

COMPOSITIONAL ABSTRACTIONS OF INTERCONNECTED DISCRETE-TIME SWITCHED SYSTEMS

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ABSTRACT. In this paper, we introduce a compositional method for the construction of finite abstractions of interconnected discrete-time switched systems. Particularly, we use a notion of so-called alternating simulation function as a relation between each switched subsystem and its finite abstraction. Based on some small-gain type conditions, we use those alternating simulation functions to construct compositionally an overall alternating simulation function as a relation between an interconnection of finite abstractions and that of switched subsystems. This overall alternating simulation function allows one to quantify the mismatch between the output behavior of the interconnection of switched subsystems and that of their finite abstractions. Additionally, we provide an approach to construct finite abstractions together with their corresponding alternating simulation functions for discrete-time switched subsystems under standard assumptions ensuring incremental input-to-state stability of a switched subsystem. Finally, we apply our results to a model of road traffic by constructing compositionally a finite abstraction of the network containing 50 cells of 1000 meters each. We use the constructed finite abstractions as substitutes to design controllers compositionally keeping the density of traffic lower than 30 vehicles per cell.

1. INTRODUCTION

Switched systems serve as an important modeling framework accurately describing several engineering systems in which physical processes have various operational modes [1]. Despite considerable number of studies that have been conducted regarding stability of switched systems, the fast grow in computational technology requires us to make same progress with respect to more sophisticated objectives such as those expressed as linear temporal logic (LTL) formulae [2]. One particular technique to address complex objectives is based on the construction of finite abstractions (a.k.a. symbolic models) of switched systems. In the finite abstractions, each abstract state represents a collection of continuous states of the switched system. Since finite abstractions are finite, one can algorithmically solves controller synthesis problems by resorting to automata-theoretic approaches [3, 4]. In general, there exist two types of finite abstractions: *sound* ones whose behaviors (approximately) contain those of the concrete systems and *complete* ones whose behaviors are (approximately) equivalent to those of the concrete systems [5].

In recent years, there have been several results on the construction of complete finite abstractions of switched systems. The work by [6] provides a finite abstraction that is related to the original incrementally stable switched system by establishing an approximate bisimulation relation between them. Recently, the result in [6] has been extended to the case of multi-rate symbolic models in [7], multi-scale symbolic models in [8], and to switched systems with aperiodic time sampling in [9].

All the proposed results in [6–10] take a monolithic view of switched systems when abstracting the entire system. However, the computational complexity of constructing finite abstractions scales exponentially with the number of state variables in the concrete switched system. Hence, the construction of finite abstractions for large-scale interconnected switched systems is mostly a complex task from a computational point of view. A convenient method to cope with this challenge is to first construct finite abstractions of the switched subsystems individually and then establish a compositional scheme that allows to construct a finite abstraction of the overall network using those individual finite abstractions.

In the past few years, several results have used the compositional framework for constructing complete finite abstractions of networks of control subsystems. Based on the notion of interconnection-compatible approximate bisimulation relation, [11] provides networks of finite abstractions that approximate networks

of stabilizable linear control systems. This work was extended in [12] to networks of incrementally input-to-state stable nonlinear control systems using the notion of approximate bisimulation relation. The work in [13] introduces a new system relation, called approximate disturbance bisimulation relation, as the basis for the compositional construction of finite abstractions. The results in [14, 15] provide techniques to construct compositionally finite abstractions of networks of nonlinear control systems using dissipativity and general small-gain type conditions, respectively. There are also other results in the literature [16–18] which provide sound finite abstractions of interconnected systems, compositionally, without requiring any stability property or condition on the gains of subsystems. Unfortunately, non of the compositional results in [11–18] provide a compositional framework for the construction of finite abstractions for interconnected switched systems.

The main contribution of this work is to provide for the first time a compositional methodology for the construction of finite abstractions of interconnected switched systems. The proposed approach leverages sufficient small-gain type conditions to establish the compositionality results which rely on the existence of alternating simulation functions as relations between switched subsystems and their finite abstractions. In particular, based on some small-gain type conditions, we use those alternating simulation functions to construct compositionally an overall alternating simulation function as a relation between an interconnection of finite abstractions and that of original switched subsystems. The existence of such an overall alternating simulation function enables one to quantify the mismatch between the output behavior of the interconnection of switched subsystems and that of their finite abstractions. Furthermore, under standard assumptions ensuring incremental input-to-state stability of a switched system (i.e., existence of a common incremental input-to-state Lyapunov function, or multiple incremental input-to-state Lyapunov functions with dwell-time), we show that one can construct finite abstractions of switched systems in general nonlinear settings. Finally, we apply our results to a model of road traffic by constructing compositionally a finite abstraction of a network containing 50 cells of 1000 meters each. We use the constructed finite abstractions as substitutes to design controllers compositionally maintaining the density of traffic lower than 30 vehicles per cell. Notation and some technical notions used in the sequel are reported in the Appendix.

2. PRELIMINARIES

2.1. Discrete-Time Switched Systems. In this paper we study discrete-time switched systems of the following form.

Definition 1. A discrete-time switched system Σ is defined by the tuple $\Sigma = (\mathbb{X}, P, \mathbb{W}, F, \mathbb{Y}, h)$, where

- \mathbb{X}, \mathbb{W} , and \mathbb{Y} are the state set, internal input set, and output set, respectively, and are assumed to be subsets of normed vector spaces with appropriate finite dimensions;
- $P = \{1 \cdots, m\}$ is the finite set of modes;
- $F = \{f_1, \dots, f_m\}$ is a collection of set-valued maps $f_p : \mathbb{X} \times \mathbb{W} \rightrightarrows \mathbb{X}$ for all $p \in P$;
- $h : \mathbb{X} \rightarrow \mathbb{Y}$ is the output map.

