Milchior in [17] by showing that it is decidable whether or not a weakly $k-$recognizable subset of $\mathbb{R}$ is definable in $\langle \mathbb{R}, +, <, 1\rangle$, where "weak" is defined as a natural condition on the states of a deterministic automaton.

We give a short outline of our paper. Section 2 gathers basic definitions and notation. In Section 3 we recall the main useful definitions and results from [2] in order to make the paper selfcontained. In Section 4 we show that the conjunction of conditions (RSP), (RB) and (FI) characterizes the $\langle \mathbb{R}, +, <, 1\rangle$-definable relations. In Section 5 we deal with the self-definable criterion of $\langle \mathbb{R}, +, <, 1\rangle$-definability. We introduce the crucial notion of quasi-singular point and show that it is definable in $\langle \mathbb{R}, +, <, 1, X\rangle$. We also provide an alternative, inductive, formulation of $\langle \mathbb{R}, +, <, 1\rangle$-definability for X: every relation obtained from X by assigning fixed real values to arbitrary components contains finitely many quasi-singular points. We then show how to extend the results to the case of $\langle \mathbb{R}, +, <, \mathbb{Z}\rangle$. In Section 6 we show that the self-definable criterion of $\langle \mathbb{R}, +, <, 1\rangle$-definability (resp. $\langle \mathbb{R}, +, <, \mathbb{Z}\rangle$-definability) of a relation $X \subseteq \mathbb{R}^n$ can be turned into an effective criterion provided that X is definable in a suitable decidable theory, and apply the result to the class of k-recognizable relations.

Other related work.

Muchnik's approach, namely expressing in the theory of the structure a property of the structure itself, can be used in other settings. We refer the interested reader to the discussion in [22, Section 4.6] and also to [19, 1, 17] for examples of such structures. A similar method has already been used in 1966, see [15, Thm 2.2.] where the authors are able to express in Presburger theory whether or not a Presburger subset is the Parikh image of a context-free language.

The theory of (expansions of) dense ordered groups has been studied extensively in model theory, in particular in connection with o-minimality, see e.g. [11, 12]. Let us also mention a recent series of results by Hieronymi which deal with expansions of $\langle \mathbb{R}, +, <, \mathbb{Z}\rangle$, and in particular with the frontier of decidability for such expansions, see, e.g., [16] and its bibliography.

2. Preliminaries

Throughout this work we assume the vector space $\mathbb{R}^n$ is provided with the metric L_∞ (i.e., $|x| = \max_{1\leq i\leq n} |x_i|$). Let $B(x, r)$ denote the open ball centered at $x \in \mathbb{R}^n$ and of radius $r > 0$. Given $x, y \in \mathbb{R}^n$ let $[x, y]$ (resp. (x, y)) denote the closed segment (resp. open segment) with extremities x, y. We use also notations such as $[x, y)$ or $(x, y]$ for half-open segments.

Let us specify our logical conventions and notations. We work within first-order predicate calculus with equality. We identify formal symbols and their interpretations. We are mainly concerned with the structures $\langle \mathbb{R}, +, <, 1\rangle$ and $\langle \mathbb{R}, +, <, \mathbb{Z}\rangle$. Given a structure $\mathcal{M}$ with domain D and $X \subseteq D^n$, we say that X is *definable in* $\mathcal{M}$, or $\mathcal{M}$*-definable*, if there exists a formula $\varphi(x_1, \ldots, x_n)$ in the signature of $\mathcal{M}$ such that $\varphi(a_1, \ldots, a_n)$ holds in $\mathcal{M}$ if and only if $(a_1, \ldots, a_n) \in X$ (this corresponds to the usual notion of *definability without parameters*).

The $\langle \mathbb{R}, +, <, 1\rangle$-theory admits quantifier elimination in the following sense, which can be interpreted geometrically as saying that a $\langle \mathbb{R}, +, <, 1\rangle$-definable relation is a finite union of closed and open polyhedra.

Theorem 2.1. [14, Thm 1] Every formula in $\langle \mathbb{R}, +, <, 1\rangle$ is equivalent to a finite Boolean combination of inequalities between linear combinations of variables with coefficients in $\mathbb{Z}$ (or, equivalently, in $\mathbb{Q}$).

In particular every nonempty $\langle \mathbb{R}, +, <, 1\rangle$-definable relation contains a point with rational components.

3. Local properties of real relations

Most of the definitions and results in this section are taken from [2]. These are variants of notions and results already known in computational geometry, see e.g. [3, 7] for the case of $\langle \mathbb{R}, +, <, 1\rangle$-definable relations. We only give formal proofs for the new results. In the whole section we fix $n \geq 1$ and $X \subseteq \mathbb{R}^n$.

3.1. Strata

The following clearly defines an equivalence relation.

Definition 3.1. Given $x, y \in \mathbb{R}^n$ we write $x \sim_X y$ or simply $x \sim y$ when X is understood, if there exists a real $r > 0$ such that the translation $w \mapsto w + y - x$ is a one-to-one mapping from $B(x, r) \cap X$ onto $B(y, r) \cap X$.

Example 3.2. Let X be a closed subset of the plane delimited by a square. There are ten $\sim_X$-equivalence classes: the set of points interior to the square, the set of points interior to its complement, the four vertices and the four open edges.

Let $\mathcal{C}l(x)$ denote the $\sim$-equivalence class to which x belongs.

Definition 3.3.

1. Given a non-zero vector $v \in \mathbb{R}^n$ and a point $y \in \mathbb{R}^n$, let $L_v(y) = \{y + \alpha v \mid \alpha \in \mathbb{R}\}$ be the line passing through y in the direction v. More generally, if $X \subseteq \mathbb{R}^n$ let $L_v(X)$ denote the set $\bigcup_{x \in X} L_v(x)$.

2. A non-zero vector $v \in \mathbb{R}^n$ is an X-*stratum* at x (or simply a *stratum* when X is understood) if there exists a real $r > 0$ such that

$$B(x, r) \cap L_v(X \cap B(x, r)) \subseteq X. \tag{1}$$

   This can be seen as saying that inside the ball $B(x, r)$, the relation X is a union of lines parallel to v. By convention the zero vector is also considered as a stratum.

3. The set of X-strata at x is denoted $\mathrm{Str}_X(x)$ or simply $\mathrm{Str}(x)$.

Proposition 3.4. [2, Proposition 3.4] For every $x \in \mathbb{R}^n$ the set *Str*(x) is a vector subspace of $\mathbb{R}^n$.

Definition 3.5. The *dimension* $\dim(x)$ of a point $x \in \mathbb{R}^n$ is the dimension of the subspace $\mathrm{Str}(x)$. We say that x is a *d-point* if $d = \dim(x)$. Moreover if $d = 0$ then x is said to be X-*singular*, or simply *singular*, and otherwise it is *nonsingular*.

Example 3.6. (Example 3.2 continued) Let $x \in \mathbb{R}^2$. If x belongs to the interior of the square or of its complement, then $\mathrm{Str}(x) = \mathbb{R}^2$. If x is one of the four vertices of the square then we have $\mathrm{Str}(x) = \{0\}$, i.e., x is singular. Finally, if x belongs to an open edge of the square but is not a vertex, then $\mathrm{Str}(x)$ has dimension 1, and two points of opposite edges have the same strata subspace, while two points of adjacent edges have different strata subspaces.

It can be shown that all strata at x can be defined with respect to a common value r in expression (1).

Proposition 3.7. [2, Proposition 3.9] For every $x \in \mathbb{R}^n$ there exists a real $r > 0$ such that for every $v \in$ *Str*$(x) \setminus \{0\}$ we have

$$B(x, r) \cap L_v(X \cap B(x, r)) \subseteq X.$$

Definition 3.8. A X*-safe radius* (or simply a *safe radius* when X is understood) for x is a real $r > 0$ satisfying the condition of Proposition 3.7. Clearly if r is safe then so are all $0 < s \leq r$. By convention every real is a safe radius if $\mathrm{Str}(x) = \{0\}$.

Example 3.9. (Example 3.2 continued) For an element x in the interior of the square or the interior of its complement a safe radius is the (minimal) distance from x to the edges of the square. If x is a vertex then $\mathrm{Str}(x) = \{0\}$ and every $r > 0$ is safe for x. In all other cases r can be chosen as the minimal distance of x to a vertex.

Remark 3.10. If $x \sim y$ then $\mathit{Str}(x) = \mathit{Str}(y)$, therefore given an $\sim$-equivalence class E, we may define $\mathrm{Str}(E)$ as the set of common strata of all $x \in E$.

Observe that the converse is false. In Example 3.2 for instance, points in the interior and points in the complement of the interior of the square have the same set of strata, namely $\mathbb{R}^2$, but are not $\sim -$equivalent.

It is possible to combine the notions of strata and of safe radius.

Lemma 3.11. [2, Lemma 3.13] Let $x \in \mathbb{R}^n$ and r be a safe radius for x. Then for all $y \in B(x, r)$ we have $\mathit{Str}(x) \subseteq \mathit{Str}(y)$.

Example 3.12. (Example 3.2 continued) Consider a point x on an (open) edge of the square and a safe radius r for x. For every point y in $B(x, r)$ which is not on the edge we have $\mathrm{Str}(x) \subsetneq \mathrm{Str}(y) = \mathbb{R}^2$. For all other points we have $\mathrm{Str}(x) = \mathrm{Str}(y)$.

Inside a ball whose radius is safe for the center, all points along a stratum are $\sim$-equivalent.

Lemma 3.13. Let x be non-singular, $v \in \mathrm{Str}(x) \setminus \{0\}$, and r be safe for x. For every $z \in B(x, r)$ we have $L_v(z) \cap B(x, r) \subseteq \mathcal{C}l(z)$.

