# Special case of Rota's basis conjecture on graphic matroids

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**Abstract**

Gian-Carlo Rota conjectured that for any $n$ bases $B_1, B_2, \ldots, B_n$ in a matroid of rank $n$, there exist $n$ disjoint transversal bases of $B_1, B_2, \ldots, B_n$. The conjecture for graphic matroids corresponds to the problem of an edge-decomposition as follows; If an $n$-vertex edge-colored connected multigraph $G$ has $n-1$ colors and the graph induced by the edges colored with $c$ is a spanning tree for each color $c$, then $G$ has $n-1$ mutually edge-disjoint rainbow spanning trees. In this paper, we prove that edge-colored graphs where the edges colored with $c$ induce a spanning star for each color $c$ can be decomposed into rainbow spanning trees.

**Mathematics Subject Classifications:** 05C05, 05C70

## 1 Introduction

The matroids are an abstraction of a concept of independecy or dependency. The matroids derived from graphs are vital example of the matroids, and lead to more general results in graph theory. A *matroid* $M$ is an ordered pair $(E, \mathcal{B})$ consisting of a finite set $E$ and a non-empty collection $\mathcal{B}$ of subsets of $E$ satisfying the basis exchange axiom. More precisely, $\mathcal{B}$ satisfies the following: If $B_1$ and $B_2$ are members of $\mathcal{B}$ and $x \in B_1 \setminus B_2$, then there is an element $y \in B_2 \setminus B_1$ such that $(B_1 \setminus \{x\}) \cup \{y\} \in \mathcal{B}$. We call an element in $\mathcal{B}$ a *basis* for $M$. It follows from the basis exchange axiom that all members of $\mathcal{B}$ have the same cardinality. We define the *rank* of $M$ to be the cardinality of a basis in $\mathcal{B}$. Two

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matroids $M = (E, \mathcal{B})$ and $M' = (E', \mathcal{B}')$ are *isomorphic*, written by $M \cong M'$, if there is a bijection $\varphi : E \to E'$ such that $X \subseteq E$ is independent in $M$ if and only if $\varphi(X) \subseteq E'$ is independent in $M'$.

Let $E$ be the set of column labels of an $m \times n$ matrix $A$ over the filed $\mathbb{R}$ of real numbers. Let $\mathcal{B}$ be the subsets of $E$ whose set of corresponding column vectors of $A$ is a basis of $V(m, \mathbb{R})$. Then $M = (E, \mathcal{B})$ is a matroid, called a *real-representable matroid.* A *transversal basis* of $n$ bases $B_1, B_2, \dots, B_n$ in a matroid of rank $n$ is a basis $e_1, e_2, \dots, e_n$ for some $e_i \in B_i$ for each $i \in \{1, 2, \dots, n\}$.

The following conjecture, so-called Rota's basis conjecture, was posed by Gian-Carlo Rota.

**Conjecture 1** (Rota's basis conjecture [13])**.** For given $n$ bases $B_1, B_2, \dots, B_n$ in a matroid of rank $n$, there exist $n$ disjoint transversal bases of $B_1, B_2, \dots, B_n$.

Huang and Rota [13] proved that if Alon-Tarsi conjecture on Latin squares holds for $n \times n$ Latin square for an even integer $n$, then Conjecture 1 holds for real-representable matroids of rank $n$. Drisko [7] and Glynn [12] proved that Alon-Tarsi conjecture is true for $n = p + 1$ and $n = p - 1$ if $p$ is an odd prime. Hence Conjecture 1 is true for real-representable matroids of rank $n = p \pm 1$. In 1994, Wild [16] proved Conjecture 1 is true for strongly base-orderable matroids and the result implies that Conjecture 1 is true for cycle matroids of series-parallel graphs. In 2006, Geelen and Humphries [9] proved that Conjecture 1 is true for paving matroids, where a paving matroid $M$ of rank $n$ is a matroid in which each circuit has size $n$ or $n + 1$. Cheung [6] computationally proved that the conjecture holds for matroids of rank at most 4. As far as we are aware, there are no other results ensuring the existence of $n$ disjoint transversal bases.

There is a natural approach to Conjecture 1, which is to find many disjoint transversal bases from given $n$ bases in a matroid of rank $n$. In 2007, Geelen and Webb [10] proved that there exist $\Omega(\sqrt{n})$ disjoint transversal bases. In 2019, this was improved by Dong and Geelen [8] and they proved that there exist $\Omega(n/\log n)$ disjoint transversal bases. Recently, Bucić, Kwan, Pokrovskiy, and Sudakov [4] proved that there exist $(1/2 - o(1))n$ disjoint transversal bases. In [15], it was proven that the conjecture holds asymptotically. As described, there are some results about the conjecture but it remains open.

The class of the matroids derived from finite graphs is one of the fundamental classes of matroids. For a graph $G$, let $V(G)$ and $E(G)$ denote the vertex set and the edge set of $G$, respectively. A *forest* is a graph with no cycle and a *tree* is a connected forest. We say that a subgraph $T$ in $G$ is a *spanning tree* in $G$ if $T$ is a tree and $V(T) = V(G)$. The matroid derived from a graph consists as follows: We construct a matroid from the edge set $E(G)$. Let $\mathcal{B}$ be the collection of edge sets of maximal forests in $G$. Then $M(G) = (E(G), \mathcal{B})$ is a matroid. We call it the *cycle matroid* of $G$, written by $M(G)$. If $G$ has $\omega(G)$ components, then the rank of the cycle matroid $M(G)$ is $|V(G)| - \omega(G)$. In particular, if $G$ is connected, then the rank of $M(G)$ is $|V(G)| - 1$, and the set of bases of $M(G)$ is the set of edges of the spanning trees in $G$. A matroid that is isomorphic to the cycle matroid of a graph is called *graphic*. Note that if $M$ is a graphic matroid, then there is a connected graph $G$ such that $M \cong M(G)$.

In this paper, we consider Rota's basis conjecture for graphic matroids. The main purpose of this paper is to ensure the existence of $n$ disjoint transversal bases in Conjecture 1 for graphic matroids by assuming graphical conditions. In order to solve the problem, we use graph-theoritical approaches that are completely different from previous approaches explained in the previous paragraphs. Moreover, our approaches do not depend on any other results. Let us introduce basic terms of graph theory. We only consider finite graphs. Let $K_{n,m}$ be a complete bipartite graph with the size of one partite set $n$ and the size of the other partite set $m$. For a positive integer $n$, $K_{1,n}$ is called a *star* and the vertex of the star with degree $n$ is called its *center*. Note that a center of $K_{1,n}$ is unique for $n \geqslant 2$. We say that a subgraph $T$ in a graph $G$ is a *spanning star* in $G$ if $T$ is a spanning tree and a star.