The discrete-time switched system Σ is described by difference inclusions of the form

$$\Sigma : \begin{cases} \mathbf{x}(k+1) \in f_{p(k)}(\mathbf{x}(k), \omega(k)), \\ \mathbf{y}(k) = h(\mathbf{x}(k)), \end{cases} \quad (1)$$

where $\mathbf{x} : \mathbb{N} \rightarrow \mathbb{X}$, $\mathbf{y} : \mathbb{N} \rightarrow \mathbb{Y}$, $\mathbf{p} : \mathbb{N} \rightarrow P$, and $\omega : \mathbb{N} \rightarrow \mathbb{W}$ are the state signal, output signal, switching signal, and internal input signal, respectively. We denote by Σ_p system (1) with constant switching signal $\mathbf{p}(k) = p \in P \forall k \in \mathbb{N}$. We use $\mathbf{X}_{x_0, \bar{\mathbf{p}}, \bar{\omega}}$ and $\mathbf{Y}_{x_0, \bar{\mathbf{p}}, \bar{\omega}}$ to denote the sets of infinite state and output runs of Σ , respectively, associated with infinite switching sequence $\bar{\mathbf{p}} = \{p_0, p_1, \dots\}$, infinite internal input sequence $\bar{\omega} = \{\omega_0, \omega_1, \dots\}$, and initial state $x_0 \in \mathbb{X}$.

Let $\phi_k, k \in \mathbb{N}_{\geq 1}$, denote the time when the k -th switching instant occurs and define $\Phi := \{\phi_k : k \in \mathbb{N}_{\geq 1}\}$ as the set of switching instants. We assume that signal \mathbf{p} satisfies a dwell-time condition [19] (i.e. there exists $k_d \in \mathbb{N}_{\geq 1}$, called the dwell-time, such that for all consecutive switching time instants $\phi_k, \phi_{k+1} \in \Phi$, $\phi_{k+1} - \phi_k \geq k_d$).

System Σ is called deterministic if $|f_p(x, w)| \leq 1 \forall x \in \mathbb{X}, \forall p \in P, \forall w \in \mathbb{W}$, and non-deterministic otherwise. System Σ is called blocking if $\exists x \in \mathbb{X}, \forall p \in P, \forall w \in \mathbb{W}$ where $|f_p(x, w)| = 0$ and non-blocking if $|f_p(x, w)| \neq 0$

$\forall x \in \mathbb{X}, \exists p \in P, \exists w \in \mathbb{W}$. System Σ is called finite if \mathbb{X} and \mathbb{W} are finite sets and infinite otherwise. In this paper, we only deal with non-blocking systems.

3. TRANSITION SYSTEMS AND ALTERNATING SIMULATION FUNCTIONS

In this section, we introduce a notion of so-called transition systems to provide an alternative description of switched systems that can be later directly related to their finite abstractions.

Definition 2. Given a discrete-time switched system $\Sigma = (\mathbb{X}, P, \mathbb{W}, F, \mathbb{Y}, h)$, we define the associated transition system $T(\Sigma) = (X, U, W, \mathcal{F}, Y, \mathcal{H})$ where:

- $X = \mathbb{X} \times P \times \{0, \dots, k_d - 1\}$ is the state set;
- $U = P$ is the external input set;
- $W = \mathbb{W}$ is the internal input set;
- \mathcal{F} is the transition function given by $(x', p', l') \in \mathcal{F}((x, p, l), u, w)$ if and only if $x' \in f_p(x, w), u = p$ and the following scenarios hold:
 - $l < k_d - 1, p' = p$ and $l' = l + 1$: switching is not allowed because the time elapsed since the latest switch is strictly smaller than the dwell time;
 - $l = k_d - 1, p' = p$ and $l' = k_d - 1$: switching is allowed but no switch occurs;
 - $l = k_d - 1, p' \neq p$ and $l' = 0$: switching is allowed and a switch occurs;
- $Y = \mathbb{Y}$ is the output set;
- $\mathcal{H} : X \rightarrow Y$ is the output map defined as $\mathcal{H}(x, p, l) = h(x)$.

We use $T(\mathbf{X})_{z_0, \bar{u}, \bar{w}}$ and $T(\mathbf{Y})_{z_0, \bar{u}, \bar{w}}$ to denote the sets of infinite state and output runs of $T(\Sigma)$, respectively, associated with infinite external input sequence $\bar{u} = \{u_0, u_1, \dots\}$, infinite internal input sequence $\bar{w} = \{w_0, w_1, \dots\}$, and initial state $z_0 = (x_0, p_0, l_0) \in X$, where $u_0 = p_0$ and $l_0 = 0$.

In the next proposition, we show that sets $\mathbf{Y}_{x_0, \bar{p}, \bar{w}}$ and $T(\mathbf{Y})_{z_0, \bar{u}, \bar{w}}$, where $\bar{p} = \bar{u}$ and $z_0 = (x_0, p_0, 0)$, are equivalent.

