CONVERGENCE OF THE COMPLETE ELECTROMAGNETIC FLUID SYSTEM TO THE FULL COMPRESSIBLE MAGNETOHYDRODYNAMIC EQUATIONS

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ABSTRACT. The full compressible magnetohydrodynamic equations can be derived formally from the complete electromagnetic fluid system in some sense as the dielectric constant tends to zero. This process is usually referred as magnetohydrodynamic approximation in physical books. In this paper we justify this singular limit rigorously in the framework of smooth solutions for well-prepared initial data.

1. Introduction and Main Results

Electromagnetic dynamics studies the motion of an electrically conducting fluid in the presence of an electromagnetic field. In electromagnetic dynamics the fluid and the electromagnetic field are connected closely with each other, hence the fundamental system of electromagnetic dynamics usually contains the hydrodynamical equations and the electromagnetic ones. The complete electromagnetic fluid system includes the conservation of mass, momentum, and energy to the fluid, the Maxwell system to the electromagnetic field, and the conservation of electric charge, which take the forms $([5, 14, 21])$ $([5, 14, 21])$ $([5, 14, 21])$ $([5, 14, 21])$

$$
\partial_t \rho + \text{div}(\rho \mathbf{u}) = 0,\tag{1.1}
$$

$$
\rho(\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) + \nabla P = \text{div} \Psi(\mathbf{u}) + \rho_e \mathbf{E} + \mu_0 \mathbf{J} \times \mathbf{H},
$$
\n(1.2)

$$
\rho \frac{\partial e}{\partial \theta} (\partial_t \theta + \mathbf{u} \cdot \nabla \theta) + \theta \frac{\partial P}{\partial \theta} \text{div } \mathbf{u}
$$

= $\text{div} (\kappa \nabla \theta) + \Psi(\mathbf{u}) : \nabla \mathbf{u} + (\mathbf{J} - \rho_e \mathbf{u}) \cdot (\mathbf{E} + \mu_0 \mathbf{u} \times \mathbf{H}),$ (1.3)

$$
\mathbf{P} = \mathbf{P} \mathbf{P} \mathbf{P} \mathbf{P} \tag{4.6}
$$

$$
\epsilon \partial_t \mathbf{E} - \operatorname{curl} \mathbf{H} + \mathbf{J} = 0,\tag{1.4}
$$

$$
\partial_t \mathbf{H} + \frac{1}{\mu_0} \text{curl } \mathbf{E} = 0, \tag{1.5}
$$

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$$
\partial_t(\rho_e) + \operatorname{div} \mathbf{J} = 0,\tag{1.6}
$$

$$
\epsilon \operatorname{div} \mathbf{E} = \rho_e, \quad \operatorname{div} \mathbf{H} = 0. \tag{1.7}
$$

The system (1.1) – (1.7) consists of 14 equations in 12 unknowns, namely, the mass density ρ , the velocity $\mathbf{u} = (u_1, u_2, u_3)$, the absolute temperature θ , the electric field $\mathbf{E} = (E_1, E_2, E_3)$, the magnetic field $\mathbf{H} = (H_1, H_2, H_3)$, and the electric charge density ρ_e . The quantity $\Psi(\mathbf{u})$ is the viscous stress tensor given by

$$
\Psi(\mathbf{u}) = 2\mu \mathbb{D}(\mathbf{u}) + \lambda \operatorname{div} \mathbf{u} \mathbf{I}_3, \quad \mathbb{D}(\mathbf{u}) = (\nabla \mathbf{u} + \nabla \mathbf{u}^\top)/2, \tag{1.8}
$$

where \mathbf{I}_3 denotes the 3 \times 3 identity matrix, and $\nabla \mathbf{u}^{\top}$ the transpose of the matrix ∇ u. The pressure $P = P(\rho, \theta)$ and the internal energy $e = e(\rho, \theta)$ are smooth functions of ρ and θ of the flow, and satisfy the Gibbs relation

$$
\theta \mathrm{d}S = \mathrm{d}e + P \,\mathrm{d}\left(\frac{1}{\rho}\right),\tag{1.9}
$$

for some smooth function (entropy) $S = S(\rho, \theta)$ which expresses the first law of the thermodynamics. The current density J is expressed by Ohm's law, i.e.,

$$
\mathbf{J} - \rho_e \mathbf{u} = \sigma (\mathbf{E} + \mu_0 \mathbf{u} \times \mathbf{H}), \qquad (1.10)
$$

where ρ_e **u** is called the convection current. The symbol $\Psi(\mathbf{u})$: $\nabla \mathbf{u}$ denotes the scalar product of two matrices:

$$
\Psi(\mathbf{u}) : \nabla \mathbf{u} = \sum_{i,j=1}^{3} \frac{\mu}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)^2 + \lambda |\text{div } \mathbf{u}|^2 = 2\mu |\mathbb{D}(\mathbf{u})|^2 + \lambda |\text{tr } \mathbb{D}(\mathbf{u})|^2. \tag{1.11}
$$

The viscosity coefficients μ and λ of the fluid satisfy $\mu > 0$ and $2\mu + 3\lambda > 0$. The parameters $\epsilon > 0$ is the dielectric constant, $\mu_0 > 0$ the magnetic permeability, $\kappa > 0$ the heat conductivity, and $\sigma > 0$ the electric conductivity coefficient, respectively. In general, the conductivity σ may be a tensor depending on the present magnetic field. However, in this paper, we shall suppose that the Hall effect is negligible and σ is a scalar quantity. If the Hall effect is to be taken into account, [\(1.10\)](#page-1-1) must be replaced by

$$
\mathbf{J} - \rho_e \mathbf{u} = \sigma (\mathbf{E} + \mu_0 \mathbf{u} \times \mathbf{H}) - \frac{\sigma \sigma_0 \mu_0}{n_e e_0} (\mathbf{E} + \mu_0 \mathbf{u} \times \mathbf{H}) \times \mathbf{H},
$$

where σ_0 is the electric conductivity in the absence of a magnetic field, n_e the electron number density, and e_0 the charge of an electron. For simplicity, in the following consideration, we shall assume that $\mu, \lambda, \epsilon, \mu_0, \kappa$, and σ are constants.

Mathematically, it is very difficult to study the properties of solutions to the electromagnetic fluid system (1.1) – (1.7) . The reason is that, as pointed out by Kawashima [\[21\]](#page-24-1), the system of the electromagnetic quantities (E, H, ρ_e) in the system (1.1) – (1.7) , which is regarded as a first-order hyperbolic system, is neither symmetric hyperbolic nor strictly hyperbolic in the three-dimensional case. The same difficulty also occurs in the first-order hyperbolic system of (E, H) which is obtained from the above system by eliminating ρ_e with the aid of the first equation of [\(1.7\)](#page-1-0). Therefore, the classic hyperbolic-parabolic theory (for example [\[37\]](#page-25-0)) can not be applied here. There are only a few mathematical results on the electromagnetic fluid system (1.1) – (1.7) in some special cases. Kawashima [\[21\]](#page-24-1) obtained the global existence of smooth solutions in the two-dimensional case when the initial data are a small perturbation of some given constant state. Umeda, Kawashima and Shizuta [\[35\]](#page-25-1) obtained the global existence and time decay of smooth solutions to the linearized equations of the system (1.1) – (1.7) in the three-dimensional case near some given constant equilibria. Based on the above arguments, it is desirable to introduce some simplifications without sacrificing the essential feature of the phenomenon.

As it was pointed out in [\[14\]](#page-24-0), the assumption that the electric charge density $\rho_e \simeq 0$ is physically very reasonable for the study of plasmas. In this situation, the convection current ρ_e **u** is negligible in comparison with the conduction current $\sigma(\mathbf{E}+\mu_0\mathbf{u}\times\mathbf{H})$, thus we can eliminate the terms involving ρ_e in the electromagnetic fluid system (1.1) – (1.7) and obtain the following simplified system:

$$
\partial_t \rho + \text{div}(\rho \mathbf{u}) = 0,\tag{1.12}
$$

$$
\rho(\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) + \nabla P = \text{div} \Psi(\mathbf{u}) + \mu_0 \mathbf{J} \times \mathbf{H},
$$
\n(1.13)

$$
\rho \frac{\partial e}{\partial \theta} (\partial_t \theta + \mathbf{u} \cdot \nabla \theta) + \theta \frac{\partial P}{\partial \theta} \text{div } \mathbf{u} = \kappa \Delta \theta + \Psi(\mathbf{u}) : \nabla \mathbf{u} + \mathbf{J} \cdot (\mathbf{E} + \mu_0 \mathbf{u} \times \mathbf{H}), \quad (1.14)
$$

$$
\epsilon \partial_t \mathbf{E} - \operatorname{curl} \mathbf{H} + \mathbf{J} = 0,\tag{1.15}
$$

$$
\partial_t \mathbf{H} + \frac{1}{\mu_0} \text{curl } \mathbf{E} = 0, \quad \text{div } \mathbf{H} = 0,
$$
\n(1.16)

with

$$
\mathbf{J} = \sigma(\mathbf{E} + \mu_0 \mathbf{u} \times \mathbf{H}). \tag{1.17}
$$

We remark here that the assumption $\rho_e \simeq 0$ is quite different from the assumption of exact neutrality $\rho_e = 0$, which would lead to the superfluous condition div $\mathbf{E} = 0$ by [\(1.7\)](#page-1-0).

Formally, if we take the dielectric constant $\epsilon = 0$ in [\(1.15\)](#page-2-0), i.e. the displacement current $\epsilon \partial_t \mathbf{E}$ is negligible, then we obtain that $\mathbf{J} = \text{curl } \mathbf{H}$. Thanks to [\(1.17\)](#page-2-1), we can eliminate the electric field \bf{E} in [\(1.13\)](#page-2-2), [\(1.14\)](#page-2-3), and [\(1.16\)](#page-2-4) and finally obtain the system

$$
\partial_t \rho + \text{div}(\rho \mathbf{u}) = 0,\tag{1.18}
$$

$$
\rho(\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) + \nabla P = \text{div} \Psi(\mathbf{u}) + \mu_0 \text{curl} \mathbf{H} \times \mathbf{H},
$$
\n(1.19)

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$$
\rho \frac{\partial e}{\partial \theta} (\partial_t \theta + u \cdot \nabla \theta) + \theta \frac{\partial P}{\partial \theta} \text{div } \mathbf{u} = \kappa \Delta \theta + \Psi(\mathbf{u}) : \nabla \mathbf{u} + \frac{1}{\sigma} |\text{curl } \mathbf{H}|^2, \qquad (1.20)
$$

$$
\partial_t \mathbf{H} - \text{curl}(\mathbf{u} \times \mathbf{H}) = -\frac{1}{\sigma \mu_0} \text{curl} (\text{curl } \mathbf{H}), \quad \text{div } \mathbf{H} = 0.
$$
 (1.21)

The equations (1.18) – (1.21) are the so-called full compressible magnetohydrodynamic equations, see [\[5,](#page-23-0) [28,](#page-24-2) [30\]](#page-25-2). It should be pointed that although it has been completely eliminated in the limit equations (1.18) – (1.21) , the electric field **E** still plays an essentially important role in the phenomena under consideration. In fact, it determines the electric current $\sigma(\mathbf{E} + \mu_0 \mathbf{u} \times \mathbf{H})$ which generates the magnetic field H . The electric field E and the magnetic field H satisfy the relation

$$
\mathbf{E} = \frac{1}{\sigma} \operatorname{curl} \mathbf{H} - \mu_0 \mathbf{u} \times \mathbf{H}.
$$

The above formal derivation is usually referred as magnetohydrodynamic approximation, see [\[5,](#page-23-0) [14\]](#page-24-0). In [\[23\]](#page-24-3), Kawashima and Shizuta justified this limit process rigorously in the two-dimensional case for local smooth solutions, i.e., $\mathbf{u} = (u_1, u_2, 0)$, $$ this situation, we can obtain that $\rho_e = 0$ and the system (1.1) – (1.7) is reduced to (1.12) – (1.17) . Later, in [\[24\]](#page-24-4), they also obtained the global convergence of the limit in the two-dimensional case under the assumption that both the initial data of the electromagnetic fluid equations and those of the compressible magnetohydrodynamic equations are a small perturbation of some given constant state in some Sobolev spaces in which the global smooth solution can be obtained. Recently, we studied the magnetohydrodynamic approximation for the isentropic electromagnetic fluid system in a three-dimensional period domain and deduced the isentropic compressible magnetohydrodynamic equations [\[18\]](#page-24-5).

The purpose of this paper is to give a rigorous derivation of the full compressible magnetohydrodynamic equations (1.18) – (1.21) from the electromagnetic fluid system (1.12) – (1.17) as the dielectric constant ϵ tends to zero. For the sake of simplicity and clarity of presentation, we shall focus on the ionized fluids obeying the perfect gas relations

$$
P = \Re \rho \theta, \quad e = c_V \theta,
$$
\n(1.22)

where the parameters \Re > 0 and c_V > 0 are the gas constant and the heat capacity at constant volume, respectively. We consider the system (1.12) – (1.17) in a periodic domain of \mathbb{R}^3 , i.e., the torus $\mathbb{T}^3 = (\mathbb{R}/(2\pi\mathbb{Z}))^3$.

