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Relational Semantics for Kleene Logic and Action Logic

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Abstract *Kleene algebras* and *action logic* were proposed to be solutions to the finite axiomatization problem of the algebra of regular sets (of strings). They are treated here as nonclassical logics—with Hilbert-style axiomatizations and semantics. We also provide intuitive accounts in terms of *information states* of the semantics which provide further insights into the formalisms. The three types of "Kripke-style" semantics which we define develop insights from *gaggle theory*, and from our *four-valued* and *generalized Kripke semantics* for the minimal substructural logic. Soundness and completeness are proven each time.

1 Introduction

Finite state automata (FSAs) are important in computer science due to their wide applicability—despite the fact that they are not full-blown models of computation. FSAs and their equivalents—regular languages (regular sets) and regular expressions—have been extensively investigated for decades. Regular expressions contain three operations, which are called "regular operations." These are *union* ($\vee$ or $+$), *concatenation* ($\circ$, $\cdot$, or simply juxtaposition), and the *Kleene star* (*). There are two distinguished elements, namely, the *empty language* ($\boldsymbol{F}$ or $\varnothing$) and the *empty string language* ($\boldsymbol{t}$ or $\{\varepsilon\}$). The set of regular languages is closed under these operations (and also under some other operations including intersection and complementation).[1] *Kleene algebras*$_{10}$ ($\mathbf{KA}_{10}$) were proposed in Kozen [24] as a solution to the problem of the finite axiomatization of the algebra of regular sets by finitely many equations.[2] Although $\mathbf{KA}_{10}$ is only a quasi variety, it can prove the equality of any two regular languages that correspond to the same minimal

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deterministic FSA. *Action logic* (**ACT**) was introduced in Pratt [25] as a solution to the same problem. **ACT** is a variety (over an extended vocabulary) and it is complete for the equality problem of regular languages. Another natural interpretation of **ACT** is that it talks about actions and "program execution." This interpretation is closely related to the interpretation of one of the sorts in dynamic logic. Both $\mathbf{KA}_{10}$ and **ACT** have other models beyond regular sets, notably, relation algebras.

Our interest in these logics stems from their *lack of conjunction* (meet) and their *lack of a set theoretical semantics*. Nonclassical logics have been supplied with set theoretical semantics from the late 50s–early 60s. First, some normal modal logics and intuitionistic logic were given such semantics by Kripke. Then in the early 70s Routley and Meyer defined relational semantics for various relevance logics, and many other nonclassical logics received set theoretical semantics along the same lines afterward. The theory of generalized Galois logic (gaggle theory) provides a unified framework for relational semantics for various nonclassical logics. Our paper might be viewed—beyond giving semantics for particular logics—as a further extension and development of gaggle theory.

To make our paper self-contained, in Section 2 we introduce syntactic calculi for $\mathbf{KL}_{10}$ and **ACT**, and we elaborate on why these logics are interesting from the point of view of representation theory. In Section 3—as a background for the semantics we are to define—we recall the basic principles of gaggle theory. Section 4 contains our semantics for $\mathbf{KL}_{10}$ together with the proof of adequacy. We also show that a natural interpretation—in terms of information states—emerges from a view akin to Shannon's concept of information. Section 5 concerns the undefinability of transitive reflexive closure in first-order logic. Section 6 introduces ideas from our four-valued approach and provides a sound and complete semantics for **ACT**. We also point out the modifications that are necessary and sufficient to have adequacy for $\mathbf{KL}_{10}$. We end this section by showing that a "dual" to Shannon's view on information, which perhaps could be called the "common view," provides additional explanation of the logics and algebras. In Section 7 we make precise the intuitive idea that the * operation is related to $\circ$ and not merely yet another connective of the logic. The semantics we define uses a single (ternary) relation to model both $\circ$ and *. Lastly, Section 8 highlights the results that show how to extend gaggle theory in various ways—what we consider to be the primary contribution of our paper.

2 $\mathbf{KL}_{10}$ and ACT Defined

$\mathbf{KA}_{10}$ was introduced in the form of an algebra. We define $\mathbf{KL}_{10}$ as a Hilbert-style calculus, so that the Lindenbaum algebra of $\mathbf{KL}_{10}$ turns out to be a $\mathbf{KA}_{10}$. $\mathbf{KL}_{10}$ is a 0-order logic, with binary connectives $\vee, \circ, \rightarrow$, a unary connective *, and constants $\boldsymbol{t}$ and $\boldsymbol{F}$.[3]

The connectives may be informally interpreted as *extensional disjunction* ($\vee$), *intensional conjunction* (fusion) ($\circ$), and the *Kleene star* (*). $\rightarrow$ is *entailment*, and we allow this connective to occur only as the main connective of a formula. $\boldsymbol{t}$ is *intensional truth* and $\boldsymbol{F}$ is *extensional falsity*.[4]

The axioms and rules of $\mathbf{KL}_{10}$ are as follows. (Here $\varphi, \psi, \chi, \ldots$ are restricted to be wffs that do not contain $\rightarrow$.)

(A1) $\varphi \to \varphi$
(A2) $\varphi \to (\varphi \vee \psi)$
(A3) $\varphi \to (\psi \vee \varphi)$
(A4) $\boldsymbol{F} \to \varphi$
(A5) $(\boldsymbol{t} \circ \varphi) \to \varphi$
(A6) $\varphi \to (\boldsymbol{t} \circ \varphi)$
(A7) $(\varphi \circ \boldsymbol{t}) \to \varphi$
(A8) $\varphi \to (\varphi \circ \boldsymbol{t})$
(A9) $(\varphi \circ (\psi \circ \chi)) \to ((\varphi \circ \psi) \circ \chi)$
(A10) $((\varphi \circ \psi) \circ \chi) \to (\varphi \circ (\psi \circ \chi))$
(A11) $(\varphi \circ \boldsymbol{F}) \to \boldsymbol{F}$
(A12) $(\boldsymbol{F} \circ \varphi) \to \boldsymbol{F}$
(A13) $(\varphi \circ (\psi \vee \chi)) \to ((\varphi \circ \psi) \vee (\varphi \circ \chi))$
(A14) $((\varphi \circ \psi) \vee (\varphi \circ \chi)) \to (\varphi \circ (\psi \vee \chi))$
(A15) $((\varphi \vee \psi) \circ \chi) \to ((\varphi \circ \chi) \vee (\psi \circ \chi))$
(A16) $((\varphi \circ \psi) \vee (\chi \circ \psi)) \to ((\varphi \vee \chi) \circ \psi)$
(A17) $(\boldsymbol{t} \vee (\varphi \circ \varphi^*)) \to \varphi^*$
(A18) $(\boldsymbol{t} \vee (\varphi^* \circ \varphi)) \to \varphi^*$
(R1) $\varphi \to \psi, \chi \to \psi \,//\, (\varphi \vee \chi) \to \psi$
(R2) $\varphi \to \psi, \chi \to \zeta \,//\, (\varphi \circ \chi) \to (\psi \circ \zeta)$
(R3) $(\varphi \circ \psi) \to \psi \,//\, (\varphi^* \circ \psi) \to \psi$
(R4) $(\varphi \circ \psi) \to \varphi \,//\, (\varphi \circ \psi^*) \to \varphi$
(R5) $\varphi \to \psi, \psi \to \chi \,//\, \varphi \to \chi$

Definition 2.1 $\mathfrak{A} = \langle A; \vee, \circ, {}^*, \boldsymbol{t}, \boldsymbol{F} \rangle$ is a *Kleene algebra*$_{10}$ when[5]

(i) $\langle A; \vee, \boldsymbol{F} \rangle$ is a join semi-lattice with bottom,
(ii) $\langle A; \circ, \boldsymbol{t} \rangle$ is a monoid,
(iii) $a \circ \boldsymbol{F} = \boldsymbol{F} = \boldsymbol{F} \circ a$,
(iv) $a \circ (b \vee c) = (a \circ b) \vee (a \circ c) \qquad (b \vee c) \circ a = (b \circ a) \vee (c \circ a)$,
(v) $\boldsymbol{t} \vee (a \circ a^*) \leq a^* \qquad \boldsymbol{t} \vee (a^* \circ a) \leq a^*$,
(vi) $a \circ b \leq b \;\Rightarrow\; a^* \circ b \leq b \qquad b \circ a \leq b \;\Rightarrow\; b \circ a^* \leq b$.

Two formulas φ and ψ are in the equivalence relation, which is used in the definition of the Lindenbaum algebra, whenever $\vdash \varphi \to \psi$ and $\vdash \psi \to \varphi$. (We leave the proof of the next lemma to the interested reader.)

Lemma 2.2 *The Lindenbaum algebra of* $\mathbf{KL}_{10}$ *is a* $\mathbf{KA}_{10}$.

ACT extends the set of connectives by *implications* ($\to$ and $\leftarrow$), which are the (right and left) residuals of fusion. The introduction of residuation is what allows equational axiomatization. A Hilbert-style calculus for **ACT** is the following collection of axiom schemes and rules.

(A1) $\varphi \to \varphi$
(A2) $\varphi \to (\varphi \vee \psi)$
(A3) $\varphi \to (\psi \vee \varphi)$
(A4) $\boldsymbol{F} \to \varphi$
(A5) $((\varphi \circ \psi) \circ \chi) \to (\varphi \circ (\psi \circ \chi))$
(A6) $(\varphi \circ (\psi \circ \chi)) \to ((\varphi \circ \psi) \circ \chi)$
(A7) $(\boldsymbol{t} \circ \varphi) \to \varphi$

(A8) $\varphi \to (\boldsymbol{t} \circ \varphi)$
(A9) $(\varphi \circ \boldsymbol{t}) \to \varphi$
(A10) $\varphi \to (\varphi \circ \boldsymbol{t})$
(A11) $((\varphi \to \psi) \circ \varphi) \to \psi$
(A12) $\varphi \to (\psi \to (\varphi \circ \psi))$
(A13) $(\varphi \circ (\psi \leftarrow \varphi)) \to \psi$
(A14) $\psi \to ((\varphi \circ \psi) \leftarrow \varphi)$
(A15) $(\boldsymbol{t} \vee (\varphi^* \circ \varphi^*) \vee \varphi) \to \varphi^*$
(A16) $(\varphi \to \varphi)^* \to (\varphi \to \varphi)$
(R1) $\varphi \to \psi, \psi \to \chi \,//\, \varphi \to \chi$
(R2) $\varphi \to \psi \,//\, (\chi \to \varphi) \to (\chi \to \psi)$
(R3) $\varphi \to \psi \,//\, (\varphi \leftarrow \chi) \to (\psi \leftarrow \chi)$
(R4) $\varphi \to \psi, \chi \to \psi \,//\, (\varphi \vee \chi) \to \psi$
(R5) $\varphi \to \psi, \chi \to \zeta \,//\, \varphi \circ \chi \to \psi \circ \zeta$
(R6) $\varphi \to \psi \,//\, \varphi^* \to \psi^*$
(R7) $\varphi \to \psi, \varphi \,//\, \psi$

The equivalence relation to define the Lindenbaum algebra of **ACT** is—again—generated by $\varphi \to \psi$ and $\psi \to \varphi$ both being theorems. For our purposes it is not of primary importance to formulate the algebra with equations only; therefore, we give the Lindenbaum algebra in a form that is more suited to semantical proofs. (The proof of the following lemma is left to the reader.)

Lemma 2.3 *The Lindenbaum algebra of* **ACT** *is* $\mathfrak{A} = \langle A; \vee, \circ, \to, \leftarrow, {}^*, \boldsymbol{t}, \boldsymbol{F}\rangle$ *where*[6]

(i) $\langle A; \vee, \boldsymbol{F}\rangle$ *is a join semi-lattice with a least element,*
(ii) $\langle A; \circ, \boldsymbol{t}\rangle$ *is a monoid,*
(iii) $a \leq b \to c \quad \Leftrightarrow \quad a \circ b \leq c \quad \Leftrightarrow \quad b \leq c \leftarrow a$,
(iv) $\boldsymbol{t} \vee (a^* \circ a^*) \vee a \leq a^*$,
(v) $a^* \leq (a \vee b)^*$,
(vi) $(a \to a)^* \leq a \to a$.

A common feature of Kripke-style semantics for various logics is the use of (some kind of) *filters* in the canonical model. In the case of normal modal logics (that have Boolean negation) these are *ultrafilters*; without Boolean negation these are *prime filters*. (In both cases conjunction is $\cap$ and disjunction is $\cup$.) If conjunction and disjunction are not in the language but there is a fusionlike operation and maybe its residual(s), then *cones* (upward closed subsets) may be used instead of filters. If a logic has both conjunction and disjunction but they do not distribute over each other, then the semantics might become more complicated; in particular, *ideals* might be used too. Nonetheless, the intensional connectives are defined on the filters, or on some combination of filters and ideals. The ideals are primarily exploited to define a *closure operation* on the union of sets of filters; that is, conjunction is unchangeably modeled as $\cap$ and disjunction is the closure of $\cup$.[7]

$\mathbf{KL}_{10}$ and **ACT** do not have conjunction; therefore, there are no filters whatsoever in their Lindenbaum algebras. This means that if we are to define a semantics for these logics then we should rely not on filters or a combination of filters and ideals but either *purely on ideals* or on a *combination of cones and ideals*. (Of course, yet another possibility is to extend these logics to include conjunction; however, our aim is to give semantics for them as they are.)