Proof:
Let $z' \in L_v(z) \cap B(x, r)$, and $s > 0$ be such that both $B(z, s), B(z', s)$ are included in $B(x, r)$. For every $w \in B(0, s)$ we have $z' + w \in L_v(z + w)$ thus $z + w \in X \leftrightarrow z' + w \in X$. □

3.2. Relativization to affine subspaces

We relativize the notion of singularity and strata to an affine subspace $S \subseteq \mathbb{R}^n$. The next definition should come as no surprise.

Definition 3.14. Given a subset $X \subseteq \mathbb{R}^n$, an affine subspace $S \subseteq \mathbb{R}^n$ and a point $x \in S$, we say that a vector $v \in \mathbb{R}^n \setminus \{0\}$ parallel to S is an (X, S)*-stratum for the point* x if for all sufficiently small $r > 0$ it holds
$$B(x, r) \cap L_v(X \cap B(x, r) \cap S) \subseteq X. \tag{2}$$

By convention the zero vector is also considered as a (X, S)-stratum. The set of (X, S)-strata of x is denoted $\mathrm{Str}_{(X,S)}(x)$. We define the equivalence relation $x \sim_{(X,S)} y$ on S as follows: $x \sim_{(X,S)} y$ if and only if there exists a real $r > 0$ such that $x + w \in X \leftrightarrow y + w \in X$ for every $w \in \mathbb{R}^n$ parallel to S and such that $|w| < r$. A point $x \in S$ is (X, S)-*singular* if it has no (X, S)-stratum. For simplicity when S is the space $\mathbb{R}^n$ we maintain the previous terminology and speak of X-strata and X-singular points. We say that a real $r > 0$ is (X, S)-*safe* if (2) holds for every nonzero $(X, S)-$stratum v.

Remark 3.15. Singularity and nonsingularity do not go through restriction to affine subspaces. E.g., in the real plane, let $X = \{(x, y) \mid y < 0\}$ and $S = \{(x, y) \mid x = 0\}$. Then the origin is not $X-$singular but it is $(X, S)-$singular. All other elements of S admit $(0, 1)$ as an $(X, S)-$stratum thus they are not $(X, S)-$singular. The opposite situation may occur. In the real plane, let $X = \{(x, y) \mid y < 0\} \cup S$. Then the origin is $X-$singular but it is not $(X, S)-$singular.

3.2.1. Relativization of the space of strata

Lemma 3.16. Let S be an affine hyperplane of $\mathbb{R}^n$ and $x \in S$. Let V be the vector subspace generated by $\mathrm{Str}_X(x) \setminus \mathrm{Str}_{(X,S)}(x)$. If $V \neq \{0\}$ then $\mathrm{Str}_X(x) = V + \mathrm{Str}_{(X,S)}(x)$, and otherwise $\mathrm{Str}_X(x) \subseteq \mathrm{Str}_{(X,S)}(x)$.

Proof:
It is clear that if $V = \{0\}$ then every X-stratum of S is an (X, S)-stratum.

Now assume there exists $v \in \mathrm{Str}_X(x) \setminus \mathrm{Str}_{(X,S)}(x)$. It suffices to prove that all $w \in \mathrm{Str}_{(X,S)}(x)$ belong to $\mathrm{Str}_X(x)$. Let $s > 0$ be simultaneously $(X, S)-$safe and $X-$safe for x. Let $0 < s' < s$ be such that $L_v(z) \cap S \subseteq B(x, s)$ for every $z \in B(x, s')$. Let $y_1, y_2 \in B(x, s')$ be such that $y_1 - y_2$ and w are parallel. It suffices to prove the equivalence $y_1 \in X \leftrightarrow y_2 \in X$. Let y_1' (resp. y_2') denote the intersection point of $L_v(y_1)$ and S (resp. $L_v(y_2)$ and S). We have $y_1, y_1' \in B(x, s)$, $v \in \mathrm{Str}_X(x)$, and s is $X-$safe for x, thus $y_1 \in X \leftrightarrow y_1' \in X$. Similarly we have $y_2 \in X \leftrightarrow y_2' \in X$. Now $y_1', y_2' \in B(x, s)$, $y_1' - y_2'$ and w are parallel, and $w \in \mathrm{Str}_{(X,S)}(x)$, which implies $y_1' \in X \leftrightarrow y_2' \in X$ and thus finally $y_1 \in X \leftrightarrow y_2 \in X$. ☐

Corollary 3.17. Let S be an hyperplane of $\mathbb{R}^n$ with underlying vector subspace V, and let $x \in S$ be non-singular. If $\mathrm{Str}_X(x) \setminus V$ is nonempty then $\mathrm{Str}_{(X,S)}(x) = \mathrm{Str}_X(x) \cap V$.

3.2.2. Relativization of the $\sim$-relation

Lemma 3.18. Let S be an hyperplane of $\mathbb{R}^n$, $y, z \in S$, and $v \neq \{0\}$ be a common $X-$stratum of y, z not parallel to S. If $y \sim_{(X,S)} z$ then $y \sim_X z$.

Proof:
Assume $y \sim_{(X,S)} z$, and let $r > 0$ be $(X, S)-$ and $X-$ safe both for y and z. Since v is not parallel to S, there exists $s > 0$ such that for every $w \in \mathbb{R}^n$ with $|w| < s$, the intersection point of $L_v(y + w)$ (resp. $L_v(z + w)$) and S exists because $\dim(S) = n - 1$ and belongs to $B(y, r)$ (resp. $B(z, r)$). It suffices to show that $y + w \in X \leftrightarrow z + w \in X$. Let $y + w'$ be the intersection point of $L_v(y + w)$ and S.

By our hypothesis on s, $y + w'$ belongs to $B(y, r)$. Moreover r is $X-$safe for y, $v \in \mathrm{Str}_X(y)$, and $w' - w$ is parallel to v, therefore $y + w \in X \leftrightarrow y + w' \in X$. Similarly we have $z + w \in X \leftrightarrow$

$z+w' \in X$. Now $|w'| < r$, thus by our assumptions $y \sim_{(X,S)} z$ we have $y+w' \in X \leftrightarrow z+w' \in X$ and therefore $y+w \in X \leftrightarrow z+w \in X$. □

Next we consider a particular case for S which plays a crucial role in expressing the characterisation stated in the main theorem. It is also a tool for reasoning by induction in Section 4.3.

Definition 3.19. Given an index $0 \le i < n$ and a real $c \in \mathbb{R}$ consider the hyperplane

$$H = \mathbb{R}^i \times \{c\} \times \mathbb{R}^{n-i-1}.$$

The intersection $X \cap H$ is called a *section* of X. It is a *rational section* if c is a rational number. We define $\pi_H : \mathbb{R}^n \to \mathbb{R}^{n-1}$ as $\pi_H(x_1, \ldots, x_n) = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$.

The following facts are easy consequences of the above definitions: for all $x, y \in H$ and v a vector parallel to H we have:

1. $x \sim_{(X,H)} y$ if and only if $\pi_H(x) \sim_{\pi_H(X)} \pi_H(y)$
2. $v \in \text{Str}_{(X,H)}(x)$ if and only if $\pi_H(v) \in \text{Str}_{\pi_H(X)}(\pi_H(x))$. In particular x is $(X,H)-$singular if and only if $\pi_H(x)$ is $\pi_H(X)-$singular.

3.3. Intersection of lines and equivalence classes

In this section we describe the intersection of a $\sim$-class E with a line parallel to some $v \in \text{Str}(E)$. It relies on the notion of adjacency of $\sim$-classes.

Definition 3.20. Let E be a nonsingular $\sim$- class and let v be one of its strata. A point x is $v-$*adjacent to* E if there exists $\epsilon > 0$ such that for all $0 < \alpha \le \epsilon$ we have $x + \alpha v \in E$.

Example 3.21. (Example 3.2 continued) We specify Example 3.2 by choosing the square as the unit square with vertices $(0,0), (0,1), (1,0)$ and $(1,1)$. All elements of the bottom open edge of the square belong to the same $\sim$-class E. The vector $v = (1,0)$ is a stratum of E. The vertex $(0,0)$ is $v-$adjacent to E. Similarly every element of E is also $v-$adjacent to E. However the vertex $(1,0)$ is not $v-$adjacent to E (but it is $(-v)-$adjacent to E).

The notion of adjacency is a property of the $\sim$-class.

Lemma 3.22. [2, Lemma 5.2] Let F be a $\sim$-class.

1. For all $x, y \in F$, all nonzero vectors v and all $\sim$-classes E, x is v-adjacent to E if and only if y is v-adjacent to E.
2. For each vector v there exists a most one $\sim$-class E such that F is v-adjacent to E.

Consequently, if for some $x \in F$ and some vector v, x is v-adjacent to E it makes sense to say that the class F is v-adjacent to E.

Lemma 3.23. [2, Corollary 5.6] Let $x \in \mathbb{R}^n$ be non-singular, $E = \mathcal{C}l(x)$ and let $v \in \mathit{Str}(x) \setminus \{0\}$. The set $L_v(x) \cap E$ is a union of disjoint open segments (possibly infinite in one or both directions) of $L_v(x)$, i.e., of the form $(y - \alpha v, y + \beta v)$ with $0 < \alpha, \beta \leq \infty$ and $y \in E$.

If $\alpha < \infty$ (resp. $\beta < \infty$) then the point $y - \alpha v$ (resp. $y + \beta v$) belongs to a $\sim$-class $F \neq E$ such that $\dim(F) < \dim(E)$ and F is v-adjacent (resp. $(-v)-$adjacent) (or simply *adjacent* when v is understood) to E.

4. Characterizations of $\langle \mathbb{R}, +, <, 1 \rangle$-definable relations

4.1. Characterization in $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$-definable relations

We recall our previous characterization of $\langle \mathbb{R}, +, <, 1 \rangle$-definable among $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$-definable relations.