An *edge-colored graph* is a graph with an edge-coloring. For an edge-colored graph $G$, $C(G)$ denotes the set of colors used in $G$. An edge-colored graph is *rainbow* if no two edges have the same color. The following conjecture is Rota's basis conjecture for graphic matroids.

**Conjecture 2.** Let $G$ be an edge-colored connected multigraph with order $n \geqslant 3$. Suppose that $G$ has $n-1$ colors and the graph induced by the edges colored with $c$ is a spanning tree for each color $c$. Then $G$ has $n-1$ mutually edge-disjoint rainbow spanning trees.

It seems that the study of Conjecture 2 has not well developed yet. In this situation, we take new approaches to Conjecture 2 by considering constructions of edge-colored graphs, and we solve Conjecture 2 when edges colored with $c$ induce a star for each color $c$. The approaches would play an important role to solve not only Conjecture 2 but also Conjecture 1.

**Theorem 3.** *Let $G$ be an edge-colored connected multigraph with order $n$. Suppose that $G$ has $n-1$ colors and the graph induced by the edges colored with $c$ is a spanning star for each color $c$. Then $G$ has $n-1$ mutually edge-disjoint rainbow spanning trees.*

Technical terms for the proof of Theorem 3 are defined in Section 2.

We introduce topics related to Cojecture 2 and Theorem 3. Conjecture 2 is a problem of an edge decomposition of an edge-colored graph into rainbow spanning trees. There are some such decomposition problems in the case when an edge-colored graph is a complete graph as follows.

**Conjecture 4** (Brualdi and Hollingsworth [3])**.** Let $m \geqslant 3$ be an integer and let $K_{2m}$ be an edge-colored complete graph. Suppose that the graph induced by the edges colored with $c$ is a perfect matching for each color $c$. Then the complete graph has $m$ mutually edge-disjoint rainbow spanning trees.

Conjecture 4 was solved in [3] replacing "$m$ edge-disjoint" with "two edge-disjoint". Recently, Conjecture 4 was solved for sufficient large $m$ in [11] (in fact [11] obtained a stronger conclusion than Conjecture 4). Furthermore, there are other results about finding some edge-disjoint rainbow spanning trees in an edge-colored graph: Kaneko, Kano, and

Suzuki [14] showed that every properly edge-colored complete graph $K_n$ has three edge-disjoint rainbow spanning trees for every integer $n \geqslant 6$. Akbari and Alipour [1] showed that every edge-colored complete graph $K_n$ such that no color appears more than $n/2$ times has two edge-disjoint rainbow spanning trees for every integer $n \geqslant 5$. Carraher, Hartke, and Horn [5] showed that every edge-colored complete graph $K_n$ such that no color appears more that $n/2$ times has at least $\lfloor n/(1000 \log n) \rfloor$ edge-disjoint rainbow spanning trees for every integer $n \geqslant 1000000$.

As described, there are many studies of an edge decomposition of an edge-colored complete graph however compared to them, the studies of edge-decomposition of edge-colored non-complete graphs are not still developed. We hope that our result helps to make progress on non-complete cases of such problems.

## 2 Proof

In this section, we prove Theorem 3. It is easy to see that Theorem 3 is true for $n = 2$. Hence we may assume that a graph $G$ has order at least three in the rest of the paper.

For two colors $c$ and $c'$ in $C(G)$, *swapping* $c$ and $c'$ is the operation that the edges colored with $c$ (respectively $c'$) are recolored with $c'$ (respectively $c$).

### 2.1 Preliminaries

We prepare some definitions and results to prove Theorem 3.

For $n \geqslant 1$, let $\mathcal{G}_n$ be the set of edge-colored multigraphs with order $n$ and having $n-1$ colors such that the graph induced by the edges colored with $c$ is a spanning star for each color $c$. For $G \in \mathcal{G}_n$, we may assume that $V(G) = \{v_0, v_1, \ldots, v_{n-1}\}$, and $C(G) = \{c_1, c_2, \ldots, c_{n-1}\}$ in the rest of the paper. For $v \in V(G)$ and $c \in C(G)$, we say that $c$ *belongs* to $v$ in $G$ if $v$ is a center of a monochromatic star whose edges are colored by $c$.

We define two functions on the color set of $G \in \mathcal{G}_n$ as follows: For $G \in \mathcal{G}_n$, the function $f_G : \{1, 2, \ldots, n-1\} \to \{0, 1, \ldots, n-1\}$ satisfies that $c_i \in C(G)$ belongs to $v_{f_G(i)}$. For a rainbow spanning subgraph $T$ in $G$ cotaining all colors in $C(G)$, $e_T(c)$ denotes the edge in $T$ colored with $c$ and we define the function $g_{G,T} : \{1, 2, \ldots, n-1\} \to \{0, 1, \ldots, n-1\}$ such that $v_{g_{G,T}(i)}$ is incident with $e_T(c_i)$ and $c_i$ does not belong to $v_{g_{G,T}(i)}$. Note that $v_{g_{G,T}(i)}$ is just the endpoint of $e_T(c_i)$ that is not the center of the star induced by the edges colored with $c_i$.

**Example 5.** Let $G$ and $T$ be the graph in the left of Fig. 1 and the rainbow spanning tree in $G$ in the right of Fig. 1, respectively. Let $c_1, c_2$, and $c_3$ be colors in $G$ and black lines, dotted lines, and doublet lines represent edges of the color $c_1$, the color $c_2$, and the color $c_3$, respectively. Then $f_G(1) = f_G(2) = f_G(3) = 0$ and $g_{G,T}(1) = 1$, $g_{G,T}(2) = 2$, $g_{G,T}(3) = 3$.