Below for simplicity of presentation, we take the physical constants $\mathfrak{R}, c_V, \sigma$, and μ_0 to be one. To emphasize the unknowns depending on the small parameter ϵ , we rewrite the electromagnetic fluid system (1.12) – (1.17) as

$$
\partial_t \rho^{\epsilon} + \text{div} \left(\rho^{\epsilon} \mathbf{u}^{\epsilon} \right) = 0, \tag{1.23}
$$

$$
\rho^{\epsilon}(\partial_t \mathbf{u}^{\epsilon} + \mathbf{u}^{\epsilon} \cdot \nabla \mathbf{u}^{\epsilon}) + \nabla(\rho^{\epsilon} \theta^{\epsilon}) = \text{div} \Psi(\mathbf{u}^{\epsilon}) + (\mathbf{E}^{\epsilon} + \mathbf{u}^{\epsilon} \times \mathbf{H}^{\epsilon}) \times \mathbf{H}^{\epsilon},\tag{1.24}
$$

$$
\rho^{\epsilon}(\partial_t \theta^{\epsilon} + \mathbf{u}^{\epsilon} \cdot \nabla \theta^{\epsilon}) + \rho^{\epsilon} \theta^{\epsilon} \text{div } \mathbf{u}^{\epsilon} = \kappa \Delta \theta^{\epsilon} + \Psi(\mathbf{u}^{\epsilon}) : \nabla \mathbf{u}^{\epsilon} + |\mathbf{E}^{\epsilon} + \mathbf{u}^{\epsilon} \times \mathbf{H}^{\epsilon}|^2, (1.25)
$$

$$
\epsilon \partial_t \mathbf{E}^{\epsilon} - \operatorname{curl} \mathbf{H}^{\epsilon} + \mathbf{E}^{\epsilon} + \mathbf{u}^{\epsilon} \times \mathbf{H}^{\epsilon} = 0, \qquad (1.26)
$$

$$
\partial_t \mathbf{H}^{\epsilon} + \operatorname{curl} \mathbf{E}^{\epsilon} = 0, \quad \operatorname{div} \mathbf{H}^{\epsilon} = 0,
$$
\n(1.27)

where $\Psi(\mathbf{u}^{\epsilon})$ and $\Psi(\mathbf{u}^{\epsilon})$: $\nabla \mathbf{u}^{\epsilon}$ are defined through [\(1.8\)](#page-1-2) and [\(1.11\)](#page-1-3) with **u** replaced by \mathbf{u}^{ϵ} . The system (1.23) – (1.27) is supplemented with the initial data

$$
(\rho^{\epsilon}, \mathbf{u}^{\epsilon}, \theta^{\epsilon}, \mathbf{E}^{\epsilon}, \mathbf{H}^{\epsilon})|_{t=0} = (\rho_0^{\epsilon}(x), \mathbf{u}_0^{\epsilon}(x), \theta_0^{\epsilon}(x), \mathbf{E}_0^{\epsilon}(x), \mathbf{H}_0^{\epsilon}(x)), \quad x \in \mathbb{T}^3.
$$
 (1.28)

We also rewrite the limiting equations [\(1.18\)](#page-2-5)–[\(1.21\)](#page-3-0) (recall that $\Re = c_V = \sigma =$ $\mu_0 = 1$) as

$$
\partial_t \rho^0 + \text{div}(\rho^0 \mathbf{u}^0) = 0,\tag{1.29}
$$

$$
\rho^{0}(\partial_{t}\mathbf{u}^{0} + \mathbf{u}^{0} \cdot \nabla \mathbf{u}^{0}) + \nabla(\rho^{0}\theta^{0}) = \operatorname{div}\Psi(\mathbf{u}^{0}) + \operatorname{curl}\mathbf{H}^{0} \times \mathbf{H}^{0},
$$
\n(1.30)

$$
\rho^0(\partial_t \theta^0 + \mathbf{u}^0 \cdot \nabla \theta^0) + \rho^0 \theta^0 \text{div } \mathbf{u}^0 = \kappa \Delta \theta^0 + \Psi(\mathbf{u}^0) : \nabla \mathbf{u}^0 + |\text{curl } \mathbf{H}^0|^2, \qquad (1.31)
$$

$$
\partial_t \mathbf{H}^0 - \text{curl}(\mathbf{u}^0 \times \mathbf{H}^0) = -\text{curl}(\text{curl} \mathbf{H}^0), \quad \text{div } \mathbf{H}^0 = 0,
$$
\n(1.32)

where $\Psi(\mathbf{u}^0)$ and $\Psi(\mathbf{u}^0)$: $\nabla \mathbf{u}^0$ are defined through [\(1.8\)](#page-1-2) and [\(1.11\)](#page-1-3) with **u** replaced by \mathbf{u}^0 . The system (1.29) – (1.32) is equipped with the initial data

$$
(\rho^{0}, \mathbf{u}^{0}, \theta^{0}, \mathbf{H}^{0})|_{t=0} = (\rho_{0}^{0}(x), \mathbf{u}_{0}^{0}(x), \theta_{0}^{0}(x), \mathbf{H}_{0}^{0}(x)), \quad x \in \mathbb{T}^{3}.
$$
 (1.33)

Notice that the electric field \mathbf{E}^0 is induced according to the relation

$$
\mathbf{E}^0 = \operatorname{curl} \mathbf{H}^0 - \mathbf{u}^0 \times \mathbf{H}^0 \tag{1.34}
$$

by moving the conductive flow in the magnetic field.

Before stating our main results, we recall the local existence of smooth solutions to the problem (1.29) – (1.33) . Since the system (1.29) – (1.32) is parabolichyperbolic, the results in [\[37\]](#page-25-0) imply that

Proposition 1.1 ([\[37\]](#page-25-0)). Let $s > 7/2$ be an integer and assume that the initial data $(\rho_0^0, \mathbf{u}_0^0, \theta_0^0, \mathbf{H}_0^0)$ satisfy

$$
\rho_0^0, \mathbf{u}_0^0, \theta_0^0, \mathbf{H}_0^0 \in H^{s+2}(\mathbb{T}^3), \quad \text{div } \mathbf{H}_0^0 = 0,
$$

$$
0 < \bar{\rho} = \inf_{x \in \mathbb{T}^3} \rho_0^0(x) \le \rho_0^0(x) \le \bar{\bar{\rho}} = \sup_{x \in \mathbb{T}^3} \rho_0^0(x) < +\infty,
$$

$$
0 < \bar{\theta} = \inf_{x \in \mathbb{T}^3} \theta_0^0(x) \le \theta_0^0(x) \le \bar{\bar{\theta}} = \sup_{x \in \mathbb{T}^3} \theta_0^0(x) < +\infty
$$

for some positive constants $\bar{\rho}, \bar{\bar{\rho}}, \bar{\theta},$ and $\bar{\bar{\theta}}.$ Then there exist positive constants T_* (the maximal time interval, $0 < T_* \leqslant +\infty$), and $\hat{\rho}, \tilde{\rho}, \hat{\theta}, \tilde{\theta}$, such that the problem [\(1.29\)](#page-4-2)– [\(1.33\)](#page-4-4) has a unique classical solution $(\rho^0, \mathbf{u}^0, \theta^0, \mathbf{H}^0)$ satisfying div $\mathbf{H}^0 = 0$ and

$$
\rho^0 \in C^l([0, T_*), H^{s+2-l}(\mathbb{T}^3)), \quad \mathbf{u}^0, \theta^0, \mathbf{H}^0 \in C^l([0, T_*), H^{s+2-2l}(\mathbb{T}^3)), \quad l = 0, 1;
$$
\n
$$
0 < \hat{\rho} = \inf_{(x, t) \in \mathbb{T}^3 \times [0, T_*)} \rho^0(x, t) \leqslant \rho^0(x, t) \leqslant \tilde{\rho} = \sup_{(x, t) \in \mathbb{T}^3 \times [0, T_*)} \rho^0(x, t) < +\infty,
$$
\n
$$
0 < \hat{\theta} = \inf_{(x, t) \in \mathbb{T}^3 \times [0, T_*)} \theta^0(x, t) \leqslant \theta^0(x, t) \leqslant \tilde{\theta} = \sup_{(x, t) \in \mathbb{T}^3 \times [0, T_*)} \theta^0(x, t) < +\infty.
$$

The main results of this paper can be stated as follows.

Theorem 1.2. Let $s > 7/2$ be an integer and $(\rho^0, \mathbf{u}^0, \theta^0, \mathbf{H}^0)$ be the unique classical solution to the problem (1.29) – (1.33) given in Proposition [1.1.](#page-4-5) Suppose that the initial data $(\rho_0^{\epsilon}, \mathbf{u}_0^{\epsilon}, \theta_0^{\epsilon}, \mathbf{E}_0^{\epsilon}, \mathbf{H}_0^{\epsilon})$ satisfy

$$
\rho_0^{\epsilon}, \mathbf{u}_0^{\epsilon}, \theta_0^{\epsilon}, \mathbf{E}_0^{\epsilon}, \mathbf{H}_0^{\epsilon} \in H^s(\mathbb{T}^3), \text{ div } \mathbf{H}_0^{\epsilon} = 0, \text{ inf }_{x \in \mathbb{T}^3} \rho_0^{\epsilon}(x) > 0, \text{ inf }_{x \in \mathbb{T}^3} \theta_0^{\epsilon}(x) > 0,
$$

and

$$
\begin{aligned} \n\| (\rho_0^{\epsilon} - \rho_0^0, \mathbf{u}_0^{\epsilon} - \mathbf{u}_0^0, \theta_0^{\epsilon} - \theta_0^0, \mathbf{H}_0^{\epsilon} - \mathbf{H}_0^0) \|_s \\
&+ \sqrt{\epsilon} \left\| \mathbf{E}_0^{\epsilon} - (\text{curl}\, \mathbf{H}_0^0 - \mathbf{u}_0^0 \times \mathbf{H}_0^0) \right\|_s \le L_0 \epsilon, \n\end{aligned} \tag{1.35}
$$

for some constant $L_0 > 0$. Then, for any $T_0 \in (0, T_*)$, there exist a constant $L > 0$, and a sufficient small constant $\epsilon_0 > 0$, such that for any $\epsilon \in (0, \epsilon_0]$, the problem [\(1.23\)](#page-4-0)–[\(1.28\)](#page-4-6) has a unique smooth solution $(\rho^{\epsilon}, \mathbf{u}^{\epsilon}, \theta^{\epsilon}, \mathbf{E}^{\epsilon}, \mathbf{H}^{\epsilon})$ on $[0, T_0]$ enjoying

$$
\|(\rho^{\epsilon} - \rho^{0}, \mathbf{u}^{\epsilon} - \mathbf{u}^{0}, \theta^{\epsilon} - \theta^{0}, \mathbf{H}^{\epsilon} - \mathbf{H}^{0})(t)\|_{s} + \sqrt{\epsilon} \|\{\mathbf{E}^{\epsilon} - (\text{curl}\,\mathbf{H}^{0} - \mathbf{u}^{0} \times \mathbf{H}^{0})\}(t)\|_{s} \le L\epsilon, \quad t \in [0, T_{0}].
$$
 (1.36)

Here $\|\cdot\|_{s}$ denotes the norm of Sobolev space $H^{s}(\mathbb{T}^{3})$.

Remark 1.1. The inequality [\(1.36\)](#page-5-0) implies that the sequences $(\rho^{\epsilon}, \mathbf{u}^{\epsilon}, \theta^{\epsilon}, \mathbf{H}^{\epsilon})$ converge strongly to $(\rho^0, \mathbf{u}^0, \theta^0, \mathbf{H}^0)$ in $L^{\infty}(0,T; H^s(\mathbb{T}^3))$ and \mathbf{E}^{ϵ} converge strongly to \mathbf{E}^0 in $L^\infty(0,T;H^s(\mathbb{T}^3))$ but with different convergence rates, where \mathbf{E}^0 is defined by [\(1.34\)](#page-4-7).

Remark 1.2. Theorem [1.2](#page-5-1) still holds for the case with general state equations with minor modifications. Furthermore, our results also hold in the whole space \mathbb{R}^3 . Indeed, neither the compactness of \mathbb{T}^3 nor Poincaré-type inequality is used in our arguments.

Remark 1.3. In the two-dimensional case, our result is similar to that of [\[23\]](#page-24-3) (see Remark 5.1 of [\[23\]](#page-24-3)). In addition, if we assume that the initial data are a small perturbation of some given constant state in the Sobolev norm $H^s(\mathbb{T}^3)$ for $s >$

 $3/2 + 2$, we can extend the local convergence result stated in Theorem [1.2](#page-5-1) to a global one.

Remark 1.4. For the local existence of solutions $(\rho^0, \mathbf{u}^0, \theta^0, \mathbf{H}^0)$ to the problem [\(1.29\)](#page-4-2)–[\(1.33\)](#page-4-4), the assumption on the regularity of initial data $(\rho_0^0, \mathbf{u}_0^0, \theta_0^0, \mathbf{H}_0^0)$ belongs to $H^s(\mathbb{T}^3)$, $s > 7/2$, is enough. Here we have added more regularity assumption in Proposition [1.1](#page-4-5) to obtain more regular solutions which are needed in the proof of Theorem [1.2.](#page-5-1)

Remark 1.5. The viscosity and heat conductivity terms in the system (1.23) – (1.27) play a crucial role in our uniformly bounded estimates (in order to control some undesirable higher-order terms). In the case of $\lambda = \mu = \kappa = 0$, the original system [\(1.23\)](#page-4-0)–[\(1.27\)](#page-4-1) are reduced to the so-call non-isentropic Euler-Maxwell system. Our arguments can not be applied to this case directly, for more details, see [\[19\]](#page-24-6).

We give some comments on the proof of Theorem [1.2.](#page-5-1) The main difficulty in dealing with the zero dielectric constant limit problem is the oscillatory behavior of the electric field as pointed out in [\[18\]](#page-24-5), besides the singularity in the Maxwell equations, there exists an extra singularity caused by the strong coupling of the electromagnetic field (the nonlinear source term) in the momentum equation. Moreover, comparing to the isentropic case studied in [\[18\]](#page-24-5), we have to circumvent additional difficulties in the derivation of uniform estimates induced by the nonlinear differential terms (such as $\Psi(\mathbf{u}^{\epsilon})$: $\nabla \mathbf{u}^{\epsilon}$) and higher order nonlinear terms (such as $|\mathbf{E}^{\epsilon} + \mathbf{u}^{\epsilon} \times \mathbf{H}^{\epsilon}|^2$ involving $\mathbf{u}^{\epsilon}, \mathbf{E}^{\epsilon}$, and \mathbf{H}^{ϵ} in the temperature equation. In this paper, we shall overcome all these difficulties and derive rigorously the full compressible magnetohydrodynamic equations from the electromagnetic fluid equations by adapting the elaborate nonlinear energy method developed in [\[18,](#page-24-5) [32\]](#page-25-3). First, we derive the error system (2.1) – (2.5) by utilizing the original system (1.23) – (1.27) and the limit equations (1.29) – (1.32) . Next, we study the estimates of H^s -norm to the error system. To do so, we shall make full use of the special structure of the error system, Sobolev imbedding, the Moser-type inequalities, and the regularity of limit equations. In particular, very refined analyses are carried out to deal with the higher order nonlinear terms in the system (2.1) – (2.5) . Finally, we combine these obtained estimates and apply Gronwall's type inequality to get the desired results. We remark that in the isentropic case in [\[18\]](#page-24-5), the density is controlled by the pressure, while in our case the density is controlled through the viscosity terms in the momentum equations.

It should be pointed out that there are a lot of works on the studies of compressible magnetohydrodynamic equations by physicists and mathematicians due to its physical importance, complexity, rich phenomena, and mathematical challenges. Below we just mention some mathematical results on the full compressible magnetohydrodynamic equations (1.18) – (1.21) , we refer the interested reader to $[1,28,30,33]$ $[1,28,30,33]$ $[1,28,30,33]$ $[1,28,30,33]$ for many discussions on physical aspects. For the one-dimensional planar compressible magnetohydrodynamic equations, the existence of global smooth solutions with small initial data was shown in $[22]$. In $[11, 34]$ $[11, 34]$, Hoff and Tsyganov obtained the global existence and uniqueness of weak solutions with small initial energy. Under some technical conditions on the heat conductivity coefficient, Chen and Wang [\[2,](#page-23-2)[3,](#page-23-3)[36\]](#page-25-6) obtained the existence, uniqueness, and Lipschitz continuous dependence of global strong solutions with large initial data, see also [\[7,](#page-23-4) [8\]](#page-24-9) on the global existence and uniqueness of global weak solutions, and [\[6\]](#page-23-5) on the global existence and uniqueness of large strong solutions with large initial data and vaccum. For the full multi-dimensional compressible magnetohydrodynamic equations, the existence of variational solutions was established in [\[4,](#page-23-6) [9,](#page-24-10) [13\]](#page-24-11), while a unique local strong solution was obtained in [\[10\]](#page-24-12). The low Mach number limit is a very interesting topic in magnetohydrodynamics, see [\[20,](#page-24-13) [27,](#page-24-14) [29,](#page-24-15) [31\]](#page-25-7) in the framework of the so-called variational solutions, and [\[15–](#page-24-16)[17\]](#page-24-17) in the framework of the local smooth solutions with small density and temperature variations, or large density/entropy and temperature variations.

Before ending this introduction, we give some notations and recall some basic facts which will be frequently used throughout this paper.

