New infinite families of congruences modulo 8 for partitions with even parts distinct

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Abstract

Let $ped(n)$ denote the number of partitions of an integer n wherein even parts are distinct. Recently, Andrews, Hirschhorn and Sellers, Chen, and Cui and Gu have derived a number of interesting congruences modulo 2, 3 and 4 for $ped(n)$. In this paper we prove several new infinite families of congruences modulo 8 for $ped(n)$. For example, we prove that for $\alpha \geqslant 0$ and $n \geqslant 0$,

$$ped\left(3^{4\alpha+4}n + \frac{11 \times 3^{4\alpha+3} - 1}{8}\right) \equiv 0 \pmod{8}.$$

Keywords: partition; congruence; regular partition

1 Introduction

Let $ped(n)$ denote the function which enumerates the number of partitions of n wherein even parts are distinct (and odd parts are unrestricted). For a positive integer t we say that a partition is t-regular if no part is divisible by t. Andrews, Hirschhorn and Sellers [1] found the generating function for $ped(n)$:

$$\sum_{n=0}^{\infty} ped(n)q^n = \prod_{n=1}^{\infty} \frac{1+q^{2n}}{1-q^{2n-1}} = \frac{f_4}{f_1}, \tag{1}$$

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where here and throughout this paper, and for any positive integer k, f_k is defined by

$$f_k := \prod_{n=1}^{\infty}(1 - q^{kn}). \tag{2}$$

From (1) it is easy to see that $ped(n)$ equals the number of 4-regular partitions of n. In recent years many congruences for the number of regular partitions have been discovered (see for example, Cui and Gu [3, 4], Dandurand and Penniston [5], Furcy and Penniston [7], Gordon and Ono [8], Keith [10], Lin and Wang [11], Lovejoy and Penniston [12], Penniston [13, 14], Webb [15], Xia and Yao [16, 17], and Yao[18]).

Numerous congruence properties are known for the function $ped(n)$. For example, Andrews, Hirschhorn and Sellers [1] proved that for $\alpha \geqslant 1$ and $n \geqslant 0$,

$$ped(3n+2) \equiv 0 \pmod{2}, \tag{3}$$

$$ped(9n+4) \equiv 0 \pmod{4}, \tag{4}$$

$$ped(9n+7) \equiv 0 \pmod{12}, \tag{5}$$

$$ped\left(3^{2\alpha+2}n + \frac{11 \times 3^{2\alpha+1} - 1}{8}\right) \equiv 0 \pmod{2}, \tag{6}$$

$$ped\left(3^{2\alpha+1}n + \frac{17 \times 3^{2\alpha} - 1}{8}\right) \equiv 0 \pmod{6}, \tag{7}$$

$$ped\left(3^{2\alpha+2}n + \frac{19 \times 3^{2\alpha+1} - 1}{8}\right) \equiv 0 \pmod{6}. \tag{8}$$

Recently, Chen [2] obtained many interesting congruences modulo 2 and 4 for $ped(n)$ using the theory of Hecke eigenforms and Cui and Gu [3] found infinite families of wonderful congruences modulo 2 for the function $ped(n)$.

The aim of this paper is to establish several new infinite families of congruences modulo 8 for $ped(n)$ by employing some results of Andrews, Hirschhorn and Sellers [1], and Cui and Gu [3]. The main results of this paper can be stated as the following theorems.

Theorem 1. *For $\alpha \geqslant 0$ and $n \geqslant 0$,*

$$ped\left(3^{2\alpha}n + \frac{3^{2\alpha} - 1}{8}\right) \equiv ped(n) \pmod{4}, \tag{9}$$

$$ped\left(3^{4\alpha}n + \frac{3^{4\alpha} - 1}{8}\right) \equiv 5^{\alpha} ped(n) \pmod{8}, \tag{10}$$

$$ped\left(3^{4\alpha+4}n + \frac{11 \times 3^{4\alpha+3} - 1}{8}\right) \equiv 0 \pmod{8}, \tag{11}$$

$$ped\left(3^{4\alpha+4}n + \frac{19 \times 3^{4\alpha+3} - 1}{8}\right) \equiv 0 \pmod{8}. \tag{12}$$

In view of (9) and the facts $ped(1) = 1$, $ped(2) = 2$, $ped(3) = 3$, $ped(4) = 4$, we obtain the following corollary.