Our first semantics (for $\mathbf{KL}_{10}$) is based on ideals only. Thus we start with solving the problem of representing the intensional connectives on ideals. Of course, we also change the usual modeling of disjunction; we trade the customary $\cup$ for a $\cap$, which makes perfect sense once we have moved from filters to ideals. For insights we rely on gaggle theory turned on its head, so to speak. More precisely, we show how to extend gaggle theory to join semi-lattices with a technical trick, namely, by using Zorn's lemma to define suitable ideals.

Another problem to solve is how to model *. One way to look at this operation is simply as yet another operation in the algebra. Since there is no distribution pattern to base the proper modeling on according to gaggle theory, we will let the behavior of * on the extremal element guide us in choosing the modeling.[8]

3 Gaggles

Gaggle theory provides semantics for a wide range of nonclassical logics in a systematic way. The regularity of the semantics is based on the *distribution* patterns of operations (over the lattice operations meet and join) and the interrelations of operations via abstract *residuation*. We want here only to introduce the main ideas of gaggle theory in order to explain how our present representation results are related to and go beyond it. Hence, we do not give a comprehensive description, and we use illustrations, which involve binary operations only.[9]

A gaggle contains a *distributive lattice* and a *family* of operations. Each operation is required to distribute *into* meet or join *over* meet or join. The last sentence is not a misprint. For example, let us consider $\supset$—classical implication. $(\varphi \vee \psi) \supset \chi$ is equivalent to $(\varphi \supset \chi) \wedge (\psi \supset \chi)$, and $\chi \supset (\varphi \wedge \psi)$ is equivalent to $(\chi \supset \varphi) \wedge (\chi \supset \psi)$. In other words, $\supset$ distributes into $\wedge$ over $\vee$ in its first and over $\wedge$ in its second argument place. From the distribution pattern a certain tonicity of an operation follows, because both meet and join allow the definition of a partial order—in a lattice, of one and the same partial order. A family of operations is not simply a collection of operations but a collection of operations of the *same arity*, where the operations are connected via abstract *residuation* laws. A concrete example of (abstract) residuation is the relation that links the (binary intensional) operations $\circ$, $\rightarrow$, and $\leftarrow$ in the Lambek calculus into a family. These operations are related in the Lindenbaum algebra as $a \circ b \leq c$ if and only if $a \leq b \rightarrow c$ if and only if $b \leq c \leftarrow a$. Roughly speaking, residuation allows moving back and forth between the left- and right-hand side of inequations with the operations changed appropriately.

Given a family of operations on a distributive lattice, gaggle theory provides rules as to how to define n-ary operations from an $n+1$-place accessibility relation on a frame and how to define the canonical accessibility relation. Instead of stating the rules we give an illustration on a familiar example of $\circ$ and $\rightarrow$ from relevance logic. $\circ$ distributes into *join* over join in both places. Therefore, the operation defined from a ternary relation is the existential image operation; that is, the definition is an *existentially* quantified *conjunction*. $\rightarrow$ distributes into *meet*; hence it is represented via a *universally* quantified *disjunction* that has the *complemented* "converse" of this relation—where the "converse" corresponds to picking another argument place to "stand for" the implication than the one that was inhabited by fusion. Set memberships in the definitions are positive when an operation distributes into join over join or into meet over meet; they are negated otherwise. That is,

$$\begin{aligned} A \circ B &= \{ z \mid \exists xy(Rxyz \wedge x \in A \wedge y \in B) \}, \\ B \to C &= \{ x \mid \forall yz(\overset{1 \rightleftarrows 3}{\overline{R}zyx} \vee y \notin B \vee z \in C) \}. \end{aligned}$$

These definitions are easily seen to be (or to be equivalent to) the usual ones—we chose their form only to emphasize the points of the previous paragraph. ($1 \rightleftarrows 3$ indicates "converse", i.e., the relation is as $\overline{R}$ except that the first and third arguments are interchanged.)

The canonical accessibility relation is defined as a *universally* quantified *disjunction* if the operation distributes into join and as an *existentially* quantified *conjunction* otherwise. Set memberships in the definitions are negative when an operation distributes into join over join or into meet over meet; they are negated otherwise. Continuing with the same example, the canonical accessibility relation is defined as

$$\forall ab\, (a \notin x \vee b \notin y \vee a \circ b \in z). \tag{R}$$

Of course, we tacitly assumed in this example that conjunction and disjunction are modeled in the usual way, that is, as *intersection* and *union*, and the objects of the canonical model are (proper) *prime filters*. As a first step toward turning gaggles upside down we state a small representation theorem.

Theorem 3.1 *Let $\mathfrak{A} = \langle A; \wedge, \vee, \circledcirc \rangle$ be a distributive lattice with an operation $\circledcirc$ which has a distribution pattern $\wedge, \wedge \to \wedge$. (That is, $a \circledcirc (b \wedge c) = (a \circledcirc b) \wedge (a \circledcirc c)$ and $(a \wedge b) \circledcirc c = (a \circledcirc c) \wedge (b \circledcirc c)$.) The powerset of a structure (U, R) with intersection, union, and the existential image operation of R contains a homomorphic image of $\mathfrak{A}$. The canonical structure is the set of prime ideals with R defined as in (R) above.*

Proof We leave some of the details of the proof to be filled in by the reader; however, we fix some ideas. The powerset is certainly a distributive lattice, so a subset closed under $\cap$, $\cup$, and $\circledcirc$ (the operation defined from R) is suitable as a carrier set. To see that the distribution pattern is preserved it is enough to note that meet is union, and the existential quantifier distributes over disjunction. $\square$

This theorem provides an important insight on how we are to model the operation $\circ$ and * when there are only ideals. (The above operation is like an intensional disjunction that appears in some substructural logics.)

4 Semantics Motivated by Gaggle Theory

A semantics for Kleene logic is obtained from a *structure* $\mathfrak{F}$ defined as follows.

Definition 4.1 $\mathfrak{F} = \langle U, \overline{R}_\circ, R_*, I \rangle$ where the elements of the four-tuple satisfy the following conditions.

(1) $\varnothing \neq U, \quad \varnothing \neq I \subseteq U, \quad \overline{R}_\circ \subseteq U \times U \times U, \quad R_* \subseteq U \times U$

(2) $\exists\beta(\overline{R}_\circ\beta\alpha\alpha \wedge \beta \notin I) \qquad (\overline{R}_\circ\alpha\beta\gamma \wedge \alpha \notin I \wedge \gamma \in A) \Rightarrow \beta \in A$

$\exists\beta(\overline{R}_\circ\alpha\beta\alpha \wedge \beta \notin I) \qquad (\overline{R}_\circ\alpha\beta\gamma \wedge \beta \notin I \wedge \gamma \in A) \Rightarrow \alpha \in A$

(3) $(\overline{R}_\circ\alpha\beta\varepsilon \wedge \overline{R}_\circ\varepsilon\gamma\delta) \Rightarrow \exists\zeta(\overline{R}_\circ\beta\gamma\zeta \wedge \overline{R}_\circ\alpha\zeta\delta)$

$(\overline{R}_\circ\beta\gamma\varepsilon \wedge \overline{R}_\circ\alpha\varepsilon\delta) \Rightarrow \exists\zeta(\overline{R}_\circ\alpha\beta\zeta \wedge \overline{R}_\circ\zeta\gamma\delta)$

(4) $(\alpha R_{*}\beta \wedge \alpha \in A) \ \Rightarrow\ \beta \in I$

(5) $(\overline{R}_{\circ}\alpha\beta\gamma \wedge \delta R_{*}\gamma \wedge \varepsilon R_{*}\alpha \wedge \delta \in A) \ \Rightarrow\ (\varepsilon \notin A \wedge \beta \in A)$

$(\overline{R}_{\circ}\alpha\beta\gamma \wedge \delta R_{*}\gamma \wedge \varepsilon R_{*}\beta \wedge \delta \in A) \ \Rightarrow\ (\varepsilon \notin A \wedge \alpha \in A)$

(6) $(\overline{R}_{\circ}\delta\varepsilon\beta \wedge \delta \notin B \wedge \beta \in B) \ \Rightarrow\ \exists\vartheta(\vartheta R_{*}\varepsilon \wedge \overline{R}_{\circ}\delta\vartheta\beta)$

$(\overline{R}_{\circ}\delta\varepsilon\beta \wedge \varepsilon \notin B \wedge \beta \in B) \ \Rightarrow\ \exists\vartheta(\vartheta R_{*}\delta \wedge \overline{R}_{\circ}\vartheta\varepsilon\beta)$

Informally, U is a set of *information states*, and I is a nonempty subset of *distinguished* states. Informally speaking these are the logical states. $\overline{R}_{\circ}$ is a ternary and R_{*} is a binary *accessibility relation* on U. The role of the rest of the conditions will become clear in the soundness proof.

Definition 4.2 Let $\mathfrak{F}$ be a structure. A $\mathbf{KL}_{10}$-*model* $\mathfrak{M}$ (based on $\mathfrak{F}$) contains a set of subsets of U ($M \subseteq \mathcal{P}(U)$) which has as its elements the sets defined in (7) and (8) and which is closed under the operations defined in (9)–(11). (The (meta)variables A and B range over $\mathcal{P}(U)$.)

(7) $\boldsymbol{t} =_{\mathrm{def}} I$

(8) $\boldsymbol{F} =_{\mathrm{def}} U$

(9) $A \vee B =_{\mathrm{def}} A \cap B$

(10) $A \circ B =_{\mathrm{def}} \{\gamma : \forall\alpha\beta((\overline{R}_{\circ}\alpha\beta\gamma \wedge \alpha \notin A) \ \Rightarrow\ \beta \in B)\}$

(11) $A^{*} =_{\mathrm{def}} \{\beta : \exists\alpha(\alpha R_{*}\beta \wedge \alpha \in A)\}$

The set I, which models the intensional truth $\boldsymbol{t}$, is sometimes created from a special situation. In the present semantics postulating the set outright avoids some complications. The bottom element of a lattice is usually the empty set of (prime) filters, so "dually" here it is the set of all the information states.

(9) is straightforward if one thinks in terms of ideals. (10) relies essentially on the insights from Theorem 3.1 together with the observation that just as $\exists$ distributes over $\vee$, so does $\forall$ over $\wedge$. Note that (10) and (11) are *very* different despite the fact that $\circ$ and * are related—for example, as are concatenation and the Kleene star in regular languages. However, as we already mentioned, * does not distribute over $\vee$ and does not preserve the bottom element—unlike $\circ$ does. The latter suggests that (if there were $\wedge$ in the logic) a $\forall\alpha(\cdots \vee \cdots)$-type definition would be appropriate (on filters). Thus, using $\exists\alpha(\cdots \wedge \cdots)$ is indeed the correct "necessity sort of" definition.

Theorem 4.3 (Soundness) *A* $\mathbf{KL}_{10}$-*model* $\mathfrak{M}$ *is a* $\mathbf{KA}_{10}$.

Proof In effect, we show that the Lindenbaum algebra is freely generated by (the equivalence classes of) the propositional variables with respect to the class of $\mathbf{KL}_{10}$-models. Namely, we show that (i)–(vi) (from Definition 2.1) hold in any $\mathbf{KL}_{10}$ model.

1 Disjunction is interpreted via set intersection; therefore, associativity, commutativity, and idempotence are immediate. With $\boldsymbol{F} = U$, $\boldsymbol{F}$ is clearly the bottom element.

2 We prove that conditions (2)–(3) are sufficient to ensure that $\circ$ is associative with I being its identity. First, let us assume that $\gamma \in I \circ A$, that is, $\forall\alpha\beta((\overline{R}_{\circ}\alpha\beta\gamma \wedge \alpha \notin I) \ \Rightarrow\ \beta \in A)$. Let us also assume (to the contrary) that $\gamma \notin A$. By universal instantiation from the previous formula we have that

$\forall\alpha((\overline{R}_\circ\alpha\gamma\gamma \wedge \alpha \notin I) \Rightarrow \gamma \in A)$. (2) provides that $\exists\beta(\overline{R}_\circ\beta\alpha\alpha \wedge \beta \notin I)$. From the apparent contradiction we obtain $\gamma \in A$. For the other direction we assume that $\gamma \in A$, and additionally that $\overline{R}_\circ\alpha\beta\gamma$ and $\alpha \notin I$. One of the other frame conditions listed in (2) is $\forall\alpha\beta\gamma((\overline{R}_\circ\alpha\beta\gamma \wedge \alpha \notin I) \Rightarrow (\gamma \in A \Rightarrow \beta \in A))$. By two detachments $\beta \in A$. Eliminating the last two assumptions we have that $\forall\alpha\beta((\overline{R}_\circ\alpha\beta\gamma \wedge \alpha \notin I) \Rightarrow \beta \in A)$, that is, $\gamma \in I \circ A$. ($A \circ I = A$ may be proven analogously using the next two conditions in (2).)