(1) We denote by $\langle \cdot, \cdot \rangle$ the standard inner product in $L^2(\mathbb{T}^3)$ with $\langle f, f \rangle =$ $||f||^2$, by H^k the standard Sobolev space $W^{k,2}$ with norm $||\cdot||_k$. The notation $|| (A_1, A_2, \ldots, A_l) ||_k$ means the summation of $||A_i||_k$ from $i = 1$ to $i = l$. For a multiindex $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, we denote $\partial_x^{\alpha} = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \partial_{x_3}^{\alpha_3}$ and $|\alpha| = |\alpha_1| + |\alpha_2| + |\alpha_3|$. For an integer m, the symbol D_x^m denotes the summation of all terms ∂_x^{α} with the multi-index α satisfying $|\alpha| = m$. We use C_i , δ_i , K_i , and K to denote the constants which are independent of ϵ and may change from line to line. We also omit the spatial domain \mathbb{T}^3 in integrals for convenience.

(2) We shall frequently use the following Moser-type calculus inequalities (see [\[25\]](#page-24-18)):

(i) For
$$
f, g \in H^s(\mathbb{T}^3) \cap L^\infty(\mathbb{T}^3)
$$
 and $|\alpha| \le s$, $s > 3/2$, it holds that

$$
\|\partial_x^{\alpha}(fg)\| \le C_s(\|f\|_{L^\infty}\|D_x^s g\| + \|g\|_{L^\infty}\|D_x^s f\|).
$$
(1.37)

(ii) For $f \in H^s(\mathbb{T}^3), D_x^1 f \in L^\infty(\mathbb{T}^3), g \in H^{s-1}(\mathbb{T}^3) \cap L^\infty(\mathbb{T}^3)$ and $|\alpha| \leq s$, $s > 5/2$, it holds that

$$
\|\partial_x^{\alpha}(fg) - f\partial_x^{\alpha}g\| \leq C_s (\|D_x^1 f\|_{L^{\infty}} \|D_x^{s-1} g\| + \|g\|_{L^{\infty}} \|D_x^s f\|). \tag{1.38}
$$

(3) Let $s > 3/2$, $f \in C^{s}(\mathbb{T}^{3})$, and $u \in H^{s}(\mathbb{T}^{3})$, then for each multi-index α , $1 \leqslant |\alpha| \leqslant s$, we have ([\[25,](#page-24-18) [26\]](#page-24-19)):

$$
\|\partial_x^{\alpha}(f(u))\| \leq C(1 + \|u\|_{L^{\infty}}^{|\alpha|-1})\|u\|_{|\alpha|};\tag{1.39}
$$

moreover, if $f(0) = 0$, then ([\[12\]](#page-24-20))

$$
\|\partial_x^{\alpha}(f(u))\| \leqslant C(\|u\|_s)\|u\|_s. \tag{1.40}
$$

This paper is organized as follows. In Section [2,](#page-8-1) we utilize the primitive system (1.23) – (1.27) and the target system (1.29) – (1.32) to derive the error system and state the local existence of the solution. In Section [3](#page-11-0) we give the a priori energy estimates of the error system and present the proof of Theorem [1.2.](#page-5-1)

2. Derivation of the error system and local existence

In this section we first derive the error system from the original system [\(1.23\)](#page-4-0)– (1.27) and the limiting equations (1.29) – (1.32) , then we state the local existence of solution to this error system.

Setting $N^{\epsilon} = \rho^{\epsilon} - \rho^0$, $\mathbf{U}^{\epsilon} = \mathbf{u}^{\epsilon} - \mathbf{u}^0$, $\Theta^{\epsilon} = \theta^{\epsilon} - \theta^0$, $\mathbf{F}^{\epsilon} = \mathbf{E}^{\epsilon} - \mathbf{E}^0$, and $\mathbf{G}^{\epsilon} = \mathbf{H}^{\epsilon} - \mathbf{H}^0$, and utilizing the system (1.23) – (1.27) and the system (1.29) – (1.32) with (1.34) , we obtain that

$$
\partial_t N^{\epsilon} + (N^{\epsilon} + \rho^0) \operatorname{div} \mathbf{U}^{\epsilon} + (\mathbf{U}^{\epsilon} + \mathbf{u}^0) \cdot \nabla N^{\epsilon} = -N^{\epsilon} \operatorname{div} \mathbf{u}^0 - \nabla \rho^0 \cdot \mathbf{U}^{\epsilon}, \qquad (2.1)
$$

\n
$$
\partial_t \mathbf{U}^{\epsilon} + [(\mathbf{U}^{\epsilon} + \mathbf{u}^0) \cdot \nabla] \mathbf{U}^{\epsilon} + \nabla \Theta^{\epsilon} + \frac{\Theta^{\epsilon} + \theta^0}{N^{\epsilon} + \rho^0} \nabla N^{\epsilon} - \frac{1}{N^{\epsilon} + \rho^0} \operatorname{div} \Psi(\mathbf{U}^{\epsilon})
$$

\n
$$
= -(\mathbf{U}^{\epsilon} \cdot \nabla) \mathbf{u}^0 - \left[\frac{\Theta^{\epsilon} + \theta^0}{N^{\epsilon} + \rho^0} - \frac{\theta^0}{\rho^0} \right] \nabla \rho^0 + \left[\frac{1}{N^{\epsilon} + \rho^0} - \frac{1}{\rho^0} \right] \operatorname{div} \Psi(\mathbf{u}^0)
$$

\n
$$
- \frac{1}{\rho^0} \operatorname{curl} \mathbf{H}^0 \times \mathbf{H}^0 + \frac{1}{N^{\epsilon} + \rho^0} [\mathbf{F}^{\epsilon} + \mathbf{u}^0 \times \mathbf{G}^{\epsilon} + \mathbf{U}^{\epsilon} \times \mathbf{H}^0] \times \mathbf{H}^0
$$

\n
$$
+ \frac{1}{N^{\epsilon} + \rho^0} (\mathbf{U}^{\epsilon} \times \mathbf{G}^{\epsilon}) \times (\mathbf{G}^{\epsilon} + \mathbf{H}^0), \qquad (2.2)
$$

\n
$$
\partial_t \Theta^{\epsilon} + [(\mathbf{U}^{\epsilon} + \mathbf{u}^0) \cdot \nabla] \Theta^{\epsilon} + (\Theta^{\epsilon} + \theta^0) \operatorname{div} \mathbf{U}^{\epsilon} - \frac{\kappa}{N^{\epsilon} + \rho^0} \Delta \Theta^{\epsilon}
$$

\n
$$
= -(\mathbf{U}^{\epsilon} \cdot \nabla) \theta^0 - \Theta^{\
$$

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$$
+\frac{1}{N^{\epsilon}+\rho^{0}}|\mathbf{F}^{\epsilon}+\mathbf{U}^{\epsilon}\times\mathbf{G}^{\epsilon}|^{2}+\frac{1}{N^{\epsilon}+\rho^{0}}|\mathbf{u}^{0}\times\mathbf{G}^{\epsilon}+\mathbf{U}^{\epsilon}\times\mathbf{H}^{0}|^{2}+\frac{2}{N^{\epsilon}+\rho^{0}}(\mathbf{F}^{\epsilon}+\mathbf{U}^{\epsilon}\times\mathbf{G}^{\epsilon})\cdot[\operatorname{curl}\mathbf{H}^{0}+\mathbf{u}^{0}\times\mathbf{G}^{\epsilon}+\mathbf{U}^{\epsilon}\times\mathbf{H}^{0}]+\frac{2}{N^{\epsilon}+\rho^{0}}\operatorname{curl}\mathbf{H}^{0}\cdot(\mathbf{u}^{0}\times\mathbf{G}^{\epsilon}+\mathbf{U}^{\epsilon}\times\mathbf{H}^{0})+\left[\frac{1}{N^{\epsilon}+\rho^{0}}-\frac{1}{\rho^{0}}\right]|\operatorname{curl}\mathbf{H}^{0}|^{2}, \qquad (2.3)
$$

 $\epsilon \partial_t \mathbf{F}^{\epsilon} - \operatorname{curl} \mathbf{G}^{\epsilon} = -[\mathbf{F}^{\epsilon} + \mathbf{U}^{\epsilon} \times \mathbf{H}^0 + \mathbf{u}^0 \times \mathbf{G}^{\epsilon}] - \mathbf{U}^{\epsilon} \times \mathbf{G}^{\epsilon}$ $-\epsilon \partial_t \text{curl } \mathbf{H}^0 + \epsilon \partial_t (\mathbf{u}^0 \times \mathbf{H}^0)$ (2.4)

$$
\partial_t \mathbf{G}^{\epsilon} + \operatorname{curl} \mathbf{F}^{\epsilon} = 0, \quad \operatorname{div} \mathbf{G}^{\epsilon} = 0,
$$
\n(2.5)

with initial data

$$
(N^{\epsilon}, \mathbf{U}^{\epsilon}, \Theta^{\epsilon}, \mathbf{F}^{\epsilon}, \mathbf{G}^{\epsilon})|_{t=0} := (N_0^{\epsilon}, \mathbf{U}_0^{\epsilon}, \Theta_0^{\epsilon}, \mathbf{F}_0^{\epsilon}, \mathbf{G}_0^{\epsilon})
$$

=
$$
(\rho_0^{\epsilon} - \rho_0^0, \mathbf{u}_0^{\epsilon} - \mathbf{u}_0^0, \theta_0^{\epsilon} - \theta_0^0, \mathbf{E}_0^{\epsilon} - (\text{curl } \mathbf{H}_0^0 - \mathbf{u}_0^0 \times \mathbf{H}_0^0), \mathbf{H}_0^{\epsilon} - \mathbf{H}_0^0).
$$
 (2.6)

Denote

$$
\mathbf{W}^{\epsilon} = \begin{pmatrix} N^{\epsilon} \\ \mathbf{U}^{\epsilon} \\ \Theta^{\epsilon} \\ \mathbf{F}^{\epsilon} \\ \mathbf{G}^{\epsilon} \end{pmatrix}, \quad \mathbf{W}_{0}^{\epsilon} = \begin{pmatrix} N_{0}^{\epsilon} \\ \mathbf{U}_{0}^{\epsilon} \\ \Theta_{0}^{\epsilon} \\ \mathbf{F}_{0}^{\epsilon} \\ \mathbf{G}^{\epsilon} \end{pmatrix}, \quad \mathbf{D}^{\epsilon} = \begin{pmatrix} \mathbf{I}_{5} & \mathbf{0} \\ \mathbf{0} & \left(\epsilon \mathbf{I}_{3} & \mathbf{0}\right) \\ \mathbf{0} & \mathbf{I}_{3} \end{pmatrix},
$$
\n
$$
\mathbf{A}_{i}^{\epsilon} = \begin{pmatrix} (\mathbf{U}^{\epsilon} + \mathbf{u}^{0})_{i} & (N^{\epsilon} + \rho^{0})e_{i}^{T} & 0 \\ \frac{\Theta^{\epsilon} + \theta^{0}}{N^{\epsilon} + \rho^{0}}e_{i} & (\mathbf{U}^{\epsilon} + \mathbf{u}^{0})_{i}\mathbf{I}_{3} & e_{i} \\ 0 & (\Theta^{\epsilon} + \theta^{0})e_{i}^{T} & (\mathbf{U}^{\epsilon} + \mathbf{u}^{0})_{i} \end{pmatrix}, \quad \mathbf{A}_{i}^{\epsilon} = \begin{pmatrix} \mathbf{0} & \mathbf{B}_{i} \\ \frac{\Theta^{\epsilon} + \theta^{0}}{N^{\epsilon} + \rho^{0}}e_{i} & (\mathbf{U}^{\epsilon} + \mathbf{u}^{0})_{i} & 0 \\ 0 & \left(\frac{\Theta}{B_{i}^{T}} & \mathbf{0}\right) \\ 0 & \frac{\Theta}{B_{i}^{T}} & 0 \end{pmatrix},
$$
\n
$$
\mathbf{A}_{i}^{\epsilon} = \begin{pmatrix} 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{\mu}{N^{\epsilon} + \rho^{0}}(e_{i}e_{j}^{T}\mathbf{I}_{3} + e_{i}^{T}e_{j}) + \frac{\lambda}{N^{\epsilon} + \rho^{0}}e_{j}^{T}e_{i} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{\kappa}{N^{\epsilon} + \rho^{0}}e_{i}e
$$

$$
\mathbf{S}^{\epsilon}(\mathbf{W}^{\epsilon}) = \begin{pmatrix} -N^{\epsilon} \operatorname{div} \mathbf{u}^{0} - \nabla \rho^{0} \cdot \mathbf{U}^{\epsilon} \\ \mathbf{R}_{1}^{\epsilon} \\ \mathbf{R}_{2}^{\epsilon} \\ \mathbf{R}_{3}^{\epsilon} \\ \mathbf{0} \end{pmatrix},
$$

where \mathbf{R}_1^{ϵ} , \mathbf{R}_2^{ϵ} , and \mathbf{R}_3^{ϵ} denote the right-hand side of [\(2.2\)](#page-8-2), [\(2.3\)](#page-9-1), and [\(2.4\)](#page-9-2), respectively. (e_1, e_2, e_3) is the canonical basis of \mathbb{R}^3 , \mathbf{I}_d $(d = 3, 5)$ is a $d \times d$ unit matrix, y_i denotes the *i*-th component of $y \in \mathbb{R}^3$, and

$$
B_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
$$

Using these notations we can rewrite the problem (2.1) – (2.6) in the form

$$
\begin{cases}\n\mathbf{D}^{\epsilon}\partial_{t}\mathbf{W}^{\epsilon} + \sum_{i=1}^{3} \mathbf{A}_{i}^{\epsilon}\mathbf{W}_{x_{i}}^{\epsilon} + \sum_{i,j=1}^{3} \mathbf{A}_{ij}^{\epsilon}\mathbf{W}_{x_{i}x_{j}}^{\epsilon} = \mathbf{S}^{\epsilon}(\mathbf{W}^{\epsilon}), \\
\mathbf{W}^{\epsilon}|_{t=0} = \mathbf{W}_{0}^{\epsilon}.\n\end{cases}
$$
\n(2.7)

It is not difficult to see that the system for W^{ϵ} in [\(2.7\)](#page-10-0) can be reduced to a quasilinear symmetric hyperbolic-parabolic one. In fact, if we introduce

$$
\mathbf{A}^\epsilon = \left(\begin{array}{cccc} \left(\begin{array}{ccc} \frac{\Theta^\epsilon + \theta^0}{(N^\epsilon + \rho^0)^2} & \mathbf{0} & 0 \\ \mathbf{0} & \mathbf{I}_3 & \mathbf{0} \\ 0 & \mathbf{0} & \frac{1}{\Theta^\epsilon + \theta^0} \end{array}\right) & \mathbf{0} \\ 0 & \mathbf{0} & \mathbf{I}_6 \end{array}\right),
$$

which is positively definite when $||N^{\epsilon}||_{L^{\infty}_{T}L^{\infty}_{x}} \leq \hat{\rho}/2$ and $||\Theta^{\epsilon}||_{L^{\infty}_{T}L^{\infty}_{x}} \leq \hat{\theta}/2$, then $\tilde{\mathbf{A}}_0^{\epsilon} = \mathbf{A}^{\epsilon} \mathbf{D}^{\epsilon}$ and $\tilde{\mathbf{A}}_i^{\epsilon} = \mathbf{A}^{\epsilon} \mathbf{A}_i^{\epsilon}$ are positive symmetric on $[0, T]$ for all $1 \leq i \leq 3$. Moreover, the assumptions that $\mu > 0$, $2\mu + 3\lambda > 0$, and $\kappa > 0$ imply that

$$
\mathcal{A}^{\epsilon}=\sum_{i,j=1}^{3}\mathbf{A}^{\epsilon}\mathbf{A}_{ij}^{\epsilon}\mathbf{W}_{x_{i}x_{j}}^{\epsilon}
$$

is an elliptic operator. Thus, we can apply the result of Vol'pert and Hudiaev [\[37\]](#page-25-0) to obtain the following local existence for the problem [\(2.7\)](#page-10-0).