Corollary 2. *For $\alpha \geqslant 0$ and $i = 0,\ 1,\ 2,\ 3$ we have that*

$$ped\left(\frac{t_i \times 3^{2\alpha} - 1}{8}\right) \equiv i \pmod 4, \tag{13}$$

where $t_0 = 33$, $t_1 = 9$, $t_2 = 17$ and $t_3 = 25$.

Replacing α by 2α in (10), we find that for $\alpha \geqslant 0$,

$$ped\left(3^{8\alpha}n + \frac{3^{8\alpha} - 1}{8}\right) \equiv ped(n) \pmod 8. \tag{14}$$

Employing (14) and the facts $ped(1) = 1$, $ped(2) = 2$, $ped(3) = 3$, $ped(4) = 4$, $ped(10) = 29$, $ped(5) = 6$, $ped(253) = 5178754681431$ and $ped(8) = 16$, we obtain the following congruences modulo 8.

Corollary 3. *For $\alpha \geqslant 0$ and $0 \leqslant j \leqslant 7$ we have that*

$$ped\left(\frac{s_j \times 3^{8\alpha} - 1}{8}\right) \equiv j \pmod 8, \tag{15}$$

where $s_0 = 65$, $s_1 = 9$, $s_2 = 17$, $s_3 = 25$, $s_4 = 33$, $s_5 = 81$, $s_6 = 41$ and $s_7 = 2025$.

Utilizing the generating functions of $ped(9n+4)$, $ped(9n+7)$ discovered by Andrews, Hirschhorn and Sellers [1] and the p-dissection identities of two Ramanujan's theta functions due to Cui and Gu [3], we will prove the following theorem.

Theorem 4. *Let p be a prime such that $p \equiv 5,\ 7 \pmod 8$ and $1 \leqslant i \leqslant p-1$. Then for $n \geqslant 0$ and $\alpha \geqslant 1$,*

$$ped\left(9p^{2\alpha}n + \frac{(72i + 33p)p^{2\alpha-1} - 1}{8}\right) \equiv 0 \pmod 8 \tag{16}$$

and

$$ped\left(9p^{2\alpha}n + \frac{(72i + 57p)p^{2\alpha-1} - 1}{8}\right) \equiv 0 \pmod 8. \tag{17}$$

2 Proof of Theorem 1

Andrews, Hirschhorn and Sellers [1] established the following results for $ped(3n+1)$:

$$\sum_{n=0}^{\infty} ped(9n+1)q^n = \frac{f_2^2 f_3^4 f_4}{f_1^5 f_6^2} + 24q\frac{f_2^3 f_3^3 f_4 f_6^3}{f_1^{10}}, \tag{18}$$

$$\sum_{n=0}^{\infty} ped(9n+4)q^n = 4\frac{f_2 f_3 f_4 f_6}{f_1^4} + 48q\frac{f_2^2 f_4 f_6^6}{f_1^9} \tag{19}$$

and

$$\sum_{n=0}^{\infty} ped(9n+7)q^n = 12\frac{f_2^4 f_3^6 f_4}{f_1^{11}}. \tag{20}$$