**Definition 6.** Let $G \in \mathcal{G}_n$. We say that $G$ is *good* if $G$ satisfies the following two conditions:

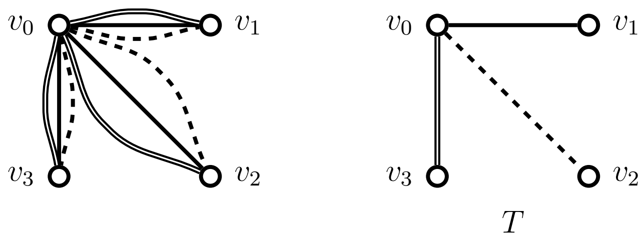


Figure 1: A graph $G$ in $\mathcal{G}_4$ and a rainbow spanning tree $T$ in $G$.

(i) $f_G(i) \geqslant f_G(j)$ for any $1 \leqslant i < j \leqslant n-1$.

(ii) There is an integer $i$ with $0 \leqslant i \leqslant n-2$ such that there exist some colors belonging to $v_j$ for $0 \leqslant j \leqslant i$ and no color belongs to $v_k$ for $i+1 \leqslant k \leqslant n-1$.

**Example 7.** Let $c_1, c_2$, and $c_3$ be colors in the graphs in Fig. 2 and black lines, dotted lines, and doublet lines represent edges of the color $c_1$, the color $c_2$, and the color $c_3$, respectively. In Fig. 2, the left graph is good but the center and right ones are not good since the center one does not satisfy the condition (i), and the right one does not satisfy the condition (ii).

The following proposition is obtained from the definition of goodness and important for the proof of Theorem 3.

**Proposition 8.** *Let $G$ be a good graph. For two integers $j, k$ with $1 \leqslant k < j \leqslant n-1$, we have $f_G(k) - f_G(j) \leqslant j - k$.*

*Proof.* If $f_G(k) = f_G(j)$, then the proposition holds. Hence we may assume that $f_G(k) > f_G(j)$ by the definition (i) of goodness. By the definition (ii) of goodness, for each $i$ with $f_G(j) < i < f_G(k)$, there is a color belonging to $v_i$ and such a color is contained in $\{c_{k+1}, \ldots, c_{j-1}\}$ by the definition (i) of goodness. Since $|\{i \mid f_G(j) < i < f_G(k)\}| = f_G(k) - f_G(j) - 1$ and $|\{i \mid k < i < j\}| = j - k - 1$, we obtain $f_G(k) - f_G(j) \leqslant j - k$. □

It is easy to obtain the following proposition from the definition of goodness. Hence we omit the proof.

**Proposition 9.** *Let $G$ be a good graph. Then $f_G(n-1) = 0$.*

**Proposition 10.** *If Theorem 3 is true for every good graph in $\mathcal{G}_n$, then Theorem 3 is true for every graph in $\mathcal{G}_n$.*

*Proof.* Let $G$ be an edge-colored graph in $\mathcal{G}_n$. The following hold:

- A graph $G'$ obtained from $G$ by swapping two colors has $n-1$ mutually edge-disjoint rainbow spanning trees if and only if $G$ has $n-1$ mutually edge-disjoint rainbow spanning trees.

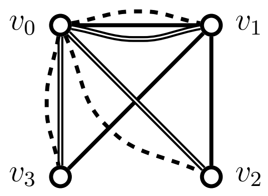


Figure 2: The left graph is good and the center and right ones are not good where black lines, dotted lines, doublet lines represent edges of the color $c_1$, the color $c_2$, and the color $c_3$, respectively.

- A graph $G'$ obtained from $G$ by changing the indices of vertices has $n-1$ mutually edge-disjoint rainbow spanning trees if and only if $G$ has $n-1$ mutually edge-disjoint rainbow spanning trees.

If $G$ is not a good graph, then we can obtain a good graph by swapping some colors and changing indices of $v_i$. Hence the proposition holds. $\square$

### 2.2 Proof of the main theorem

We continue to use the same notations as in Subsection 2.1. We state our main theorem again.

**Theorem 11** (cf. Theorem 3)**.** *For $G \in \mathcal{G}_n$, $G$ has $n-1$ mutually edge-disjoint rainbow spanning trees.*

*Proof.* By Proposition 10, we may assume that $G$ is good. Since we never change a graph $G$ in the rest of the proof, for convenience, we will write $g_T(\cdot)$ and $f(\cdot)$ instead of $g_{G,T}(\cdot)$ and $f_G(\cdot)$, respectively. We define a function $h : \{1, 2, \ldots, n-1\} \times \{1, 2, \ldots, n-1\} \to \mathbb{Z}$ as follows: For $1 \leqslant i \leqslant n-1$ and $1 \leqslant j \leqslant n-1$, if $f(j) \geqslant j$, then

$$\begin{cases} h(i,j) = i+j-2 & \text{if } i+j-1 \leqslant f(j), \\ h(i,j) = i+j-1 \pmod n & \text{otherwise}, \end{cases} \tag{1}$$

and if $f(j) \leqslant j-1$, then

$$\begin{cases} h(i,j) = i+j-1 \pmod n & \text{if } i+j-n \leqslant f(j), \\ h(i,j) = i+j-n & \text{otherwise}. \end{cases} \tag{2}$$

*Claim* 2.1*.* Let $j$ be an integer with $1 \leqslant j \leqslant n-1$. Then the following claims hold:

**Case 1.** $f(j) \geqslant j$

(i) If $1 \leqslant i \leqslant f(j) - j + 1$, then $j - 1 \leqslant h(i,j) \leqslant f(j) - 1$ and $h(i,j) = i + j - 2$.

(ii) If $f(j) - j + 2 \leqslant i \leqslant n - j$, then $f(j) + 1 \leqslant h(i,j) \leqslant n - 1$ and $h(i,j) = i + j - 1$.

(iii) If $n - j + 1 \leqslant i \leqslant n - 1$, then $0 \leqslant h(i,j) \leqslant j - 2$ and $h(i,j) = i + j - 1 - n$.

**Case 2.** $f(j) \leqslant j-1$

(i) If $1 \leqslant i \leqslant n-j$, then $j \leqslant h(i,j) \leqslant n-1$ and $h(i,j) = i+j-1$.

(ii) If $n-j+1 \leqslant i \leqslant n-j+f(j)$, then $0 \leqslant h(i,j) \leqslant f(j)-1$ and $h(i,j) = i+j-1-n$.

(iii) If $n-j+f(j)+1 \leqslant i \leqslant n-1$, then $f(j)+1 \leqslant h(i,j) \leqslant j-1$ and $h(i,j) = i+j-n$.

Moreover, $h(i,j) \neq h(i',j)$ for any $1 \leqslant i \neq i' \leqslant n-1$ and for any $1 \leqslant j \leqslant n-1$.