Proposition 3. Consider $\Sigma, T(\Sigma), \bar{p} = \{p_0, p_1, \dots\} = \bar{u}, \bar{w} = \{w_0, w_1, \dots\}$, and $x_0 \in \mathbb{X}$. Then, $\mathbf{Y}_{x_0, \bar{p}, \bar{w}} = T(\mathbf{Y})_{z_0, \bar{u}, \bar{w}}$, where $z_0 = (x_0, p_0, 0)$.

The proof is straightforward and omitted here due to lack of space.

From now on, we use Σ and $T(\Sigma)$ interchangeably.

In the following, we introduce a notion of so-called alternating simulation functions, inspired by Definition 1 in [20], which quantitatively relates transition systems with internal inputs.

Definition 4. Consider $T(\Sigma) = (X, U, W, \mathcal{F}, Y, \mathcal{H})$ and $\hat{T}(\hat{\Sigma}) = (\hat{X}, \hat{U}, \hat{W}, \hat{\mathcal{F}}, \hat{Y}, \hat{\mathcal{H}})$ where $\hat{W} \subseteq W$ and $\hat{Y} \subseteq Y$. A function $\mathcal{S} : X \times \hat{X} \rightarrow \mathbb{R}_{\geq 0}$ is called an alternating simulation function from $\hat{T}(\hat{\Sigma})$ to $T(\Sigma)$ if $\forall (x, p, l) \in X$ and $\forall (\hat{x}, p, l) \in \hat{X}$, one has

$$\alpha(\|\mathcal{H}(x, p, l) - \hat{\mathcal{H}}(\hat{x}, p, l)\|) \leq \mathcal{S}((x, p, l), (\hat{x}, p, l)), \quad (2)$$

and $\forall (x, p, l) \in X$ and $\forall (\hat{x}, p, l) \in \hat{X}, \forall \hat{u} \in \hat{U}, \forall w \in W, \forall \hat{w} \in \hat{W}, \forall (x', p', l') \in \mathcal{F}((x, p, l), \hat{u}, w) \exists (\hat{x}', p', l') \in \hat{\mathcal{F}}((\hat{x}, p, l), \hat{u}, \hat{w})$ such that one gets

$$\mathcal{S}((x', p', l'), (\hat{x}', p', l')) \leq \max\{\sigma \mathcal{S}((x, p, l), (\hat{x}, p, l)), \varrho(\|w - \hat{w}\|), \varepsilon\}, \quad (3)$$

for some $\alpha, \varrho \in \mathcal{K}_\infty, 0 < \sigma < 1$, and $\varepsilon \in \mathbb{R}_{\geq 0}$.

If Σ does not have internal inputs, which is the case for interconnected systems (cf. Definition 7), Definition 1 reduces to the tuple $\Sigma = (\mathbb{X}, P, F, \mathbb{Y}, H)$, the set-valued map f_p becomes $f_p : \mathbb{X} \rightrightarrows \mathbb{X}$, and (1) reduces to:

$$\Sigma : \begin{cases} \mathbf{x}(k+1) \in f_{p(k)}(\mathbf{x}(k)), \\ \mathbf{y}(k) = h(\mathbf{x}(k)). \end{cases} \quad (4)$$

Correspondingly, Definition 2 reduces to tuple $T(\Sigma) = (X, U, \mathcal{F}, Y, \mathcal{H})$, and the transition function \mathcal{F} is given by $(x', p', l') \in \mathcal{F}((x, p, l), u)$ if and only if $x' \in f_p(x), u = p$ and the following scenarios hold:

- $l < k_d - 1, p' = p$ and $l' = l + 1$;

- $l = k_d - 1$, $p' = p$ and $l' = k_d - 1$;
- $l = k_d - 1$, $p' \neq p$ and $l' = 0$.

Moreover, Definition 4 reduces to the following.

Definition 5. Consider $T(\Sigma) = (X, U, \mathcal{F}, Y, \mathcal{H})$ and $\hat{T}(\hat{\Sigma}) = (\hat{X}, \hat{U}, \hat{\mathcal{F}}, \hat{Y}, \hat{\mathcal{H}})$ where $\hat{Y} \subseteq Y$. A function $\tilde{\mathcal{S}} : X \times \hat{X} \rightarrow \mathbb{R}_{\geq 0}$ is called an alternating simulation function from $\hat{T}(\hat{\Sigma})$ to $T(\Sigma)$ if $\forall (x, p, l) \in X$ and $\forall (\hat{x}, p, l) \in \hat{X}$, one has

$$\tilde{\alpha}(\|\mathcal{H}(x, p, l) - \hat{\mathcal{H}}(\hat{x}, p, l)\|) \leq \tilde{\mathcal{S}}((x, p, l), (\hat{x}, p, l)), \quad (5)$$

and $\forall (x, p, l) \in X$ and $\forall (\hat{x}, p, l) \in \hat{X}$, $\forall \hat{u} \in \hat{U}$, $\forall (x', p', l') \in \mathcal{F}((x, p, l), \hat{u}) \exists (\hat{x}', p', l') \in \hat{\mathcal{F}}((\hat{x}, p, l), \hat{u})$ such that one gets

$$\tilde{\mathcal{S}}((x', p', l'), (\hat{x}', p', l')) \leq \max\{\tilde{\sigma}\tilde{\mathcal{S}}((x, p, l), (\hat{x}, p, l)), \tilde{\varepsilon}\}, \quad (6)$$

for some $\tilde{\alpha} \in \mathcal{K}_{\infty}$, $0 < \tilde{\sigma} < 1$, and $\tilde{\varepsilon} \in \mathbb{R}_{\geq 0}$.