Now we prove that $\circ$ is associative. Let $\delta \in (A \circ B) \circ C$. Then $\forall\alpha\gamma((\overline{R}_\circ\alpha\gamma\delta \wedge \alpha \notin A \circ B) \Rightarrow \gamma \in C)$. Further, let us assume that $\overline{R}_\circ\varepsilon\zeta\delta$ and $\varepsilon \notin A$ as well as $\overline{R}_\circ\gamma\vartheta\zeta$ and $\vartheta \notin C$. By eliminating and introducing a few conjunctions we have the antecedent of the second condition in (3), $(\overline{R}_\circ\gamma\vartheta\zeta \wedge \overline{R}_\circ\varepsilon\zeta\delta)$. Modus ponens yields $\exists\alpha(\overline{R}_\circ\varepsilon\gamma\alpha \wedge \overline{R}_\circ\alpha\vartheta\delta)$. Having instantiated $\exists\alpha$, and having conjoined the second subformula with $\vartheta \notin C$, we apply modus tollens, and so we get $\alpha \in A \circ B$. Then, from the definition of $\circ$ and $\overline{R}_\circ\varepsilon\gamma\alpha \wedge \varepsilon \notin A$, we obtain $\gamma \in B$. Further, $\forall\gamma\vartheta((\overline{R}_\circ\gamma\vartheta\zeta \wedge \gamma \notin B) \Rightarrow \vartheta \in C)$, that is, $\zeta \in B \circ C$; hence $\delta \in A \circ (B \circ C)$ after we introduce $\Rightarrow$ and apply the definition of $\circ$. (The other inclusion may be proven similarly using the first condition in (3).)

3 To prove that $\circ$ is normal, we first prove that $\boldsymbol{F} = \boldsymbol{F} \circ A$. From right to left the inclusion is trivial. From left to right the inclusion means that if $\gamma \in \boldsymbol{F}$, then $\gamma \in \boldsymbol{F} \circ A$. However, the latter is the same as $\forall\alpha\beta((\overline{R}_\circ\alpha\beta\gamma \wedge \beta \notin A) \Rightarrow \alpha \in \boldsymbol{F})$—which is certainly true. ($\boldsymbol{F} = A \circ \boldsymbol{F}$ is alike.)

4 We prove that $\circ$ distributes over join from both sides as follows. Let $\gamma \notin A \circ (B \vee C)$, that is, $\exists\alpha\beta(\overline{R}_\circ\alpha\beta\gamma \wedge \alpha \notin A \wedge \beta \notin (B \vee C))$. Since $\beta \notin B \vee C$ if and only if $\beta \notin B$ or $\beta \notin C$, by distribution $\gamma \notin A \circ B$ or $\gamma \notin A \circ C$, that is, γ is not a member of their intersection, which means that $\gamma \notin (A \circ B) \vee (A \circ C)$. (The other direction is by retracing the steps, and the "from right" is analogously by a certain symmetry of the definition of $\circ$.)

5 There are two equations and two quasi equations characterizing *, but we chop the two equations into "subequations." To prove $A^* \subseteq I$ we assume $\beta \in A^*$. Then $\exists\alpha(\alpha R_*\beta \wedge \alpha \in A)$. By modus ponens from (4) we obtain $\beta \in I$. For the second subequation, let us assume that $\gamma \in A^*$, that is, $\exists\delta(\delta R_*\gamma \wedge \delta \in A)$. Assume also $\overline{R}_\circ\alpha\beta\gamma$ and $\varepsilon R_*\alpha$. From (5) $\varepsilon \notin A$ and $\beta \in A$. By $\Rightarrow$ introduction $\alpha \notin A^*$, and further, $\gamma \in A^* \circ A$. (The third subequation is proven similarly to this one.)

6 Lastly, we prove that the two quasi equations hold too. Suppose that $B \subseteq B \circ A$ and $\beta \in B$. Expanding $\beta \in B \circ A$ we get $\forall\alpha\gamma((\overline{R}_\circ\alpha\gamma\beta \wedge \gamma \notin A) \Rightarrow \alpha \in B)$. Relying on the definition of $\circ$ and the conclusion we are to derive, let us assume that $\overline{R}_\circ\delta\varepsilon\beta$ as well as $\delta \notin B$. From the first condition in (6) by detachment we get $\exists\vartheta(\vartheta R_*\varepsilon \wedge \overline{R}_\circ\delta\vartheta\beta)$. Instantiating in $\beta \in B \circ A$ and detaching the antecedent we have $\vartheta \in A$. Then $\varepsilon \in A^*$; hence, $\beta \in B \circ A^*$ as we had to show. (The other inequation may be proven to hold similarly using the other condition in (6).) □

Before we turn to prove completeness we pause to provide a more detailed *informal interpretation* of the semantics in terms of actions. The background concept is that information is "proportional" to the amount of *eliminated possibilities*. This idea is famous with Shannon, who developed it quantitatively. Here we do not assume a fixed numerical space, rather we assume that information states might be "incommensurable," in which case it is still meaningful to interpret some logical operations on them. Thus, the lack of information corresponds to the whole set U of information states. Actions are represented by the information states that they exclude. Given two actions A and B the choice (that is, $A \vee B$) excludes only the states which are excluded by both A and B. These are the states which fall into the intersection of A and B. The composition of actions A and B (that is, $A \circ B$) admits an information state which can be reached from any state that is not excluded by A via $\overline{R}_\circ$ and through a state not excluded by B. Lastly, the result of the repeat of the action A excludes any information state that is accessible from an already excluded state passing along the relation R_*.

Now we prove completeness of $\mathbf{KL}_{10}$ with respect to our first canonical model that is defined as follows.

Definition 4.4 (Canonical model) The *canonical model* is $\mathfrak{M} = \langle \mathcal{I}, \overline{R}_\circ, R_*, I, h \rangle$ where $\mathcal{I}$ is the set of (nonempty) proper ideals on the Lindenbaum algebra, and the other elements of the quintuple are defined as

(12) $\overline{R}_\circ\alpha\beta\gamma \ \Leftrightarrow_{\text{def}} \ \forall ab(a \circ b \in \gamma \ \Rightarrow \ (a \in \alpha \vee b \in \beta))$,

(13) $\alpha R_*\beta \ \Leftrightarrow_{\text{def}} \ \forall a(a \in \alpha \ \Rightarrow \ a^* \in \beta)$,

(14) $I =_{\text{def}} \{\alpha : \alpha \in \mathcal{I} \wedge \boldsymbol{t} \in \alpha\}$,

(15) $U =_{\text{def}} \mathcal{I}$ (or equivalently, $U =_{\text{def}} \{\alpha : \alpha \in \mathcal{I} \wedge \boldsymbol{F} \in \alpha\}$),

(16) $h(a) =_{\text{def}} \{\alpha : \alpha \in \mathcal{I} \wedge a \in \alpha\}$.

Theorem 4.5 (Completeness) *h is an isomorphism.*

Proof First we prove that h is a homomorphism and, second, that h is 1-1.

1.1 The two constants are obviously preserved by h.

1.2 The easiest connective is $\vee$. If $\gamma \in h(a \vee b)$ then $a \vee b \in \gamma$ by the definition of h, and $a \in \gamma$ and $b \in \gamma$ since $\gamma \in \mathcal{I}$. Then γ is in both ha and hb and so also in $ha \vee hb$ using properties of $\cap$. All the steps are reversible; hence h preserves $\vee$.

1.3 We prove that $h(a \circ b) = ha \circ hb$. From left to right, let us assume that γ is a member of the set; then $a \circ b \in \gamma$. Additionally let us suppose $\overline{R}_\circ\alpha\beta\gamma$. Modus ponens gives $a \in \alpha \vee b \in \beta$, that is, $\alpha \in ha \vee \beta \in hb$. By implication introduction $\forall\alpha\beta(\overline{R}_\circ\alpha\beta\gamma \ \Rightarrow \ (\alpha \in ha \vee \beta \in hb))$, which is the same—according to the definition of $\circ$—as $\gamma \in ha \circ hb$.

For the inclusion from right to left we assume $\gamma \notin h(a \circ b)$; hence $a \circ b \notin \gamma$. We have to show that there are ideals that are in relation $\overline{R}_\circ$ with respect to γ and not elements of ha and hb. Let E be the set of pairs of ideals defined as $\{\langle I_1, I_2\rangle : \forall ab(a \circ b \in \gamma \ \Rightarrow \ a \in I_1 \vee b \in I_2)\}$. Whatever γ is, if $a \circ b$ is in γ, then $(a]$ and $(b]$ are suitable ideals, because $\circ$ is a monotone operation. Thus E is surely nonempty. As usual, E might be partially ordered by pointwise inclusion. We are to use Zorn's lemma in a somewhat unusual form.[10] For this purpose we note that ideals are subalgebras of a semi-lattice, and their intersection is an ideal too. In particular, if a chain of ideals is in E then so is the intersection of the chain. Then there is a maximal element in E.[11] To see that I_1 and I_2 from the maximal pair $\langle I_1, I_2\rangle$ are $\notin ha$ and $\notin hb$, respectively, we argue in two steps. First, if $a \circ b$ is not of the form $(a_1 \vee \cdots \vee a_n) \circ (b_1 \vee \cdots \vee b_m)$ then $I_1 \notin ha$ and $I_2 \notin hb$ by monotonicitiy of $\circ$. If $a \circ b$ is of the mentioned form then by distributivity $a \circ b$ is an $m \times n$ element disjunction where the disjuncts are $a_i \circ b_j$ $(1 \leq i \leq n, 1 \leq j \leq m)$.

For the sake of transparency, let us assume $n = 2$ and $m = 2$. (The cases when one of the indexes is 1, or one or both of them are ≥ 2 are argued similarly. We only give an illustration in Figure 1 to help the reader to see how to construct the argument for the m, n case.) Since $a \circ b \notin \gamma$ at least one of the four disjuncts is not in γ. By simple combinatorics one can choose just one of a_1 and a_2 and just one of b_1 and b_2 to account for all the disjuncts that are in γ. Hence if $a \in I_1$ or $b \in I_2$ then the pair of these two ideals is not maximal in E contradicting the assumption. It remains to introduce existential quantifiers $\exists \alpha\beta(\overline{R}_\circ\alpha\beta\gamma \wedge \alpha \notin ha \wedge \beta \notin hb)$ and to apply the definition of $\circ$ to get $\gamma \notin ha \circ hb$.

$$\begin{array}{c} \left.\begin{array}{ccc} a_1 \circ b_1 & \dots & a_1 \circ b_m \\ \vdots & \vdots & \vdots \end{array}\right\} a_1, \dots, a_{i-1} \\ \dots \qquad \cancel{a_i \circ b_j} \qquad \dots \\ \left.\begin{array}{ccc} \vdots & \vdots & \vdots \\ \underbrace{a_n \circ b_1}_{b_1, \dots, b_{j-1}} & \dots & \underbrace{a_n \circ b_m}_{b_{j+1}, \dots, b_m} \end{array}\right\} a_{i+1}, \dots, a_n \end{array}$$

Figure 1 The $m \times n$ matrix of disjuncts where $(a_i \circ b_j) \notin \gamma$.

1.4 Let us assume that $\beta \in h(a^*)$, that is, $a^* \in \beta$. By the definition of R_* and monotonicity of *, we have that $(a]R_*\beta$. Since $(a] \in ha$, $\exists\alpha(\alpha R_*\beta \wedge \alpha \in ha)$, which is the same as $\beta \in (ha)^*$. To prove the converse, suppose the latter. Then $\exists\alpha(\alpha R_*\beta \wedge \alpha \in ha)$, and further, $\exists\alpha(\alpha R_*\beta \wedge a \in \alpha)$. From the definition of R_* we get $a^* \in \beta$, and hence $\beta \in h(a^*)$, as desired.

2 To show that h is one-one, let us assume that $a \nleq b$. Then $(b] \in hb$ but $(b] \notin ha$ because $a \notin (b]$, that is, $hb \nsubseteq ha$. This completes the proof. □

Before we proceed to define our next semantics, we briefly recall a result from model theory.

5 Undefinability of Transitive (Reflexive) Closure

Transitive closure of binary relations plays an important role in many places from set theory to filtration in modal logics. Transitive closure and transitive reflexive closure are defined as follows.