*Proof.* Case 1–(i) In this case, $f(j) \geqslant i+j-1$ and so $h(i,j) = i+j-2$. It is easy to obtain that $j-1 \leqslant i+j-2 \leqslant f(j)-1$ from $1 \leqslant i \leqslant f(j)-j+1$. Hence $j-1 \leqslant h(i,j) \leqslant f(j)-1$.

Case 1–(ii) In this case, $f(j) \leqslant i+j-2$ and so $h(i,j) = i+j-1 \pmod{n}$. We obtain that $f(j)+1 \leqslant i+j-1 \leqslant n-1$ from $f(j)-j+2 \leqslant i \leqslant n-j$. Hence $f(j)+1 \leqslant h(i,j) \leqslant n-1$ and $h(i,j) = i+j-1$.

Case 1–(iii) In this case, $f(j) \leqslant n-1 \leqslant i+j-2$ and so $h(i,j) = i+j-1 \pmod{n}$. We obtain that $n \leqslant i+j-1 \leqslant n+j-2$ from $n-j+1 \leqslant i \leqslant n-1$. Hence $0 \leqslant h(i,j) \leqslant j-2$ and $h(i,j) = i+j-1-n$.

Case 2–(i) In this case, $i+j-n \leqslant 0 \leqslant f(j)$ and so $h(i,j) = i+j-1 \pmod{n}$. We obtain that $j \leqslant i+j-1 \leqslant n-1$ from $1 \leqslant i \leqslant n-j$. Hence $j \leqslant h(i,j) \leqslant n-1$ and $h(i,j) = i+j-1$.

Case 2–(ii) In this case, $f(j) \geqslant i+j-n$ and so $h(i,j) = i+j-1 \pmod{n}$. We obtain that $n \leqslant i+j-1 \leqslant n+f(j)-1$ from $n-j+1 \leqslant i \leqslant n-j+f(j)$. Hence $0 \leqslant h(i,j) \leqslant f(j)-1$ and $h(i,j) = i+j-1-n$.

Case 2–(iii) In this case, $f(j) \leqslant i+j-1-n$ and so $h(i,j) = i+j-n$. We obtain that $f(j)+1 \leqslant i+j-n \leqslant j-1$ from $n-j+f(j)+1 \leqslant i \leqslant n-1$. Hence $f(j)+1 \leqslant h(i,j) \leqslant j-1$.

For fixed $j$, the ranges of $h(i,j)$ of the cases in Case 1 (respectively Case 2) are mutually disjoint and $h(i,j)$ is a linear function of $i$ for each case. Hence $h$ is an injective mapping of $i$ from $\{1,2,\ldots,n-1\}$ into $\{0,1,\ldots,n-1\} \setminus \{f(j)\}$ for fixed $j$. □

By Claim 2.1, we obtain $f(j) \neq h(i,j)$ for any $1 \leqslant i \leqslant n-1$ and for any $1 \leqslant j \leqslant n-1$ and the following claim holds.

*Claim* 2.2. For any two integers $i$ and $j$ with $1 \leqslant i \leqslant n-1$ and $1 \leqslant j \leqslant n-1$, there is an edge $v_{f(j)}v_{h(i,j)}$ in $G$.

Let $T_i$ be the rainbow spanning subgraph of $G$ such that $E(T_i) = \bigcup_{1 \leqslant j \leqslant n-1} \{v_{f(j)}v_{h(i,j)}\}$. Then $g_{T_i}(j) = h(i,j)$. Since $n \geqslant 2$, it follows from Claim 2.1 that the following claim holds.

*Claim* 2.3. $g_{T_i}(j) \in \{i+j-1, i+j-2, i+j-n, i+j-1-n\}$ and $i+j-1-n \leqslant g_{T_i}(j) \leqslant i+j-1$.

Since $h(i,j) \neq h(i',j)$ for any $1 \leqslant i \neq i' \leqslant n-1$ and for any $1 \leqslant j \leqslant n-1$ by the last statement of Claim 2.1, the following claim holds.

*Claim* 2.4. The spanning subgraphs $T_1, T_2, \ldots, T_{n-1}$ are mutually edge-disjoint.

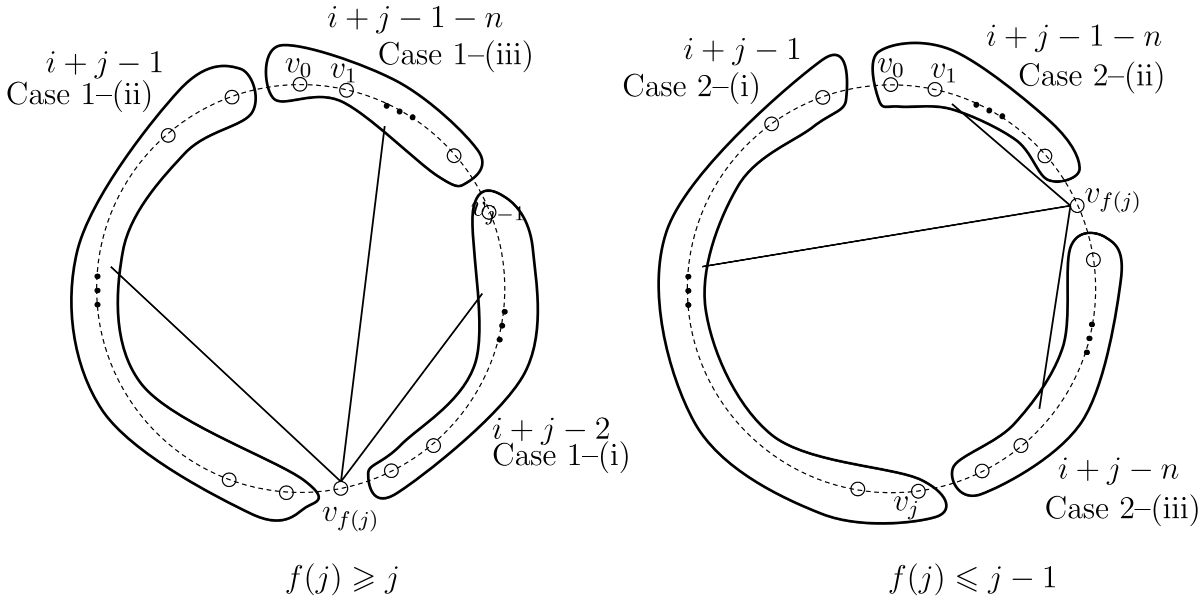


Figure 3: For a fixed integer $j$, vertices surrounded by a closed curve is a range of a formula of $g_{T_i}(j)$ and a corresponding case in Claim 2.1. For example, if $f(j) \geqslant j$, then $g_{T_i(j)} = i+j-2$ if and only if $j-1 \leqslant g_{T_i}(j) \leqslant f(j)-1$ (it corresponds to Case 1–(i) in Claim 2.1).