Theorem 4.1. [2, Theorem 6.1] Let $n \geq 1$ and let $X \subseteq \mathbb{R}^n$ be $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$-definable. Then X is $\langle \mathbb{R}, +, <, 1 \rangle$-definable if and only if the following two conditions hold

(FSP) There exist only finitely many $X-$singular points.

(DS) Every rational section of X is $\langle \mathbb{R}, +, <, 1 \rangle$-definable.

The necessity of condition (FSP) is proved by Proposition 4.6 of [2] and that of (DS) is trivial since a rational section is the intersection of two $\langle \mathbb{R}, +, <, 1 \rangle$-definable relations. The proof that conditions (FSP) and (DS) are sufficient uses several properties of $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$-definable relations which are listed in the form of a proposition below.

Proposition 4.2. Let $n \geq 1$ and $X \subseteq \mathbb{R}^n$ be $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$-definable. The following holds.

(RSP) The components of the X-singular points are rational numbers [2, Proposition 4.6].

(FI) The equivalence relation $\sim$ has finite index and thus the number of different vector spaces $\mathit{Str}(x)$ is finite when x runs over $\mathbb{R}^n$ [2, Corollary 4.5].

(RB) For all nonsingular points x, the vector space $\mathrm{Str}(x)$ has a rational basis in the sense that it can be generated by a set of vectors with rational components [2, Proposition 4.7].

4.2. Characterization in arbitrary relations

Now we aim to characterize $\langle \mathbb{R}, +, <, 1 \rangle$-definability for an arbitrary relation $X \subseteq \mathbb{R}^n$. We prove that the conditions (FSP),(DS),(RSP) are sufficient, i.e., compared to Theorem 4.1 one can remove the condition "X is $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$-definable" and add condition (RSP).

Theorem 4.3. Let $n \geq 1$ and $X \subseteq \mathbb{R}^n$. Then X is $\langle \mathbb{R}, +, <, 1 \rangle$-definable if and only if it satisfies the three conditions *(FSP), (DS), (RSP)*

(FSP) It has only finitely many singular points.

(DS) Every rational section of X is $\langle \mathbb{R}, +, <, 1 \rangle$-definable.

(RSP) Every singular point has rational components.

Observe that the three conditions are needed, as shown by the following relations which are not $\langle \mathbb{R}, +, <, 1 \rangle$-definable.

- Consider the binary relation $X = \{(x, x) \mid x \in \mathbb{Z}\}$. The singular elements of X are precisely the elements of X, thus X satisfies (RSP) but not (FSP). It satisfies (DS) because every rational section of X is either empty or equal to the singleton $\{(x, x)\}$ for some $x \in \mathbb{Z}$, thus it is $\langle \mathbb{R}, +, <, 1 \rangle$-definable.
- The binary relation $X = \mathbb{R} \times \mathbb{Z}$ has no singular point thus it satisfies (FSP) and (RSP). However it does not satisfy (DS) since, e.g., the rational section $\{0\} \times \mathbb{Z}$ is not $\langle \mathbb{R}, +, <, 1 \rangle$-definable.
- The unary relation $X = \{\sqrt{2}\}$ admits $\sqrt{2}$ as unique singular point, thus it satisfies (FSP) but not (RSP). It satisfies (DS) since every rational section of X is empty.

Now we prove Theorem 4.3.

Proof:
The necessity of the first two conditions is a direct consequence of Theorem 4.1, that of the third condition is due to Proposition 4.2.

Now we turn to the other direction which is the bulk of the proof and we proceed in two steps. First we show that properties (FSP), (DS) and (RSP) imply properties (RB) and (FI) (Claims 4.4 and 4.5) and then based on these two properties we show that there exists a $\langle \mathbb{R}, +, <, 1 \rangle$-formula defining X.

Claim 4.4. If X satisfies conditions (FSP), (DS) and (RSP) then it satisfies condition (RB).

Proof:
We prove that for every non-singular point $x \in \mathbb{R}^n$, $\mathrm{Str}(x)$ has a rational basis. If $n = 1$ this follows from the fact that for every $x \in \mathbb{R}$ the set $\mathrm{Str}(x)$ is either equal to $\{0\}$ or equal to $\mathbb{R}$, thus we assume $n \geq 2$.

For every $i \in \{1, \ldots, n\}$ let $H_i = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i = 0\}$. Let us call *rational $i-$hyperplane* any hyperplane S of the form $S = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i = c\}$ where $c \in \mathbb{Q}$. The underlying vector space of S is H_i.

Let x be a $d-$point with $d \geq 1$, i.e., a point for which $V = \mathrm{Str}(x)$ has dimension d. For $d = n$ the result is obvious. For $1 \leq d < n$ we prove the result by induction on d.

Case $d = 1$: It suffices to show that every $1-$point x has a stratum in $\mathbb{Q}^n$. Let $v \in \mathrm{Str}(x) \setminus \{0\}$, and let $r > 0$ be safe for x. We can find $i \in \{1, \ldots, n\}$ and two distinct rational $i-$hyperplanes S_1 and S_2, not parallel to v, such that $L_v(x)$ intersects S_1 (resp. S_2) inside $B(x, r)$, say at some point y_1 (resp. y_2). By Lemma 3.13 we have $y_1 \sim x$. By Corollary 3.17 it follows that

$$\mathrm{Str}_{(X,S_1)}(y_1) = \mathrm{Str}_X(y_1) \cap H_i = \mathrm{Str}_X(x) \cap H_i$$

and the rightmost expression is reduced to $\{0\}$ since $d = 1$ and $v \notin H_i$. This implies that y_1 is $(X, S_1)-$singular, i.e., that $\pi_{S_1}(y_1)$ is $\pi_{S_1}(X)-$singular. Similarly y_2 is $(X, S_2)-$singular, i.e., $\pi_{S_2}(y_2)$ is $\pi_{S_2}(X)-$singular.

By condition (DS) the rational sections $X \cap S_1$ (resp. $X \cap S_2$) are $\langle \mathbb{R}, +, <, 1\rangle$-definable, thus the $(n-1)-$ary relations $\pi_{S_1}(X)$ (resp. $\pi_{S_2}(X)$) are also $\langle \mathbb{R}, +, <, 1\rangle$-definable, and by our hypothesis (RSP) this implies that $\pi_S(y_1)$ (resp. $\pi_S(y_2)$) has rational components. Thus the same holds for y_1 and y_2, and also for $y_1 - y_2$, and the result follows from the fact that $y_1 - y_2 \in \mathrm{Str}_X(x)$.

Case $2 \leq d < n$: Let $I \subseteq \{1, \ldots, n\}$ denote the set of indices i such that $V \not\subseteq H_i$. We have $V \subseteq \bigcap_{i \in \{1,\ldots,n\} \setminus I} H_i$ thus $\dim(V) \leq n - (n - |I|) = |I|$, and it follows from our assumption $\dim(V) = d \geq 2$ that $|I| \geq 2$.

Now we prove that $V = \sum_{i \in I}(V \cap H_i)$. It suffices to prove $V \subseteq \sum_{i \in I}(V \cap H_i)$, and this in turn amounts to prove that $\dim(\sum_{i \in I}(V \cap H_i)) = d$. For every $1 \leq i \leq n$ we have

$$\dim(V + H_i) = \dim(V) + \dim(H_i) - \dim(V \cap H_i).$$

Now if $i \in I$ then $\dim(V + H_i) > \dim(H_i)$, i.e., $\dim(V + H_i) = n$, which leads to $\dim(V \cap H_i) = d + (n-1) - n = d - 1$. Thus, in order to prove $\dim(\sum_{i \in I}(V \cap H_i)) = d$ it suffices to show that there exist $i, j \in I$ such that $V \cap H_i \neq V \cap H_j$. Assume for a contradiction that for all $i, j \in I$ we have $V \cap H_i = V \cap H_j$. Then for every $i \in I$ we have

$$V \cap H_i = V \cap \bigcap_{j \in I} H_j \subseteq \bigcap_{j \notin I} H_j \cap \bigcap_{j \in I} H_j = \{0\}$$

which contradicts the fact that $\dim(V \cap H_i) = d - 1 \geq 1$.

We proved that $V = \sum_{i \in I}(V \cap H_i)$, thus it suffices to prove that for every $i \in I$, $V \cap H_i$ has a rational basis. Let v be an element of $V \setminus H_i$, and let r be safe for x. We can find a rational $i-$hyperplane S not parallel to v and such that the intersection point of S and $L_v(x)$, say y, belongs to $B(x, r)$. By Lemma 3.13 (applied to $z = x$) we have $y \sim x$. Corollary 3.17 then implies

$$\mathrm{Str}_{(X,S)}(y) = \mathrm{Str}_X(y) \cap H_i = \mathrm{Str}_X(x) \cap H_i = V \cap H_i$$

which yields

$$\mathrm{Str}_{\pi_S(X)}(y) = \pi_S(V \cap H_i).$$

Now by condition (DS), $X \cap S$ is $\langle \mathbb{R}, +, <, 1\rangle$-definable, and $\pi_S(X)$ as well. Therefore by Proposition 4.2 applied to $X \cap S$, the relation $X \cap S$ satisfies (RB) thus $\pi_S(V \cap H_i)$ has a rational basis, and this implies that $V \cap H_i$ also has a rational basis. □

Claim 4.5. If X satisfies conditions (FSP), (DS) and (RSP) then it satisfies condition (FI).

Proof:
Before proving the claim we need a simple definition.

Definition 4.6. Given $X \subseteq \mathbb{R}^n$ and a $\sim$-class E, we define the *isolated part of* E as the subset

$$Z = \{x \in E \mid L_v(x) \subseteq E \text{ for all nonzero vectors } v \in \mathrm{Str}(E)\}.$$

A subset of $\mathbb{R}^n$ is X-*isolated* (or simply *isolated* when X is understood) if it is equal to the isolated part of some $\sim$-class.