Proposition 2.1. Let $s > 7/2$ be an integer and $(\rho_0^0, \mathbf{u}_0^0, \theta_0^0, \mathbf{H}_0^0)$ satisfy the condi-tions in Proposition [1.1.](#page-4-5) Assume that the initial data $(N_0^{\epsilon}, \mathbf{U}_0^{\epsilon}, \Theta_0^{\epsilon}, \mathbf{F}_0^{\epsilon}, \mathbf{G}_0^{\epsilon})$ satisfy N_0^{ϵ} , \mathbf{U}_0^{ϵ} , Θ_0^{ϵ} , \mathbf{G}_0^{ϵ} \in $H^s(\mathbb{T}^3)$, div \mathbf{G}_0^{ϵ} = 0, and

$$
\|N^\epsilon_0\|_s\leqslant \delta,\quad \|\Theta^\epsilon_0\|_s\leqslant \delta
$$

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for some constant $\delta > 0$. Then there exist positive constants T^{ϵ} (0 $\lt T^{\epsilon}$ \leq $+\infty$) and K such that the Cauchy problem [\(2.7\)](#page-10-0) has a unique classical solution $(N^{\epsilon}, \mathbf{U}^{\epsilon}, \Theta^{\epsilon}, \mathbf{F}^{\epsilon}, \mathbf{G}^{\epsilon})$ satisfying div $\mathbf{G}^{\epsilon} = 0$ and

$$
N^{\epsilon}, \mathbf{F}^{\epsilon}, \mathbf{G}^{\epsilon} \in C^{l}([0, T^{\epsilon}), H^{s-l}), \mathbf{U}^{\epsilon}, \Theta^{\epsilon} \in C^{l}([0, T^{\epsilon}), H^{s-2l}), \ l = 0, 1;
$$

$$
\|(N^{\epsilon}, \mathbf{U}^{\epsilon}, \Theta^{\epsilon}, \mathbf{F}^{\epsilon}, \mathbf{G}^{\epsilon})(t)\|_{s} \leq K\delta, \ t \in [0, T^{\epsilon}).
$$

Note that for smooth solutions, the electromagnetic fluid system (1.23) – (1.27) with the initial data [\(1.28\)](#page-4-6) are equivalent to [\(2.1\)](#page-8-0)–[\(2.6\)](#page-9-3) or [\(2.7\)](#page-10-0) on $[0, T]$, T < $\min\{T^{\epsilon},T_*\}$. Therefore, in order to obtain the convergence of electromagnetic fluid equations (1.23) – (1.27) to the full compressible magnetohydrodynamic equations (1.29) – (1.32) , we only need to establish uniform decay estimates with respect to the parameter ϵ of the solution to the error system [\(2.7\)](#page-10-0). This will be achieved by the elaborate energy method presented in next section.

3. Uniform energy estimates and proof of Theorem [1.2](#page-5-1)

In this section we derive uniform decay estimates with respect to the parameter ϵ of the solution to the problem [\(2.7\)](#page-10-0) and justify rigorously the convergence of electromagnetic fluid system to the full compressible magnetohydrodynamic equations [\(1.29\)](#page-4-2)–[\(1.32\)](#page-4-3). Here we adapt and modify some techniques developed in [\[18,](#page-24-5)[32\]](#page-25-3) and put main efforts on the estimates of higher order nonlinear terms.

We first establish the convergence rate of the error equations by establishing the a priori estimates uniformly in ϵ . For presentation conciseness, we define

$$
\begin{aligned} \|\mathcal{E}^{\epsilon}(t)\|_{s}^{2} &:= \|(N^{\epsilon}, \mathbf{U}^{\epsilon}, \Theta^{\epsilon}, \mathbf{G}^{\epsilon})(t)\|_{s}^{2}, \\ \|\mathcal{E}^{\epsilon}(t)\|_{s}^{2} &:= \|\mathcal{E}^{\epsilon}(t)\|_{s}^{2} + \epsilon \|\mathbf{F}^{\epsilon}(t)\|_{s}^{2}, \\ \|\mathcal{E}^{\epsilon}\|_{s,T} &:= \sup_{0 < t \leq T} \|\mathcal{E}^{\epsilon}(t)\|_{s}. \end{aligned}
$$

The crucial estimate of our paper is the following decay result on the error system (2.1) – (2.5) .

Proposition 3.1. Let $s > 7/2$ be an integer and assume that the initial data $(N_0^{\epsilon},\mathbf{U}_0^{\epsilon},\Theta_0^{\epsilon},\mathbf{F}_0^{\epsilon},\mathbf{G}_0^{\epsilon})$ satisfy

$$
\|(N_0^{\epsilon}, \mathbf{U}_0^{\epsilon}, \Theta_0^{\epsilon}, \mathbf{G}_0^{\epsilon})\|_{s}^{2} + \epsilon \|\mathbf{F}_0^{\epsilon}\|_{s}^{2} = \|\mathcal{E}^{\epsilon}(t=0)\|_{s}^{2} \le M_0 \epsilon^{2}
$$
(3.1)

for sufficiently small ϵ and some constant $M_0 > 0$ independent of ϵ . Then, for any $T_0 \in (0, T_*)$, there exist two constants $M_1 > 0$ and $\epsilon_1 > 0$ depending only on T_0 , such that for all $\epsilon \in (0, \epsilon_1]$, it holds that $T^{\epsilon} \geq T_0$ and the solution $(N^{\epsilon}, \mathbf{U}^{\epsilon}, \Theta^{\epsilon}, \mathbf{F}^{\epsilon}, \mathbf{G}^{\epsilon})$ of the problem (2.1) – (2.6) , well-defined in $[0, T_0]$, enjoys that

$$
\|\mathcal{E}^{\epsilon}\|_{s,T_0} \leqslant M_1\epsilon. \tag{3.2}
$$

Once this proposition is established, the proof of Theorem [1.2](#page-5-1) is a direct procedure. In fact, we have

Proof of Theorem [1.2.](#page-5-1) Suppose that Proposition [3.1](#page-11-1) holds. According to the definition of the error functions $(N^{\epsilon}, \mathbf{U}^{\epsilon}, \Theta^{\epsilon}, \mathbf{F}^{\epsilon}, \mathbf{G}^{\epsilon})$ and the regularity of $(\rho^0, \mathbf{u}^0, \theta^0, \mathbf{H}^0)$, the error system (2.1) – (2.5) and the primitive system (1.23) – (1.27) are equivalent on $[0, T]$ for some $T > 0$. Therefore the assumption [\(1.35\)](#page-5-2) in Theorem [1.2](#page-5-1) imply the assumption (3.1) in Proposition [3.1,](#page-11-1) and hence (3.2) implies (1.36) .

Therefore, our main goal next is to prove Proposition [3.1](#page-11-1) which can be approached by the following a priori estimates. For some given $\hat{T} < 1$ and any $\tilde{T} < \hat{T}$ independent of ϵ , we denote $T \equiv T_{\epsilon} = \min{\lbrace \tilde{T}, T^{\epsilon} \rbrace}$.

Lemma 3.2. Let the assumptions in Proposition [3.1](#page-11-1) hold. Then, for all $0 < t < T$ and sufficiently small ϵ , there exist two positive constants δ_1 and δ_2 , such that

$$
\|\mathcal{E}^{\epsilon}(t)\|_{s}^{2} + \int_{0}^{t} \left\{\delta_{1} \|\nabla \mathbf{U}^{\epsilon}\|_{s}^{2} + \delta_{2} \|\nabla \Theta^{\epsilon}\|_{s}^{2} + \frac{1}{4} \|\mathbf{F}^{\epsilon}\|_{s}^{2}\right\}(\tau) d\tau
$$

$$
\leq \|\mathcal{E}^{\epsilon}(t=0)\|_{s}^{2} + C \int_{0}^{t} \left\{\left(\|\mathcal{E}^{\epsilon}\|_{s}^{2s} + \|\mathcal{E}^{\epsilon}\|_{s}^{2} + 1\right) \|\mathcal{E}^{\epsilon}\|_{s}^{2}\right\}(\tau) d\tau + C\epsilon^{2}.
$$
 (3.3)

Proof. Let $0 \leq \alpha \leq s$. In the following arguments the commutators will disappear in the case of $|\alpha| = 0$.

Applying the operator ∂_x^{α} to [\(2.1\)](#page-8-0), multiplying the resulting equation by $\partial_x^{\alpha} N^{\epsilon}$, and integrating over \mathbb{T}^3 , we obtain that

$$
\frac{1}{2} \frac{d}{dt} \langle \partial_x^{\alpha} N^{\epsilon}, \partial_x^{\alpha} N^{\epsilon} \rangle = - \langle \partial_x^{\alpha} ([(\mathbf{U}^{\epsilon} + \mathbf{u}^{0}) \cdot \nabla] N^{\epsilon}), \partial_x^{\alpha} N^{\epsilon} \rangle \n- \langle \partial_x^{\alpha} ((N^{\epsilon} + \rho^{0}) \operatorname{div} \mathbf{U}^{\epsilon}), \partial_x^{\alpha} N^{\epsilon} \rangle \n+ \langle \partial_x^{\alpha} ((-N^{\epsilon} \operatorname{div} \mathbf{u}^{0} - \nabla \rho^{0} \cdot \mathbf{U}^{\epsilon}), \partial_x^{\alpha} N^{\epsilon} \rangle. \tag{3.4}
$$

Next we bound every term on the right-hand side of [\(3.4\)](#page-12-0). By the regularity of u^0 , Cauchy-Schwarz's inequality, and Sobolev's imbedding, we have

$$
\langle \partial_x^{\alpha} \langle [(\mathbf{U}^{\epsilon} + \mathbf{u}^{0}) \cdot \nabla] N^{\epsilon} \rangle, \partial_x^{\alpha} N^{\epsilon} \rangle
$$

\n
$$
= \langle [(\mathbf{U}^{\epsilon} + \mathbf{u}^{0}) \cdot \nabla] \partial_x^{\alpha} N^{\epsilon}, \partial_x^{\alpha} N^{\epsilon} \rangle + \langle \mathcal{H}^{(1)}, \partial_x^{\alpha} N^{\epsilon} \rangle
$$

\n
$$
= -\frac{1}{2} \langle \text{div} (\mathbf{U}^{\epsilon} + \mathbf{u}^{0}) \partial_x^{\alpha} N^{\epsilon}, \partial_x^{\alpha} N^{\epsilon} \rangle + \langle \mathcal{H}^{(1)}, \partial_x^{\alpha} N^{\epsilon} \rangle
$$

\n
$$
\leq C (\|\mathcal{E}^{\epsilon}(t)\|_{s} + 1) \|\partial_x^{\alpha} N^{\epsilon}\|^{2} + \|\mathcal{H}^{(1)}\|^{2}, \tag{3.5}
$$

where the commutator

$$
\mathcal{H}^{(1)} = \partial_x^{\alpha} \left(\left[(\mathbf{U}^{\epsilon} + \mathbf{u}^0) \cdot \nabla \right] N^{\epsilon} \right) - \left[(\mathbf{U}^{\epsilon} + \mathbf{u}^0) \cdot \nabla \right] \partial_x^{\alpha} N^{\epsilon}
$$

can be bounded as follows:

$$
\|\mathcal{H}^{(1)}\| \leq C(\|D_x^1(\mathbf{U}^{\epsilon} + \mathbf{u}^0)\|_{L^{\infty}}\|D_x^s N^{\epsilon}\| + \|D_x^1 N^{\epsilon}\|_{L^{\infty}}\|D_x^{s-1}(\mathbf{U}^{\epsilon} + \mathbf{u}^0)\|)
$$

$$
\leq C(\|\mathcal{E}^{\epsilon}(t)\|_{s}^{2} + \|\mathcal{E}^{\epsilon}(t)\|_{s}).
$$
 (3.6)

Here we have used the Moser-type and Cauchy-Schwarz's inequalities, the regularity of u^0 and Sobolev's imbedding.

Similarly, the second term on the right-hand side of [\(3.4\)](#page-12-0) can bounded as follows.

$$
\langle \partial_x^{\alpha} ((N^{\epsilon} + \rho^0) \text{div } \mathbf{U}^{\epsilon}), \partial_x^{\alpha} N^{\epsilon} \rangle
$$

= $\langle (N^{\epsilon} + \rho^0) \partial_x^{\alpha} \text{div } \mathbf{U}^{\epsilon}, \partial_x^{\alpha} N^{\epsilon} \rangle + \langle \mathcal{H}^{(2)}, \partial_x^{\alpha} N^{\epsilon} \rangle$
 $\leq \eta_1 \| \nabla \partial_x^{\alpha} \mathbf{U}^{\epsilon} \|^2 + C_{\eta_1} \| \partial_x^{\alpha} N^{\epsilon} \|^2 + \| \mathcal{H}^{(2)} \|^2$ (3.7)

for any $\eta_1 > 0$, where the commutator

$$
\mathcal{H}^{(2)} = \partial_x^{\alpha}((N^{\epsilon} + \rho^0)\mathrm{div}\,\mathbf{U}^{\epsilon}) - (N^{\epsilon} + \rho^0)\partial_x^{\alpha}\mathrm{div}\,\mathbf{U}^{\epsilon}
$$

can be estimated by

$$
\|\mathcal{H}^{(2)}\| \leq C(\|D_x^1(N^{\epsilon}+\rho^0)\|_{L^{\infty}}\|D_x^s\mathbf{U}^{\epsilon}\| + \|D_x^1\mathbf{U}^{\epsilon}\|_{L^{\infty}}\|D_x^{s-1}(N^{\epsilon}+\rho^0)\|)
$$

$$
\leq C(\|\mathcal{E}^{\epsilon}(t)\|_{s}^{2} + \|\mathcal{E}^{\epsilon}(t)\|_{s}).
$$
 (3.8)

By the Moser-type and Cauchy-Schwarz's inequalities, and the regularity of \mathbf{u}^0 and ρ^0 , we can control the third term on the right-hand side of [\(3.4\)](#page-12-0) by

$$
\left| \left\langle \partial_x^{\alpha} (-N^{\epsilon} \operatorname{div} \mathbf{u}^0 - \nabla \rho^0 \cdot \mathbf{U}^{\epsilon}), \partial_x^{\alpha} N^{\epsilon} \right\rangle \right| \leqslant C (\|\partial_x^{\alpha} N^{\epsilon}\|^2 + \|\partial_x^{\alpha} \mathbf{U}^{\epsilon}\|^2). \tag{3.9}
$$

Substituting (3.5) – (3.9) into (3.4) , we conclude that

$$
\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \langle \partial_x^{\alpha} N^{\epsilon}, \partial_x^{\alpha} N^{\epsilon} \rangle \leq \eta_1 \| \nabla \partial_x^{\alpha} \mathbf{U}^{\epsilon} \|^2 + C_{\eta_1} \| \partial_x^{\alpha} N^{\epsilon} \|^2 \n+ C \big[(\|\mathcal{E}^{\epsilon}(t)\|_{s} + 1) \| \partial_x^{\alpha} N^{\epsilon} \|^2 + \|\mathcal{E}^{\epsilon}(t) \|^4_{s} + \epsilon^2 \big].
$$
\n(3.10)