By the binomial theorem it is easy to see that for all positive integers m and k,

$$f_k^{2m} \equiv f_{2k}^m \pmod{2}. \tag{21}$$

By (21) we see that

$$\frac{f_1^2}{f_2} \equiv \frac{f_2}{f_1^2} \equiv 1 \pmod{2}, \tag{22}$$

which yields

$$\frac{f_2^2}{f_1^4} \equiv \frac{f_3^4}{f_6^2} \equiv 1 \pmod{4}. \tag{23}$$

It follows from (18) and (23) that

$$\sum_{n=0}^{\infty} ped(9n+1)q^n \equiv \frac{f_4}{f_1} \pmod{4}. \tag{24}$$

In view of (1) and (24) we see that for $n \geqslant 0$,

$$ped(9n+1) \equiv ped(n) \pmod{4}. \tag{25}$$

Congruence (9) follows from (25) and mathematical induction.

Andrews, Hirschhorn and Sellers [1] also established the following 3-dissection formula of the generating function of $ped(n)$:

$$\sum_{n=0}^{\infty} ped(n)q^n = \frac{f_{12}f_{18}^4}{f_3^3 f_{36}^2} + q\frac{f_6^2 f_9^3 f_{36}}{f_3^4 f_{18}^2} + 2q^2\frac{f_6 f_{18} f_{36}}{f_3^3}. \tag{26}$$

Fortin, Jacob and Mathieu [6], and Hirschhorn and Sellers [9] independently derived the following 3-dissection formula of the generating function of overpartitions:

$$\frac{f_2}{f_1^2} = \frac{f_6^4 f_9^6}{f_3^8 f_{18}^3} + 2q\frac{f_6^3 f_9^3}{f_3^7} + 4q^2\frac{f_6^2 f_{18}^3}{f_3^6}. \tag{27}$$

Combining (1), (18), (26), (27) we deduced that

$$\begin{aligned}
\sum_{n=0}^{\infty} ped(9n+1)q^n &\equiv \frac{f_3^4}{f_6^2}\frac{f_2^2}{f_1^4}\frac{f_4}{f_1} \\
&\equiv \frac{f_3^4}{f_6^2}\left(\frac{f_6^4 f_9^6}{f_3^8 f_{18}^3} + 2q\frac{f_6^3 f_9^3}{f_3^7} + 4q^2\frac{f_6^2 f_{18}^3}{f_3^6}\right)^2 \left(\frac{f_{12}f_{18}^4}{f_3^3 f_{36}^2} + q\frac{f_6^2 f_9^3 f_{36}}{f_3^4 f_{18}^2} + 2q^2\frac{f_6 f_{18} f_{36}}{f_3^3}\right) \\
&\equiv \frac{f_6^6 f_9^{12} f_{12}}{f_3^{15} f_{18}^2 f_{36}^2} + q\frac{f_6^8 f_9^{15} f_{36}}{f_3^{16} f_{18}^8} + 4q\frac{f_6^5 f_9^9 f_{12} f_{18}}{f_3^{14} f_{36}^2} + 6q^2\frac{f_6^7 f_9^{12} f_{36}}{f_3^{15} f_{18}^5} \\
&\qquad + 4q^2\frac{f_6^4 f_9^6 f_{12} f_{18}^4}{f_3^{13} f_{36}^2} + 4q^3\frac{f_6^6 f_9^9 f_{36}}{f_3^{14} f_{18}^2} \pmod{8}.
\end{aligned} \tag{28}$$

Extracting those terms associated with powers q^{3n+1} on both sides of (28), then dividing by q and replacing q^3 by q, we find that

$$\sum_{n=0}^{\infty} ped(27n+10)q^n \equiv \frac{f_2^8 f_3^{15} f_{12}}{f_1^{16} f_6^8} + 4\frac{f_2^5 f_3^9 f_4 f_6}{f_1^{14} f_{12}^2} \pmod 8. \tag{29}$$

By the binomial theorem and (22) we have

$$\frac{f_2^8}{f_1^{16}} \equiv \frac{f_3^{16}}{f_6^8} \equiv 1 \pmod 8, \tag{30}$$

which yields

$$\frac{f_2^8 f_3^{15} f_{12}}{f_1^{16} f_6^8} \equiv \frac{f_{12}}{f_3} \pmod 8. \tag{31}$$