The next result shows that the existence of an alternating simulation function for transition systems without internal inputs implies the existence of an approximate alternating simulation relation between them as defined in [5].

Proposition 6. Consider $T(\Sigma) = (X, U, \mathcal{F}, Y, \mathcal{H})$ and $\hat{T}(\hat{\Sigma}) = (\hat{X}, \hat{U}, \hat{\mathcal{F}}, \hat{Y}, \hat{\mathcal{H}})$ where $\hat{Y} \subseteq Y$. Assume $\tilde{\mathcal{S}}$ is an alternating simulation function from $\hat{T}(\hat{\Sigma})$ to $T(\Sigma)$ as in Definition 5. Then, relation $R \subseteq X \times \hat{X}$ defined by

$$R = \left\{ ((x, p, l), (\hat{x}, p, l)) \in X \times \hat{X} \mid \tilde{\mathcal{S}}((x, p, l), (\hat{x}, p, l)) \leq \tilde{\varepsilon} \right\}$$

is an $\hat{\varepsilon}$ -approximate alternating simulation relation, defined in [5], from $\hat{T}(\hat{\Sigma})$ to $T(\Sigma)$ with $\hat{\varepsilon} = \tilde{\alpha}^{-1}(\tilde{\varepsilon})$.

4. COMPOSITIONALITY RESULT

In this section, we analyze networks of discrete-time switched subsystems and leverage sufficient small-gain type conditions under which one can construct an alternating simulation function from a network of finite abstractions to the concrete network by using alternating simulation functions of the subsystems. In the following, we define first a network of discrete-time switched subsystems.

4.1. Interconnected Systems. We consider $N \in \mathbb{N}_{\geq 1}$ discrete-time switched subsystems

$$\Sigma_i = (\mathbb{X}_i, P_i, \mathbb{W}_i, F_i, \mathbb{Y}_i, h_i), i \in [1; N],$$

with partitioned internal inputs as

$$w_i = [w_{i1}; \dots; w_{i(i-1)}; w_{i(i+1)}; \dots; w_{iN}], \mathbb{W}_i = \prod_{j=1}^{N-1} \mathbb{W}_{ij}, \quad (7)$$

and with output map and set partitioned as

$$h_i(x_i) = [h_{i1}(x_i); \dots; h_{iN}(x_i)], \mathbb{Y}_i = \prod_{j=1}^N \mathbb{Y}_{ij}. \quad (8)$$

We interpret the outputs y_{ii} as external ones, whereas y_{ij} with $i \neq j$ are internal ones which are used to define the interconnected switched systems. In particular, we assume that $w_{ij} = y_{ji}$, if there is connection from switched subsystem Σ_j to Σ_i , otherwise we set $h_{ji} \equiv 0$. Next, given input-output structure as in (7) and (8), we define the interconnection of switched subsystems.

Definition 7. Consider $N \in \mathbb{N}_{\geq 1}$ switched subsystems $\Sigma_i = (\mathbb{X}_i, P_i, \mathbb{W}_i, F_i, \mathbb{Y}_i, h_i)$, $i \in [1; N]$, with the input-output structure given by (7) and (8). The interconnected switched system $\Sigma = (\mathbb{X}, P, F, \mathbb{Y}, h)$, denoted by $\mathcal{I}(\Sigma_1, \dots, \Sigma_N)$, is defined by $\mathbb{X} = \prod_{i=1}^N \mathbb{X}_i$, $P = \prod_{i=1}^N P_i$, $F = \prod_{i=1}^N F_i$, $\mathbb{Y} = \prod_{i=1}^N \mathbb{Y}_{ii}$, and map $h(x) := [h_{11}(x_1); \dots; h_{NN}(x_N)]$, where $x = [x_1; \dots; x_N]$, and subject to the constraint:

$$\forall i, j \in [1; N], i \neq j, w_{ij} = y_{ji}, \mathbb{Y}_{ji} \subseteq \mathbb{W}_{ij}. \quad (9)$$

Similarly, given transition subsystem $T_i(\Sigma_i), i \in [1; N]$, one can also define the network of those transition subsystems as $\mathcal{I}(T_1(\Sigma_1), \dots, T_N(\Sigma_N))$.

Next subsection provides one of the main results of the paper on the compositional construction of abstractions for networks of switched systems.

4.2. Compositional Abstractions of Interconnected Switched Systems. In this subsection, we assume that we are given N discrete-time switched subsystems $\Sigma_i = (\mathbb{X}_i, P_i, \mathbb{W}_i, F_i, \mathbb{Y}_i, h_i), i \in [1; N]$, or equivalently, $T_i(\Sigma_i) = (X_i, U_i, W_i, \mathcal{F}_i, Y_i, \mathcal{H}_i)$, together with their corresponding abstractions $\hat{T}_i(\hat{\Sigma}_i) = (\hat{X}_i, \hat{U}_i, \hat{W}_i, \hat{\mathcal{F}}_i, \hat{Y}_i, \hat{\mathcal{H}}_i)$ and alternating simulation functions \mathcal{S}_i from $\hat{T}_i(\hat{\Sigma}_i)$ to $T_i(\Sigma_i)$. Moreover, for σ_i, α_i , and ϱ_i associated with $\mathcal{S}_i, \forall i \in [1; N]$, appeared in Definition 4, we define

$$\gamma_{ij}(s) := \begin{cases} \sigma_i s & \text{if } i = j, \\ \varrho_i \circ \alpha_j^{-1}(s) & \text{if } i \neq j, \end{cases} \quad \forall s \in \mathbb{R}_{\geq 0}, \forall i, j \in [1; N]. \quad (10)$$

We raise the next small-gain assumption to establish the main compositionality results of the paper.