We shall show that $T_1, T_2, \dots, T_{n-1}$ are spanning trees in $G$. Suppose that $T_i$ is not a spanning tree for some $1 \leqslant i \leqslant n-1$. Since $|E(T_i)| = n-1$, $T_i$ has a cycle $C$. For $1 \leqslant j \leqslant n-1$, we orient the edge of $E(T_i)$ colored with $c_j$ from the center of the star colored with $c_j$ to the other end-vertex, i.e. $e_{T_i}(c_j)$ is assigned a direction from $v_{f(j)}$ to $v_{g_{T_i}(j)}$. Fig. 3 illustrates the definition of $g_{T_i}(j)$ for a fixed integer $j$. In Fig. 3, we arrange the vertices in a circle and write the number of $g_{T_i}(j)$ outside of the circles.

*Claim* 2.5. For every $1 \leqslant j \neq k \leqslant n-1$, $g_{T_i}(j) \neq g_{T_i}(k)$ except for $\{j,k\} = \{1, n-1\}$. If $g_{T_i}(1) = g_{T_i}(n-1)$, then $f(1) \neq f(n-1)$, $f(1) \geqslant i$, and $g_{T_i}(1) = g_{T_i}(n-1) = i-1$.

*Proof.* Suppose that $g_{T_i}(j) = g_{T_i}(k)$ for some $1 \leqslant j \neq k \leqslant n-1$. We may assume that $j > k$. We divide the proof into two cases. By the definition (i) of goodness, we obtain $f(k) \geqslant f(j)$.

*Subclaim* 2.5.1. The following claims hold;

(i) $g_{T_i}(j) \in \{i+j-2, i+j-1-n, i+j-n\}$.

(ii) $g_{T_i}(k) \in \{i+k-1, i+k-2, i+k-n\}$.

*Proof.* (i) Suppose not. By Claim 2.3, $g_{T_i}(j) = i+j-1$. By Claim 2.3, $g_{T_i}(k) \leqslant i+k-1$. Since $j > k$, $g_{T_i}(j) = i+j-1 > i+k-1 \geqslant g_{T_i}(k)$, which contradicts to $g_{T_i}(j) = g_{T_i}(k)$. Hence $g_{T_i}(j) \neq i+j-1$.

(ii) Suppose not. By Claim 2.3, $g_{T_i}(k) = i+k-n-1$. Since $j > k$, $g_{T_i}(k) = i+k-n-1 < i+j-n-1 \leqslant g_{T_i}(j)$, which contradicts to $g_{T_i}(j) = g_{T_i}(k)$. □

**Case 1.** $f(j) \geqslant j$.

By the definition (i) of goodness and our assumptions, $f(k) \geqslant f(j) \geqslant j > k$ and so we obtain $f(k) > k$.

Suppose $g_{T_i}(j) = i + j - 2$. Then $f(j) \geqslant i + j - 1$ by (1). Suppose $g_{T_i}(k) \neq i + k - 1$. By Claim 2.3, $g_{T_i}(k) \leqslant i + k - 2$. Since $j > k$, $g_{T_i}(j) = i + j - 2 > i + k - 2 \geqslant g_{T_i}(k)$, which contradicts to $g_{T_i}(j) = g_{T_i}(k)$. Hence $g_{T_i}(k) = i + k - 1$ and so $f(k) \leqslant i + k - 2$ by (1). Then we obtain $f(k) \leqslant i+k-2 < i+j-1 \leqslant f(j)$, which contradicts to $f(k) \geqslant f(j)$.

Hence we may assume $g_{T_i}(j) = i + j - 1 - n$. If $g_{T_i}(j) \in \{i + k - 2, i + k - 1\}$, then it follows from $g_{T_i}(j) = g_{T_i}(k)$ that $j \geqslant k + n - 1$, which contradicts to $j \leqslant n - 1$. Hence we may assume $g_{T_i}(k) = i + k - n$ and so $f(k) \leqslant i + k - n - 1$ by (2). Since $f(k) > k$, we obtain $k < i + k - n - 1$ and so $n + 1 < i$, which contradicts to $i \leqslant n - 1$. Hence the proof of the case is complete.

**Case 2.** $f(j) \leqslant j - 1$.

Suppose $g_{T_i}(j) = i + j - n$. Suppose further $g_{T_i}(k) = i + k - n$. Since $j > k$, $g_{T_i}(k) = i + k - n < i + j - n = g_{T_i}(j)$, which contradicts to $g_{T_i}(j) = g_{T_i}(k)$. Hence $g_{T_i}(k) \in \{i + k - 1, i + k - 2\}$. If $g_{T_i}(k) = i + k - 1$, then $j = k + n - 1$, which contradicts to $j \leqslant n - 1$. Hence $g_{T_i}(k) = i + k - 2$. Then we obtain $j = k + n - 2$, which implies $k = 1$ and $j = n - 1$ and $g_{T_i}(k) = g_{T_i}(j) = i - 1$. Moreover, by (1) and (2), we obtain $f(k) \geqslant i + k - 1 = i$, $f(j) \leqslant i + j - n - 1 = i - 2$ and so $f(k) \neq f(j)$.

Suppose $g_{T_i}(j) = i + j - 1 - n$. By (2), $f(j) \geqslant i + j - n$. If $g_{T_i}(k)$ is equal to either $i + k - 1$ or $i + k - 2$, then $j \geqslant k + n - 1$, which contradicts to $j \leqslant n - 1$. Hence we may assume $g_{T_i}(k) = i + k - n$ and so $f(k) \leqslant i + k - n - 1$ by (2). Then we obtain $j = k + 1$. However, $f(k) \leqslant i+k-n-1 = i+j-n-2 < f(j)$, which contradicts to $f(k) \geqslant f(j)$. □

We define the types (A1), (A2), (B1), and (B2) of $c_j \in C(G)$ according to the value of $g_{T_i}(j)$.