Definition 5.1 Let ϱ be a binary relation on X. The *transitive closure* of ϱ (denoted by ϱ^+) is inductively defined as

(1) $\langle x, y\rangle \in \varrho \;\Rightarrow\; \langle x, y\rangle \in \varrho^+$,
(2) $\langle x, y\rangle, \langle y, z\rangle \in \varrho^+ \;\Rightarrow\; \langle x, z\rangle \in \varrho^+$,

where x, y, z range over X.
The *transitive reflexive closure* of ϱ (denoted by ϱ^*) is inductively defined as

(3) $\langle x, x\rangle \in \varrho^*$,
(4) $\langle x, y\rangle \in \varrho \;\Rightarrow\; \langle x, y\rangle \in \varrho^*$,
(5) $\langle x, y\rangle, \langle y, z\rangle \in \varrho^* \;\Rightarrow\; \langle x, z\rangle \in \varrho^*$,

where $x, y, z \in X$. (Of course, if ϱ^+ is already defined from ϱ then ϱ^* is just $\varrho^+ \cup \{\, \langle x, x\rangle : x \in X \,\}$.)

Despite the seeming simplicity of these definitions, it is useful to underscore that the definitions are *inductive* (*recursive*). There are two possible misunderstandings concerning the definability of ϱ^*. First, a confusion might be caused by the following two formulas:

$$\forall x. P(x, x)$$

$$\forall xyz((P(x, y) \wedge P(y, z)) \Rightarrow P(x, z))$$

which *do state* that P (or, more accurately, the interpretation of this two-place predicate) is a reflexive and transitive relation. However, these are sentences (closed wffs) and they *do not define* a binary relation at all.

Another potential misunderstanding is that given a binary relation ϱ, it is possible to define ϱ^* by the formula

$$\langle x, z\rangle \in \varrho^* \Leftrightarrow \exists y_0 \ldots y_n : x = y_0 \wedge y_n = z \wedge \forall i. 0 \leq i < n \Rightarrow \langle y_i, y_{i+1}\rangle \in \varrho.$$

Of course, for this formula to be a definition of ϱ^* the index n would have to be thought of as tacitly quantified (i.e., bound by $\exists n \in \mathbb{N}$). However, according to a well-known metatheorem of first-order logic there is no wff with one free variable that is true of models of arbitrary finite size but false on models of infinite size. In other words, the seemingly first-order formula above, in fact, is not such and cannot be turned into a true first-order formula.

The *first-order undefinability* of the (reflexive) transitive closure of a binary relation does not follow simply from these observations; however, it may be proven using pebble games (or any of the equivalent methods—cf. Ebbinghaus and Flum [20]). This undefinability result is, of course, not an issue for $\mathbf{KL}_{10}$ and **ACT** itself, since these are not first-order logics, but it is useful to keep the undefinability in mind when we build the semantics, especially when we want to use the same relation to represent $\circ$ and *. Now we turn to define our second semantics.

6 Semantics Motivated by Four-valued Approach

The idea behind our four-valued approach to substructural logics was to use filters and ideals in the semantics. The filters were thought of as theories, whereas ideals were considered countertheories, and a proposition consisted of a set of theories and a set of countertheories. Moreover, theories and countertheories jointly characterized a proposition by being sets of sentences which were implied and which implied a particular sentence. To conform to our terminology in the preceding sections we will continue to talk about 'information states' instead of 'theories'.[12]

Definition 6.1 A *structure* for action logic is $\mathfrak{F} = \langle U^+, U^-, \sqsubseteq, R_\circ, \overline{R}_*, \imath, \lesssim\rangle$ where the elements satisfy the conditions (1)–(10). $\mathfrak{F}$ is called a $*$-*structure* if it additionally satisfies (11) and (12).

(1) $U^+ \neq \varnothing \neq U^-$; $\imath \in U^+$; $\sqsubseteq, \overline{R}_* \subseteq U^{+^2}$; $R_\circ \subseteq U^{+^3}$; $\lesssim\ \subseteq U^+ \times U^-$

(2) $(\alpha \lesssim x \wedge \alpha \sqsubseteq \alpha') \Rightarrow \alpha' \lesssim x$ and $\exists\alpha\forall x.\alpha \lesssim x$

(3) $\exists!\, \alpha\, \forall\beta. \beta \sqsubseteq \alpha$ (this α is denoted by $\top$)

(4) $(\alpha \sqsubseteq \beta \wedge \beta \sqsubseteq \gamma) \Rightarrow \alpha \sqsubseteq \gamma$ and $\alpha \sqsubseteq \alpha$

(5) $R_\circ{\downarrow}{\downarrow}{\uparrow}$ and ${\uparrow}\bar{R}_*{\downarrow}$, i.e.,

$(R_\circ\alpha\beta\gamma \wedge \alpha' \sqsubseteq \alpha \wedge \beta' \sqsubseteq \beta \wedge \gamma \sqsubseteq \gamma') \Rightarrow R_\circ\alpha'\beta'\gamma'$

$(\alpha\bar{R}_*\beta \wedge \alpha \sqsubseteq \alpha' \wedge \beta' \sqsubseteq \beta) \Rightarrow \alpha'\bar{R}_*\beta'$

(6) $\exists\alpha(R_\circ\alpha\beta\beta \wedge \iota \sqsubseteq \alpha)$ and $(R_\circ\alpha\beta\gamma \wedge \iota \sqsubseteq \alpha) \Rightarrow \beta \sqsubseteq \gamma$

$\exists\alpha(R_\circ\beta\alpha\beta \wedge \iota \sqsubseteq \alpha)$ and $(R_\circ\alpha\beta\gamma \wedge \iota \sqsubseteq \beta) \Rightarrow \alpha \sqsubseteq \gamma$

(7) $(R_\circ\alpha\beta\eta \wedge R_\circ\eta\gamma\delta) \Rightarrow \exists\varepsilon(R_\circ\beta\gamma\varepsilon \wedge R_\circ\alpha\varepsilon\delta)$

$(R_\circ\beta\gamma\varepsilon \wedge R_\circ\alpha\varepsilon\delta) \Rightarrow \exists\eta(R_\circ\alpha\beta\eta \wedge R_\circ\eta\gamma\delta)$

(8) $(R_\circ\alpha\beta\gamma \wedge \alpha \in [\mathsf{T})) \Rightarrow \gamma \in [\mathsf{T})$

$(R_\circ\alpha\beta\gamma \wedge \beta \in [\mathsf{T})) \Rightarrow \gamma \in [\mathsf{T})$

(9) $(\iota \sqsubseteq \beta \wedge \alpha\bar{R}_*\beta) \Rightarrow \alpha \in A$

(10) $\alpha\bar{R}_*\beta \Rightarrow \beta \sqsubseteq \alpha$

(11) $(R_\circ\alpha\beta\gamma \wedge \forall\delta(\delta\bar{R}_*\alpha \Rightarrow \delta \in A) \wedge \forall\varepsilon(\varepsilon\bar{R}_*\beta \Rightarrow \varepsilon \in A) \wedge \vartheta\bar{R}_*\gamma) \Rightarrow \vartheta \in A$

(12) $\exists\beta\gamma(R_\circ\alpha\beta\gamma \wedge \beta \in A \wedge \gamma \notin A) \Rightarrow \exists\delta\varepsilon\vartheta(\delta\bar{R}_*\alpha \wedge R_\circ\delta\varepsilon\vartheta \wedge \varepsilon \in A \wedge \vartheta \notin A)$

We borrowed from the four-valued approach the idea that states are of two kinds. However, below we primarily rely on sets of positive states, moreover, only on those that are *stable*. Stability is to be defined via the relation $\leqslant$ which links positive and negative states and—by going a type level up—sets of positive states to sets of negative ones.

Definition 6.2 Let $X \subseteq U^+$ and $Y \subseteq U^-$. $\Rsh$ and $\Lsh$ ($\Rsh\colon \mathcal{P}(U^+) \longrightarrow \mathcal{P}(U^-)$, $\Lsh\colon \mathcal{P}(U^-) \longrightarrow \mathcal{P}(U^+)$) are defined as

(13) ${}^{\Rsh}X =_{\mathrm{def}} \{ y\colon \forall\alpha(\alpha \in X \Rightarrow \alpha \leqslant y) \}$,

(14) $Y^{\Lsh} =_{\mathrm{def}} \{ \alpha\colon \forall y(y \in Y \Rightarrow \alpha \leqslant y) \}$.

A set of positive states X is *stable* when $X = ({}^{\Rsh}X)^{\Lsh}$. The set of stable subsets of U^+ is denoted by $\mathcal{P}(U^+)^{\curlyvee}$. A *model* is a $*$-structure with a valuation added—$\mathfrak{M} = \langle \mathfrak{F}, h \rangle$, where h maps wffs into stable sets in accordance with (15)–(21). In other words, $\mathfrak{M} = \langle M, \vee, \circ, \rightarrow, \leftarrow, {}^*, \boldsymbol{t}, \boldsymbol{F} \rangle$, where $M \subseteq \mathcal{P}(U^+)^{\curlyvee}$, M contains $[\iota)$ and $[\mathsf{T})$, and M is closed under the operations defined in (17)–(21).

Definition 6.3 Let $A, B \in \mathcal{P}(U^+)^{\curlyvee}$.

(15) $h(\boldsymbol{t}) = [\iota)$

(16) $h(\boldsymbol{F}) = [\mathsf{T})$

(17) $h(A \vee B) = ({}^{\Rsh}h(A) \cap {}^{\Rsh}h(B))^{\Lsh}$

(18) $h(A \circ B) = \{ \gamma : \exists\alpha \in h(A)\, \exists\beta \in h(B).\, R_\circ\alpha\beta\gamma \}$

(19) $h(A \rightarrow B) = \{ \gamma : \forall\beta \in h(A)\, \forall\alpha \notin h(B).\, \bar{R}_\circ\gamma\beta\alpha \}$

(20) $h(A \leftarrow B) = \{ \gamma : \forall\beta \in h(B)\, \forall\alpha \notin h(A).\, \bar{R}_\circ\beta\gamma\alpha \}$

(21) $h(A^*) = \{ \beta\colon \forall\alpha \notin h(A).\, \alpha R_*\beta \}$

Of course, in general, the clauses themselves do not guarantee that having started with stable sets the resulting sets are stable too. Therefore, now we prove that certain relationships hold between the various types of structures and closure under the operations we have just defined. First we introduce the notion of upward closed sets.

Definition 6.4 Let $\langle W, \leq \rangle$ be a pre-ordered set. The set of *upward closed subsets of* W (denoted by $\mathcal{P}(W)^{\uparrow}$) and the *upward closed subset generated by* x (denoted by $[x)$) are defined as

(22) $V \in \mathcal{P}(W)^{\uparrow} \quad \text{iff} \quad V \in \mathcal{P}(W) \wedge \forall xy((x \in V \wedge x \leq y) \Rightarrow y \in V)$

(23) $y \in [x) \quad \text{iff} \quad x \leq y\,.$

Lemma 6.5 *Let $\mathfrak{F}$ be a structure. If $X \in \mathcal{P}(U^+)^{\curlyvee}$ then $X \in \mathcal{P}(U^+)^{\uparrow}$.*

Proof Let us suppose the antecedent and $X \notin \mathcal{P}(U^+)^{\uparrow}$. Then $\exists \alpha\beta(\alpha \in X \wedge \alpha \sqsubseteq \beta \wedge \beta \notin X)$. Since X is stable, $\forall y(y \in {}^{\overset{\rightarrow}{\ulcorner}}X \Rightarrow \alpha \lesssim y)$. However, $\alpha \lesssim y$ and $\alpha \sqsubseteq \beta$ implies $\beta \lesssim y$; hence $\beta \in ({}^{\overset{\rightarrow}{\ulcorner}}X)^{\overset{\leftarrow}{\urcorner}}$, which is a contradiction. □

Lemma 6.6 (Hereditary operations) *If $h(A), h(B) \in \mathcal{P}(U^+)^{\uparrow}$, then $h(A \vee B)$, $h(A \circ B), h(A \rightarrow B), h(A \leftarrow B), h(A^*) \in \mathcal{P}(U^+)^{\uparrow}$. Also, $h(\boldsymbol{t})$ and $h(\boldsymbol{F})$ are upward closed.*

Proof For $\vee$ the claim follows from the proof of the preceding lemma, which also established that $Y^{\overset{\leftarrow}{\urcorner}} \in \mathcal{P}(U^+)^{\uparrow}$ (for $Y \subseteq U^-$). The cases of $\circ, \rightarrow, \leftarrow$, and * follow from the tonicity conditions imposed on $R_{\circ}$ and $\bar{R}_{*}$. (We omit the details.) $h(\boldsymbol{t})$ is an element of $\mathcal{P}(U^+)^{\uparrow}$ by its definition, and so is $h(\boldsymbol{F})$. □

Definition 6.7 Let $\mathfrak{F}$ be a structure. $\alpha, \beta \in U^+$ are *indistinguishable* whenever $\alpha \lesssim x \Leftrightarrow \beta \lesssim x$ $(\forall x \in U^-)$. $\mathfrak{F}$ is *distinguished* if and only if there are no distinct $\alpha, \beta \in U^+$ which are indistinguishable.