Applying the operator ∂_x^{α} to [\(2.2\)](#page-8-2), multiplying the resulting equation by $\partial_x^{\alpha} \mathbf{U}^{\epsilon}$, and integrating over \mathbb{T}^3 , we obtain that

$$
\begin{split} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\langle\partial_x^{\alpha}\mathbf{U}^{\epsilon},\partial_x^{\alpha}\mathbf{U}^{\epsilon}\rangle+\langle\partial_x^{\alpha}([(\mathbf{U}^{\epsilon}+\mathbf{u}^{0})\cdot\nabla]\mathbf{U}^{\epsilon}),\partial_x^{\alpha}\mathbf{U}^{\epsilon}\rangle\\ &+\langle\partial_x^{\alpha}\nabla\Theta^{\epsilon},\partial_x^{\alpha}\mathbf{U}^{\epsilon}\rangle+\left\langle\partial_x^{\alpha}\left(\frac{\Theta^{\epsilon}+\theta^0}{N^{\epsilon}+\rho^0}\nabla N^{\epsilon}\right),\partial_x^{\alpha}\mathbf{U}^{\epsilon}\right\rangle\\ &-\left\langle\partial_x^{\alpha}\left(\frac{1}{N^{\epsilon}+\rho^0}\mathrm{div}\,\Psi(\mathbf{U}^{\epsilon})\right),\partial_x^{\alpha}\mathbf{U}^{\epsilon}\right\rangle\\ =&-\langle\partial_x^{\alpha}\left[(\mathbf{U}^{\epsilon}\cdot\nabla)\mathbf{u}^0\right],\partial_x^{\alpha}\mathbf{U}^{\epsilon}\rangle-\left\langle\partial_x^{\alpha}\left\{\frac{1}{\rho^0}\mathrm{curl}\,\mathbf{H}^0\times\mathbf{H}^0\right\},\partial_x^{\alpha}\mathbf{U}^{\epsilon}\right\rangle\\ &+\left\langle\partial_x^{\alpha}\left\{\left[\frac{\Theta^{\epsilon}+\theta^0}{N^{\epsilon}+\rho^0}-\frac{\theta^0}{\rho^0}\right]\nabla\rho^0\right\},\partial_x^{\alpha}\mathbf{U}^{\epsilon}\right\rangle\end{split}
$$

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$$
+\left\langle \partial_x^{\alpha} \left\{ \left[\frac{1}{N^{\epsilon} + \rho^0} - \frac{1}{\rho^0} \right] \mathrm{div} \Psi(\mathbf{u}^0) \right\}, \partial_x^{\alpha} \mathbf{U}^{\epsilon} \right\rangle + \left\langle \partial_x^{\alpha} \left\{ \frac{1}{N^{\epsilon} + \rho^0} [\mathbf{F}^{\epsilon} + \mathbf{u}^0 \times \mathbf{G}^{\epsilon} + \mathbf{U}^{\epsilon} \times \mathbf{H}^0] \times \mathbf{H}^0 \right\}, \partial_x^{\alpha} \mathbf{U}^{\epsilon} \right\rangle + \left\langle \partial_x^{\alpha} \left\{ \frac{1}{N^{\epsilon} + \rho^0} [\mathbf{F}^{\epsilon} + \mathbf{u}^0 \times \mathbf{G}^{\epsilon} + \mathbf{U}^{\epsilon} \times \mathbf{H}^0] \times \mathbf{G}^{\epsilon} \right\}, \partial_x^{\alpha} \mathbf{U}^{\epsilon} \right\rangle + \left\langle \partial_x^{\alpha} \left\{ \frac{1}{N^{\epsilon} + \rho^0} (\mathbf{U}^{\epsilon} \times \mathbf{G}^{\epsilon}) \times (\mathbf{G}^{\epsilon} + \mathbf{H}^0) \right\}, \partial_x^{\alpha} \mathbf{U}^{\epsilon} \right\rangle := \sum_{i=1}^7 \mathcal{R}^{(i)}.
$$
 (3.11)

We first bound the terms on the left-hand side of [\(3.11\)](#page-14-0). Similar to [\(3.5\)](#page-12-1) we infer that

$$
\langle \partial_x^{\alpha} ([(\mathbf{U}^{\epsilon} + \mathbf{u}^{0}) \cdot \nabla] \mathbf{U}^{\epsilon}), \partial_x^{\alpha} \mathbf{U}^{\epsilon} \rangle
$$

\n
$$
= \langle [(\mathbf{U}^{\epsilon} + \mathbf{u}^{0}) \cdot \nabla] \partial_x^{\alpha} \mathbf{U}^{\epsilon}, \partial_x^{\alpha} \mathbf{U}^{\epsilon} \rangle + \langle \mathcal{H}^{(3)}, \partial_x^{\alpha} \mathbf{U}^{\epsilon} \rangle
$$

\n
$$
= -\frac{1}{2} \langle \text{div} (\mathbf{U}^{\epsilon} + \mathbf{u}^{0}) \partial_x^{\alpha} \mathbf{U}^{\epsilon}, \partial_x^{\alpha} \mathbf{U}^{\epsilon} \rangle + \langle \mathcal{H}^{(3)}, \partial_x^{\alpha} \mathbf{U}^{\epsilon} \rangle
$$

\n
$$
\leq C (\|\mathcal{E}^{\epsilon}(t)\|_{s} + 1) \|\partial_x^{\alpha} \mathbf{U}^{\epsilon}\|^{2} + \|\mathcal{H}^{(3)}\|^{2}, \tag{3.12}
$$

where the commutator

$$
\mathcal{H}^{(3)} = \partial_x^{\alpha} ([(\mathbf{U}^{\epsilon} + \mathbf{u}^0) \cdot \nabla] \mathbf{U}^{\epsilon}) - [(\mathbf{U}^{\epsilon} + \mathbf{u}^0) \cdot \nabla] \partial_x^{\alpha} \mathbf{U}^{\epsilon}
$$

can be bounded by

$$
\|\mathcal{H}^{(3)}\| \leq C(\|D_x^1(\mathbf{U}^{\epsilon} + \mathbf{u}^0)\|_{L^{\infty}}\|D_x^s \mathbf{U}^{\epsilon}\| + \|D_x^1 \mathbf{U}^{\epsilon}\|_{L^{\infty}}\|D_x^{s-1}(\mathbf{U}^{\epsilon} + \mathbf{u}^0)\|)
$$

$$
\leq C(\|\mathcal{E}^{\epsilon}(t)\|_{s}^{2} + \|\mathcal{E}^{\epsilon}(t)\|_{s}).
$$
 (3.13)

By Holder's inequality, we have

$$
\langle \partial_x^{\alpha} \nabla \Theta^{\epsilon}, \partial_x^{\alpha} \mathbf{U}^{\epsilon} \rangle \leq \eta_2 \| \partial_x^{\alpha} \nabla \Theta^{\epsilon} \|^2 + C_{\eta_2} \| \partial_x^{\alpha} \mathbf{U}^{\epsilon} \|^2 \tag{3.14}
$$

for any $\eta_2 > 0$. For the fourth term on the left-hand side of [\(3.11\)](#page-14-0), similar to [\(3.7\)](#page-13-1), we integrate by parts to deduce that

$$
\left\langle \partial_x^{\alpha} \left(\frac{\Theta^{\epsilon} + \theta^0}{N^{\epsilon} + \rho^0} \nabla N^{\epsilon} \right), \partial_x^{\alpha} \mathbf{U}^{\epsilon} \right\rangle
$$

\n
$$
= \left\langle \frac{\Theta^{\epsilon} + \theta^0}{N^{\epsilon} + \rho^0} \partial_x^{\alpha} \nabla N^{\epsilon}, \partial_x^{\alpha} \mathbf{U}^{\epsilon} \right\rangle + \left\langle \mathcal{H}^{(4)}, \partial_x^{\alpha} \mathbf{U}^{\epsilon} \right\rangle
$$

\n
$$
= -\left\langle \partial_x^{\alpha} N^{\epsilon}, \text{div} \left(\frac{\Theta^{\epsilon} + \theta^0}{N^{\epsilon} + \rho^0} \partial_x^{\alpha} \mathbf{U}^{\epsilon} \right) \right\rangle + \left\langle \mathcal{H}^{(4)}, \partial_x^{\alpha} \mathbf{U}^{\epsilon} \right\rangle
$$

\n
$$
\leq \eta_3 \| \nabla \partial_x^{\alpha} \mathbf{U}^{\epsilon} \|^2 + C_{\eta_3} \| \partial_x^{\alpha} N^{\epsilon} \|^2 + C \| \mathcal{E}^{\epsilon}(t) \|_{s}^{4} + \| \mathcal{H}^{(4)} \|^2
$$
(3.15)

for any $\eta_3 > 0$, where the commutator

$$
\mathcal{H}^{(4)} = \partial_x^{\alpha} \left(\frac{\Theta^{\epsilon} + \theta^0}{N^{\epsilon} + \rho^0} \nabla N^{\epsilon} \right) - \frac{\Theta^{\epsilon} + \theta^0}{N^{\epsilon} + \rho^0} \partial_x^{\alpha} \nabla N^{\epsilon}
$$

can be bounded as follows by using [\(1.38\)](#page-7-0) and [\(1.39\)](#page-8-3), and Cauchy-Schwarz's inequality:

$$
\|\mathcal{H}^{(4)}\| \leq C \left(\left\| D_x^1 \left(\frac{\Theta^{\epsilon} + \theta^0}{N^{\epsilon} + \rho^0} \right) \right\|_{L^{\infty}} \| D_x^s N^{\epsilon} \| + \| D_x^1 N^{\epsilon} \|_{L^{\infty}} \left\| D_x^{s-1} \left(\frac{\Theta^{\epsilon} + \theta^0}{N^{\epsilon} + \rho^0} \right) \right\| \right) \leq C (\|\mathcal{E}^{\epsilon}(t)\|_{s}^{2(s+1)} + \|\mathcal{E}^{\epsilon}(t)\|_{s}^{2} + \|\mathcal{E}^{\epsilon}(t)\|_{s}).
$$
\n(3.16)

For the fifth term on the left-hand side of [\(3.11\)](#page-14-0), we integrate by parts to deduce

$$
- \left\langle \partial_x^{\alpha} \left(\frac{1}{N^{\epsilon} + \rho^0} \text{div} \, \Psi(\mathbf{U}^{\epsilon}) \right), \partial_x^{\alpha} \mathbf{U}^{\epsilon} \right\rangle
$$

= $-\left\langle \frac{1}{N^{\epsilon} + \rho^0} \partial_x^{\alpha} \text{div} \, \Psi(\mathbf{U}^{\epsilon}), \partial_x^{\alpha} \mathbf{U}^{\epsilon} \right\rangle - \left\langle \mathcal{H}^{(5)}, \partial_x^{\alpha} \mathbf{U}^{\epsilon} \right\rangle, \tag{3.17}$

where the commutator

$$
\mathcal{H}^{(5)} = \partial_x^{\alpha} \left(\frac{1}{N^{\epsilon} + \rho^0} \mathrm{div} \, \Psi(\mathbf{U}^{\epsilon}) \right) - \frac{1}{N^{\epsilon} + \rho^0} \partial_x^{\alpha} \mathrm{div} \, \Psi(\mathbf{U}^{\epsilon}).
$$

By the Moser-type and Cauchy-Schwarz inequalities, the regularity of ρ^0 and the positivity of $N^{\epsilon} + \rho_0$, the definition of $\Psi(\mathbf{U}^{\epsilon})$ and Sobolev's imbedding, we find that

$$
\begin{split}\n&\left|\langle \mathcal{H}^{(5)}, \partial_x^{\alpha} \mathbf{U}^{\epsilon} \rangle\right| \\
&\leq \|\mathcal{H}^{(5)}\| \cdot \|\partial_x^{\alpha} \mathbf{U}^{\epsilon}\| \\
&\leq C \left(\left\| D_x^1 \left(\frac{1}{N^{\epsilon} + \rho^0} \right) \right\|_{L^{\infty}} \|\text{div}\,\Psi(\mathbf{U}^{\epsilon})\|_{s-1} + \|\text{div}\,\Psi(\mathbf{U}^{\epsilon})\|_{L^{\infty}} \left\| \frac{1}{N^{\epsilon} + \rho^0} \right\|_{s} \right) \|\partial_x^{\alpha} \mathbf{U}^{\epsilon}\| \\
&\leq \eta_4 \|\nabla \mathbf{U}^{\epsilon}\|_{s}^2 + C_{\eta_4} (\|\mathcal{E}^{\epsilon}(t)\|_{s}^2 + 1) (\|\partial_x^{\alpha} \mathbf{U}^{\epsilon}\|^2 + \|\partial_x^{\alpha} N^{\epsilon}\|^2 + \|\mathcal{E}^{\epsilon}(t)\|_{s}^s)\n\end{split} \tag{3.18}
$$

for any $\eta_4 > 0$, where we have used the assumption $s > 3/2 + 2$ and the imbedding $H^l(\mathbb{T}^3) \hookrightarrow L^{\infty}(\mathbb{R}^3)$ for $l > 3/2$. By virtue of the definition of $\Psi(\mathbf{U}^{\epsilon})$ and partial integrations, the first term on the right-hand side of [\(3.17\)](#page-15-0) can be rewritten as

$$
-\left\langle \frac{1}{N^{\epsilon} + \rho^{0}} \partial_{x}^{\alpha} \operatorname{div} \Psi(U^{\epsilon}), \partial_{x}^{\alpha} U^{\epsilon} \right\rangle
$$

\n
$$
= 2\mu \left\langle \frac{1}{N^{\epsilon} + \rho^{0}} \partial_{x}^{\alpha} \mathbb{D}(U^{\epsilon}), \partial_{x}^{\alpha} \mathbb{D}(U^{\epsilon}) \right\rangle + \lambda \left\langle \frac{1}{N^{\epsilon} + \rho^{0}} \partial_{x}^{\alpha} \operatorname{div} U^{\epsilon}, \partial_{x}^{\alpha} \operatorname{div} U^{\epsilon} \right\rangle
$$

\n
$$
+ 2\mu \left\langle \nabla \left(\frac{1}{N^{\epsilon} + \rho^{0}} \right) \otimes \partial_{x}^{\alpha} U^{\epsilon}, \partial_{x}^{\alpha} \mathbb{D}(U^{\epsilon}) \right\rangle
$$

\n
$$
+ \lambda \left\langle \nabla \left(\frac{1}{N^{\epsilon} + \rho^{0}} \right) \cdot \partial_{x}^{\alpha} U^{\epsilon}, \partial_{x}^{\alpha} \operatorname{div} U^{\epsilon} \right\rangle
$$

\n
$$
:= \sum_{i=1}^{4} \mathcal{I}^{(i)}.
$$
 (3.19)

Recalling the facts that $\mu > 0$ and $2\mu + 3\lambda > 0$, and the positivity of $N^{\epsilon} + \rho_0$, the first two terms $\mathcal{I}^{(1)}$ and $\mathcal{I}^{(2)}$ can be bounded as follows:

$$
\mathcal{I}^{(1)} + \mathcal{I}^{(2)} = \int \frac{1}{N^{\epsilon} + \rho^{0}} \{ 2\mu |\partial_{x}^{\alpha} \mathbb{D}(\mathbf{U}^{\epsilon})|^{2} + \lambda |\partial_{x}^{\alpha} \text{tr} \mathbb{D}(\mathbf{U}^{\epsilon})|^{2} \} dx
$$

\n
$$
\geq 2\mu \int \frac{1}{N^{\epsilon} + \rho^{0}} \left(|\partial_{x}^{\alpha} \mathbb{D}(\mathbf{U}^{\epsilon})|^{2} - \frac{1}{3} |\partial_{x}^{\alpha} \text{tr} \mathbb{D}(\mathbf{U}^{\epsilon})|^{2} \right) dx
$$

\n
$$
= \mu \int \frac{1}{N^{\epsilon} + \rho^{0}} \left(|\partial_{x}^{\alpha} \nabla \mathbf{U}^{\epsilon}|^{2} + \frac{1}{3} |\partial_{x}^{\alpha} \text{div } \mathbf{U}^{\epsilon}|^{2} \right) dx
$$

\n
$$
\geq \mu \int \frac{1}{N^{\epsilon} + \rho^{0}} |\partial_{x}^{\alpha} \nabla \mathbf{U}^{\epsilon}|^{2} dx.
$$
 (3.20)

By virtue of Cauchy-Schwarz's inequality, the regularity of ρ^0 and the positivity of $N^{\epsilon} + \rho_0$, the terms $\mathcal{I}^{(3)}$ and $\mathcal{I}^{(4)}$ can be bounded by

$$
|\mathcal{I}^{(3)}| + |\mathcal{I}^{(4)}| \le \eta_5 \|\nabla \partial_x^{\alpha} \mathbf{U}^{\epsilon}\|^2 + C_{\eta_5} (\|\mathcal{E}^{\epsilon}(t)\|_{s}^2 + 1) (\|\partial_x^{\alpha} \mathbf{U}^{\epsilon}\|^2 + \|\partial_x^{\alpha} N^{\epsilon}\|^2) \tag{3.21}
$$

for any $\eta_5 > 0$, where the assumption $s > 3/2 + 2$ has been used.