- If $g_{T_i}(j) = i + j - 1$, then $c_j$ is of type (A1),
- if $g_{T_i}(j) = i + j - n$, then $c_j$ is of type (A2),
- if $g_{T_i}(j) = i + j - 2$, then $c_j$ is of type (B1), and
- if $g_{T_i}(j) = i + j - 1 - n$, then $c_j$ is of type (B2).

We remark that $c_j$ is of type (A1) or (A2) if and only if $f(j) < g_{T_i}(j)$, similarly, $c_j$ is of type (B1) or (B2) if and only if $f(j) > g_{T_i}(j)$.

*Claim* 2.6. The cycle $C$ is a directed cycle.

*Proof.* Suppose that $C$ is not a directed cycle. By Claim 2.5, $C$ has a unique vertex having indegree two in $C$ and $C$ contains $v_{f(1)}v_{i-1}$ and $v_{f(n-1)}v_{i-1}$. Since $f(n-1) = 0$ by Proposition 9, $i - 1 \geqslant 1$ and so $i \geqslant 2$. Moreover, $C$ consists of a vertex $w$ whose outdegree is two in $C$ (possibly $w \in \{v_{f(1)}, v_{f(n-1)}\}$) and two internally disjoint directed paths $P$ and $Q$ from $w$ to $v_{i-1}$ in $C$. We may assume that $P$ contains $v_{f(n-1)}$ and $Q$ contains $v_{f(1)}$. Write $P = v_{f(p_r)}v_{f(p_{r-1})} \dots v_{f(p_1)}v_{i-1}$ and $Q = v_{f(q_s)}v_{f(q_{s-1})} \dots v_{f(q_1)}v_{i-1}$, where $v_{f(p_r)} = v_{f(q_s)} = w$, $v_{f(p_1)} = v_{f(n-1)}$, and $v_{f(q_1)} = v_{f(1)}$.

*Subclaim* 2.6.1. The following claims hold;

(i) If $p_2$ exists, then $c_{p_2}$ is of type (B2) and $f(p_2) \leqslant i-2$.

(ii) If $q_2$ exists, then $c_{q_2}$ is of type (A1) and $f(q_2) \geqslant i$.

*Proof.* (i) Since $g_{T_i}(p_2) = f(p_1) = f(n-1) = 0$, the color $c_{p_2}$ is of type (B1) or (B2). We obtain that if $c_{p_2}$ is of type (B1), then $i + p_2 - 2 = 0$ and so $1 \leqslant p_2 = 2 - i$ and hence $i \leqslant 1$, which contradicts to $i \geqslant 2$. Hence $c_{p_2}$ is of type (B2) and $p_2 = n + 1 - i$. By Proposition 8, we obtain

$$\begin{aligned} f(p_2) &= f(p_2) - f(n-1) \\ &\leqslant n - 1 - p_2 \\ &= n - 1 - (n + 1 - i) \\ &= i - 2. \end{aligned}$$

(ii) Since $g_{T_i}(q_2) = f(1) > f(q_2)$, the color $c_{q_2}$ is of type (A1) or (A2). Since $f(1) \geqslant i$ by Claim 2.5, we obtain $g_{T_i}(q_2) = f(1) \geqslant i$. Hence $c_{q_2}$ is of type (A1). By Proposition 8, we obtain

$$\begin{aligned} q_2 - 1 &\geqslant f(1) - f(q_2) \\ &= g_{T_i}(q_2) - f(q_2) \\ &= i + q_2 - 1 - f(q_2). \end{aligned}$$

From the above inequality, $f(q_2) \geqslant i$. □

*Subclaim* 2.6.2. The following claims hold;

(i) If $r \geqslant 3$, then $f(p_{r'}) \leqslant f(p_2)$ for $3 \leqslant r' \leqslant r$.

(ii) If $s \geqslant 3$, then $f(q_{s'}) \geqslant f(q_2)$ for $3 \leqslant s' \leqslant s$.

*Proof.* (i) Suppose that $f(p_{r'}) > f(p_2)$ for some $3 \leqslant r' \leqslant r$ (see the left graph in Fig. 4). By the definition (i) of goodness, $p_{r'} < p_2$. We may assume that $r'$ is the smallest integer in $\{3, \ldots, r\}$ satisfying $f(p_{r'}) > f(p_2)$. By the choice of $r'$, we obtain $f(p_{r'-1}) \leqslant f(p_2) < f(p_{r'})$ and so $f(p_{r'}) > f(p_{r'-1}) = g_{T_i}(p_{r'})$. This implies that $c_{p_{r'}}$ is of type (B1) or (B2). Suppose that $c_{p_{r'}}$ is of type (B2). Since $p_{r'} < p_2$, it follows from Claim 2.3 that $g_{T_i}(p_{r'}) = i + p_{r'} - 1 - n < i + p_2 - 1 - n \leqslant g_{T_i}(p_2) = 0$ and so $g_{T_i}(p_{r'}) < 0$, which contradicts to $g_{T_i}(p_{r'}) \geqslant 0$. Hence $c_{p_{r'}}$ is of type (B1) and we obtain

$$\begin{aligned} f(p_{r'-1}) - f(n-1) &= g_{T_i}(p_{r'}) - g_{T_i}(p_2) \\ &= i + p_{r'} - 2 - (i + p_2 - 1 - n) \\ &= p_{r'} - p_2 + n - 1. \end{aligned}$$

By Proposition 8, $f(p_{r'-1}) - f(n-1) \leqslant n - 1 - p_{r'-1}$ and this together with the above equation implies $p_{r'} \leqslant p_2 - p_{r'-1}$. Since $f(p_{r'-1}) \leqslant f(p_2)$ and $P$ is a directed path, we obtain $p_2 \leqslant p_{r'-1}$ and so $p_{r'} \leqslant 0$, which contradicts to $p_{r'} \geqslant 1$.