Assumption 8. Assume that functions γ_{ij} defined in (10) satisfy

$$\gamma_{i_1 i_2} \circ \gamma_{i_2 i_3} \circ \dots \circ \gamma_{i_{r-1} i_r} \circ \gamma_{i_r i_1} < \mathcal{I}_d, \quad (11)$$

$\forall (i_1, \dots, i_r) \in \{1, \dots, N\}^r$, where $r \in \{1, \dots, N\}$.

The next theorem provides a compositional approach on the construction of abstractions of networks of discrete-time switched subsystems and that of the corresponding alternating simulation functions.

Theorem 9. Consider the interconnected transition system $T(\Sigma) = (X, U, \mathcal{F}, Y, \mathcal{H})$ induced by $N \in \mathbb{N}_{\geq 1}$ transition subsystems $T_i(\Sigma_i), \forall i \in [1; N]$. Assume that each $T_i(\Sigma_i)$ and its abstraction $\hat{T}_i(\hat{\Sigma}_i)$ admit an alternating simulation function \mathcal{S}_i as in Definition 4. Let Assumption 8 hold. Then, there exist $\delta_i \in \mathcal{K}_{\infty}$ such that

$$\tilde{\mathcal{S}}((x, p, l), (\hat{x}, p, l)) := \max_{i \in [1; N]} \{\delta_i^{-1} \circ \mathcal{S}_i((x_i, p_i, l_i), (\hat{x}_i, p_i, l_i))\}$$

is an alternating simulation function from $\hat{T}(\hat{\Sigma}) = \mathcal{I}(\hat{T}_1(\hat{\Sigma}_1), \dots, \hat{T}_N(\hat{\Sigma}_N))$ to $T(\Sigma) = \mathcal{I}(T_1(\Sigma_1), \dots, T_N(\Sigma_N))$.

5. CONSTRUCTION OF FINITE ABSTRACTIONS

In this section, we consider $\Sigma = (\mathbb{X}, P, \mathbb{W}, F, \mathbb{Y}, h)$ as an infinite, deterministic switched system, and assume its output map h satisfies the following general Lipschitz-like assumption: there exists an $\ell \in \mathcal{K}_{\infty}$ such that $\|h(x) - h(x')\| \leq \ell(\|x - x'\|)$ for all $x, x' \in \mathbb{X}$. In addition, the existence of an alternating simulation function between $T(\Sigma)$ and its finite abstraction is established under the assumption that Σ_p is incrementally input-to-state stable (δ -ISS) [21] as defined next.

Definition 10. System Σ_p is δ -ISS if there exist functions $V_p : \mathbb{X} \times \mathbb{X} \rightarrow \mathbb{R}_{\geq 0}$, $\underline{\alpha}_p, \bar{\alpha}_p, \rho_p \in \mathcal{K}_{\infty}$, and constant $0 < \kappa_p < 1$, such that for all $x, \hat{x} \in \mathbb{X}$, and for all $w, \hat{w} \in \mathbb{W}$

$$\underline{\alpha}_p(\|x - \hat{x}\|) \leq V_p(x, \hat{x}) \leq \bar{\alpha}_p(\|x - \hat{x}\|), \quad (12)$$

$$V_p(f_p(x, w), f_p(\hat{x}, \hat{w})) \leq \kappa_p V_p(x, \hat{x}) + \rho_p(\|w - \hat{w}\|). \quad (13)$$

We say that $V_p, \forall p \in P$, are multiple δ -ISS Lyapunov functions for system Σ if it satisfies (12) and (13). Moreover, if $V_p = V_{p'}, \forall p, p' \in P$, we omit the index p in (12), (13), and say that V is a common δ -ISS Lyapunov function for system Σ . We refer interested readers to [1] for more details on common and multiple Lyapunov functions for switched systems.

Now, we show how to construct a finite abstraction $\hat{T}(\hat{\Sigma})$ of transition system $T(\Sigma)$ associated to the switched system Σ in which Σ_p is δ -ISS.