Clearly, a subset of $\mathcal{P}(U^-)$ together with the indistinguishability relation gives rise to a partition on U^+. Furthermore, any structure satisfying (2) and divided out by this equivalence relation yields a distinguished structure which again satisfies (2). Informally, in a distinguished frame any two positive states differ from each other in regard to which negative states they are linked.

Definition 6.8 Let $\mathfrak{F}$ be a structure for action logic. $\mathfrak{F}$ is

(24) *inverted* iff ${}^{\overset{\rightarrow}{\ulcorner}}[\alpha) \subseteq {}^{\overset{\rightarrow}{\ulcorner}}[\beta) \Rightarrow \alpha \sqsubseteq \beta$,

(25) *complete* iff $\exists!\,\beta.\ \bigcap_{i \in I} {}^{\overset{\rightarrow}{\ulcorner}}[\alpha_i) = {}^{\overset{\rightarrow}{\ulcorner}}[\beta)$,

(26) $R_{\circ}^{\circ}$*-fit* iff $\exists!\,\gamma'.\,[\gamma') = \{\,\gamma : \exists\alpha\beta(R_{\circ}\alpha\beta\gamma \wedge \alpha \in [\alpha') \wedge \beta \in [\beta'))\,\}$,

(27) $R_{\circ}^{\rightarrow}$*-fit* iff $\exists!\,\alpha'.\,[\alpha') = \{\,\alpha : \forall\beta\gamma\,((R_{\circ}\alpha\beta\gamma \wedge \beta \in [\beta')) \Rightarrow \gamma \in [\gamma'))\,\}$,

(28) $R_{\circ}^{\leftarrow}$*-fit* iff $\exists!\,\beta'.\,[\beta') = \{\,\beta : \forall\alpha\gamma\,((R_{\circ}\alpha\beta\gamma \wedge \alpha \in [\alpha')) \Rightarrow \gamma \in [\gamma'))\,\}$,

(29) $\bar{R}_{*}$*-fit* iff $\exists!\,\beta'.\,[\beta') = \{\,\beta : \forall\alpha(\alpha\bar{R}_{*}\beta \Rightarrow \alpha \in [\alpha'))\,\}$.

Lemma 6.9 *Let $\mathfrak{F}$ be a distinguished complete inverted structure. Then $X \in \mathcal{P}(U^+)^{\curlyvee}$ if and only if $\exists\alpha.\ X = [\alpha)$.*[13]

Proof From right to left let us assume that $X = [\alpha)$. By (2) $y \in {}^{\overset{\rightarrow}{\ulcorner}}[\alpha) \Leftrightarrow \alpha \lesssim y$. Were $X \notin \mathcal{P}(U^+)^{\curlyvee}$ then there would be a $\beta \in ({}^{\overset{\rightarrow}{\ulcorner}}X)^{\overset{\leftarrow}{\urcorner}}$ such that $\beta \notin X$, that is,

$\alpha \not\sqsubseteq \beta$. By the definition of $\overset{\leftarrow}{\urcorner}\, \forall y(y \in \overset{\to}{\ulcorner}X \Rightarrow \beta \leqslant y)$. Replacing the antecedent we get $\forall y(\alpha \leqslant y \Rightarrow \beta \leqslant y)$, which means that $\overset{\to}{\ulcorner}[\alpha) \subseteq \overset{\to}{\ulcorner}[\beta)$. Since $\mathfrak{F}$ is inverted it follows that $\alpha \sqsubseteq \beta$—a contradiction.

For the other direction let us assume that X is stable, but for no α holds that $X = [\alpha)$. $X \in \mathcal{P}(U^+)^{\curlyvee}$ implies $X \in \mathcal{P}(U^+)^{\uparrow}$. $\overset{\to}{\ulcorner}X = \bigcap_{\alpha \in X} \overset{\to}{\ulcorner}[\alpha)$, which by the frame being complete is $\overset{\to}{\ulcorner}[\beta)$ (for some β). If there is a γ such that $\overset{\to}{\ulcorner}[\beta) \subseteq \overset{\to}{\ulcorner}[\gamma)$—which puts γ into $(\overset{\to}{\ulcorner}[\beta))^{\overset{\leftarrow}{\urcorner}}$—then indeed $\gamma \in X$ because $\beta \sqsubseteq \gamma$ by the invertedness of the frame. □

The properties of structures that we introduced so far are *sufficient* for the stability of constants and operations.

Lemma 6.10 (Stability of constants and operations) *Let $\mathfrak{F}$ be a distinguished structure. Then*

(i) $h(\boldsymbol{F}) = [\mathsf{T}) \in \mathcal{P}(U^+)^{\curlyvee}$;
(ii) $h(\boldsymbol{t}) = [\,\iota\,) \in \mathcal{P}(U^+)^{\curlyvee}$ *if* $\mathfrak{F}$ *is inverted.*

Let $\mathfrak{F}$ be a distinguished complete inverted structure and $h(A), h(B) \in \mathcal{P}(U^+)^{\curlyvee}$. Then

(iii) $h(A \vee B) \in \mathcal{P}(U^+)^{\curlyvee}$;
(iv) $h(A \circ B) \in \mathcal{P}(U^+)^{\curlyvee}$ *if* $\mathfrak{F}$ *is* $R^{\circ}_{\circ}$*-fit;*
(v) $h(A \to B) \in \mathcal{P}(U^+)^{\curlyvee}$ *if* $\mathfrak{F}$ *is* $R^{\to}_{\circ}$*-fit;*
(vi) $h(A \leftarrow B) \in \mathcal{P}(U^+)^{\curlyvee}$ *if* $\mathfrak{F}$ *is* $R^{\leftarrow}_{\circ}$*-fit;*
(vii) $h(A^*) \in \mathcal{P}(U^+)^{\curlyvee}$ *if* $\mathfrak{F}$ *is* $\overline{R}_*$*-fit.*

Proof

1 (3) together with the assumption that the frame is distinguished guarantees that $[\mathsf{T})$ is stable.

2 From Lemma 6.9 follows that $[\,\iota\,) \in \mathcal{P}(U^+)^{\curlyvee}$ since ι generates $[\,\iota\,)$.

3 It is easy to see that (iii) follows from the completeness of a structure; indeed, a special case of the completeness condition, namely, $I = \mathbf{2}$ suffices.

4 The conditions (26), (27), (28), and (29) guarantee, respectively, that (iv), (v), (vi), and (vii) are true. □

Theorem 6.11 (Soundness) *All the equations of the Lindenbaum algebra of action logic hold in a model $\mathfrak{M}$.*

Proof

1 Levi's definition of $\vee$ (cf. Birkhoff [7]) guarantees that the stable subsets of U^+ form a join semilattice. Since $h(\boldsymbol{F}) = [\mathsf{T})\ \overset{\to}{\ulcorner}h(\boldsymbol{F}) \cap \overset{\to}{\ulcorner}h(A) = \overset{\to}{\ulcorner}h(A)$.

2 The conditions in (7) make $\circ$ associative, and from (6) it follows that $[\,\iota\,)$ is the left-right lower-upper identity of the operation. (We leave the details to be filled in by the reader.)

3 Residuation follows from the definitions of $\circ$ and $\to$, $\leftarrow$ (18)–(20). (Again, we leave the details to be filled in by the reader.)

4 We show that $h(\boldsymbol{F} \circ A) = h(\boldsymbol{F})$, because our way of modeling $\boldsymbol{F}$ is somewhat unusual. Let us assume that $\gamma \in h(\boldsymbol{F} \circ A)$, that is, $\exists \alpha\beta(R_{\circ}\alpha\beta\gamma \wedge \alpha \in h(\boldsymbol{F}) \wedge \beta \in h(A))$.

From one of the conditions in (8) we immediately get that $\gamma \in [\mathsf{T})$ which means $\gamma \in h(\boldsymbol{F})$. The other inclusion is obvious and $h(A \circ \boldsymbol{F}) = h(\boldsymbol{F})$ is alike.

5 To prove $h(\boldsymbol{t}) \subseteq h(A^*)$, assume $\beta \in h(\boldsymbol{t})$. Additionally, let us also suppose $\alpha \bar{R}_* \beta$. Condition (9) implies $\alpha \in h(A)$; hence $\beta \in h(A^*)$.

Next let $\beta \in h(A)$, and suppose $\alpha \bar{R}_* \beta$. (10) implies $\beta \sqsubseteq \alpha$, however, since $h(A) \in \mathcal{P}(U^+)^\uparrow$. Also $\alpha \in h(A)$ and so $\beta \in h(A^*)$.

For the third part of the equation, let us assume that $\gamma \in h(A^* \circ A^*)$, that is, $\exists \alpha\beta(R_\circ \alpha\beta\gamma \wedge \alpha \in h(A^*) \wedge \beta \in h(A^*))$. To prove that $\gamma \in h(A^*)$ let us also assume that $\vartheta \bar{R}_* \gamma$—from which we intend to derive $\vartheta \in h(A)$. The definition of h for * gives $\forall \delta(\delta \bar{R}_* \alpha \Rightarrow \delta \in h(A))$ and $\forall \delta(\delta \bar{R}_* \beta \Rightarrow \delta \in h(A))$. Conjoining the last two formulas with the $R_\circ$ and $\bar{R}_*$ formulas we get the antecedent of (11); by detachment, then $\vartheta \in h(A)$. Eliminating the last assumption and applying the definition of h on * we get that $\gamma \in h(A^*)$.

6 Let us assume that $\gamma \in h(A^*)$, that is, $\forall \beta(\beta \bar{R}_* \gamma \Rightarrow \beta \in h(A))$. Let us also assume that $\beta \bar{R}_* \gamma$. By modus ponens $\beta \in h(A)$ and since $h(A) \subseteq h(A \vee B)$, it follows that $\beta \in h(A \vee B)$. Then $\gamma \in h(A \vee B)$.

7 We prove the last inclusion contrapositively. Thus, $\alpha \notin h(A \to A)$ if and only if $\exists \beta\gamma\,(R_\circ \alpha\beta\gamma \wedge \beta \in h(A) \wedge \gamma \notin h(A))$. This is the antecedent of the implication in (12), and so by modus ponens, $\exists \delta\varepsilon\vartheta\,(\delta \bar{R}_* \alpha \wedge R_\circ \delta\varepsilon\vartheta \wedge \varepsilon \in h(A) \wedge \vartheta \notin h(A))$. Using the definition of h, the latter is the same as $\exists \delta(\delta \bar{R}_* \alpha \wedge \delta \notin h(A \to A))$; further, $\alpha \notin h(A \to A)$. □

The following is an immediate consequence of Lemma 6.10 and Theorem 6.11—it summarizes the relationship between the properties of structures and models of **ACT**.

Corollary 6.12 *Let $\mathfrak{F}$ be a distinguished, inverted, complete $R_\circ^{\leftarrow,\circ,\to}$-fit and $\bar{R}_*$-fit $*$-structure. Then $\mathfrak{M} = \langle M, \vee, \circ, \leftarrow, \to, {}^*, [\imath), [\mathsf{T}), h\rangle$ (where $M \subseteq \mathcal{P}(U^+)^\curlyvee$ and $\mathrm{Im}(h) \subseteq M$) is a model for* **ACT** *whenever h maps propositional variables into $\mathcal{P}(U^+)^\curlyvee$ and is extended to complex formulas according to (17)–(21).*

Our earlier representation results in Bimbó and Dunn [6] (for the minimal substructural logic *LS*) contained *two different semantics*. One of the representations relied on both positive and negative states. Another one, which we called "generalized Kripke semantics," used only positive states. In the present representation we "unified" the two approaches (by slightly modifying the one that used pairs). Although we have two sets, the set of positive and that of negative states, we now show that the canonical structure and model may be defined both ways.

In the first case we define only the canonical model for **ACT**.

Definition 6.13 The *canonical model* is $\mathfrak{M} = \langle \mathcal{C}, \mathcal{I}, \subseteq, R_\circ, \bar{R}_*, [\boldsymbol{t}), \between, h\rangle$ where the elements are as follows.