Substituting (3.12) – (3.21) into (3.11) , we conclude that

$$
\frac{1}{2} \frac{d}{dt} \langle \partial_x^{\alpha} \mathbf{U}^{\epsilon}, \partial_x^{\alpha} \mathbf{U}^{\epsilon} \rangle + \int \frac{\mu}{N^{\epsilon} + \rho^{0}} |\nabla \partial_x^{\alpha} \mathbf{U}^{\epsilon}|^{2} dx - (\eta_{1} + \eta_{3} + \eta_{4} + \eta_{5}) || \nabla \partial_x^{\alpha} \mathbf{U}^{\epsilon} ||^{2} \n\leq C_{\eta} \{ (\|\mathcal{E}^{\epsilon}(t)\|_{s}^{2} + 1) (\|\partial_x^{\alpha} \mathbf{U}^{\epsilon}\|^{2} + \|\partial_x^{\alpha} N^{\epsilon}\|^{2} + \|\mathcal{E}^{\epsilon}(t)\|_{s}^{s}) \} \n+ \eta_{2} \|\partial_x^{\alpha} \nabla \Theta^{\epsilon} \|^{2} + \sum_{i=1}^{7} \mathcal{R}^{(i)}
$$
\n(3.22)

for some constant $C_{\eta} > 0$ depending on η_i $(i = 1, \ldots, 5)$.

We have to estimate the terms on the right-hand side of [\(3.22\)](#page-16-1). In view of the regularity of $(\rho^0, \mathbf{u}^0, \mathbf{H}^0)$, the positivity of $N^{\epsilon} + \rho^0$ and Cauchy-Schwarz's inequality, the first two terms $\mathcal{R}^{(1)}$ and $\mathcal{R}^{(2)}$ can be controlled by

$$
\left|\mathcal{R}^{(1)}\right| + \left|\mathcal{R}^{(2)}\right| \leq C(\left|\mathcal{E}^{\epsilon}(t)\right|_{s}^{2} + 1)(\left|\left|\partial_{x}^{\alpha} N^{\epsilon}\right|\right|^{2} + \left|\left|\partial_{x}^{\alpha} \mathbf{U}^{\epsilon}\right|\right|^{2}).\tag{3.23}
$$

For the terms $\mathcal{R}^{(3)}$ and $\mathcal{R}^{(4)}$, by the regularity of ρ^0 and \mathbf{u}^0 , the positivity of $N^{\epsilon} + \rho^{0}$, Cauchy-Schwarz's inequality and [\(1.40\)](#page-8-4), we see that

$$
\left|\mathcal{R}^{(3)}\right| + \left|\mathcal{R}^{(4)}\right| \leqslant C(\left\|\mathcal{E}^{\epsilon}(t)\right\|_{s}^{2} + C\left\|\partial_{x}^{\alpha} \mathbf{U}^{\epsilon}\right\|^{2}).\tag{3.24}
$$

For the fifth term $\mathcal{R}^{(5)}$, we utilize the positivity of $N^{\epsilon} + \rho^0$ to deduce that

$$
\mathcal{R}^{(5)} = \left\langle \partial_x^{\alpha} \mathbf{F}^{\epsilon} \times \frac{\mathbf{H}^0}{N^{\epsilon} + \rho^0}, \partial_x^{\alpha} \mathbf{U}^{\epsilon} \right\rangle + \left\langle \mathcal{H}^{(6)}, \partial_x^{\alpha} \mathbf{U}^{\epsilon} \right\rangle + \sigma \mathcal{R}^{(5_1)}
$$

$$
\leq \frac{1}{16} \| \partial_x^{\alpha} \mathbf{F}^{\epsilon} \|^2 + C \| \partial_x^{\alpha} \mathbf{U}^{\epsilon} \|^2 + \left\langle \mathcal{H}^{(6)}, \partial_x^{\alpha} \mathbf{U}^{\epsilon} \right\rangle + \mathcal{R}^{(5_1)}, \tag{3.25}
$$

where

$$
\mathcal{H}^{(6)} = \partial_x^{\alpha} \left\{ \frac{\mathbf{F}^{\epsilon}}{N^{\epsilon} + \rho^0} \times \mathbf{H}^0 \right\} - \partial_x^{\alpha} \mathbf{F}^{\epsilon} \times \frac{\mathbf{H}^0}{N^{\epsilon} + \rho^0}
$$

and

$$
\mathcal{R}^{(5_1)} = \left\langle \partial_x^{\alpha} \left\{ \frac{\sigma}{N^{\epsilon} + \rho^0} [\mathbf{u}^0 \times \mathbf{G}^{\epsilon} + \mathbf{U}^{\epsilon} \times \mathbf{H}^0] \times \mathbf{H}^0 \right\}, \partial_x^{\alpha} \mathbf{U}^{\epsilon} \right\rangle.
$$

If we make use of the Moser-type inequality, [\(1.39\)](#page-8-3) and the regularity of ρ^0 and H^0 , we obtain that

$$
\left| \langle \mathcal{H}^{(6)}, \partial_x^{\alpha} \mathbf{U}^{\epsilon} \rangle \right| \leq \left| \mathcal{H}^{(6)} \right| \cdot \left| \partial_x^{\alpha} \mathbf{U}^{\epsilon} \right|
$$

\n
$$
\leq C \left[\left\| D_x^1 \left(\frac{\mathbf{H}^0}{N^{\epsilon} + \rho^0} \right) \right\|_{L^{\infty}} \left\| \mathbf{F}^{\epsilon} \right\|_{s-1} + \left\| \mathbf{F}^{\epsilon} \right\|_{L^{\infty}} \left\| \frac{\mathbf{H}^0}{N^{\epsilon} + \rho^0} \right\|_s \right] \left\| \partial_x^{\alpha} \mathbf{U}^{\epsilon} \right\|
$$

\n
$$
\leq \eta_6 \left\| \mathbf{F}^{\epsilon} \right\|_{s-1}^2 + C_{\eta_6} \left(\left\| \mathcal{E}^{\epsilon}(t) \right\|_{s}^{2(s+1)} + 1 \right) \left\| \partial_x^{\alpha} \mathbf{U}^{\epsilon} \right\|^2
$$
 (3.26)

for any $\eta_6 > 0$. Recalling the regularity of \mathbf{u}^0 and \mathbf{H}^0 , [\(1.37\)](#page-7-1) and [\(1.39\)](#page-8-3) and Hölder's inequality, we find that

$$
\left|\mathcal{R}^{(5_1)}\right| \leqslant C(\left\|\mathcal{E}^{\epsilon}(t)\right\|_{s}^{s} + 1)(\left\|\partial_x^{\alpha} N^{\epsilon}\right\|^2 + \left\|\partial_x^{\alpha} \mathbf{U}^{\epsilon}\right\|^2 + \left\|\partial_x^{\alpha} \mathbf{G}^{\epsilon}\right\|^2). \tag{3.27}
$$

For the sixth term $\mathcal{R}^{(6)}$ we again make use of the positivity of $N^{\epsilon} + \rho^0$ and Sobolev's imbedding to infer that

$$
\mathcal{R}^{(6)} = \left\langle \partial_x^{\alpha} \mathbf{F}^{\epsilon} \times \frac{\mathbf{G}^{\epsilon}}{N^{\epsilon} + \rho^{0}}, \partial_x^{\alpha} \mathbf{U}^{\epsilon} \right\rangle + \left\langle \mathcal{H}^{(7)}, \partial_x^{\alpha} \mathbf{U}^{\epsilon} \right\rangle + \mathcal{R}^{(6_1)} \n\leq \frac{1}{16} \|\partial_x^{\alpha} \mathbf{F}^{\epsilon}\|^2 + C \|\mathcal{E}^{\epsilon}(t)\|_{s}^{2} \|\partial_x^{\alpha} \mathbf{U}^{\epsilon}\|^2 + \left\langle \mathcal{H}^{(7)}, \partial_x^{\alpha} \mathbf{U}^{\epsilon} \right\rangle + \mathcal{R}^{(6_1)},
$$
\n(3.28)

where

$$
\mathcal{H}^{(7)} = \partial_x^{\alpha} \left\{ \frac{\mathbf{F}^{\epsilon}}{N^{\epsilon} + \rho^0} \times \mathbf{G}^{\epsilon} \right\} - \partial_x^{\alpha} \mathbf{F}^{\epsilon} \times \frac{\mathbf{G}^{\epsilon}}{N^{\epsilon} + \rho^0}
$$

and

$$
\mathcal{R}^{(6_1)} = \left\langle \partial_x^{\alpha} \left\{ \frac{\sigma}{N^{\epsilon} + \rho^0} [\mathbf{u}^0 \times \mathbf{G}^{\epsilon} + \mathbf{U}^{\epsilon} \times \mathbf{H}^0] \times \mathbf{G}^{\epsilon} \right\}, \partial_x^{\alpha} \mathbf{U}^{\epsilon} \right\rangle.
$$

From the Hölder's and Moser-type inequalities we get

$$
\left| \langle \mathcal{H}^{(7)}, \partial_x^{\alpha} \mathbf{U}^{\epsilon} \rangle \right|
$$

\n
$$
\leq \left| \mathcal{H}^{(7)} \right| \cdot \left| \partial_x^{\alpha} \mathbf{U}^{\epsilon} \right|
$$

\n
$$
\leq C \left[\left\| D_x^1 \left(\frac{\mathbf{G}^{\epsilon}}{N^{\epsilon} + \rho^0} \right) \right\|_{L^{\infty}} \left\| \mathbf{F}^{\epsilon} \right\|_{s-1} + \left\| \mathbf{F}^{\epsilon} \right\|_{L^{\infty}} \left\| \frac{\mathbf{G}^{\epsilon}}{N^{\epsilon} + \rho^0} \right\|_{s} \right] \left\| \partial_x^{\alpha} \mathbf{U}^{\epsilon} \right\|
$$

\n
$$
\leq \eta_7 \left\| \mathbf{F}^{\epsilon} \right\|_{s-1}^2 + C_{\eta_7} \left(\left\| \mathcal{E}^{\epsilon}(t) \right\|_{s}^{2(s+1)} + 1 \right) \left\| \partial_x^{\alpha} \mathbf{U}^{\epsilon} \right\|^2
$$
(3.29)

for any $\eta_7 > 0$, while for the term $\mathcal{R}^{(6_1)}$ one has the following estimate

$$
\left|\mathcal{R}^{6_1}\right| \leq C(\left|\mathcal{E}^{\epsilon}(t)\right|_{s}^{2} + \left|\mathcal{E}^{\epsilon}(t)\right|_{s} + 1)\left|\mathcal{E}^{\epsilon}(t)\right|_{s}^{2}.
$$
 (3.30)

For the last term $\mathcal{R}^{(7)}$, recalling the formula $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a}$ and applying (1.37) , (1.39) , and Hölder's inequality, we easily deduce that

$$
\left| \mathcal{R}^{(7)} \right| = \left| \left\langle \partial_x^{\alpha} \left\{ \frac{1}{N^{\epsilon} + \rho^{0}} \{ [\mathbf{U}^{\epsilon} \cdot (\mathbf{G}^{\epsilon} + \mathbf{H}^{0})] \mathbf{G}^{\epsilon} - [\mathbf{G}^{\epsilon} \cdot (\mathbf{G}^{\epsilon} + \mathbf{H}^{0})] \mathbf{U}^{\epsilon} \} \right\} , \partial_x^{\alpha} \mathbf{U}^{\epsilon} \right\rangle \right|
$$

\$\leq C(\|\mathcal{E}^{\epsilon}(t)\|_{s}^{s} + 1) \|\mathcal{E}^{\epsilon}(t)\|_{s}^{4} + \|\mathcal{E}^{\epsilon}(t)\|_{s}^{3} . \tag{3.31}

Substituting (3.23) – (3.31) into (3.22) , we conclude that

$$
\frac{1}{2} \frac{d}{dt} \langle \partial_x^{\alpha} \mathbf{U}^{\epsilon}, \partial_x^{\alpha} \mathbf{U}^{\epsilon} \rangle + \int \frac{\mu}{N^{\epsilon} + \rho^{0}} |\nabla \partial_x^{\alpha} \mathbf{U}^{\epsilon}|^{2} dx - (\eta_{1} + \eta_{3} + \eta_{4} + \eta_{5}) || \nabla \partial_x^{\alpha} \mathbf{U}^{\epsilon} ||^{2} \n\leq \tilde{C}_{\eta} [(\|\mathcal{E}^{\epsilon}(t)\|_{s}^{2s} + 1) \|\mathcal{E}^{\epsilon}(t)\|_{s}^{4} + \|\mathcal{E}^{\epsilon}(t)\|_{s}^{3} + \|\mathcal{E}^{\epsilon}(t)\|_{s}^{2}] \n+ \eta_{2} \|\partial_x^{\alpha} \nabla \Theta^{\epsilon} \|^{2} + \left(\eta_{6} + \eta_{7} + \frac{1}{8}\right) \|\mathbf{F}^{\epsilon}\|_{s}^{2}.
$$
\n(3.32)

for some constant $\tilde{C}_\eta > 0$ depending on η_i $(i = 1, ..., 7)$.