Definition 11. Consider a transition system $T(\Sigma) = (X, U, W, \mathcal{F}, Y, \mathcal{H})$, associated to the switched system $\Sigma = (\mathbb{X}, P, \mathbb{W}, F, \mathbb{Y}, h)$, where \mathbb{X}, \mathbb{W} are assumed to be finite unions of boxes. Let Σ_p be δ -ISS as in Definition 10. Then one can construct a finite transition system $\hat{T}(\hat{\Sigma}) = (\hat{X}, \hat{U}, \hat{W}, \hat{\mathcal{F}}, \hat{Y}, \hat{\mathcal{H}})$ where:

- $\hat{X} = \hat{\mathbb{X}} \times P \times \{0, \dots, k_d - 1\}$, where $\hat{\mathbb{X}} = [\mathbb{X}]_\eta$ and $0 < \eta \leq \text{span}(\mathbb{X})$ is the state set quantization parameter;
- $\hat{U} = U = P$ is the external input set;
- $\hat{W} = [\mathbb{W}]_\varpi$, where $0 \leq \varpi \leq \text{span}(\mathbb{W})$ is the internal input set quantization parameter.
- $(\hat{x}', p', l') \in \hat{\mathcal{F}}((\hat{x}, p, l), \hat{u}, \hat{w})$ if and only if $\hat{x}' \in \hat{f}_p(\hat{x}, \hat{w}) \Leftrightarrow \|f_p(\hat{x}, \hat{w}) - \hat{x}'\| \leq \eta$, $\hat{u} = p$ and the following scenarios hold:
 - $l < k_d - 1$, $p' = p$ and $l' = l + 1$;
 - $l = k_d - 1$, $p' = p$ and $l' = k_d - 1$;
 - $l = k_d - 1$, $p' \neq p$ and $l' = 0$;
- $\hat{Y} = \{\mathcal{H}(\hat{x}, p, l) | (\hat{x}, p, l) \in \hat{X}\}$;
- $\hat{\mathcal{H}} : \hat{X} \rightarrow \hat{Y}$ is the output map defined as $\hat{\mathcal{H}}(\hat{x}, p, l) = \mathcal{H}(\hat{x}, p, l) = h(\hat{x})$;

Remark 12. In the context of networks of subsystems, \hat{W} should be constructed in such a way that it satisfies (7) and (9) in the compositional setting with respect to outputs sets of other finite transition subsystems.

We impose the following assumptions on function V_p in Definition 10 which are used to prove some of the main results later.

Assumption 13. There exists $\mu \geq 1$ such that

$$\forall x, y \in \mathbb{X}, \forall p, p' \in P, V_p(x, y) \leq \mu V_{p'}(x, y). \quad (14)$$

Assumption 14. For all $p \in P$, there exists a \mathcal{K}_∞ function γ_p such that

$$\forall x, y, z \in \mathbb{X}, V_p(x, y) \leq V_p(x, z) + \gamma_p(\|y - z\|). \quad (15)$$

Now, we establish the relation between $T(\Sigma)$ and $\hat{T}(\hat{\Sigma})$, introduced above, via the notion of alternating simulation function as in Definition 4.

Theorem 15. Consider a switched system $\Sigma = (\mathbb{X}, P, \mathbb{W}, F, \mathbb{Y}, h)$ with its equivalent transition system $T(\Sigma) = (X, U, W, \mathcal{F}, Y, \mathcal{H})$. Let Σ_p be δ -ISS as in Definition 10. Consider a finite transition system $\hat{T}(\hat{\Sigma}) = (\hat{X}, \hat{U}, \hat{W}, \hat{\mathcal{F}}, \hat{Y}, \hat{\mathcal{H}})$ constructed as in Definition 11. Assume that Assumptions 13 and 14 hold. Let $\epsilon > 1$. If, $\forall p \in P$, $k_d \geq \epsilon \frac{\ln(\mu)}{\ln(\frac{1}{\kappa_p})} + 1$, then function \mathcal{V} defined as

$$\mathcal{V}((x, p, l), (\hat{x}, p, l)) := \kappa_p^{\frac{l}{\epsilon}} V_p(x, \hat{x}) \quad (16)$$

is an alternating simulation function from $\hat{T}(\hat{\Sigma})$ to $T(\Sigma)$.

Remark 16. If Σ admits a common δ -ISS Lyapunov function satisfying Assumption 14, then function \mathcal{V} in Theorem 15 reduces to $\mathcal{V}((x, p, l), (\hat{x}, p, l)) := V(x, \hat{x})$.

6. CASE STUDY

The chosen switched system Σ here is the model of a circular road around a city (Highway) divided into 50 cells of 1000 meters each. The road has 25 entries and 50 ways out in such a way that cell q has an entry and exit if $q \in Q_1 = \{q \text{ is odd } | q \in [1; 50]\}$ and has an exit and no entry if $q \in Q_2 = \{q \text{ is even } | q \in [1; 50]\}$. The entries are controlled by traffic signals, denoted $s_r, r \in [1; 25]$, that enable (green light) or not (red light) the vehicles to pass. In Σ , the dynamic we want to observe is the density of traffic, given in vehicles per cell, for each cell q of the road. The state of switched system Σ is a 50-dimensional vector and its set of modes can be understood as all possible linear combination of traffic signals s_r . More formally, since each traffic signal s_r can have two modes (1 for red light and 2 for green), one can consider the modes of system Σ as $p \in P = \{1, 2\}^{25}$. During the sampling time interval $\tau = \frac{10}{60 \times 60}$ in hours (h), we assume that 12 vehicles can pass the entry controlled by a traffic signal s_r when it is green. Moreover, 10% of vehicles that are in cells $q \in Q_1$, and 35% of vehicles that are in cells $q \in Q_2$ go out using available exits. As explained in [22], the evolution of the density \mathbf{x} of all cells are described by the interconnected discrete-time switched model:

$$\Sigma : \begin{cases} \mathbf{x}(k+1) &= A\mathbf{x}(k) + B_{\mathbf{p}(k)}, \\ \mathbf{y}(k) &= \mathbf{x}(k), \end{cases}$$