(1) $\mathcal{C} =_{\mathrm{def}} \{\, X \colon X \neq \varnothing \wedge X \in \mathcal{P}(A)^\uparrow \,\}$

(2) $\mathcal{I} =_{\mathrm{def}} \{\, X \colon X \in \mathcal{P}(A) \wedge \forall ab((a \in X \wedge b \leq a) \Rightarrow b \in X) \wedge \forall ab(a, b \in X \Rightarrow a \vee b \in X) \,\}$

(3) $\subseteq =_{\mathrm{def}} \subseteq \restriction \mathcal{C}$

(4) $R_\circ \alpha\beta\gamma \;\Leftrightarrow_{\mathrm{def}}\; \forall ab((a \in \alpha \wedge b \in \beta) \Rightarrow a \circ b \in \gamma)$

(5) $\alpha \bar{R}_{*} \beta \;\Leftrightarrow_{\text{def}}\; \forall a(a^{*} \in \beta \;\Rightarrow\; a \in \alpha)$

(6) $\alpha \between x \;\Leftrightarrow_{\text{def}}\; \exists a(a \in \alpha \wedge a \in x)$

(7) $h(a) =_{\text{def}} \{\, X \colon X \in \mathcal{C} \wedge a \in X \,\}$

Informally, $\mathcal{C}$ is the set of *cones* on the Lindenbaum algebra of **ACT**, whereas $\mathcal{I}$ is the set of *ideals*.[14] $R_{\circ}$ and $\bar{R}_{*}$ might look familiar when $\circ$ and * are thought of as fusion and necessity. $\between$ is the nonempty intersection of pairs of cones and ideals; that is, a cone α and an ideal x are in this relation whenever they *overlap*. h maps formulas into upward closed subsets of $\mathcal{C}$ such that each element contains the formula in question.[15]

Theorem 6.14 (Completeness) *The canonical valuation is an isomorphism.*

Proof First we prove that h maps all formulas into stable sets. By definition, ha is a principal cone of cones generated by a principal cone, namely, $[[a))$. Since $\mathcal{I}$ contains ideals (without any restrictions) $^{\vec{\ulcorner}}ha$ is an upward closed set—a principal cone (with respect to $\subseteq$)—of ideals generated by $(a]$. In other words, $^{\vec{\ulcorner}}ha = [(a])$, and so $(^{\vec{\ulcorner}}ha)^{\overleftarrow{\urcorner}} = ha$.

The rest of the proof falls into two parts, namely, showing that h preserves the operations and constants and then that h is 1-1.

1.1 $\alpha \in h\boldsymbol{t}$ if and only if $\boldsymbol{t} \in \alpha$ by definition; $h\boldsymbol{t}$ is stable since it is of the form ha. $\alpha \in h\boldsymbol{F}$ if and only if $\boldsymbol{F} \in \alpha$, and so, $h\boldsymbol{F}$ is stable too.

1.2 To show $h(a \vee b) = ha \vee hb$ we suppose that $\gamma \notin h(a \vee b)$. Since $a \vee b \notin \gamma$, let us consider $(a \vee b]$. Certainly, $\neg\, \gamma \between (a \vee b]$, but $[a) \between (a \vee b]$ and $[b) \between (a \vee b]$. Then $(a \vee b] \in {}^{\vec{\ulcorner}}ha \cap {}^{\vec{\ulcorner}}hb$, and so $\exists x(x \in {}^{\vec{\ulcorner}}ha \cap {}^{\vec{\ulcorner}}hb \wedge \neg\, \gamma \between x)$. Thus, $\gamma \notin ha \vee hb$. For the converse inclusion, let us assume that the last formula holds. From $\exists x(x \in {}^{\vec{\ulcorner}}ha \wedge x \in {}^{\vec{\ulcorner}}hb \wedge \neg\gamma \between x)$ and the definition of $^{\vec{\ulcorner}}$ we get $\forall\beta(\beta \in ha \Rightarrow \beta \between x)$ and $\forall\beta(\beta \in hb \Rightarrow \beta \between x)$. Since $[a) \in ha$ and $[b) \in hb$, $a, b \in x$ and so $a \vee b \in x$ (because of $x \in \mathcal{I}$). But from $\neg\gamma \between x$ follows that $a \vee b \notin \gamma$ and so $\gamma \notin h(a \vee b)$.

1.3 Let $\gamma \in h(a \circ b)$. Then $a \circ b \in \gamma$, and by the definition of $R_{\circ}$, $R_{\circ}[a)[b)\gamma$. $[a) \in ha$ and $[b) \in hb$; therefore, $\exists\alpha\beta(R_{\circ}\alpha\beta\gamma \wedge \alpha \in ha \wedge \beta \in hb)$, that is, $\gamma \in ha \circ hb$. For the other direction let $\gamma \in ha \circ hb$, which means that for some α, β, $R_{\circ}\alpha\beta\gamma$ and $\alpha \in ha$ and $\beta \in hb$. Then, from $a \in \alpha$, $b \in \beta$ and the definition of $R_{\circ}$ $a \circ b \in \gamma$; thus $\gamma \in h(a \circ b)$.

1.4 Let $\alpha \in h(b \to c)$, that is, $b \to c \in \alpha$. Let us also assume that $R_{\circ}\alpha\beta\gamma$ and $b \in \beta$. In the Lindenbaum algebra $(b \to c) \circ b \leq c$, so $c \in \gamma$, and $\gamma \in hc$. Then $\alpha \in hb \to hc$. For the other inclusion, let $\alpha \notin h(b \to c)$, that is, $b \to c \notin \alpha$. Let us consider the cone $[b)$ for β, and for γ take $\{\, c \colon \exists ab'(a \in \alpha \wedge b' \in [b) \wedge a \circ b' \leq c) \,\}$—which is a cone by monotonicity of $\circ$. Let us assume that $c \in \gamma$. Then $x \circ b \leq c$ for some $x \in \alpha$. But then $x \leq b \to c$, which is a contradiction. Therefore, $\exists\beta\gamma\,(R_{\circ}\alpha\beta\gamma \wedge \beta \in hb \wedge \gamma \notin hc)$, that is, $\alpha \notin hb \to hc$. (The case of $\leftarrow$ is alike.)

1.5 Let $\alpha \in h(a^{*})$ and $\beta \bar{R}_{*}\alpha$. Then $a^{*} \in \alpha$ and $\forall(b^{*} \in \alpha \;\Rightarrow\; b \in \beta)$. Since $a \in \beta$, $\beta \in h(a)$, and $\alpha \in (ha)^{*}$. For the converse suppose that $\alpha \notin h(a^{*})$, that is, $a^{*} \notin \alpha$. For β consider $\{\, c \colon \exists b(b^{*} \in \alpha \wedge b \leq c) \,\}$, which is a cone because

* is isotone; then $\beta\bar{R}_*\alpha$. If a were an element of β then there would be a $c^* \in \alpha$ such that $c \leq a$. However, then it follows that $a^* \in \alpha$ contradicting the original assumption. Hence $\exists\beta(\beta\bar{R}_*\alpha \wedge \beta \notin ha)$, and so $\alpha \notin (ha)^*$ as we had to show.

2 The separation of any two (i.e., distinct) elements is obvious. If $a \nleq b$, then $b \notin [a)$; hence $[a) \notin hb$, but certainly, $[a) \in ha$. Therefore, $ha \nsubseteq hb$. □

Now we prove a second completeness theorem based on a canonical frame which is like a structure in the generalized Kripke semantics in [6]. One of the novelties of the four-valued semantics for first-degree entailment was that it allowed formulas to be true *and* false at the same time—cf. Dunn [10] and [11]. We develop this idea further by *identifying* the set of positive and negative states, that is, U^+ and U^-. In other words, it is not determined a priori, so to speak, if an information state is positive or negative. Rather the role that the state plays determines its "sign." Thus it is not a "typo" that $\mathcal{C}$ is repeated in the following definition.

Definition 6.15 The *canonical structure* $\mathfrak{F}$ is $\langle \mathcal{C}, \mathcal{C}, \subseteq, R_\circ, \bar{R}_*, [\boldsymbol{t}), \subseteq \rangle$ where $\mathcal{C}$, $\subseteq$, $R_\circ$, $\bar{R}_*$, $[\boldsymbol{t})$ are as in Definition 6.13.

This canonical structure is very much like the one in the generalized Kripke semantics for minimal substructural logic. Since U^+ and U^-, as well as $\sqsubseteq$ and $\leqslant$ are identified, the structure could be presented in a simpler form as $\langle \mathcal{C}, \subseteq, R_\circ, \bar{R}_*, [\boldsymbol{t}) \rangle$. This way the similarity to the original Kripke semantics for modal logics would become more obvious—with the exception that a relation is added.[16]

Lemma 6.16 *$X \in \mathcal{P}(\mathcal{C})^{\curlyvee}$ if and only if $X \in \mathcal{P}(\mathcal{C})^{\uparrow}$ and $\bigcap X \in X$.*[17]

Proof From right to left, $\bigcap X \in X$ implies that $\bigcap X = \alpha$, and since $X \in \mathcal{P}(\mathcal{C})^{\uparrow}$ and $\bigcap X \in X$, in fact, $X = [\alpha)$. Then ${}^{\ulcorner\!\!\rightarrow}X = (\alpha]$, and so $({}^{\ulcorner\!\!\rightarrow}X)^{\leftarrow\!\!\urcorner} = X$.

From left to right, let X be stable. $\bigcap X$ is a cone, and it is not empty, because the semi-lattice of the Lindenbaum algebra has a greatest element too. Furthermore, $(\bigcap X]$ equates to ${}^{\ulcorner\!\!\rightarrow}X$, since these are precisely the cones that are included in all elements of X. Then $[\bigcap X)$ is $({}^{\ulcorner\!\!\rightarrow}X)^{\leftarrow\!\!\urcorner}$, and so if $\bigcap X$ were not a member of X then X would not be stable. Since it is stable, $X = [\bigcap X)$ proving both $\bigcap X \in X$ and $X \in \mathcal{P}(\mathcal{C})^{\uparrow}$. □

This lemma is *more general* and *more specific* at the same time than Lemmas 5.2 and 5.3 in [6]. This lemma uses cones not filters, but now we deal with a bounded semi-lattice, whereas in [6] we had an unbounded lattice.[18]

Lemma 6.17 *The canonical structure (as defined in 6.15) is a structure (as defined in 6.1).*

Proof

1 Clearly all the conditions in (1) hold.

2 If $\alpha \supseteq x$ and $\alpha \subseteq \alpha'$ then $\alpha' \supseteq x$. Further, $\forall x.\,[\boldsymbol{F}) \supseteq x$; thus (2) holds.

3 $[\boldsymbol{F}) \in \mathcal{C}$, and clearly, $[\boldsymbol{F}) = \mathsf{T}$.

4 $\subseteq$ is transitive and reflexive as needed.

5 We show that $\bar{R}_*$ has the right tonicity—$R_\circ$ is similar modulo some changes. Let us assume $\alpha\bar{R}_*\beta$, $\alpha \subseteq \alpha'$, and $\beta' \subseteq \beta$. Then $\forall a(a^* \in \beta \;\Rightarrow\; a \in \alpha)$. But $a \in \alpha$

implies $a \in \alpha'$ and $a^* \in \beta'$ implies $a^* \in \beta$. That is, $\forall a(a^* \in \beta' \Rightarrow a \in \alpha')$, which is $\alpha' \bar{R}_* \beta'$.

6 By monotonicity of $\circ$, the definition of $R_\circ$, and because $\boldsymbol{t} \circ a = a$ $R_\circ[\boldsymbol{t})\beta\beta$ as well as $R_\circ\beta[\boldsymbol{t})\beta$. For the two other conditions, let us suppose $[\boldsymbol{t}) \subseteq \alpha$ and $b \in \beta$. $\boldsymbol{t} \in \alpha$, and $b \in \beta$ imply (by the definition of $R_\circ$) $b \circ \boldsymbol{t} = b \in \gamma$. Switching the argument places of $R_\circ$ does not affect the reasoning; thus all four formulas in (6) are true.

7 Let us assume $R_\circ\alpha\beta\eta \wedge R_\circ\eta\gamma\delta$. To construct an appropriate ε let us define $\{e : \exists ad(a \in \alpha \wedge a \circ d \in \delta \wedge d \leq e)\}$. Monotonicity ensures that ε is a cone and $R_\circ\alpha\varepsilon\delta$. To verify that $R_\circ\beta\gamma\varepsilon$, let us assume that $a \in \alpha$, $b \in \beta$, and $c \in \gamma$. By the assumptions $a \circ b \in \eta$ and $(a \circ b) \circ c = a \circ (b \circ c) \in \delta$; therefore, $b \circ c \in \varepsilon$. Then indeed, $R_\circ\beta\gamma\varepsilon$. (The other condition is provable by permuting α, β, and γ.)

8 Let us assume that $R_\circ\alpha\beta\gamma$ and $\alpha \in [\mathsf{T})$, that is, $\alpha = [\boldsymbol{F})$. Since $\boldsymbol{F} \in \alpha$, for any $b \in \beta$ $\boldsymbol{F} \circ b \in \gamma$, but $\boldsymbol{F} \circ b = \boldsymbol{F}$ and so $\gamma = [\boldsymbol{F})$. (The condition for the second argument place of $R_\circ$ may be shown to hold similarly.)

9 If $[\boldsymbol{t}) \subseteq \beta$ then $\boldsymbol{t} \in \beta$. By the definition of $\bar{R}_*$ if $a^* \in \beta$ then $a \in \alpha$ whenever $\alpha\bar{R}_*\beta$. Then $\boldsymbol{F} \in \alpha$, which implies $\alpha = [\boldsymbol{F})$. However, for all A $[\boldsymbol{F}) \in A$ whenever $A \in \mathcal{P}(\mathfrak{C})^\uparrow$ and $A \neq \emptyset$.