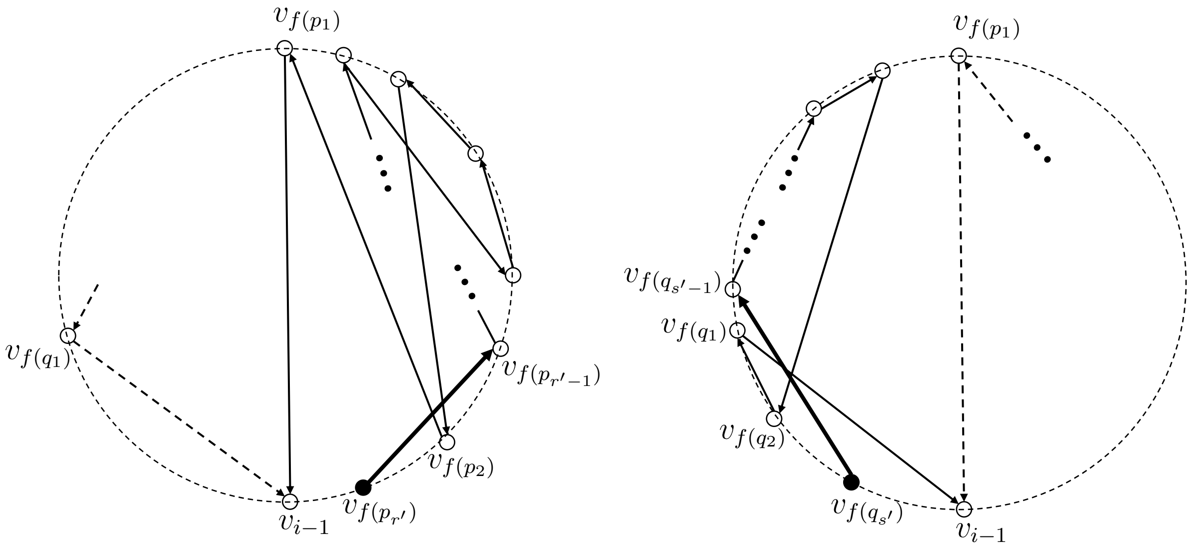


Figure 4: Vertices are labeled with the same ordering as Fig. 3. In the left graph, the black arrows are oriented edges contained in $P$, dotted arrows are oriented edges contained in $Q$, and the black vertex is $v_{f(p_{r'})}$. In the right graph, the black arrows are oriented edges contained in $Q$, dotted arrows are oriented edges contained in $P$, and the black vertex is $v_{f(q_{s'})}$.

(ii) Suppose that $f(q_{s'}) < f(q_2)$ for some $3 \leqslant s' \leqslant s$ (see the right graph in Fig. 4). By the definition (i) of goodness, $q_2 < q_{s'}$. We may assume that $s'$ is the smallest integer in $\{3, \dots, s\}$ satisfying $f(q_{s'}) < f(q_2)$. By the choice of $s'$, we obatin $f(q_{s'-1}) \geqslant f(q_2) > f(q_{s'})$ and so $f(q_{s'}) < f(q_{s'-1}) = g_{T_i}(q_{s'})$. This implies that $c_{q_{s'}}$ is of type (A1) or (A2). Suppose that $c_{q_{s'}}$ is of type (A1). Since $q_2 < q_{s'}$, $f(q_{s'-1}) = g_{T_i}(q_{s'}) = i + q_{s'} - 1 > i + q_2 - 1 = g_{T_i}(q_2) = f(1)$ and so $f(q_{s'-1}) > f(1)$, which contradicts to the definition (i) of goodness. Hence $c_{q_{s'}}$ is of type (A2) and we obtain

$$\begin{aligned} f(1) - f(q_{s'-1}) &= g_{T_i}(q_2) - g_{T_i}(q_{s'}) \\ &= i + q_2 - 1 - (i + q_{s'} - n) \\ &= q_2 - q_{s'} + n - 1. \end{aligned}$$

By Proposition 8, $f(1) - f(q_{s'-1}) \leqslant q_{s'-1} - 1$ and this together with the above equation implies $q_2 - q_{s'-1} + n \leqslant q_{s'}$. Since $f(q_{s'-1}) \geqslant f(q_s)$ and $Q$ is a directed path, we obtain $q_2 > q_{s'-1}$ and so $q_{s'} \geqslant n$, which contradicts to $q_{s'} \leqslant n - 1$. □

By Subclaims 2.6.1 and 2.6.2, $0 \leqslant f(p_{r'}) \leqslant i - 2$ for $2 \leqslant r' \leqslant r$ and $i \leqslant f(q_{s'}) \leqslant f(1)$ for $2 \leqslant s' \leqslant s$. However, this contradicts to $v_{f(p_r)} = v_{f(q_s)}$. □

By Claim 2.6, $C$ is a directed cycle. Let $j_1, j_2, \ldots, j_\ell$ be the integers such that $C = v_{f(j_1)}v_{f(j_2)}\ldots v_{f(j_\ell)}v_{f(j_1)}$ i.e. $g_{T_i}(j_s) = f(j_{s+1})$ for every $1 \leqslant s \leqslant \ell$, where $\ell = |E(C)|$ and $j_{\ell+1} = j_1$. We may assume that $j_1$ is the largest integer in $j_1, j_2, \ldots, j_\ell$. By the definition (i) of goodness and the choice of $j_1$, $f(j_1) < f(j_k)$ for any $2 \leqslant k \leqslant \ell$. Note that $c_{j_1}$ is of type (A1) or (A2) and $c_{j_\ell}$ is of type (B1) or (B2).

*Claim* 2.7. The length of $C$ is at least three, i.e. $\ell \geqslant 3$.

*Proof.* Suppose that $\ell = 2$. Then $c_{j_2}$ is of type (B1) or (B2). We divide the proof into two cases.

**Case 1.** $c_{j_1}$ is of type (A1).

Suppose that $c_{j_2}$ is of type (B1). Then $f(j_2) - f(j_1) = g_{T_i}(j_1) - g_{T_i}(j_2) = i + j_1 - 1 - (i + j_2 - 2) = j_1 - j_2 + 1$, which contradicts Proposition 8. Thus, $c_{j_2}$ is of type (B2). Then $f(j_2) - f(j_1) = g_{T_i}(j_1) - g_{T_i}(j_2) = i + j_1 - 1 - (i + j_2 - 1 - n) = j_1 - j_2 + n > n$, which contradicts to $f(j_2) \leqslant n - 1$.