It follows from (21) that

$$\frac{f_2^5 f_3^9 f_4 f_6}{f_1^{14} f_{12}^2} \equiv \frac{f_{12}}{f_3} \pmod 2. \tag{32}$$

Substituting (31) and (32) into (29), we see that

$$\sum_{n=0}^{\infty} ped(27n+10)q^n \equiv 5\frac{f_{12}}{f_3} \pmod 8, \tag{33}$$

which implies that

$$\sum_{n=0}^{\infty} ped(81n+10)q^n \equiv 5\frac{f_4}{f_1} \pmod 8 \tag{34}$$

and for $n \geqslant 0$,

$$ped(81n+37) \equiv 0 \pmod 8, \tag{35}$$

$$ped(81n+64) \equiv 0 \pmod 8. \tag{36}$$

Thanks to (1) and (34), we see that for $n \geqslant 0$,

$$ped(81n+10) \equiv 5ped(n) \pmod 8. \tag{37}$$

Congruence (10) follows from (37) and mathematical induction. Replacing n by $81n+37$ in (10) and employing (35), we obtain (11). Replacing n by $81n+64$ in (10) and using (36), we deduce (12). The proof is complete.

3 Proof of Theorem 4

Thanks to (19) and (21), we have

$$\sum_{n=0}^{\infty} ped(9n+4)q^n \equiv 4f_2\psi(q^3) \pmod 8, \tag{38}$$

where $\psi(q)$ is defined by

$$\psi(q) := \frac{f_2^2}{f_1}. \tag{39}$$

In their nice paper [3], Cui and Gu established p-dissection formulas for f_1 and $\psi(q)$. They proved that for any odd prime p,

$$\psi(q) = \sum_{k=0}^{\frac{p-3}{2}} q^{\frac{k^2+k}{2}} f\left(q^{\frac{p^2+(2k+1)p}{2}}, q^{\frac{p^2-(2k+1)p}{2}}\right) + q^{\frac{p^2-1}{8}}\psi(q^{p^2}) \tag{40}$$

and for any prime $p \geqslant 5$,

$$f_1 = \sum_{\substack{k=\frac{1-p}{2},\\ k\neq \frac{\pm p-1}{6}}}^{\frac{p-1}{2}} (-1)^k q^{\frac{3k^2+k}{2}} f\left(-q^{\frac{3p^2+(6k+1)p}{2}}, -q^{\frac{3p^2-(6k+1)p}{2}}\right) + (-1)^{\frac{\pm p-1}{6}} q^{\frac{p^2-1}{24}} f_{p^2}, \tag{41}$$

where

$$\frac{\pm p-1}{6} := \begin{cases} \dfrac{p-1}{6} & \text{if } p \equiv 1 \pmod 6, \\ \dfrac{-p-1}{6} & \text{if } p \equiv -1 \pmod 6 \end{cases} \tag{42}$$

and the Ramanujan theta function $f(a,b)$ is defined by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \tag{43}$$

where $|ab| < 1$.

Let $a(n)$ be defined by

$$\sum_{n=0}^{\infty} a(n)q^n := f_2\psi(q^3). \tag{44}$$

It follows from (38) and (44) that for $n \geqslant 0$,

$$ped(9n+4) \equiv 4a(n) \pmod 8. \tag{45}$$

Substituting (40) and (41) into (44), we see that for any prime $p \equiv 5,\ 7 \pmod 8$,

$$\sum_{n=0}^{\infty} a(n)q^n \tag{46}$$

$$= \left(\sum_{\substack{m=\frac{1-p}{2}, \\ m\neq \frac{\pm p-1}{6}}}^{\frac{p-1}{2}} (-1)^m q^{3m^2+m} f\left(-q^{3p^2+(6m+1)p}, -q^{3p^2-(6m+1)p}\right) + (-1)^{\frac{\pm p-1}{6}} q^{\frac{p^2-1}{12}} f_{2p^2} \right)$$