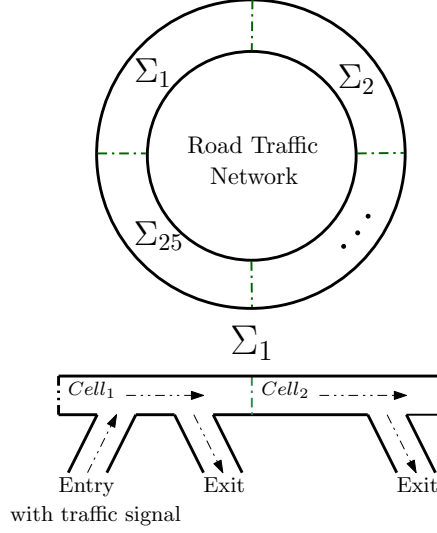


FIGURE 1. Model of a road traffic network in a circular highway composed of 25 identical links, each link has two cell.

where $A \in \mathbb{R}^{50 \times 50}$ is a matrix with elements $\{A\}_{q,q} = 0.9 - \frac{\tau v}{d}$ if $q \in Q_1$ and $\{A\}_{q,q} = 0.65 - \frac{\tau v}{d}$ if $q \in Q_2$, $\{A\}_{q+1,q} = \{A\}_{1,50} = \frac{\tau v}{d}$, $\forall q \in [1; 50]$, and all other elements are identically zero, where $d = 1$ and $v = 120$ are the length in kilometers (km) and the flow speed of the vehicles in kilometers per hour (km/h), respectively. The vector $B_p \in \mathbb{R}^{50}$ is defined as $B_p = [b_{1p_1}; \dots; b_{25p_{25}}]$ such that $b_{ip_i} = [0; 0]$ if $p_i = 1$, and $b_{ip_i} = [0; 12]$ if $p_i = 2$, $\forall i \in [1, 25]$, $[p_1; \dots; p_{25}] \in P = \{1, 2\}^{25}$, where P is the set of modes of Σ . Now, in order to apply the compositionality result, we introduce subsystems Σ_i , $\forall i \in [1; 25]$. Each subsystems Σ_i represents the dynamic of one link of the entire highway, where each link contains 2 cells, one entry, and two exits as illustrated in Fig 1. The subsystems Σ_i is described by

$$\Sigma_i : \begin{cases} \mathbf{x}_i(k+1) &= A_i \mathbf{x}_i(k) + D_i w_i(k) + B_{ip_i}(k), \\ \mathbf{y}_i(k) &= C_i \mathbf{x}_i(k), \end{cases}$$

where, $\forall i \in [1; 25]$,

$$A_i = \begin{bmatrix} 0.9 - \frac{\tau v}{d} & 0 \\ \frac{\tau v}{d} & 0.65 - \frac{\tau v}{d} \end{bmatrix}, D_i = \begin{bmatrix} \frac{\tau v}{d} \\ 0 \end{bmatrix}, B_{i1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \\ B_{i2} = \begin{bmatrix} 12 \\ 0 \end{bmatrix}, C_i = \begin{bmatrix} C_{ii} \\ C_{i(i+1)} \end{bmatrix}, C_{ii} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, C_{i(i+1)} = [0 \quad 1],$$

$\omega_i(k) = C_{(i-1)i} \mathbf{x}_{i-1}(k)$ (with $C_{01} := C_{N(N+1)}$, and $\mathbf{x}_0 := \mathbf{x}_N$, $N = 25$), and the set of modes is $P_i = \{1, 2\}$. Clearly, one can verify that $\Sigma = \mathcal{I}(\Sigma_1, \dots, \Sigma_{25})$.

Note that, for any $i \in [1; 25]$, conditions (12) and (13) are satisfied with $V_{ip_i}(x_i, \hat{x}_i) = \|x_i - \hat{x}_i\|$, $\underline{\alpha}_{ip_i} = \bar{\alpha}_{ip_i} = \mathcal{I}_d$, $\kappa_{ip_i} = 0.65$, $\rho_{ip_i} = 0.33\mathcal{I}_d$, $\forall p_i \in P_i$. Furthermore, condition (15) is satisfied with $\gamma_{ip_i} = \mathcal{I}_d$, $\forall p_i \in P_i$. Moreover, since $V_{ip_i} = V_{ip'_i}$, $\forall p_i, p'_i \in P_i$, and according to Remark 16, function $\mathcal{V}_i((x_i, p_i, l_i), (\hat{x}_i, p_i, l_i)) = \|x_i - \hat{x}_i\|$ is an alternating simulation function from $\hat{T}_i(\hat{\Sigma}_i)$ to $T_i(\Sigma_i)$. Note that for the construction of finite abstractions, we have chosen the finite set $\hat{W}_i = \{C_{(i-1)i} \hat{x}_{i-1} | \hat{x}_{i-1} \in \hat{\mathbb{X}}_{i-1}\}$, $\forall i \in [1; 25]$, (with $C_{01} := C_{N(N+1)}$, $\hat{x}_0 := \hat{x}_N$, and $\hat{\mathbb{X}}_0 := \hat{\mathbb{X}}_N$, $N = 25$). Now, by employing (10), we have $\gamma_{ij} < \mathcal{I}_d$, $\forall i, j \in [1; 25]$, hence the small-gain condition (11) is satisfied. Using the results in Theorem 9 with $\delta_i^{-1} = \mathcal{I}_d$, $\forall i \in [1; 25]$, one can verify that $\mathcal{V}((x, p, l), (\hat{x}, p, l)) = \max_i \|x_i - \hat{x}_i\|$ is an alternating simulation function from $\mathcal{I}(\hat{T}_1(\hat{\Sigma}_1), \dots, \hat{T}_{25}(\hat{\Sigma}_{25}))$ to $\mathcal{I}(T_1(\Sigma_1), \dots, T_{25}(\Sigma_{25}))$.