Example 4.7. Let $X \subseteq \mathbb{R}^2$ be defined as $X = L_1 \cup L_2$ where L_1 denotes the horizontal axis and L_2 denotes the open half-line $L_2 = \{(x_1, x_2) \mid x_2 = 1 \text{ and } x_1 > 0\}$. In this case there are three $\sim$-classes, namely $E_1 = X$, $E_2 = \{(0,1)\}$ and $E_3 = \mathbb{R}^2 \setminus (E_1 \cup E_2)$. Let us describe the isolated part for each of these $\sim$-classes. A point belongs to the isolated part of a $\sim$-class if whatever stratum is chosen, all points in the direction are trapped in the class. For instance for the $\sim$-class $E_1 = X$, we can show that the isolated part is obtained by deletion of the half-line L_2 of X, whose points are clearly not trapped. Indeed the subspace $\mathrm{Str}(E_1)$ is generated by the vector $(1,0)$. Therefore for every $v \in \mathrm{Str}(E_1)$, if $x \in L_1$ then $L_v(x) = L_1 \subseteq E_1$, and if $x \in L_2$ then the line $L_v(x)$ intersects E_2 thus $L_v(x) \not\subseteq E_1$. This shows the isolated part of E_1 is equal to L_1. The $\sim$-class E_2 has dimension 0 thus obviously it is equal to its isolated part. Finally the isolated part of E_3 is empty since the vector $v = (0,1)$ is a stratum of E_3 and for every $x \in E_3$ the line $L_v(x)$ intersects E_1.

Lemma 4.8. Let $X \subseteq \mathbb{R}^n$ satisfy (FSP), (DS) and (RSP). We have

1. Let E be a $\sim$-class and Z be its isolated part. Then Z is a finite union of affine subspaces with underlying vector subspace $\mathrm{Str}(E)$ each containing a point with rational components.

2. There exist finitely many isolated subsets.

Proof:
By induction on n. For $n = 1$ if X is equal to $\mathbb{R}$ or to the empty set, the only isolated set is X and it obviously satisfies (1). Otherwise a nonempty isolated set Z consists of equivalent points of a $\sim$-class of dimension 0, i.e., it is a union of singular points. Now by (FSP) and (DS) there exist finitely many such points and they have rational components, which implies (1) and (2).

Now let $n \geq 1$. All isolated sets Z included in a $\sim$-class E of dimension 0 satisfy (1), and moreover there are finitely many such sets Z. Thus it suffices to consider the case where $Z \neq \emptyset$ and $\mathrm{Str}(E) \neq \{0\}$.

Let $v \in \mathrm{Str}(E) \setminus \{0\}$ and let $i \in \{1, \ldots, n\}$ be such that $v \notin H_i$. For every $z \in Z$ we have $L_v(z) \subseteq Z$, thus Z intersects the hyperplane H_i. All elements of $Z \cap H_i$ are $\sim_X$-equivalent thus they are also $\sim_{(X,H_i)}$-equivalent. Furthermore for every $x \in Z \cap H_i$ we have $\mathrm{Str}_{(X,H_i)}(x) = \mathrm{Str}_X(x) \cap H_i$ by Corollary 3.17 and thus for every $w \in \mathrm{Str}_X(x) \cap H_i$ we have $L_w(x) \subseteq Z \cap H_i$. This shows that $\pi_{H_i}(x)$ belongs to a $\pi_{H_i}(X)-$isolated set, hence $\pi_{H_i}(Z)$ is included in a $\pi_{H_i}(X)-$isolated set, say $W \subseteq \pi_{H_i}(H_i)$.

Now by condition (DS) the set $\pi_{H_i}(X)$ is $\langle \mathbb{R}, +, <, 1 \rangle$-definable, thus by Theorem 4.1 it satisfies also (FSP). By our induction hypothesis it follows that W can be written as $W = \bigcup_{j=1}^{p} W_j$, where either all W_j's are parallel affine subspaces with underlying vector space $\pi_{H_i}(\mathrm{Str}(E))$ each containing some point with rational components (by (1)), or each W_j is reduced to a point with rational components (by (1)). Every W_j which intersects $\pi_{H_i}(Z)$ satisfies $W_j \subseteq \pi_{H_i}(Z)$, which shows that $\pi_{H_i}(Z) = \bigcup_{j\in J} W_j$ for some $J \subseteq \{1,\dots,p\}$. That is, we have $Z \cap H_i = \bigcup_{j\in J} W'_j$ where each $W'_j = \pi^{-1}_{H_i}(W_j)$. Observe that if x belongs to W_j and has rational components then the point $x' = \pi^{-1}_{H_i}(x)$ also has rational components. Now $Z = (Z \cap H_i) + \mathrm{Str}(E)$ thus $Z = \bigcup_{j\in J}(W'_j + \mathrm{Str}(E))$. Since the underlying vector space of each W'_j is included in $\mathrm{Str}(E)$, this proves (1).

Concerning (2) we observe that Z is completely determined by $Z \cap H_i$, i.e., $\pi_{H_i}(Z)$. By our induction hypothesis there are finitely many $\pi_{H_i}(X)-$isolated parts $W = \bigcup_{j=1}^{p} W_j$ and each X-isolated part is determined by a subset of the form $\bigcup_{j\in J} W_j$ for some $J \subseteq \{1,\dots,p\}$. This proves point (2). □

Now we turn to the proof of Claim 4.5. Lemma 4.8 shows that the number of $\sim$-classes having a nonempty isolated part is finite. It thus suffices to prove that for every $0 \le d \le n$ there exist finitely many d-classes E having an empty isolated part.

For $d = 0$ the result follows from (FSP) and the fact that each $0-$class is a union of singular points. For $d = n$ there exist at most two $d-$classes, which correspond to elements in the interior of X or the interior of its complement.

For $0 \le d < n$ we reason by induction on d. Observe first that if a $\sim$-class E has dimension d and has an empty isolated part, then there exist $x \in E$ and $v \in \mathrm{Str}(E) \setminus \{0\}$ such that $L_v(x) \nsubseteq E$. By Lemma 3.23 this implies that there exist $y \in L_v(x)$ and a $\sim$-class F such that $y \in F$, F is adjacent to E, $\dim(F) < \dim(E)$, and $[x, y) \subseteq \mathcal{C}l(x)$. Now by our induction hypothesis there exist finitely many $\sim$-classes with dimension less than d. Thus in order to prove the claim, it suffices to show that there are finitely many d-classes to which some d'-class with $d' < d$ is adjacent.

In order to meet a contradiction, assume that there exists a $d'-$class F which is adjacent to infinitely many d-classes, say E_j with $j \in J$. We may furthermore assume that for each class E_j there is no integer $d' < d'' < d$ such that some d''-class is adjacent to E_j. Because of Lemma 3.22 it is enough to fix an element y in F and investigate the classes to which it is adjacent.

We first consider the case $d' = 0$.

Because of condition (FSP), for some real $s > 0$ the point y is the unique singular point in $B(y, s)$. Moreover for every $j \in J$, F is adjacent to E_j, thus there exists a point $x_j \in E_j$ such that $[x_j, y) \subseteq E_j$. Let HL_j denote the open halfline with endpoint y and containing x_j. Observe that we necessarily have $HL_j \cap B(y, s) \subseteq \mathcal{C}l(x_j)$. Indeed, by Lemma 3.23 the condition $HL_j \cap B(y, s) \subsetneq \mathcal{C}l(x_j)$ implies that there exists a point $z = y + \alpha(x_j - y) \in B(y, s)$ such that $\alpha > 1$ and $\dim(z) < d$. Since y is the unique singular point in $B(y, s)$ this implies $\dim(z) > 0$ but then because of $[x_j, z) \subseteq \mathcal{C}l(x_j)$ the maximality condition stipulated for d' is violated.

Let z_j be the point on HL_j at distance $\frac{s}{2}$ from y and let z be adherent to the set $\{z_j \mid j \in J\}$. The point z is nonsingular since y is the unique singular point in the ball $B(y, s)$. Let $v \in \mathrm{Str}(z) \setminus \{0\}$. Consider some $\ell \in \{1,\dots,n\}$, some rational $\ell-$hyperplane S such that $z \notin S$ and some real $0 < t < \frac{s}{2}$ such that $L_v(B(z, t)) \cap S \subseteq B(z, \frac{s}{2})$. The ball $B(z, t)$ contains infinitely many non $\sim$-equivalent points, and by Lemma 3.18 their projections on S in the direction v are non $\sim_{(X,S)}$-equivalent. But by condition (DS) the relation $X \cap S$ is $\langle \mathbb{R}, +, <, 1\rangle$-definable, thus $\pi_S(X)$ satisfies condition (FI) of Proposition 4.2, a contradiction.

Now we consider the case where $d' > 0$.

Choose some $v \in \text{Str}(y)$ and let r be a safe radius for y. We can find $0 < s < r$, $k \in \{1, \ldots, n\}$ and some $k-$hyperplane S not parallel to v such that $L_v(B(y, s)) \cap S \subseteq B(y, r)$. By definition of y, $B(y, s)$ intersects infinitely many pairwise distinct $d-$classes. Given two non $\sim$-equivalent d-points $z_1, z_2 \in B(y, s)$, and their respective projections w_1, w_2 over S along the direction v, we have $w_1 \not\sim_{(X,S)} w_2$ by Lemma 3.18. This implies that there exist infinitely many $\sim_{(X,S)}$-classes. However by condition (DS), the relation $X \cap S$ is $\langle \mathbb{R}, +, <, 1\rangle$-definable, thus $\pi_S(X)$ satisfies condition (FI) of Proposition 4.2, a contradiction. $\square$

Now we turn to the proof of Theorem 4.3. Observe that X is equal to the union of $\sim$-classes of its elements, thus by Claim 4.5, in order to prove that X is $\langle \mathbb{R}, +, <, 1\rangle$-definable it suffices to prove that all $\sim_X$-classes are $\langle \mathbb{R}, +, <, 1\rangle$-definable. More precisely, we prove that each $\sim$-class E is definable from $\sim$-classes F with smaller dimension, i.e., that E is definable in the expansion of $\langle \mathbb{R}, +, <, 1\rangle$ obtained by adding a predicate for each such F. We proceed by induction on the dimension d of Str(E).