$$\times \left(\sum_{k=0}^{\frac{p-3}{2}} q^{\frac{3(k^2+k)}{2}} f\left(q^{\frac{3(p^2+(2k+1)p)}{2}}, q^{\frac{3(p^2-(2k+1)p)}{2}}\right) + q^{\frac{3(p^2-1)}{8}} \psi(q^{3p^2}) \right). \tag{47}$$

Now, we consider the congruence

$$3m^2 + m + \frac{3(k^2+k)}{2} \equiv \frac{11(p^2-1)}{24} \pmod p, \tag{48}$$

where $-\frac{p-1}{2} \leqslant m \leqslant \frac{p-1}{2}$ and $0 \leqslant k \leqslant \frac{p-1}{2}$. Congruence (48) can be rewritten as follows

$$2(6m+1)^2 + (6k+3)^2 \equiv 0 \pmod p. \tag{49}$$

Since $p \equiv 5,\ 7 \pmod 8$, we have that -2 is a ratic nonresidue modulo p and hence (49) is equivalent to

$$6m+1 \equiv 6k+3 \equiv 0 \pmod p. \tag{50}$$

Thus, $m = \frac{\pm p-1}{6}$ and $k = \frac{p-1}{2}$. Extracting those terms associated with powers $q^{pn+\frac{11(p^2-1)}{24}}$ on both sides of (46) and employing the fact that Congruence (48) holds if and only if $m = \frac{\pm p-1}{6}$ and $k = \frac{p-1}{2}$, we have

$$\sum_{n=0}^{\infty} a\left(pn + \frac{11(p^2-1)}{24}\right) q^{pn+\frac{11(p^2-1)}{24}} = (-1)^{\frac{\pm p-1}{6}} q^{\frac{11(p^2-1)}{24}} f_{2p^2} \psi(q^{3p^2}). \tag{51}$$

Dividing $q^{\frac{11(p^2-1)}{24}}$ on both sides of (51) and then replacing q^p by q, we get

$$\sum_{n=0}^{\infty} a\left(pn + \frac{11(p^2-1)}{24}\right) q^n = (-1)^{\frac{\pm p-1}{6}} f_{2p} \psi(q^{3p}), \tag{52}$$

which implies that

$$\sum_{n=0}^{\infty} a\left(p^2n + \frac{11(p^2-1)}{24}\right) q^n = (-1)^{\frac{\pm p-1}{6}} f_2 \psi(q^3) \tag{53}$$

and

$$a\left(p(pn+i)+\frac{11(p^2-1)}{24}\right)=0 \tag{54}$$

for $n\geqslant 0$ and $1\leqslant i\leqslant p-1$. Combining (44) and (53), we have

$$a\left(p^2n+\frac{11(p^2-1)}{24}\right)\equiv a(n) \pmod 2. \tag{55}$$

By (55) and mathematical induction, we find that for $n\geqslant 0$ and $\alpha\geqslant 0$,

$$a\left(p^{2\alpha}n+\frac{11(p^{2\alpha}-1)}{24}\right)\equiv a(n) \pmod 2. \tag{56}$$

Replacing n by $p(pn+i)+\frac{11(p^2-1)}{24}$ $(1\leqslant i\leqslant p-1)$ in (56) and using (54), we deduce that for $n\geqslant 0$ and $\alpha\geqslant 1$,

$$a\left(p^{2\alpha}n+\frac{(24i+11p)p^{2\alpha-1}-11}{24}\right)\equiv 0 \pmod 2. \tag{57}$$

Finally, replacing n by $p^{2\alpha}n+\frac{(24i+11p)p^{2\alpha-1}-11}{24}$ $(1\leqslant i\leqslant p-1)$ in (45) and using (57), we get (16).