10 Let $\alpha\bar{R}_*\beta$. Then if $a \in \beta$, so is a^* (because $a \leq a^*$ and $\beta \in \mathfrak{C}$). However, $\alpha\bar{R}_*\beta$ and $a^* \in \beta$ imply $a \in \alpha$; therefore, $\beta \subseteq \alpha$. □

Lemma 6.18 *The canonical structure $\mathfrak{F}$ is distinguished, complete, and inverted as well as an $R_\circ^{\leftarrow,\circ,\rightarrow}$-fit and $\bar{R}_*$-fit structure.*

Proof

1 Let α and β be two cones, that is, $\alpha \neq \beta$. By reflexivity $\alpha \subseteq \alpha$ and $\beta \subseteq \beta$. If $\alpha \subseteq \beta$ and $\beta \subseteq \alpha$ would hold, then α would be equal to β. Therefore, either $\alpha \nsubseteq \beta$ or $\beta \nsubseteq \alpha$, and so $\exists\gamma \in \mathfrak{C}$ distinguishing $^{\Rsh}\{\alpha\}$ from $^{\Rsh}\{\beta\}$.

2 Let $^{\Rsh}[\alpha) \subseteq {}^{\Rsh}[\beta)$, that is, whenever $\gamma \subseteq \alpha$ also $\gamma \subseteq \beta$. By universal instantiation $\alpha \subseteq \alpha \Rightarrow \alpha \subseteq \beta$; hence $\alpha \subseteq \beta$ as we had to show.

3 Let $\alpha_{i \in I}$ be a collection of cones. For each i $^{\Rsh}[\alpha_i)$ is $(\alpha_i]$. The intersection of cocones is a cocone.[19] Since the Lindenbaum algebra has a top element $\boldsymbol{T}$ $[\boldsymbol{T}) \in (\alpha_i]$ for all i, the intersection of all $(\alpha_i]$s is not empty. For any collection of cones their intersection is a cone; moreover, $\bigcap_{i \in I}(\alpha_i] = \big(\bigcap_{i \in I}\alpha_i\big]$, that is, $(\beta]$ for some $\beta \in \mathfrak{C}$. Since $\subseteq$ is a partial order β is unique. Then, indeed, $\exists!\beta \bigcap_{i \in I} {}^{\Rsh}[\alpha_i) = {}^{\Rsh}[\beta)$.

4 We define γ to be $\{c : \exists ab(a \in \alpha \wedge b \in \beta \wedge a \circ b \leq c)\}$. γ is a cone and clearly minimal among those cones for which $R_\circ\alpha\beta\gamma$. γ is unique to have this property and so the structure is $R_\circ^\circ$-fit.

5 The cases of the two implications are "symmetric" and so we detail only one of them. Let us consider the principal cones of cones $[\beta)$ and $[\gamma)$. We define α to be the set $\{c : \exists a \leq c \forall b \in \beta . a \circ b \in \gamma\}$. By monotonicity of $\circ$, $\alpha \in \mathfrak{C}$. Furthermore, α is obviously minimal among the cones for which $R_\circ\alpha\beta\gamma$ and at the same time $\beta' \in [\beta)$ ensures $\gamma' \in [\gamma)$. Therefore, indeed $[\alpha) = \{\alpha' : \forall\beta'\gamma'((R_\circ\alpha'\beta'\gamma' \wedge \beta' \in [\beta)) \Rightarrow \gamma' \in [\gamma))\}$. Since α is the only minimal such set, it is unique.

6 Let us take $[\alpha)$, a principal cone of cones. We define β to be $\{ c : \exists a (a \in \alpha \wedge a^* \leq c) \}$. Since * is isotone, β is upward closed, and clearly the minimal set which ensures that $\alpha' \overline{R}_* \beta \;\Rightarrow\; \alpha' \in [\alpha)$ is true. The tonicity of $\overline{R}_*$ then gives that $[\beta) = \{ \beta' : \forall \alpha' (\alpha' \overline{R}_* \beta' \;\Rightarrow\; \alpha' \in [\alpha)) \}$. Antisymmetry of $\subseteq$ ensures that β is unique. □

Since some of the conditions imposed on a structure involve propositions in an essential way, we prove completeness after we introduce the notion of admissible assignments. We limit the set of potential assignments because, as we saw before, a model is built of stable sets of information states (rather than arbitrary sets of possible worlds—as in ordinary modal logics). In other words, we want to make sure that A ranges over sets that may enter a model (in particular, when it comes to conditions (11) and (12)).

Definition 6.19 (Admissible assignments) s is an *admissible assignment* on the canonical structure $\mathfrak{F}$ if $s(p) \in \mathcal{P}(\mathfrak{C})^{\curlyvee}$, $s(\boldsymbol{t}) = [[\boldsymbol{t}))$, and $s(\boldsymbol{F}) = [[\boldsymbol{F}))$.

In Lemma 6.18 we proved that the canonical structure has all properties defined in (24)–(29), and in Lemma 6.10 we proved these properties to guarantee the stability of the operations. Therefore, any admissible assignment s on the canonical structure can be extended to a valuation h_s; however, the so obtained valuation *need not* be the canonical valuation.

Lemma 6.20 *Let s be an admissible assignment and h_s its extension to a valuation. If $\forall \alpha (\alpha \in h_s(A) \Rightarrow a \in \alpha)$ then $\forall \beta (\beta \in h_s(A^*) \Rightarrow a^* \in \beta)$.*

Proof We argue contrapositively that the statement holds. Suppose $\exists \beta (\beta \in h_s(A^*) \wedge a^* \notin \beta)$. By the definition of $h_s(A^*)$, $\forall \alpha (\alpha \overline{R}_* \beta \Rightarrow \alpha \in h_s(A))$. Consider the cone α, which is minimal with respect to inclusion among those for which $\alpha \overline{R}_* \beta$ is true. Assume, nonetheless, that $a \in \alpha$. This means that for some $b \leq a$, $b^* \in \beta$. But by monotonicity then $a^* \in \beta$ which contradicts to the conjunct $a^* \notin \beta$. Hence, $\exists \alpha (\alpha \in h_s(A) \wedge a \notin \alpha)$ which concludes the proof. □

Lemma 6.21 ($*$-structure) *The canonical structure $\mathfrak{F}$ with a valuation h_s (where h_s is the valuation that extends an admissible assignment s) is a $*$-structure.*

Proof Lemma 6.17 provides the first half of the proof. It remains to show that given a valuation h_s the conditions (11) and (12) are also satisfied on the canonical structure.

Let us assume that $\gamma \in h_s(A^* \circ A^*)$, and let a be an element of all $\delta \in h_s(A)$. Then by Lemma 6.20 $a^* \in \alpha, \beta$ (where $R_\circ \alpha \beta \gamma$ and $\alpha, \beta \in h_s(A^*)$). By the definition of $R_\circ$, $a^* \circ a^* \in \gamma$, that is, $a^* \in \gamma$. Further, by the definition of $\overline{R}_*$, it follows that $a \in \vartheta$ whenever $\vartheta \overline{R}_* \gamma$. Since the argument may be repeated without any change for the generators of the minimal cone in $h_s(A)$, $\vartheta \in h_s(A)$, and so $\forall \vartheta (\vartheta \overline{R}_* \gamma \;\Rightarrow \vartheta \in h_s(A))$, which is the same as $\gamma \in h_s(A^*)$.

For the last condition (12) let us suppose the antecedent. Let us assume that A is $[\varepsilon)$. There is an a such that $a \in \varepsilon$ but $a \notin \gamma$. Since $R_\circ \alpha \beta \gamma$ implies $R_\circ \alpha \varepsilon \gamma$, by the definition of $R_\circ$ we know that $\boldsymbol{t} \notin \alpha$. We define δ as the set $\{ c : \exists b (b^* \in \alpha \wedge b \leq c) \}$. * is isotone; hence $\delta \in \mathfrak{C}$. Further, in order to have $R_\circ \delta \varepsilon \vartheta$ we define $\vartheta = \{ c : \exists ab (a \in \delta \wedge b \in \varepsilon \wedge a \circ b \leq c) \}$. The monotonicity of $\circ$

ensures that $\vartheta \in \mathcal{C}$, too. Let us assume, for a reductio, that $\varepsilon \subseteq \vartheta$. Since, for any a, $a = \boldsymbol{t} \circ a$, the inclusion implies $\boldsymbol{t} \in \delta$, which is a contradiction ($\boldsymbol{t} = \boldsymbol{t}^* \notin \alpha, \delta$). □

Definition 6.22 The *canonical valuation h* is defined as

$$h(a) = \{ C : C \in \mathcal{C} \wedge a \in C \}.$$

Theorem 6.23 (Completeness) *The canonical valuation is an isomorphism.*

Proof From Lemmas 6.17 and 6.18 we know that the canonical structure is a distinguished, inverted, complete, as well as an $R_{\circ}^{\leftarrow,\circ,\rightarrow}$-fit and $\overline{R}_{*}$-fit structure. Clearly, $h(p) \in \mathcal{P}(\mathcal{C})^{\curlyvee}$ by Lemma 6.16. Therefore, by Lemma 6.10, h is indeed a valuation provided it is a homomorphism. From Theorem 6.14 we can extract for all the operations—except for $\vee$—that h is a homomorphism and also preserves the constants. (The definitions of the operations (save that of $\vee$) are the same as before.)

Now we prove that h preserves $\vee$ too. Let us assume that $\gamma \in ha \vee hb$, that is, $\forall x (x \in {}^{\ulcorner\!\!\!\rightarrow} ha \cap {}^{\ulcorner\!\!\!\rightarrow} hb \Rightarrow x \subseteq \gamma)$. By definition of h, $\gamma \in ha$ if and only if $a \in \gamma$, that is, $ha = [[\, a\,))$; similarly, $hb = [[\, b\,))$. Since the least upper bound of $\{ a, b \}$ is $a \vee b$, $x \in {}^{\ulcorner\!\!\!\rightarrow} ha \cap {}^{\ulcorner\!\!\!\rightarrow} hb$ whenever $x \subseteq [a \vee b)$. Therefore, $\gamma \supseteq [a \vee b)$, and so $a \vee b \in \gamma$. The steps may be traced back, and so $h(a \vee b) = ha \vee hb$.

Then by Lemma 6.21 the canonical structure with the canonical valuation h is a $*$-structure. Thus it only remains to be shown that h is 1-1, for which Theorem 6.14 step 2 suffices. □

The following claim, a consequence of Lemma 6.21 and of the proof of the completeness theorem (Theorem 6.23), may be viewed as an analogue of the claim that a normal modal logic is canonical. The latter means that if the Lindenbaum algebra of a logic is in a class of algebras, then the complex algebra of the canonical structure of the Lindenbaum algebra is in that class too. To put it differently, any assignment of (the equivalence classes of) propositional variables into elements of the complex algebra of the canonical structure extends to a homomorphism. That is, under arbitrary valuations the complex algebra of the canonical structure contains a homomorphic image of the Lindenbaum algebra of the logic. The disanalogy between the case of **ACT** and modal logics stems from what the *canonical embedding algebra* for **ACT** turns out to be and what the *complex algebra of the canonical structure* would be. (In the case of normal modal logics these two algebras are the same.)

We contend that as a first approximation to the notion of canonicity (when a **BA** is not a reduct of the Lindenbaum algebra) the algebra suitable as an embedding algebra is *the algebra of stable sets of the canonical structure*. Then, of course, the more restricted notion of assignment that we introduced under the name "admissible assignment" is immediately justified.

Corollary 6.24 **ACT** *is sound on the canonical structure whenever h_s is a valuation that extends an admissible assignment s.*

Now we interpret informally the semantics we gave in this section. The propositions are characterized by the *information states* that are *implied* by a proposition. We should point out that there might be information states that are neither excluded nor implied by a certain proposition; thus the role of the two functions $\ulcorner\!\!\!\rightarrow$ and $\urcorner\!\!\!\leftarrow$ is to allow us to move from a set of information states that are implied to those that are excluded—via $\ulcorner\!\!\!\rightarrow$, and vice versa—via $\urcorner\!\!\!\leftarrow$. The information states that are excluded by $A \vee B$—a choice between A and B—are exactly those that are excluded by both A and B; that is, in $^{\urcorner}A \cap {}^{\urcorner}B$. Then we get to the set of states that are implied by moving back, so to speak, which is what $({}^{\urcorner}A \cap {}^{\urcorner}B)^{\ulcorner}$ means. The interpretation of sequencing actions is straightforward. If the relation $R_\circ$ allows passing from states α and β which are implied by the propositions A and B, respectively, to a state γ, then γ is implied by $A \circ B$. In the case of $\to$, $\leftarrow$, and * it is easier to consider when an information state is not implied by those complex propositions. An information state is not implied by the action 'B if ever A' ($A \to B$) when A implies a state from which one can get to a state—via $R_\circ$—that is not implied by B. Dually, the action 'had A then B' ($B \leftarrow A$) implies an information state α when from any state implied by A there is a path through $R_\circ$ and α to some state in B. Lastly, an information state is not implied by the iteration of A whenever that state can be reached via $\bar{R}_*$ from a state not implied by A. In other words, state β is implied by A^* when β is reachable from α only if α is an information state implied by the proposition A.