If $d = 0$ then E is a union of singular points, and by (FSP) and (RSP) it follows that E is a finite subset of $\mathbb{Q}^n$ thus is $\langle \mathbb{R}, +, <, 1\rangle$-definable.

Assume now $0 < d \leq n$. By Claim 4.4 there exists a rational basis $V(E) = \{v_1, \ldots, v_d\}$ of Str(E). Let $Z \subseteq E$ be the isolated part of E and let $E' = E \setminus Z$. By Lemma 4.8 (1), Z is a finite union of parallel affine subspaces with underlying vector space $V(E)$ each containing a point with rational components, thus Z is $\langle \mathbb{R}, +, <, 1\rangle$-definable. It remains to prove that E' is $\langle \mathbb{R}, +, <, 1\rangle$-definable. We use the following characterization of E'.

Lemma 4.9. For every $x \in \mathbb{R}^n$, we have $x \in E'$ if and only if there exist $1 \leq p \leq d$ and a sequence of pairwise distinct elements $x_0, \ldots, x_p \in \mathbb{R}^n$ such that $x_0 = x$ and

1. for every $0 \leq k \leq p - 1$, $x_{k+1} - x_k \in V(E)$ and $[x_k, x_{k+1})$ does not intersect any $\sim$-class of strictly smaller dimension than $\dim(E)$
2. if $F = \mathcal{C}l(x_p)$ then F is $(x_{p-1} - x_p)$-adjacent to E and $\dim(F) < \dim(E)$.

Proof:
We first prove that the conditions are sufficient. We prove by backward induction that $[x_k, x_{k+1}) \subseteq E$ for every $0 \leq k \leq p - 1$. This will imply that $x = x_0 \in E$, and the fact that $x_p - x$ belongs to Str(E) and $\dim(F) < \dim(E)$ will lead to $x \in E'$. Set $k = p - 1$. By Point 2 of Lemma 3.22 the element x_p is $(x_{p-1} - x_p)$-adjacent to E, thus $[x_{p-1}, x_p)$ intersects E. Moreover $[x_{p-1}, x_p)$ does not intersect any $\sim$-class G such that $\dim(G) < \dim(E)$, thus by Lemma 3.23 we have $[x_{p-1}, x_p) \subseteq E$. For $0 \leq k < p - 1$, by our induction hypothesis we have $x_{k+1} \in E$. Moreover $[x_k, x_{k+1})$ does not intersect any $\sim$-class G such that $\dim(G) < \dim(E)$, thus $[x_k, x_{k+1}) \subseteq E$ again by Lemma 3.23.

We prove the necessity. By definition of E' and Lemma 3.23 there exist $v \in \text{Str}(E)$ and $y \in L_v(x)$ such that $[x, y) \subseteq E$ and $y \notin E$. Decompose $v = \alpha_1 v_{i_1} + \cdots + \alpha_p v_{i_p}$ where $0 < i_1 < \cdots < i_p \leq d$ and $\alpha_1, \cdots, \alpha_p \neq 0$. We can assume w.l.o.g that y is chosen such that p is minimal and furthermore that α_p is minimal too. For $0 \leq k < p$ set $x_k = x + \alpha_1 v_{i_1} + \cdots + \alpha_k v_{i_k}$. By minimality of p and α_p, the segments $[x_0, x_1), \ldots, [x_{p-1}, x_p)$ intersect no class of dimension less than $\dim(E)$. Then $y = x_p$ is $(x_{p-1} - x_p)$-adjacent to E. $\square$

In order to prove that E' is $\langle \mathbb{R}, +, <, 1\rangle$-definable it suffices to show that we can express in $\langle \mathbb{R}, +, <, 1\rangle$ the existence of a sequence $x_0, \ldots, x_p \in \mathbb{R}^n$ which satisfies both conditions of Lemma 4.9. Observe that $V(E)$ is finite and each of its element is $\langle \mathbb{R}, +, <, 1\rangle$-definable, thus we can express in $\langle \mathbb{R}, +, <, 1\rangle$ the fact that a segment is parallel to some element of $V(E)$. Moreover by (FI) there exist finitely many $\sim$-classes F such that $\dim(F) < \dim(E)$, and all such classes are $\langle \mathbb{R}, +, <, 1\rangle$-definable by our induction hypothesis. This allows us to express condition (1) in $\langle \mathbb{R}, +, <, 1\rangle$. For (2) we use again the fact that there are only finitely many classes F to consider and that all of them are $\langle \mathbb{R}, +, <, 1\rangle$-definable. □

4.3. An alternative noneffective formulation.

In this section we re-formulate Theorem 4.3 in terms of (generalized) projections of X by building on the notion of generalized section which extends that of section, in the sense that it allows us to fix several components.

Definition 4.10. Given $n \geq 1$ and $X \subseteq \mathbb{R}^n$, a *generalized section of* X is a relation of the form

$$X_{s,a} = \{(x_1, \ldots, x_n) \in X \mid x_{s_1} = a_1, \ldots, x_{s_r} = a_r\} \tag{3}$$

where $r > 0$, $s = (s_1, \ldots, s_r)$ is an $r-$tuple of integers with $1 \leq s_1 < \cdots < s_r \leq n$, and $a = (a_1 \ldots, a_r)$ is an $r-$tuple of reals. When $r = 0$ we define $X_{s,a} = X$ by convention, i.e., X is a generalized section of itself. If $r > 0$ then the section is said to be *proper*. If all elements of a are rational numbers then $X_{s,a}$ is called a *rational generalized section of* X.

In the above definition, each $X_{s,a}$ is a subset of $\mathbb{R}^n$. If we remove the r fixed components $x_{s_1}, \ldots, x_{s_r}$ we can see $X_{s,a}$ as a subset of $\mathbb{R}^{n-r}$, which will be called a *generalized projection* of X (resp. a *rational generalized projection* of X if $X_{s,a}$ is a rational generalized section of X).

Proposition 4.11. For every $n \geq 1$, a relation $X \subseteq \mathbb{R}^n$ is $\langle \mathbb{R}, +, <, 1\rangle$-definable if and only if every rational generalized projection of X has finitely many singular points and these points have rational components.

Proof:
The proof goes by induction on n. The case $n = 1$ is obvious. Assume now $n > 1$.

Let X be $\langle \mathbb{R}, +, <, 1\rangle$-definable and let Y be a rational generalized projection of X. If $Y = X$ then the result follows from Theorem 4.3. If Y is proper then Y is definable in $\langle \mathbb{R}, +, <, 1, X\rangle$ thus it is also $\langle \mathbb{R}, +, <, 1\rangle$-definable, and the result follows from our induction hypothesis.

Conversely assume that every rational generalized projection of X has finitely many singular points and they have rational components. We show that X satisfies all three conditions of Theorem 4.3. Conditions (FSP) and (RSP) follow from our hypothesis and the fact that X is a rational generalized projection of itself. It remains to prove condition (DS) namely that every rational section of X is $\langle \mathbb{R}, +, <, 1\rangle$-definable. This amounts to proving that every rational projection Z of X is $\langle \mathbb{R}, +, <, 1\rangle$-definable. Now every generalized projection Y of Z is also a generalized projection of X, thus by our induction hypothesis Y has finitely many singular points and they have rational components. Since Z is a proper projection of X, by our induction hypothesis it follows that Z is $\langle \mathbb{R}, +, <, 1\rangle$-definable. □

5. A definable criterion for $\langle \mathbb{R}, +, <, 1 \rangle$-definability in suitable structures

In this section we prove that for every $n \geq 1$ and $X \subseteq \mathbb{R}^n$, if every nonempty $\langle \mathbb{R}, +, <, 1, X \rangle$-definable relation contains a point with rational components then we can state a variant of Proposition 4.11 which is expressible in $\langle \mathbb{R}, +, <, 1, X \rangle$. This means that there exists a $\langle \mathbb{R}, +, <, 1, X \rangle$-sentence (uniform in X) which expresses the fact that X is $\langle \mathbb{R}, +, <, 1 \rangle$-definable. This provides a *definable* criterion for $\langle \mathbb{R}, +, <, 1 \rangle$-definability, similar to Muchnik's result [18] for definability in Presburger Arithmetic. We also extend these ideas to the case of $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$-definability.

In this section $\mathcal{S}_X$ stands for the structure $\langle \mathbb{R}, +, <, 1, X \rangle$.

5.1. Quasi-singular points

We aim to express $\langle \mathbb{R}, +, <, 1 \rangle$-definability of a relation $X \subseteq \mathbb{R}^n$ in the structure $\mathcal{S}_X$ itself. A natural approach is to express the conditions of Proposition 4.11 as an $\mathcal{S}_X$-sentence, however the formulation involves the set $\mathbb{Q}$ as well as the set of X-singular elements. On the one hand $\mathbb{Q}$ is not necessarily $\mathcal{S}_X$-definable, and on the other hand the naive definition of $X-$singularity involves the operation of multiplication which is also not necessarily $\mathcal{S}_X$-definable. For the special case where X is $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$-definable, we introduced in [2, Lemma 4.9] an ad hoc definition. Yet this definition does not necessarily hold when the relation X is no longer assumed to be $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$-definable. In order to overcome this difficulty we introduce a weaker property but that is still definable in $\mathcal{S}_X$. This proves to be sufficient to establish our result.