Applying the operator ∂_x^{α} to [\(2.3\)](#page-9-1), multiplying the resulting equation by $\partial_x^{\alpha} \Theta^{\epsilon}$, and integrating over \mathbb{T}^3 , we arrive at

$$
\frac{1}{2} \frac{d}{dt} \langle \partial_x^{\alpha} \Theta^{\epsilon}, \partial_x^{\alpha} \Theta^{\epsilon} \rangle + \langle \partial_x^{\alpha} \{ [(U^{\epsilon} + u^{0}) \cdot \nabla] \Theta^{\epsilon} \}, \partial_x^{\alpha} \Theta^{\epsilon} \rangle \n+ \langle \partial_x^{\alpha} \{ (\Theta^{\epsilon} + \theta^{0}) \operatorname{div} U^{\epsilon} \}, \partial_x^{\alpha} \Theta^{\epsilon} \rangle - \langle \partial_x^{\alpha} \{ \frac{\kappa}{N^{\epsilon} + \rho^{0}} \Delta \Theta^{\epsilon} \} , \partial_x^{\alpha} \Theta^{\epsilon} \rangle \n= - \langle \partial_x^{\alpha} \{ (U^{\epsilon} \cdot \nabla) \theta^{0} - \Theta^{\epsilon} \operatorname{div} u^{0} \}, \partial_x^{\alpha} \Theta^{\epsilon} \rangle \n+ \langle \partial_x^{\alpha} \{ \left[\frac{\kappa}{N^{\epsilon} + \rho^{0}} - \frac{\kappa}{\rho^{0}} \right] \Delta \theta^{0} \} , \partial_x^{\alpha} \Theta^{\epsilon} \rangle \n+ \langle \partial_x^{\alpha} \{ \left[\frac{2\mu}{N^{\epsilon} + \rho^{0}} - \frac{2\mu}{\rho^{0}} \right] | \mathbb{D}(u^{0}) |^{2} \} , \partial_x^{\alpha} \Theta^{\epsilon} \rangle \n+ \langle \partial_x^{\alpha} \{ \left[\frac{\lambda}{N^{\epsilon} + \rho^{0}} - \frac{\lambda}{\rho^{0}} \right] (tr \mathbb{D}(u^{0}))^{2} \} , \partial_x^{\alpha} \Theta^{\epsilon} \rangle \n+ \langle \partial_x^{\alpha} \{ \left[\frac{1}{N^{\epsilon} + \rho^{0}} - \frac{1}{\rho^{0}} \right] | \operatorname{curl} H^{0} |^{2} \} , \partial_x^{\alpha} \Theta^{\epsilon} \rangle \n+ \langle \partial_x^{\alpha} \{ \frac{2\mu}{N^{\epsilon} + \rho^{0}} | \mathbb{D}(U^{\epsilon}) |^{2} + \frac{\lambda}{N^{\epsilon} + \rho^{0}} | \operatorname{tr} \mathbb{D}(U^{\epsilon}) |^{2} \} , \partial_x^{\alpha} \Theta^{\epsilon} \rangle \n+ \langle \partial_x^{\alpha} \{ \frac{2\mu}{N^{\epsilon} + \rho^{0}} | \mathbb{D}(U^{\epsilon}) : \mathbb
$$

$$
+\left\langle \partial_x^{\alpha} \left\{ \frac{2(\mathbf{U}^{\epsilon} \times \mathbf{G}^{\epsilon})}{N^{\epsilon} + \rho^{0}} \cdot \left[\operatorname{curl} \mathbf{H}^{0} + \mathbf{u}^{0} \times \mathbf{G}^{\epsilon} + \mathbf{U}^{\epsilon} \times \mathbf{H}^{0} \right] \right\}, \partial_x^{\alpha} \Theta^{\epsilon} \right\rangle
$$

+
$$
\left\langle \partial_x^{\alpha} \left\{ \frac{2}{N^{\epsilon} + \rho^{0}} \operatorname{curl} \mathbf{H}^{0} \cdot (\mathbf{u}^{0} \times \mathbf{G}^{\epsilon} + \mathbf{U}^{\epsilon} \times \mathbf{H}^{0}) \right\}, \partial_x^{\alpha} \Theta^{\epsilon} \right\rangle
$$

:=
$$
\sum_{i=1}^{13} \mathcal{S}^{(i)}.
$$
 (3.33)

We first bound the terms on the left-hand side of [\(3.33\)](#page-19-0). Similar to [\(3.5\)](#page-12-1), we have

$$
\langle \partial_x^{\alpha} \langle [(\mathbf{U}^{\epsilon} + \mathbf{u}^{0}) \cdot \nabla] \Theta^{\epsilon} \rangle, \partial_x^{\alpha} \Theta^{\epsilon} \rangle
$$

\n
$$
= \langle [(\mathbf{U}^{\epsilon} + \mathbf{u}^{0}) \cdot \nabla] \partial_x^{\alpha} \Theta^{\epsilon}, \partial_x^{\alpha} \Theta^{\epsilon} \rangle + \langle \mathcal{H}^{(8)}, \partial_x^{\alpha} \Theta^{\epsilon} \rangle
$$

\n
$$
= -\frac{1}{2} \langle \text{div} (\mathbf{U}^{\epsilon} + \mathbf{u}^{0}) \partial_x^{\alpha} \Theta^{\epsilon}, \partial_x^{\alpha} \Theta^{\epsilon} \rangle + \langle \mathcal{H}^{(8)}, \partial_x^{\alpha} \Theta^{\epsilon} \rangle
$$

\n
$$
\leq C (\|\mathcal{E}^{\epsilon}(t)\|_{s} + 1) \|\partial_x^{\alpha} \Theta^{\epsilon}\|^{2} + \|\mathcal{H}^{(8)}\|^{2}, \tag{3.34}
$$

where the commutator

$$
\mathcal{H}^{(8)} = \partial_x^{\alpha} \left(\left[(\mathbf{U}^{\epsilon} + \mathbf{u}^0) \cdot \nabla \right] \Theta^{\epsilon} \right) - \left[(\mathbf{U}^{\epsilon} + \mathbf{u}^0) \cdot \nabla \right] \partial_x^{\alpha} \Theta^{\epsilon}
$$

can be bounded by

$$
\|\mathcal{H}^{(8)}\| \leq C(\|D_x^1(\mathbf{U}^{\epsilon} + \mathbf{u}^0)\|_{L^{\infty}}\|D_x^s \mathbf{U}^{\epsilon}\| + \|D_x^1 \mathbf{U}^{\epsilon}\|_{L^{\infty}}\|D_x^{s-1}(\mathbf{U}^{\epsilon} + \mathbf{u}^0)\|)
$$

\$\leq C(\|\mathcal{E}^{\epsilon}(t)\|_{s}^{2} + \|\mathcal{E}^{\epsilon}(t)\|_{s}). \qquad (3.35)

The second term on the left-hand side of [\(3.33\)](#page-19-0) can bounded, similar to [\(3.7\)](#page-13-1), as follows:

$$
\langle \partial_x^{\alpha} ((\Theta^{\epsilon} + \theta^0) \text{div } \mathbf{U}^{\epsilon}), \partial_x^{\alpha} \Theta^{\epsilon} \rangle
$$

= $\langle (\Theta^{\epsilon} + \rho^0) \partial_x^{\alpha} \text{div } \mathbf{U}^{\epsilon}, \partial_x^{\alpha} \Theta^{\epsilon} \rangle + \langle \mathcal{H}^{(9)}, \partial_x^{\alpha} \Theta^{\epsilon} \rangle$
 $\leq \eta_{8} \|\nabla \partial_x^{\alpha} \mathbf{U}^{\epsilon}\|^2 + C_{\eta_{8}} \|\partial_x^{\alpha} N^{\epsilon}\|^2 + \|\mathcal{H}^{(9)}\|^2$ (3.36)

for any $\eta_8 > 0$, where the commutator

$$
\mathcal{H}^{(9)} = \partial_x^{\alpha}((\Theta^{\epsilon} + \rho^{0})\mathrm{div}\,\mathbf{U}^{\epsilon}) - (\Theta^{\epsilon} + \theta^{0})\partial_x^{\alpha}\mathrm{div}\,\mathbf{U}^{\epsilon}
$$

can be controlled as

$$
\|\mathcal{H}^{(9)}\| \leq C(\|D_x^1(\Theta^\epsilon + \theta^0)\|_{L^\infty}\|D_x^s \mathbf{U}^\epsilon\| + \|D_x^1 \mathbf{U}^\epsilon\|_{L^\infty}\|D_x^{s-1}(\Theta^\epsilon + \theta^0)\|)
$$

$$
\leq C(\|\mathcal{E}^\epsilon(t)\|_{s}^2 + \|\mathcal{E}^\epsilon(t)\|_{s}).
$$
 (3.37)

For the fourth term on the left-hand side of [\(3.33\)](#page-19-0), we integrate by parts to deduce that

$$
- \left. \kappa \left\langle \partial_x^\alpha \left(\frac{1}{N^\epsilon + \rho^0} \Delta \mathbf{U}^\epsilon \right), \partial_x^\alpha \Theta^\epsilon \right\rangle \right.
$$

$$
= -\kappa \left\langle \frac{1}{N^{\epsilon} + \rho^{0}} \Delta \partial_{x}^{\alpha} \Theta^{\epsilon}, \partial_{x}^{\alpha} \Theta^{\epsilon} \right\rangle - \kappa \left\langle \mathcal{H}^{(10)}, \partial_{x}^{\alpha} \Theta^{\epsilon} \right\rangle
$$

\n
$$
= \kappa \left\langle \frac{1}{N^{\epsilon} + \rho^{0}} \nabla \partial_{x}^{\alpha} \Theta^{\epsilon}, \nabla \partial_{x}^{\alpha} \Theta^{\epsilon} \right\rangle
$$

\n
$$
+ \kappa \left\langle \nabla \left(\frac{1}{N^{\epsilon} + \rho^{0}} \right) \nabla \partial_{x}^{\alpha} \Theta^{\epsilon}, \partial_{x}^{\alpha} \Theta^{\epsilon} \right\rangle - \kappa \left\langle \mathcal{H}^{(10)}, \partial_{x}^{\alpha} \Theta^{\epsilon} \right\rangle, \tag{3.38}
$$

where

$$
\mathcal{H}^{(10)} = \partial_x^{\alpha} \left(\frac{1}{N^{\epsilon} + \rho^0} \Delta \Theta^{\epsilon} \right) - \frac{1}{N^{\epsilon} + \rho^0} \Delta \partial_x^{\alpha} \Theta^{\epsilon}.
$$

By the Moser-type and Hölder's inequalities, the regularity of ρ^0 , the positivity of $N^{\epsilon} + \rho_0$ and [\(1.39\)](#page-8-3), we find that

$$
\left| \langle \mathcal{H}^{(10)}, \partial_x^{\alpha} \Theta^{\epsilon} \rangle \right| \leq \left\| \mathcal{H}^{(10)} \right\| \cdot \left\| \partial_x^{\alpha} \Theta^{\epsilon} \right\|
$$

\n
$$
\leq C \left(\left\| D_x^1 \left(\frac{1}{N^{\epsilon} + \rho^0} \right) \right\|_{L^{\infty}} \left\| \Delta \Theta^{\epsilon} \right\|_{s-1} + \left\| \Delta \Theta^{\epsilon} \right\|_{L^{\infty}} \left\| \frac{1}{N^{\epsilon} + \rho^0} \right\|_s \right) \left\| \partial_x^{\alpha} \Theta^{\epsilon} \right\|
$$

\n
$$
\leq \eta_9 \left\| \nabla \Theta^{\epsilon} \right\|_s^2 + C_{\eta_9} \left(\left\| \mathcal{E}^{\epsilon}(t) \right\|_s^s + 1 \right) \left(\left\| \partial_x^{\alpha} \Theta^{\epsilon} \right\|^{2} + \left\| \partial_x^{\alpha} N^{\epsilon} \right\|^{2} \right) \tag{3.39}
$$

and

$$
\left| \left\langle \nabla \left(\frac{1}{N^{\epsilon} + \rho^{0}} \right) \nabla \partial_{x}^{\alpha} \Theta^{\epsilon}, \partial_{x}^{\alpha} \Theta^{\epsilon} \right\rangle \right|
$$

\n
$$
\leq \eta_{10} \|\nabla \partial_{x}^{\alpha} \Theta^{\epsilon}\|^{2} + C_{\eta_{10}} \left\| \nabla \left(\frac{1}{N^{\epsilon} + \rho^{0}} \right) \right\|_{L^{\infty}}^{2} \|\partial_{x}^{\alpha} \Theta^{\epsilon}\|^{2}
$$

\n
$$
\leq \eta_{10} \|\nabla \partial_{x}^{\alpha} \Theta^{\epsilon}\|^{2} + C_{\eta_{10}} (\|\mathcal{E}^{\epsilon}(t)\|_{s}^{2} + 1) \|\partial_{x}^{\alpha} \Theta^{\epsilon}\|^{2}
$$
\n(3.40)

for any $\eta_9 > 0$ and $\eta_{10} > 0$, where we have used the assumption $s > 3/2 + 2$ in the derivation of [\(3.39\)](#page-20-0) and the imbedding $H^l(\mathbb{T}^3) \hookrightarrow L^{\infty}(\mathbb{R}^3)$ for $l > 3/2$.

Now, we estimate every term on the right-hand side of [\(3.33\)](#page-19-0). By virtue of the regularity of θ^0 and \mathbf{u}^0 , and Cauchy-Schwarz's inequality, the first term $\mathcal{S}^{(1)}$ can be estimated as follows:

$$
\left| \mathcal{S}^{(1)} \right| \leqslant C \big(\|\mathcal{E}^{\epsilon}(t)\|_{s}^{2} + 1\big) \big(\|\partial_x^{\alpha} \Theta^{\epsilon}\|^{2} + \|\partial_x^{\alpha} \mathbf{U}^{\epsilon}\|^{2} \big). \tag{3.41}
$$

For the terms $\mathcal{S}^{(i)}$ $(i = 2, 3, 4, 5)$, we utilize the regularity of ρ^0 , \mathbf{u}^0 and \mathbf{H}^0 , the positivity of $N^{\epsilon} + \rho^{0}$, Cauchy-Schwarz's inequality and [\(1.40\)](#page-8-4) to deduce that

$$
\left|\mathcal{S}^{(2)}\right| + \left|\mathcal{S}^{(3)}\right| + \left|\mathcal{S}^{(4)}\right| + \left|\mathcal{S}^{(5)}\right| \leq C(\left\|\mathcal{E}^{\epsilon}(t)\right\|_{s}^{2s} + \left\|\mathcal{E}^{\epsilon}(t)\right\|_{s}^{2}) + C\left\|\partial_{x}^{\alpha}\Theta^{\epsilon}\right\|^{2},\tag{3.42}
$$

while for the sixth term $\mathcal{S}^{(6)}$, we integrate by parts, and use Cauchy-Schwarz's inequality and the positivity of $\Theta^{\epsilon} + \rho^{0}$ to obtain that

$$
\mathcal{S}^{(6)} = -\left\langle \partial_x^{\alpha-\alpha_1} \left\{ \frac{2\mu}{N^{\epsilon} + \rho^0} |\mathbb{D}(\mathbf{U}^{\epsilon})|^2 + \frac{\lambda}{N^{\epsilon} + \rho^0} |\text{tr}\mathbb{D}(\mathbf{U}^{\epsilon})|^2 \right\}, \partial_x^{\alpha-\alpha_1} \Theta^{\epsilon} \right\rangle
$$

\$\leq \eta_{11} \|\nabla \partial_x^{\alpha} \Theta^{\epsilon}\|^2 + C_{\eta_{11}} (\|\mathcal{E}^{\epsilon}(t)\|_{s}^4 + \|\mathcal{E}^{\epsilon}(t)\|_{s}^{2(s+1)}) \qquad (3.43)\$

for any $\eta_{11} > 0$, where $\alpha_1 = (1, 0, 0)$ or $(0, 1, 0)$ or $(0, 0, 1)$. Similarly, we have

$$
\left|\mathcal{S}^{(7)}\right| + \left|\mathcal{S}^{(8)}\right| \le \eta_{12} \|\nabla \partial_x^{\alpha} \Theta^{\epsilon}\|^2 + C_{\eta_{12}} (\|\mathcal{E}^{\epsilon}(t)\|_{s}^4 + \|\mathcal{E}^{\epsilon}(t)\|_{s}^{2(s+1)}) \tag{3.44}
$$

for any $\eta_{12} > 0$.