We conclude the paper by proving (17). In view of (20) and (21), we find that

$$\sum_{n=0}^{\infty} ped(9n+7)q^n\equiv 4f_1\psi(q^6) \pmod 8, \tag{58}$$

where $\psi(q)$ is defined by (39). Let $b(n)$ be defined by

$$\sum_{n=0}^{\infty} b(n)q^n := f_1\psi(q^6). \tag{59}$$

By (58) and (59), we find that for $n\geqslant 0$,

$$ped(9n+7)\equiv 4b(n) \pmod 8. \tag{60}$$

Substituting (40) and (41) into (59), we see that for any prime $p\equiv 5,\ 7 \pmod 8$,

$$\begin{aligned}\sum_{n=0}^{\infty} b(n)q^n =&\left(\sum_{\substack{m=\frac{1-p}{2},\\ m\neq\frac{\pm p-1}{6}}}^{\frac{p-1}{2}}(-1)^m q^{\frac{3m^2+m}{2}} f\left(-q^{\frac{3p^2+(6m+1)p}{2}},-q^{\frac{3p^2-(6m+1)p}{2}}\right)+(-1)^{\frac{\pm p-1}{6}}q^{\frac{p^2-1}{24}}f_{p^2}\right)\\ &\times\left(\sum_{k=0}^{\frac{p-3}{2}} q^{3(k^2+k)} f\left(q^{3(p^2+(2k+1)p)},q^{3(p^2-(2k+1)p)}\right)+q^{\frac{3(p^2-1)}{4}}\psi(q^{6p^2})\right).\end{aligned} \tag{61}$$

As above, for any prime $p \equiv 5,\ 7 \pmod 8$, $-\frac{p-1}{2} \leqslant m \leqslant \frac{p-1}{2}$ and $0 \leqslant k \leqslant \frac{p-1}{2}$, the congruence relation

$$\frac{3m^2+m}{2} + 3(k^2+k) \equiv \frac{19(p^2-1)}{24} \pmod p \tag{62}$$

holds if and only if $m = \frac{\pm p-1}{6}$ and $k = \frac{p-1}{2}$. This implies that

$$\sum_{n=0}^{\infty} b\left(pn + \frac{19(p^2-1)}{24}\right) q^n = (-1)^{\frac{\pm p-1}{6}} f_p \psi(q^{6p}). \tag{63}$$

Thanks to (63), we find that

$$\sum_{n=0}^{\infty} b\left(p^2 n + \frac{19(p^2-1)}{24}\right) q^n = (-1)^{\frac{\pm p-1}{6}} f_1 \psi(q^6) \tag{64}$$

and

$$b\left(p(pn+i) + \frac{19(p^2-1)}{24}\right) = 0 \tag{65}$$

for $n \geqslant 0$ and $1 \leqslant i \leqslant p-1$. It follows from (59) and (64) that for $n \geqslant 0$,

$$b\left(p^2 n + \frac{19(p^2-1)}{24}\right) \equiv b(n) \pmod 2. \tag{66}$$

By (66) and mathematical induction, we deduce that for $n \geqslant 0$ and $\alpha \geqslant 0$,

$$b\left(p^{2\alpha} n + \frac{19(p^{2\alpha}-1)}{24}\right) \equiv b(n) \pmod 2. \tag{67}$$

Replacing n by $p(pn+i) + \frac{19(p^2-1)}{24}$ $(1 \leqslant i \leqslant p-1)$ in (67) and employing (65), we find that

$$b\left(p^{2\alpha} n + \frac{(24i+19p)p^{2\alpha-1}-19}{24}\right) \equiv 0 \pmod 2 \tag{68}$$

for $n \geqslant 0$, $\alpha \geqslant 1$ and $1 \leqslant i \leqslant p-1$. Congruence (17) follows from (60) and (68). This completes the proof of Theorem 4.

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