Definition 5.1. Let $X \subseteq \mathbb{R}^n$, $x \in \mathbb{R}^n$, and r, s be two reals such that $0 < s < r$.

- a vector $v \in \mathbb{R}^n$ is an (r, s)-*quasi-stratum* of x if $|v| \leq s$ and $(y \in X \leftrightarrow y + v \in X)$ holds for all $y \in \mathbb{R}^n$ such that $y, y + v \in B(x, r)$.
- We say that $x \in \mathbb{R}^n$ is $X-$*quasi-singular* if it does not satisfy the following property:

  there exist reals $r, s > 0$ such that the set of (r, s)-quasi-strata of x is nonempty, closed, and is stable under $v \mapsto v/2$. (4)

It is not difficult to check that if x is not singular and r is safe, then every stratum of x is an (r, s)-quasi-stratum for $0 < s < r$. However even for r safe, there may exist (r, s)-quasi-strata of x which are not strata of x, as shown in the following example.

Example 5.2. Let $n = 2$, $X = (\mathbb{Z} \cup \{-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}\}) \times \mathbb{R}$, and $x = (0, 0)$. Then *Str*(x) is generated by the vector $(0, 1)$, and every real $r > 0$ is safe for x. Given $0 < s < r$, the (r, s)-quasi-strata of x can be characterized as follows:

- if $r > \frac{5}{2}$ then the (r, s)-quasi-strata of x are vectors of the form $(0, l)$ with $|l| \leq s$ (i.e these are the strata of x with norm at most s).

- if $r \leq \frac{5}{2}$ then these are vectors of the form $(\frac{k}{2}, l)$ where $k \in \mathbb{Z}$, $k \leq 2r$, and $|(\frac{k}{2}, \ell)| \leq s$. Note that for $s < \frac{1}{2}$, these are exactly the strata of x with norm at most s.

Lemma 5.3. Let $X \subseteq \mathbb{R}^n$. The set of quasi-singular elements of X is $\mathcal{S}_X$-definable. Moreover the property "X has finitely many quasi-singular elements" is $\mathcal{S}_X$-definable (uniformly in X) .

Proof:
The property that v is an (r, s)-stratum of x can be expressed by the formula

$$\phi(X, x, r, s, v) \equiv 0 < |v| < s \wedge \forall y\ (y, y + v \in B(x, r) \rightarrow (y \in X \leftrightarrow y + v \in X)).$$

The set of X-quasi-singular elements can be defined by the formula

$$\begin{aligned} QS(x, X) \equiv \quad & \neg\exists r \exists s\ ((0 < s < r) \wedge \exists v(\phi(X, x, r, s, v)) \wedge \\ & \forall v\ (\phi(X, x, r, s, v) \rightarrow \phi(X, x, r, s, \tfrac{v}{2})) \wedge \\ & \forall v\ (((|v| < s) \wedge \forall \epsilon > 0(\exists u\ (\phi(X, x, r, s, u) \wedge |v - u| < \epsilon))) \rightarrow \phi(X, x, r, s, v))) \end{aligned} \tag{5}$$

The finiteness of the set of quasi-singular points can be expressed by the formula

$$\begin{aligned} FS(X) \equiv \quad & (\exists t > 0\ \forall x\ (QS(x, X) \rightarrow |x| < t)) \\ & \wedge(\exists u > 0(\forall x \forall y((QS(x, X) \wedge QS(y, X) \wedge x \neq y) \rightarrow |x - y| > u))) \end{aligned} \tag{6}$$

□

Lemma 5.4. Let $X \subseteq \mathbb{R}^n$. If x is not quasi-singular then for some reals $0 < s < r$ there exists an (r, s)-quasi-stratum of x and every (r, s)-quasi-stratum of x is a stratum of x.

Proof:
In this proof "quasi-stratum" stands for "(r, s)-quasi-stratum". We consider the negation of QS. The matrix of the formula consists of four conjuncts. The second conjunct asserts that there exists a quasi-stratum. The third conjunct asserts that if a vector v is a quasi-stratum then for all integers $p \leq 0$ the vector $2^p v$ is a quasi-stratum. Also if $p \geq 0$ and $|2^p v| < s$ then $2^p v$ is a quasi-stratum. Indeed, because $B(x, r)$ is convex, if y and $y + 2v$ belong to $B(x, r)$ then $y + v$ belongs to $B(x, r)$ and we have

$$y \in X \leftrightarrow y + v \in X \leftrightarrow z = y + 2v \in X. \tag{7}$$

This generalizes to any $p \geq 0$ provided $|2^p v| < s$.

We will show that if v is quasi-stratum, then it is a stratum, i.e., if $y, z \in B(x, r)$ and $z \in L_v(y)$ then $y \in X \leftrightarrow z \in X$. To fix ideas set $z = y + 2^\ell \beta v$ for some real $0 < \beta < 1$. Let $\alpha_q = \sum_{-q<i<0} a_i 2^i$ with $a_i \in \{0, 1\}$, be a sequence of dyadic rationals converging to β. Since $|2^i v| < s$ holds for all $-q < i < 0$, every $2^i v$ is a quasi-stratum and therefore so is $\alpha_q v$. Arguing as in (7) we see that for all $t, t + \alpha_q v \in B(x, r)$ we have $t \in X \leftrightarrow t + \alpha_q v \in X$. Because $\alpha_q < 1$ this shows that all $\alpha_q v$ are quasi-strata. The last conjunct implies that βv is a quasi-stratum and again using the same argument as in (7) we get

$$y \in X \leftrightarrow y + \beta v \in X \leftrightarrow y + 2\beta v \leftrightarrow \cdots \leftrightarrow z = y + 2^\ell \beta v \in X$$

□

Lemma 5.5. For every $X \subseteq \mathbb{R}^n$, every $X-$singular element is $X-$quasi-singular.

Proof:
If $x \in \mathbb{R}^n$ is not $X-$quasi-singular then there exist $0 < s < r$ and an (r, s)-quasi-stratum, which is a stratum by Lemma 5.4. □

Lemma 5.6. If $X \subseteq \mathbb{R}^n$ is $\langle \mathbb{R}, +, <, 1 \rangle$-definable, every $X-$quasi-singular element is $X-$singular.

Proof:
We prove that if $x \in \mathbb{R}^n$ is not $X-$singular then it is not quasi-singular, i.e., that it satisfies $\neg QS(x, X)$, cf. Expression (5). We find suitable values of r, s such that the set of (r, s)-quasi-strata of x coincides with the set of strata v of x such that $|v| \leq s$. The result will then follow from the fact that Str(x) is a non-trivial vector subspace.

The relation X is $\langle \mathbb{R}, +, <, 1 \rangle$-definable thus by [2, Corollary 4.4] there exists $r > 0$ such that inside the ball $B(x, r)$, X coincides with a finite collection of cones. By cone we mean an intersection of open or closed halfspaces delimited by finitely many, say k, hyperplanes of dimension $n - 1$ and containing x. Without loss of generality we can assume that r is safe. We show that a suitable value for s is $s = \frac{1}{k} r$.

Since r is safe, every stratum v of x with $|v| \leq s$ is also an (r, s)-quasi-stratum of x. Conversely let v be an (r, s)-quasi-stratum of x, and assume for a contradiction that $v \notin$ Str(x). Then there exists a point $y \in B(x, r)$ such that the line $L_v(y)$ intersects X and its complement inside $B(x, r)$. Let h be any homothety with ratio $0 < \lambda \leq 1$ centered at x such that the segment $L_v(h(y)) \cap B(x, r)$ has length greater than r. Then, within $B(x, r)$, the line $L_v(h(y))$ decomposes into $2 \leq p \leq k$ segments which are alternatively inside and outside X. One of these segments has length at least $\frac{1}{p} r \geq s \geq |v|$. We obtain that for some $z \in L_v(h(y))$ we have $z, z + v \in B(x, r)$ and $z \in X \leftrightarrow z + v \notin X$, which contradicts our assumption that v is an (r, s)-quasi-stratum of x. □

Note that in Lemma 5.6 the condition that X is $\langle \mathbb{R}, +, <, 1 \rangle$-definable cannot be removed. Consider, e.g., $X = \mathbb{R} \times \mathbb{Q}$. Then it can be shown that for all $x \in \mathbb{R}^2$ we cannot find any reals $0 < s < r$ for which the set of (r, s)-quasi-strata of x is closed, and this implies that x is $X-$quasi-singular. However x is not $X-$singular since $(1, 0)$ is an $X-$stratum for x.

5.2. Alternative characterization of $\langle \mathbb{R}, +, <, 1 \rangle$-definability in $\mathcal{S}_X$.

We can state the following variant of Theorem 4.3 for $\mathcal{S}_X$-definable relations under the hypothesis that all nonempty $\mathcal{S}_X$-definable relations Y contain a point with rational components (recall that $\mathcal{S}_X$ stands for the structure $\langle \mathbb{R}, +, <, 1, X \rangle$ where X is some fixed but arbitrary relation). Observe that this implies that all definable finite subsets Y of $\mathbb{R}^n$ are included in $\mathbb{Q}^n$. Indeed, for all points $y \in Y$ with rational components the set $Y \setminus \{y\}$ is $\mathcal{S}_X$-definable.

Proposition 5.7. Let $n \geq 1$ and $X \subseteq \mathbb{R}^n$ be such that every nonempty $\mathcal{S}_X$-definable relation contains a point with rational components. Then X is $\langle \mathbb{R}, +, <, 1 \rangle$-definable if and only if every generalized projection of X has finitely many quasi-singular points.

Proof:
We proceed by induction on n.