Next we design a controller for Σ via finite abstractions $\hat{T}_i(\hat{\Sigma}_i)$ such that the controller maintains the density of traffic lower than 30 vehicles per cell. The idea here is to design local controllers for finite abstractions $\hat{T}_i(\hat{\Sigma}_i)$, and then use them in concrete switched subsystems Σ_i . To do so, the local controllers are designed

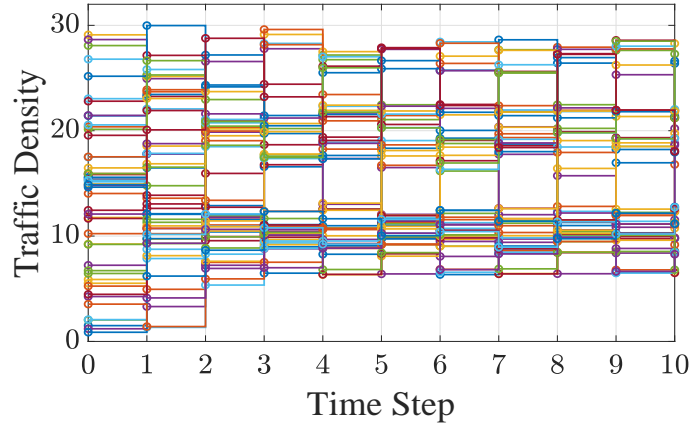


FIGURE 2. Closed-loop state trajectories of Σ consisting of 50 cells.

while assuming that the other subsystems meet their specifications. The computation times for constructing abstractions and designing controllers for Σ_i with state quantization parameter $\eta_i = 0.03$ are 10.2s and 0.014s, respectively. Figure 2 shows the closed-loop state trajectories of the of Σ consisting of 50 cells. Note that it would not have been possible to synthesize a controller for the 50-dimensional switched system Σ without applying the proposed compositional method.

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.1. **Notation.** We denote by \mathbb{R} , \mathbb{Z} , and \mathbb{N} the set of real numbers, integers, and non-negative integers, respectively. These symbols are annotated with subscripts to restrict them in the obvious way, e.g., $\mathbb{R}_{>0}$ denotes the positive real numbers. Given $N \in \mathbb{N}_{\geq 1}$, vectors $\nu_i \in \mathbb{R}^{n_i}$, $n_i \in \mathbb{N}_{\geq 1}$, and $i \in [1; N]$, we use $\nu = [\nu_1; \dots; \nu_N]$ to denote the vector in \mathbb{R}^n with $n = \sum_i n_i$ consisting of the concatenation of vectors ν_i . The closed interval in \mathbb{N} is denoted by $[a; b]$ for $a, b \in \mathbb{N}$ and $a \leq b$. We denote by $\text{diag}(A_1, \dots, A_N)$ the block diagonal matrix with diagonal matrix entries A_1, \dots, A_N . We denote the identity matrix in $\mathbb{R}^{n \times n}$ by I_n . The individual elements in a matrix $A \in \mathbb{R}^{m \times n}$, are denoted by $\{A\}_{ij}$, where $i \in [1; m]$ and $j \in [1; n]$. We denote by $\|\cdot\|$ the infinity norm. We denote by $|\cdot|$ the cardinality of a given set and by \emptyset the empty set. For any set $S \subseteq \mathbb{R}^n$ of the form of finite union of boxes, e.g., $S = \bigcup_{j=1}^M S_j$ for some $M \in \mathbb{N}$, where $S_j = \prod_{i=1}^n [c_i^j, d_i^j] \subseteq \mathbb{R}^n$ with $c_i^j < d_i^j$, and positive constant $\eta \leq \text{span}(S)$, where $\text{span}(S) = \min_{j=1, \dots, M} \eta_{S_j}$ and $\eta_{S_j} = \min\{|d_1^j - c_1^j|, \dots, |d_n^j - c_n^j|\}$, we define $[S]_\eta = \{a \in S \mid a_i = k_i \eta, k_i \in \mathbb{Z}, i = 1, \dots, n\}$. The set $[S]_\eta$ will be used as a finite approximation of the set S with precision η . Note that $[S]_\eta \neq \emptyset$ for any $\eta \leq \text{span}(S)$. We use notations \mathcal{K} and \mathcal{K}_∞ to denote different classes of comparison functions, as follows: $\mathcal{K} = \{\alpha : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0} \mid \alpha \text{ is continuous, strictly increasing, and } \alpha(0) = 0\}$; $\mathcal{K}_\infty = \{\alpha \in \mathcal{K} \mid \lim_{r \rightarrow \infty} \alpha(r) = \infty\}$. For $\alpha, \gamma \in \mathcal{K}_\infty$ we write $\alpha < \gamma$ if $\alpha(s) < \gamma(s)$ for all $s > 0$, and $\mathcal{I}_d \in \mathcal{K}_\infty$ denotes the identity function.

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