**Case 2.** $c_{j_1}$ is of type (A2).

Suppose that $c_{j_2}$ is of type (B1). Then $f(j_2) - f(j_1) = g_{T_i}(j_1) - g_{T_i}(j_2) = i + j_1 - n - (i + j_2 - 2) = j_1 - j_2 - n + 2 \leqslant 0$, which contradicts to $f(j_1) < f(j_2)$. Suppose that $c_{j_2}$ is of type (B2). Then $f(j_2) - f(j_1) = g_{T_i}(j_1) - g_{T_i}(j_2) = i + j_1 - n - (i + j_2 - 1 - n) = j_1 - j_2 + 1$, which contradicts Proposition 8. □

*Claim* 2.8. For every $2 \leqslant k \leqslant \ell$, $g_{T_i}(j_1) > g_{T_i}(j_k)$.

*Proof.* Suppose that $g_{T_i}(j_1) < g_{T_i}(j_k)$ for some $2 \leqslant k \leqslant \ell$ (see Fig. 5). We may assume that $k$ is the smallest integer in $\{2, 3, \ldots, \ell\}$ satisfying $g_{T_i}(j_1) < g_{T_i}(j_k)$. By the choice of $k$, we obtain $f(j_k) = g_{T_i}(j_{k-1}) \leqslant g_{T_i}(j_1) < g_{T_i}(j_k)$ and so $c_{j_k}$ is of type (A1) or (A2). By the choice of $j_1$, $j_{k+1} \neq j_1$. Suppose that $c_{j_1}$ is of type (A1). Since $j_1 > j_k$, it follows from Claim 2.3 that we obtain $g_{T_i}(j_1) = i + j_1 - 1 > i + j_k - 1 \geqslant g_{T_i}(j_k)$, which contradicts to $g_{T_i}(j_1) < g_{T_i}(j_k)$. Hence $c_{j_1}$ is of type (A2). Suppose that $c_{j_k}$ is of type (A2). Since $j_1 > j_k$, we obtain $g_{T_i}(j_k) = i + j_k - n < i + j_1 - n = g_{T_i}(j_1)$, which contradicts to $g_{T_i}(j_1) < g_{T_i}(j_k)$. Hence $c_{j_k}$ is of type (A1). Then we obtain

$$\begin{aligned} f(j_{k+1}) - f(j_2) &= g_{T_i}(j_k) - g_{T_i}(j_1) \\ &= i + j_k - 1 - (i + j_1 - n) \\ &= j_k - j_1 + n - 1. \end{aligned} \tag{3}$$

By Proposition 8, $f(j_{k+1}) - f(j_2) \leqslant j_2 - j_{k+1}$. This together with (3) impies

$$j_{k+1} + j_k + n - 1 \leqslant j_1 + j_2.$$

If $k = 2$, then we obtain $j_{k+1} + n - 1 \leqslant j_1$, which contradicts to $j_1 \leqslant n - 1$. Hence we may assume $k \geqslant 3$. By the choice of $k$, we obtain $f(j_k) = g_{T_i}(j_{k-1}) \leqslant g_{T_i}(j_1) = f(j_2)$. Since $C$ is a directed cycle and $k \neq 2$, we obtain $f(j_k) \neq f(j_2)$ and so $f(j_k) < f(j_2)$. This together with the definition (i) of goodness implies that $j_2 \leqslant j_k$ and we obtain $j_{k+1} + n - 1 \leqslant j_1$, which contradicts to $j_1 \leqslant n - 1$. □

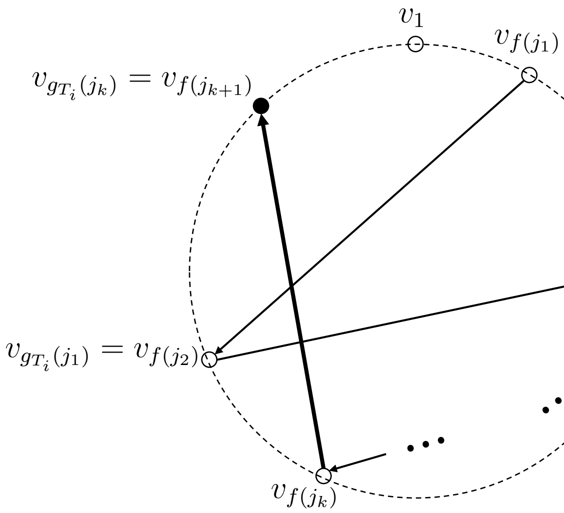


Figure 5: Vertices are labeled with the same ordering as Fig. 3. In the graph, the black arrows are the oriented edges contained in $C$ and the black vertex is $v_{g_{T_i}(j_k)}$.

Recall that $c_{j_\ell}$ is of type (B1) or (B2). By Claim 2.8, $c_{j_2}$ is of type (B1) or (B2) and by the definition (i) of goodness, we obtain $j_2 < j_\ell$. Note that $g_{T_i}(j_2) = f(j_3) > f(j_1) = g_{T_i}(j_\ell)$. Suppose that $c_{j_2}$ is of type (B2). Since $j_2 < j_\ell$, it follows from Claim 2.3 that $g_{T_i}(j_2) = i + j_2 - 1 - n < i + j_\ell - 1 - n \leqslant g_{T_i}(j_\ell)$, which contradicts to $g_{T_i}(j_2) > g_{T_i}(j_\ell)$. Hence $c_{j_2}$ is of type (B1). Suppose that $c_{j_\ell}$ is of type (B1). Since $j_2 < j_\ell$, $g_{T_i}(j_\ell) = i + j_\ell - 2 > i + j_2 - 2 = g_{T_i}(j_2)$, which contradicts to $g_{T_i}(j_2) > g_{T_i}(j_\ell)$. Hence $c_{j_\ell}$ is of type (B2).

Suppose that $c_{j_1}$ is of type (A2). By Caim 2.8, we obtain $g_{T_i}(j_1) > g_{T_i}(j_2)$ and

$$\begin{aligned} 0 < g_{T_i}(j_1) - g_{T_i}(j_2) &= i + j_1 - n - (i + j_2 - 2) \\ &= j_1 - j_2 - n + 2. \end{aligned}$$

This implies $j_1 > j_2 + n - 2$, which contradicts to $j_1 \leqslant n - 1$. Hence we have only to consider the case when $c_{j_1}$ is of type (A1).

Then

$$\begin{aligned} f(j_2) - f(j_1) &= g_{T_i}(j_1) - g_{T_i}(j_\ell) \\ &= i + j_1 - 1 - (i + j_\ell - 1 - n) \\ &= j_1 - j_\ell + n > n. \end{aligned}$$

This contradicts to $f(j_2) \leqslant n - 1$. □

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