Case $n = 1$. Assume first that X is $\langle \mathbb{R}, +, <, 1\rangle$-definable. The only generalized projection of X that need to be studied is X itself. Now by Theorem 4.3, X has finitely many singular points, and by Lemma 5.6 these are precisely its quasi-singular points.

Conversely assume that every generalized projection of X has finitely many quasi-singular points. If the generalized projection is not X, then it is a singleton and there is nothing to check. It remains to consider the case where the projection is equal to X. By Lemma 5.5 this implies that X has finitely many singular points. Now $X \subseteq \mathbb{R}$ thus the set of $X-$singular points coincides with the topological boundary $Bd(X)$ of X. It follows that $Bd(X)$ is finite, i.e., X is the union of finitely many intervals. Moreover $Bd(X)$ is $\mathcal{S}_X$-definable and by our assumption on $\mathcal{S}_X$ it follows that $Bd(X) \subseteq \mathbb{Q}$ thus every $X-$singular point is rational. The result follows from Theorem 4.3.

Case $n > 1$. Assume first that X is $\langle \mathbb{R}, +, <, 1\rangle$-definable. The relation satisfies property (FSP) of Theorem 4.3 and by Lemma 5.6 it has finitely many quasi-singular points. It thus suffices to consider proper subsets of X. Assume without loss of generality that the projections are obtained by freezing the $0 < p \leq n$ first components. For every $a = (a_1, \ldots, a_p) \in \mathbb{R}^p$, consider the projection

$$X_a = \{(x_{p+1}, \ldots, x_n) \in \mathbb{R}^{n-p} \mid (a_1, \ldots, a_p, x_{p+1}, \ldots, x_n) \in X\}.$$

Consider the set A of elements $a = (a_1, \ldots, a_p) \in \mathbb{R}^p$ such that the relation X_a has infinitely many quasi-singular points. Using expression (6), the set A is $\mathcal{S}_X$-definable, thus it is $\langle \mathbb{R}, +, <, 1\rangle$-definable because so is X. If this set were nonempty, by Theorem 2.1 it would contain an element of $\mathbb{Q}^p$, which means that there exists a rational generalized projection of X which has infinitely many quasi-singular points, a contradiction.

Conversely assume that every generalized projection of X has finitely many quasi-singular points, and let us prove that X satisfies all conditions of Theorem 4.3. Condition (DS) follows from the fact that every rational section of X is a generalized projection of X thus is $\langle \mathbb{R}, +, <, 1\rangle$-definable by our induction hypothesis. For conditions (FSP) and (RSP), we observe that X is a generalized projection of itself thus the set of $X-$quasi-singular points is finite. By Lemma 5.3 this set is $\mathcal{S}_X$-definable thus it is a subset of $\mathbb{Q}^n$ by our assumption on $\mathcal{S}_X$, and the result follows from Lemma 5.5. □

5.3. Defining $\langle \mathbb{R}, +, <, 1\rangle$-definability

The formulation of conditions in Proposition 5.7 allows us to express $\langle \mathbb{R}, +, <, 1\rangle$-definability as a sentence in the structure $\mathcal{S}_X$ itself.

Theorem 5.8. Let $n \geq 1$ and $X \subseteq \mathbb{R}^n$ be such that every nonempty $\mathcal{S}_X$-definable relation contains a point with rational components. There exists a $\mathcal{S}_X$-sentence Φ_n (which is uniform in X) which holds in $\mathcal{S}_X$ if and only if X is $\langle \mathbb{R}, +, <, 1\rangle$-definable.

Proof:
Let $[n]$ denote the set $\{1, \ldots, n\}$. By Proposition 5.7 it suffices to express the fact that every generalized projection of X has finitely many quasi-singular points. This leads us to consider all possible

generalized projections obtained by freezing a subset $[n] \setminus I$ of components as in Definition 4.10. Let $\mathbb{R}^I$ denote the product of copies of $\mathbb{R}$ indexed by I and for all $x, y \in \mathbb{R}^I$ set $|x-y|_I = |(x+z)-(y+z)|$ for any $z \in \mathbb{R}^{[n]\setminus I}$. For $x \in \mathbb{R}^I$ and $r \geq 0$ set $B_I(x,r) = \{y \in \mathbb{R}^I \mid |x-y|_I < r\}$. We use the pair (n, I) as a parameter for the predicates ϕ, QS, FS (see Lemma 5.3). The symbol ξ stands for the subvector with frozen components (we have $\xi \in \mathbb{R}^{[n]\setminus I}$). With some abuse of notation we write $\xi + z$ for $\xi \in \mathbb{R}^{[n]\setminus I}$ and $z \in \mathbb{R}^I$, that is, if $\xi = (\xi_i)_{i \in \mathbb{R}^{[n]\setminus I}}$ and $z = (z_i)_{i \in I}$ then $\xi + z = (w_1, \ldots, w_n)$ with $w_i = z_i$ if $i \in I$ and $w_i = \xi_i$ otherwise. We can define the predicates $\phi_{n,I}$, $QS_{n,I}$ and $FS_{n,I}$ as follows:

$$\begin{aligned}
\phi_{n,I}(X,\xi,x,r,s,v) \equiv\ & \xi \in \mathbb{R}^{[n]\setminus I} \wedge x, v \in \mathbb{R}^I \wedge 0 < |v|_I < s \\
& \wedge \forall y \in \mathbb{R}^I (y, y+v \in B_I(x,r) \rightarrow (\xi + y \in X \leftrightarrow \xi + y + v \in X))
\end{aligned}$$

$$\begin{aligned}
QS_{n,I}(x,\xi,X) \equiv\ & \neg \exists r \exists s ((0 < s < r) \wedge \exists v\ (\phi_{n,I}(X,\xi,x,r,s,v)) \\
& \wedge \forall v\ (\phi_{n,I}(X,\xi,x,r,s,v) \rightarrow \phi_{n,I}(X,\xi,x,r,s,\tfrac{v}{2})) \\
& \wedge \forall u\ (((|u|_I < s) \wedge \forall \epsilon > 0 (\exists v\ (\phi_{n,I}(X,\xi,x,r,s,v) \wedge |v-u|_I < \epsilon))) \\
& \qquad \rightarrow \phi_{n,I}(X,\xi,x,r,s,u)))
\end{aligned}$$

$$\begin{aligned}
FS_{n,I}(X,\xi) \equiv\ & (\exists t > 0 \forall x\ (QS_{n,I}(x,\xi,X) \rightarrow |x|_I < t) \\
& \wedge (\exists s > 0 (\forall x \forall y ((QS_{n,I}(x,\xi,X) \wedge QS_{n,I}(y,\xi,X) \wedge x \neq y) \rightarrow |x-y|_I > s)
\end{aligned}$$

This leads to the following definition of Φ_n:

$$\Phi_n \equiv \bigwedge_{I \subseteq [n]} \forall \xi \in \mathbb{R}^{[n]\setminus I}\ \ FS_{n,I}(X,\xi). \tag{8}$$

□

Remark 5.9. One can prove that Theorem 5.8 does not hold anymore if we remove the assumption that every nonempty $\mathcal{S}_X$-definable relation contains a point with rational components. Indeed consider $n = 1$ (the case $n \geq 1$ easily reduces to this case) and a singleton set $X = \{x\} \subseteq \mathbb{R}$. Then by Theorem 2.1, X is $\langle \mathbb{R}, +, <, 1 \rangle$-definable if and only if $x \in \mathbb{Q}$. Thus if there exists a $\mathcal{S}_X$-sentence Φ_n which expresses that X is $\langle \mathbb{R}, +, <, 1 \rangle$-definable, then it is easy to transform Φ_n into a $\langle \mathbb{R}, +, <, 1 \rangle$-formula $\Phi'_n(x)$ which defines $\mathbb{Q}$ in $\langle \mathbb{R}, +, <, 1 \rangle$, and this contradicts Theorem 2.1.

5.4. Extensions to $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$-definability

We extend the previous results to the case of $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$-definability. Here $\mathcal{T}_X$ stands for the structure $\langle \mathbb{R}, +, <, \mathbb{Z}, X \rangle$ with $n \geq 1$ and $X \subseteq \mathbb{R}^n$. We prove that if every nonempty $\mathcal{T}_X$-definable relation contains a point with rational components then one can express the property that X is $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$-definable with a $\mathcal{T}_X$-sentence.

The construction is based on the decomposition of any set of reals into "integer" and "fractional" sets, which allows us to reduce the $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$-definability of X, on one hand to the $\langle \mathbb{Z}, +, < \rangle$-definability of some subsets of $\mathbb{Z}^n$ and on the other hand to the $\langle \mathbb{R}, +, <, 1 \rangle$-definability of a collection of subsets of $[0, 1)^n$. In order to express these two kinds of properties in $\mathcal{T}_X$, we rely respectively on Muchnik's Theorem [18] and on Theorem 5.8.

We start with a property which holds for all relations under no particular assumption. Given a relation $X \subseteq \mathbb{R}^n$ consider the denumerable set of distinct restrictions of X to unit hypercubes, i.e.,

$$\tau_a\big(X \cap ([a_1, a_1+1), \cdots, [a_n, a_n+1))\big)$$

where $a = (a_1, \cdots, a_n) \in \mathbb{Z}^n$ and τ_a is the translation $x \mapsto x - a$. Let Δ_m denote this collection of sets where m runs over some denumerable set M. For each $m \in M$, let $\Sigma_m \subseteq \mathbb{Z}^n$ satisfy the condition

$$x \in \Sigma_m \leftrightarrow x + \Delta_m = X \cap \big([x_1, x_1, +1), \cdots, [x_n, x_n, +1)\big)$$

Observe that the decomposition

$$X = \bigcup_{m \in M} \Sigma_m + \Delta_m \tag{9}$$

is unique by construction. In the particular case of $\langle \mathbb{R}, +, <, \mathbb{Z}\rangle$-definable relations we have the following result.