For the ninth term $\mathcal{S}^{(9)}$, we rewrite it as

$$
\mathcal{S}^{(9)} = \left\langle \partial_x^{\alpha} \left\{ \frac{1}{N^{\epsilon} + \rho^0} |\mathbf{F}^{\epsilon} + \mathbf{U}^{\epsilon} \times \mathbf{G}^{\epsilon}|^2 \right\}, \partial_x^{\alpha} \Theta^{\epsilon} \right\rangle
$$

\n
$$
= \left\langle \partial_x^{\alpha} \left\{ \frac{1}{N^{\epsilon} + \rho^0} |\mathbf{F}^{\epsilon}|^2 \right\}, \partial_x^{\alpha} \Theta^{\epsilon} \right\rangle
$$

\n
$$
+ \left\langle \partial_x^{\alpha} \left\{ \frac{2}{N^{\epsilon} + \rho^0} \mathbf{F}^{\epsilon} \cdot (\mathbf{U}^{\epsilon} \times \mathbf{G}^{\epsilon}) \right\}, \partial_x^{\alpha} \Theta^{\epsilon} \right\rangle
$$

\n
$$
+ \left\langle \partial_x^{\alpha} \left\{ \frac{1}{N^{\epsilon} + \rho^0} |\mathbf{U}^{\epsilon} \times \mathbf{G}^{\epsilon}|^2 \right\}, \partial_x^{\alpha} \Theta^{\epsilon} \right\rangle
$$

\n
$$
:= \mathcal{S}^{(9_1)} + \mathcal{S}^{(9_2)} + \mathcal{S}^{(9_3)}.
$$

By Cauchy-Schwarz's inequality and Sobolev's embedding, we can bound the term $\mathcal{S}^{(9_1)}$ by

$$
\mathcal{S}^{(9_1)} = \left\langle \frac{1}{N^{\epsilon} + \rho^0} \partial_x^{\alpha} \left(|\mathbf{F}^{\epsilon}|^2 \right), \partial_x^{\alpha} \Theta^{\epsilon} \right\rangle
$$

+
$$
\sum_{\beta \le \alpha, |\beta| < |\alpha|} \left\langle \partial_x^{\alpha-\beta} \left(\frac{1}{N^{\epsilon} + \rho^0} \right) \partial_x^{\beta} |\mathbf{F}^{\epsilon}|^2, \partial_x^{\alpha} \Theta^{\epsilon} \right\rangle
$$

$$
\le \gamma_1 \| \mathbf{F}^{\epsilon} \|_{s}^4 + C_{\gamma_1} \| \partial_x^{\alpha} \Theta^{\epsilon} \|^2 (1 + \| \mathcal{E}(t) \|_{s}^{2(s+1)})
$$
(3.45)

for any $\gamma_1 > 0$. For the term $\mathcal{S}^{(9_2)}$, similar to $\mathcal{R}^{(6)}$, we have

$$
\mathcal{S}^{(9_2)} = 2 \left\langle \partial_x^{\alpha} \mathbf{F}^{\epsilon} \cdot \frac{\mathbf{U}^{\epsilon} \times \mathbf{G}^{\epsilon}}{N^{\epsilon} + \rho^0}, \partial_x^{\alpha} \Theta^{\epsilon} \right\rangle + 2 \left\langle \mathcal{H}^{(11)}, \partial_x^{\alpha} \Theta^{\epsilon} \right\rangle
$$

$$
\leq \frac{1}{16} \| \partial_x^{\alpha} \mathbf{F}^{\epsilon} \|^{2} + C \| \mathcal{E}^{\epsilon}(t) \|_{s}^{2} \| \partial_x^{\alpha} \mathbf{U}^{\epsilon} \|^{2} + 2 \left\langle \mathcal{H}^{(11)}, \partial_x^{\alpha} \Theta^{\epsilon} \right\rangle, \tag{3.46}
$$

.

where

$$
\mathcal{H}^{(11)} = \partial_x^{\alpha} \left\{ \mathbf{F}^{\epsilon} \cdot \frac{\mathbf{U}^{\epsilon} \times \mathbf{G}^{\epsilon}}{N^{\epsilon} + \rho^0} \right\} - \partial_x^{\alpha} \mathbf{F}^{\epsilon} \cdot \frac{\mathbf{U}^{\epsilon} \times \mathbf{G}^{\epsilon}}{N^{\epsilon} + \rho^0}
$$

By the Cauchy-Schwarz's and Moser-type inequalities, we obtain that

$$
2|\langle \mathcal{H}^{(11)}, \partial_x^{\alpha} \Theta^{\epsilon} \rangle|
$$

\n
$$
\leq 2\|\mathcal{H}^{(11)}\| \cdot \|\partial_x^{\alpha} \Theta^{\epsilon}\|
$$

\n
$$
\leq C \left[\left\| D_x^1 \left(\frac{\mathbf{U}^{\epsilon} \times \mathbf{G}^{\epsilon}}{N^{\epsilon} + \rho^0} \right) \right\|_{L^{\infty}} \|\mathbf{F}^{\epsilon}\|_{s-1} + \|\mathbf{F}^{\epsilon}\|_{L^{\infty}} \left\| \frac{\mathbf{U}^{\epsilon} \times \mathbf{G}^{\epsilon}}{N^{\epsilon} + \rho^0} \right\|_{s} \right] \|\partial_x^{\alpha} \Theta^{\epsilon}\|
$$

\n
$$
\leq \gamma_2 \|\mathbf{F}^{\epsilon}\|_{s-1}^2 + C_{\gamma_2} (\|\mathcal{E}^{\epsilon}(t)\|_{s}^2 + 1) \|\partial_x^{\alpha} \Theta^{\epsilon}\|^2.
$$
 (3.47)

for any $\gamma_2 > 0$. The term $S^{(9_3)}$ can be bounded as follows, using the Cauchy-Schwarz and Moser-type inequalities.

$$
\left| \mathcal{S}^{(9_3)} \right| \leqslant C \|\partial_x^{\alpha} \Theta^{\epsilon}\|^2 (1 + \| \mathcal{E}(t) \|_{s}^{2(s+1)}). \tag{3.48}
$$

By the regularity of θ^0 , \mathbf{u}^0 and \mathbf{H}^0 , the positivity of $\Theta^{\epsilon} + \rho^0$, and Cauchy-Schwarz's inequality, the first terms $\mathcal{S}^{(10)}$ and $\mathcal{S}^{(13)}$ can be bounded as follows:

$$
\left| \mathcal{S}^{(10)} \right| + \left| \mathcal{S}^{(13)} \right| \leqslant C \left(\left\| \mathcal{E}^{\epsilon}(t) \right\|_{s}^{2s} + 1 \right) \left(\left\| \partial_x^{\alpha} \Theta^{\epsilon} \right\|^{2} + \left\| \partial_x^{\alpha} \mathbf{U}^{\epsilon} \right\|^{2} + \left\| \partial_x^{\alpha} \mathbf{G}^{\epsilon} \right\|^{2} \right). \tag{3.49}
$$

In a manner similar to $S^{(9_2)}$, we can control the term $S^{(11)}$ by

$$
\left|\mathcal{S}^{(11)}\right| \leq \gamma_3 \|\mathbf{F}^{\epsilon}\|_{s-1}^2 + C_{\gamma_3}(\|\mathcal{E}^{\epsilon}(t)\|_{s}^{2s} + 1)(\|\partial_x^{\alpha}\Theta^{\epsilon}\|^2 + \|\partial_x^{\alpha}\mathbf{U}^{\epsilon}\|^2 + \|\partial_x^{\alpha}\mathbf{G}^{\epsilon}\|^2) \tag{3.50}
$$

for any $\gamma_3 > 0$. Finally, similarly to $\mathcal{S}^{(9_3)}$, the term $\mathcal{S}^{(12)}$ can be bounded by

$$
\left| \mathcal{S}^{(12)} \right| \leq C \|\partial_x^{\alpha} \Theta^{\epsilon} \|^2 (1 + \| \mathcal{E}(t) \|_s^{2(s+1)}). \tag{3.51}
$$

Substituting (3.34) – (3.51) into (3.33) , we conclude that

$$
\frac{1}{2} \frac{d}{dt} \langle \partial_x^{\alpha} \Theta^{\epsilon}, \partial_x^{\alpha} \Theta^{\epsilon} \rangle + \kappa \langle \frac{1}{N^{\epsilon} + \rho^{0}} \nabla \partial_x^{\alpha} \Theta^{\epsilon}, \nabla \partial_x^{\alpha} \Theta^{\epsilon} \rangle
$$

$$
- (\eta_9 + \eta_{10} + \eta_{11} + \eta_{12}) \|\nabla \partial_x^{\alpha} \Theta^{\epsilon} \|^2
$$

$$
\leq C_{\eta, \gamma} \left[(\|\mathcal{E}^{\epsilon}(t)\|_{s}^{2(s+1)} + \|\mathcal{E}^{\epsilon}(t)\|_{s}^{2} + \|\mathcal{E}^{\epsilon}(t)\|_{s} + 1) \|\mathcal{E}^{\epsilon}(t)\|_{s}^{2} \right]
$$

$$
+ \gamma_{1} \|\mathbf{F}^{\epsilon}\|_{s}^{4} + \langle \gamma_{2} + \gamma_{3} + \frac{1}{16} \rangle \|\mathbf{F}^{\epsilon}\|_{s}^{2} \tag{3.52}
$$

for some constant $C_{\eta,\gamma} > 0$ depending on η_i (i = 9, 10, 11, 12) and γ_j (j = 1, 2, 3).

Applying the operator ∂_x^{α} to [\(2.4\)](#page-9-2) and [\(2.5\)](#page-9-0), multiplying the results by $\partial_x^{\alpha} \mathbf{F}^{\epsilon}$ and $\partial_x^{\alpha} \mathbf{G}^{\epsilon}$ respectively, and integrating then over \mathbb{T}^3 , one obtains that

$$
\frac{1}{2} \frac{d}{dt} (\epsilon \| \partial_x^{\alpha} \mathbf{F}^{\epsilon} \|^{2} + \| \partial_x^{\alpha} \mathbf{G}^{\epsilon} \|^{2}) + \| \partial_x^{\alpha} \mathbf{F}^{\epsilon} \|^{2} \n+ \int (\text{curl } \partial_x^{\alpha} \mathbf{F}^{\epsilon} \cdot \partial_x^{\alpha} \mathbf{G}^{\epsilon} - \text{curl } \partial_x^{\alpha} \mathbf{G}^{\epsilon} \cdot \partial_x^{\alpha} \mathbf{F}^{\epsilon}) dx \n= \langle [\partial_x^{\alpha} (\mathbf{U}^{\epsilon} \times \mathbf{H}^0) + \partial_x^{\alpha} (\mathbf{u}^0 \times \mathbf{G}^{\epsilon})] - \partial_x^{\alpha} (\mathbf{U}^{\epsilon} \times \mathbf{G}^{\epsilon}), \partial_x^{\alpha} \mathbf{F}^{\epsilon} \rangle \n- \langle \epsilon \partial_x^{\alpha} \partial_t \text{curl } \mathbf{H}^0 + \epsilon \partial_x^{\alpha} \partial_t (\mathbf{u}^0 \times \mathbf{H}^0), \partial_x^{\alpha} \mathbf{F}^{\epsilon} \rangle.
$$
\n(3.53)

Following a process similar to that in [\[18\]](#page-24-5) and applying [\(3.53\)](#page-22-1), we finally obtain that

$$
\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} (\epsilon \| \partial_x^{\alpha} \mathbf{F}^{\epsilon} \|^{2} + \| \partial_x^{\alpha} \mathbf{G}^{\epsilon} \|^{2}) + \frac{3}{4} \| \partial_x^{\alpha} \mathbf{F}^{\epsilon} \|^{2} \leq C (\| \mathcal{E}^{\epsilon}(t) \|_{s}^{2} + 1) \| (\partial_x^{\alpha} \mathbf{U}^{\epsilon}, \partial_x^{\alpha} \mathbf{G}^{\epsilon}) \|^{2} + C \epsilon^{2}.
$$
\n(3.54)

Combining [\(3.10\)](#page-13-2), [\(3.32\)](#page-18-1), and [\(3.52\)](#page-22-2) with [\(3.54\)](#page-22-3), summing up α with $0 \leq |\alpha| \leq s$, using the fact that $N^{\epsilon} + \rho^0 \geq \hat{N} + \hat{\rho} > 0$, $\mathbf{F}^{\epsilon} \in C^l([0,T], H^{s-2l})$ $(l = 0, 1)$, and

choosing η_i (i = 1, ..., 12) and $\gamma_1, \gamma_2, \gamma_3$ sufficiently small, we obtain [\(3.3\)](#page-12-2). This completes the proof of Lemma 3.2. \Box

With the estimate [\(3.3\)](#page-12-2) in hand, we can now prove Proposition [3.1.](#page-11-1)

Proof of Proposition [3.1.](#page-11-1) As in [\[18,](#page-24-5) [32\]](#page-25-3), we introduce an ϵ -weighted energy functional

$$
\Gamma^{\epsilon}(t) = ||\mathcal{E}^{\epsilon}(t)||_{s}^{2}.
$$

Then, it follows from [\(3.3\)](#page-12-2) that there exists a constant $\epsilon > 0$ depending only on T, such that for any $\epsilon \in (0, \epsilon]$ and any $t \in (0, T]$,

$$
\Gamma^{\epsilon}(t) \leq C\Gamma^{\epsilon}(t=0) + C\int_0^t \left\{ \left((\Gamma^{\epsilon})^s + \Gamma^{\epsilon} + 1 \right) \Gamma^{\epsilon} \right\} (\tau) d\tau + C\epsilon^2.
$$
 (3.55)

Thus, applying the Gronwall lemma to [\(3.55\)](#page-23-7), and keeping in mind that Γ^{ϵ} ($t =$ $(0) \leq C\epsilon^2$ and Proposition [3.1,](#page-11-1) we find that there exist a $0 < T_1 < 1$ and an $\epsilon > 0$, such that $T^{\epsilon} \geq T_1$ for all $\epsilon \in (0, \epsilon]$ and $\Gamma^{\epsilon}(t) \leq C\epsilon^2$ for all $t \in (0, T_1]$. Therefore, the desired a priori estimate [\(3.2\)](#page-11-3) holds. Moreover, by the standard continuation induction argument, we can extend $T^{\epsilon} \geq T_0$ to any $T_0 < T_*$.

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