Proposition 5.10. ([10, Theorem 7], see also [13]) A relation $X \subseteq \mathbb{R}^n$ is $\langle \mathbb{R}, +, <, \mathbb{Z}\rangle$-definable if and only if in the decomposition (9) the following three conditions hold:

(FU) the set M in (9) is finite.

(IP) each Σ_m is $\langle \mathbb{Z}, +, <\rangle$-definable.

(FP) each Δ_m is $\langle \mathbb{R}, +, <, 1\rangle$-definable.

Proposition 5.11. For all $n \geq 1$ and $X \subseteq \mathbb{R}^n$, if every nonempty $\mathcal{T}_X$-definable relation contains a point with rational components then there exists a $\mathcal{T}_X$-sentence Γ_n (uniform in X) which holds if and only if X is $\langle \mathbb{R}, +, <, \mathbb{Z}\rangle$-definable

Proof:
In view of Proposition 5.10 it suffices to show that the three conditions are expressible in $\langle \mathbb{R}, +, <, \mathbb{Z}, X\rangle$. Let $\Phi(X)$ be the $\mathcal{T}_X$-formula which states that X is $\langle \mathbb{R}, +, <, 1\rangle$-definable, see equation (8).

Condition (FU): Let $x \approx y$ denote the equivalence relation which says that the two points x and y belong to the same Σ_m in the decomposition 9. It is expressed by the $\mathcal{T}_X$-formula

$$x \in \mathbb{Z}^n \wedge y \in \mathbb{Z}^n \wedge \forall z \in [0,1)^n \ (x + z \in X \leftrightarrow y + z \in X) \tag{10}$$

The finiteness of the number of classes is expressed by the $\mathcal{T}_X$-formula

$$\exists N \ \forall x \in \mathbb{Z}^n \ \exists y \in \mathbb{Z}^n \ (|y| < N \wedge y \approx x)$$

Condition (IP): By [18, Thm 1] for every $Y \subseteq \mathbb{Z}^n$ there exists a $\langle \mathbb{Z}, +, <, Y\rangle$-formula $\Psi(Y)$ (uniform in Y) which holds if and only if Y is $\langle \mathbb{Z}, +, <\rangle$-definable. Let $\Psi^*(Y)$ denote the $\langle \mathbb{R}, \mathbb{Z}, +, <, Y\rangle$-formula obtained from $\Psi(Y)$ by relativizing all quantifiers to $\mathbb{Z}$. Given $Y \subseteq \mathbb{Z}^n$ (seen as a subset of

$\mathbb{R}^n$), the formula $\Psi^*(Y)$ holds in $\langle \mathbb{R}, \mathbb{Z}, +, <, Y\rangle$ if and only if Y is $\langle \mathbb{Z}, +, <\rangle$-definable. Thus we can express in $\mathcal{T}_X$ the fact that all $\approx$-equivalence classes are $\langle \mathbb{Z}, +, <\rangle$-definable with the formula

$$\forall x \in \mathbb{Z}^n \Psi^*((y \in \mathbb{Z}^n \wedge y \approx x))$$

Condition (FP): The fact that every hypercube of X of unit side is $\langle \mathbb{R}, +, <, 1\rangle$-definable is expressed by

$$\forall x_1, \dots x_n \in \mathbb{Z}^n \; \Phi((0 \leq y_1 < 1 \wedge \cdots \wedge 0 \leq y_n < 1 \wedge (x_1 + y_1, \cdots, x_n + y_n) \in X)). \qquad \square$$

6. Application to decidability

6.1. Deciding $\langle \mathbb{R}, +, <, 1\rangle$-definability and $\langle \mathbb{R}, +, <, \mathbb{Z}\rangle$-definability

Theorem 5.8 and Proposition 5.11 prove the existence of definable criteria for $\langle \mathbb{R}, +, <, 1\rangle$-definability and $\langle \mathbb{R}, +, <, \mathbb{Z}\rangle$-definability for a given relation X, respectively. If X is definable in some decidable expansion of $\langle \mathbb{R}, +, <, 1\rangle$ (resp. $\langle \mathbb{R}, +, <, \mathbb{Z}\rangle$) then we can obtain effective criteria. This can be formulated as follows.

Theorem 6.1. Let $\mathcal{M}$ be any decidable expansion of $\langle \mathbb{R}, +, <, 1\rangle$ such that every nonempty $\mathcal{M}$-definable relation contains a point with rational components. Then it is decidable whether a $\mathcal{M}$-definable relation $X \subseteq \mathbb{R}^n$ is $\langle \mathbb{R}, +, <, 1\rangle$-definable.

Proof:
Assume that X is $\mathcal{M}$-definable by the formula $\psi(x)$. In Equation (8), if we substitute $\psi(x)$ for every occurrence of $x \in X$ then we obtain a $\mathcal{M}$-sentence which holds if and only if X is $\langle \mathbb{R}, +, <, 1\rangle$-definable, and the result follows from the decidability of $\mathcal{M}$. $\square$

Theorem 6.2. Let $\mathcal{N}$ be any decidable expansion of $\langle \mathbb{R}, +, <, \mathbb{Z}\rangle$ such that every nonempty $\mathcal{N}$-definable relation contains a point with rational components. Then it is decidable whether a $\mathcal{N}$-definable relation $X \subseteq \mathbb{R}^n$ is $\langle \mathbb{R}, +, <, \mathbb{Z}\rangle$-definable (resp. whether a $\mathcal{N}$-definable relation $X \subseteq \mathbb{R}^n$ is $\langle \mathbb{R}, +, <, 1\rangle$-definable).

Proof:
The claim about $\langle \mathbb{R}, +, <, 1\rangle$-definability follows immediately from the fact that $\mathcal{N}$ satisfies the conditions of Theorem 6.1. For $\langle \mathbb{R}, +, <, \mathbb{Z}\rangle$-definability, we use the same idea as for the proof of Theorem 6.1, but instead of Φ_n we use the sentence Γ_n of Proposition 5.11. $\square$

6.2. Application to recognizable numerical relations

We finally apply the results of Section 6.1 to the class of k-recognizable relations on reals.

Let us recall that given an integer base $k \geq 2$ and a non-negative real x, a *k-encoding* of x is any right infinite word on the alphabet $\Sigma_k = \{0, \dots, k-1\} \cup \{\star\}$ of the form $w = a_p \dots a_1 \star a_0 a_{-1} a_{-2} \dots$ such that $a_i \in \{0, \dots, k-1\}$ for every $i \leq p$ and $x = \sum_{i \leq p} a_i k^i$. The definition extends to the case of negative reals x by using the k's complement representation method where the leftmost digit equals $k-1$: a $k-$encoding of x is a right infinite word on the alphabet Σ_k of the form $w = a_p \dots a_1 \star a_0 a_{-1} a_{-2} \dots$ where $a_p = k-1$ and $x = -k^p + \sum_{i \le p-1} a_i k^i$. Note that every real has infinitely many k-encodings.

In order for an automaton to be able to process n-tuples of representations in base k of reals we prefix it, if necessary, with as few occurrences of 0 for the nonnegative components or $k-1$ to the negative components so that the n components have the same length to the left of the symbol $\star$. This does not change the numerical values represented. By identifying a real with its k-encodings, relations of arity n on $\mathbb{R}$ can thus be viewed as subsets of n-tuples of sequences on $\{0,\dots,k-1\} \cup \{\star\}$, i.e., as subsets of

$$(\{0,\dots,k-1\}^n)^* \{\overbrace{(\star,\dots,\star)}^{n\text{times}}\} (\{0,\dots,k-1\}^n)^\omega$$

Definition 6.3. A relation $X \subseteq \mathbb{R}^n$ is *k-recognizable* if the set of k-encodings of its elements is recognized by some deterministic Muller-automaton.

The collection of recognizable relations has a natural logical characterization.

Theorem 6.4. [9, Thm 5 and 6] Let $k \ge 2$ be an integer. A subset of $\mathbb{R}^n$ is k-recognizable if and only if it is definable in $\langle \mathbb{R}, \mathbb{Z}, +, <, X_k \rangle$ where $X_k \subseteq \mathbb{R}^3$ is such that $X_k(x,y,z)$ holds if and only if y is a power of k and z is the coefficient of y in some $k-$encoding of x.

Consequently, since the emptiness problem for recognizable relations is decidable, the theory of $\langle \mathbb{R}, \mathbb{Z}, +, <, X_k \rangle$ is decidable.

Moreover the class of $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$-definable relations enjoys the following characterization.

Theorem 6.5. [4, 5, 6] A subset of $\mathbb{R}^n$ is $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$-definable if and only if it is k-recognizable for every integer $k \ge 2$.

As a consequence, deciding whether a k-recognizable relation is $l-$recognizable for every base $l \ge 2$ amounts to decide whether it is $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$-definable. We can prove the following result.

Theorem 6.6. Given an integer $k \ge 2$, it is decidable whether a k-recognizable relation $X \subseteq \mathbb{R}^n$ is $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$-definable (resp. whether a $k-$recognizable relation $X \subseteq \mathbb{R}^n$ is $\langle \mathbb{R}, +, <, 1 \rangle$-definable).

Proof:
By Theorems 6.2 and 6.4 it suffices to prove that every nonempty k-recognizable relation $Y \subseteq \mathbb{R}^n$ contains an element in $\mathbb{Q}^n$. By our assumption the set of k-encodings of elements of Y is nonempty and is recognized by a finite Muller automaton, thus it contains an ultimately periodic $\omega-$word, which is the $k-$encoding of some element of $\mathbb{Q}^n$. □

Acknowledgment

We wish to thank the anonymous referees and Françoise Point for their useful comments and suggestions.

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