We conclude this section by listing the modifications that are necessary to obtain a semantics that is not only sound but also complete for $\mathbf{KL}_{10}$. Since **ACT** is a conservative extension of $\mathbf{KL}_{10}$, the conditions that concern $\to$ and $\leftarrow$ have to be dropped. But some conditions have to be added to compensate for some implied losses. In Definition 6.1, condition (7) should include that $\circ$ is additive; in an abbreviated form the formulas look like $A \circ (B \vee C) \subseteq (A \circ B) \vee (A \circ C)$ and $(B \vee C) \circ A \subseteq (B \circ A) \vee (C \circ A)$. Clause (11) should be replaced by two clauses, namely,

$$\exists \alpha\beta(R_\circ \alpha\beta\gamma \wedge \alpha \in A \wedge \forall \delta(\delta \bar{R}_* \beta \Rightarrow \delta \in A)) \Rightarrow \forall \delta(\delta \bar{R}_* \gamma \Rightarrow \delta \in A)$$

and

$$\exists \alpha\beta(R_\circ \beta\alpha\gamma \wedge \alpha \in A \wedge \forall \delta(\delta \bar{R}_* \beta \Rightarrow \delta \in A)) \Rightarrow \forall \delta(\delta \bar{R}_* \gamma \Rightarrow \delta \in A).$$

Condition (12) should be replaced similarly by two conditions, which are in shorthand form

$$A \circ B \subseteq B \Rightarrow A^* \circ B \subseteq B$$

and

$$B \circ A \subseteq B \Rightarrow B \circ A^* \subseteq B.$$

In Definition 6.8, (27) and (28) are now superfluous, as is the closure for $\to$ and $\leftarrow$ in the following definition of a model. Similarly, in Definition 6.3 and Lemmas 6.6 and 6.10, the clauses concerning the two residuals are to be omitted. In Theorem 6.11, step 3 is to be dropped, and the distributivity of $\circ$ is to be inserted—which is guaranteed by the new frame condition and the definition of $\circ$. Also steps 6 and 7 should be replaced by showing (v) and (vi) (from the definition of $\mathbf{KA}_{10}$). In the first completeness theorem (6.14) the only change is that **1.4** is omitted. As a preliminary to the second completeness theorem, in Lemma 6.17, step **7** should include a proof that the "half" distributivity of $\circ$ over $\vee$ holds. In Lemma 6.18 two cases of fitness are, obviously, to be omitted. Lastly, in Lemma 6.21 it should be proven that the new conditions (which replace (11) and (12)) hold. Having carried out all these modifications, there are no changes to be made to Theorem 6.23—because of the way we structured the presentation of the second completeness theorem. (We leave filling in the details to the reader.)

7 Star Semantics

Gaggle theory is about representing families of operations that are of the same arity and are appropriately linked (as we sketched in Section 3). Gaggles and partial gaggles may also include constants that are related to an operation in a family, for example, as are identities and combinators. In some applications constants were used to define new (with respect to the family) operations—as in the case of intuitionistic logic where negation is definable from a constant falsity and implication. Now we show that *, thought of as an operation which is "derived" from concatenation ($\circ$), can be modeled using the same relation that models fusion.

We will define the * operation in two stages, but before that we return to the Lindenbaum algebra and introduce some notation to simplify some expressions.

Definition 7.1 The nth *power* of a in the Lindenbaum algebra of **ACT** is defined as follows.

(1) $a^0 = \boldsymbol{t}$
(2) $a^{n+1} = a \circ a^n$ (whenever $n \geq 0$)

This notation is well in accordance with the usual definitions in formal language theory. a^0 corresponds to the empty string of which $\boldsymbol{t}$ is indeed an analogue. $a^1 = a \circ \boldsymbol{t} = a$ and a^n are completely usual in formal language theory except that $\circ$ is ordinarily omitted from the notation.

Lemma 7.2 *In the Lindenbaum algebra of* **ACT**,

$$\forall n \in \mathbb{N}.\ a^n \leq a^*.$$

Proof For the basis we note that $\boldsymbol{t} \leq a^*$ by a (sub)equation in (iv). If $a^n \leq a^*$, then $a \circ a^n \leq a \circ a^*$ by monotonicity, and since $a \circ a^* \leq a^*$, the proof is finished. □

Lemma 7.3 *Let $\mathfrak{A}$ be the Lindenbaum algebra of action logic. There is no x such that $a^n \leq x$, for all $n \in \mathbb{N}$, and $x < a^*$.*

Proof Consider the relational algebra $\mathfrak{R}$ which contains all the binary relations on two elements o_1 and o_2. There are 16 relations in $\mathfrak{R}$, with $\{\langle o_1, o_1\rangle, \langle o_2, o_2\rangle\} = \boldsymbol{t}$, $\varnothing = \boldsymbol{F}$, $\cup = \vee$, $\circ = \circ$, and $^* = {}^*$ (that is, composition is $\circ$ and reflexive transitive closure is *). The implications are added by the definitions $r_1 \to r_2 = \{\langle x, y\rangle : \forall z(\langle y, z\rangle \in r_1 \Rightarrow \langle x, z\rangle \in r_2)\}$, $r_2 \leftarrow r_1 = \{\langle x, y\rangle : \forall z(\langle z, x\rangle \in r_1 \Rightarrow \langle z, y\rangle \in r_2)\}$ (cf. [25]). (Complementation, converse, and intersection are as usual.)

To see that there is no x as specified in the claim, let us consider, for instance, $a = \{\langle o_1, o_2\rangle, \langle o_2, o_1\rangle\}$. Then $a^0 = \{\langle o_1, o_1\rangle, \langle o_2, o_2\rangle\}$, $a^1 = a$—of course—and $a^2 = a^0$. The only element which is greater than both of these elements is the total relation on $\{o_1, o_2\}$, indeed, the * of the element we started with. (We leave the inspection of the other elements to the interested reader.) □

Definition 7.4 A *structure* for **ACT** is $\mathfrak{F} = \langle U, \sqsubseteq, R_\circ, \leqslant, \mathsf{T}, \imath\rangle$ where the elements of the six-tuple satisfy conditions (1)–(8). If $\mathfrak{F}$ additionally satisfies (9) and (10) then it is called a $*$-*structure*.[20]

(1) $\varnothing \neq U$; $\sqsubseteq \in U^2$; $R_\circ \subseteq U^3$; $\leqslant \subseteq U^2$; $\mathsf{T}, \imath \in U$

(2) $(\alpha \leqslant \beta \wedge \alpha \sqsubseteq \alpha') \Rightarrow \alpha' \leqslant \beta$ and $\exists\alpha\forall\beta.\, \alpha \leqslant \beta$

(3) $\forall\beta.\, \beta \sqsubseteq \mathsf{T} \wedge \forall\alpha(\forall\beta.\, \beta \sqsubseteq \alpha \Rightarrow \alpha = \mathsf{T})$

(4) $(\alpha \sqsubseteq \beta \wedge \beta \sqsubseteq \gamma) \Rightarrow \alpha \sqsubseteq \gamma$ and $\alpha \sqsubseteq \alpha$

(5) $R_\circ {\downarrow\downarrow\uparrow}$, that is,
$(R_\circ \alpha\beta\gamma \wedge \alpha' \sqsubseteq \alpha \wedge \beta' \sqsubseteq \beta \wedge \gamma \sqsubseteq \gamma') \Rightarrow R_\circ \alpha'\beta'\gamma'$

(6) $\exists\alpha(R_\circ \alpha\beta\beta \wedge \iota \sqsubseteq \alpha)$ and $(R_\circ \alpha\beta\gamma \wedge \iota \sqsubseteq \alpha) \Rightarrow \beta \sqsubseteq \gamma$
$\exists\alpha(R_\circ \beta\alpha\beta \wedge \iota \sqsubseteq \alpha)$ and $(R_\circ \alpha\beta\gamma \wedge \iota \sqsubseteq \beta) \Rightarrow \alpha \sqsubseteq \gamma$

(7) $(R_\circ \alpha\beta\eta \wedge R_\circ \eta\gamma\delta) \Rightarrow \exists\varepsilon(R_\circ \beta\gamma\varepsilon \wedge R_\circ \alpha\varepsilon\delta)$
$(R_\circ \beta\gamma\varepsilon \wedge R_\circ \alpha\varepsilon\delta) \Rightarrow \exists\eta(R_\circ \alpha\beta\eta \wedge R_\circ \eta\gamma\delta)$

(8) $(R_\circ \alpha\beta\gamma \wedge \alpha = \mathsf{T}) \Rightarrow \gamma = \mathsf{T}$
$(R_\circ \alpha\beta\gamma \wedge \beta = \mathsf{T}) \Rightarrow \gamma = \mathsf{T}$

(9) $A^* \circ A^* \subseteq A^*$

(10) $[\iota) \vee A \circ B \subseteq B \Rightarrow A^* \subseteq B$

Let h be a valuation. The operations—except *—are as in Definition 6.3 (17)–(21); the constants are interpreted as in (15)–(16). Additionally, $h(A^n)$ and $h(A^*)$ are defined as follows.

Definition 7.5 We define inductively $h(A^n)$ and based on this definition we define $h(A^*)$.

(i) $h(A^0) = [\iota)$
(ii) $h(A^1) = h(A)$
(iii) $h(A^{n+1}) = \{\gamma : \exists\alpha\beta(R_\circ \alpha\beta\gamma \wedge \alpha \in h(A) \wedge \beta \in h(A^n))\}$ $(n \geq 1)$
(iv) $h(A^*) = \{\delta : \forall\gamma\,(\exists n(n \in \mathbb{N} \wedge \gamma \in h(A^n)) \Rightarrow \gamma \leqslant \delta)\}^{\urcorner}$

Lemma 7.6 (Power and * are hereditary operations) *If $h(A) \in \mathcal{P}(U)^{\uparrow}$ then $h(A^n), h(A^*) \in \mathcal{P}(U)^{\uparrow}$.*

Proof The first can be proven inductively. $h(A^0)$ and $h(A^1)$ are upward closed. $h(A^{n+1})$ is upward closed too—$h(A^n)$ is such by the hypothesis of induction, and since so is $h(A)$, the tonicity of $R_\circ$ ensures that $h(A^{n+1}) \in \mathcal{P}(U)^{\uparrow}$. The definition obviously ensures that the claim is true for $h(A^*)$. □

Lemma 7.7 (Stability of $h(A^n)$ and $h(A^*)$) *Let $\mathfrak{F}$ be a distinguished, inverted, complete $R_\circ^\circ$-fit structure. Then $h(A) \in \mathcal{P}(U)^{\curlyvee}$ implies $h(A^n) \in \mathcal{P}(U)^{\curlyvee}$ and $h(A^*) \in \mathcal{P}(U)^{\curlyvee}$.*

Proof

1 From Lemma 6.10 we know that $h(A^0)$ $(h(\boldsymbol{t}))$ is stable because of the assumption that the structure is inverted. $h(A^1) = h(A) \in \mathcal{P}(U)^{\curlyvee}$ by the assumption of the lemma. Assuming that $h(A^n)$ is stable, the definition of $h(A^{n+1})$ and the structure's $R_\circ^\circ$-fitness together with Lemma 6.10 ensure that $h(A^{n+1})$ is stable too.

2 Completeness means that for an arbitrary set of stable sets there is a unique information state which generates a cone that has the same $\ulcorner$-image as the intersection of the stable sets. Since * is the $\urcorner$-image of this set, it is stable. □

Theorem 7.8 *On a distinguished, inverted, complete, $R_\circ^{\leftarrow,\circ,\rightarrow}$-fit $*$-structure $\mathfrak{F}$ with a valuation h added (such that $h(p) \in \mathcal{P}(U)^{\curlyvee}$ and h satisfies (15)–(20) (from Section 6)) all the equations of the Lindenbaum algebra of action logic hold.*

Proof Note that we changed only the conditions relating to the *, and so the first part of the proof is the same as steps 1 – 4 in the proof of Theorem 6.11.[21]

1 Obviously, $h(\boldsymbol{t}) \subseteq h(A^*)$ and $h(A) \subseteq h(A^*)$ by the two base cases of the definition of $h(A^n)$, and further, by the definition of $h(A^*)$.

2 Since $h(A) \subseteq h(A \vee B)$, clearly, $X \cap h(A) \subseteq X \cap h(A \vee B)$ (for any X). The latter suffices for $h(A^*) \subseteq h((A \vee B)^*)$ to hold.

3 The last two conditions take care of the remaining equalities. □

Definition 7.9 The *canonical structure* $\mathfrak{F}$ is $\langle \mathcal{C}, \subseteq, R_\circ, \subseteq, [\boldsymbol{F}), [\boldsymbol{t}) \rangle$ where $\mathcal{C}, \subseteq$, and $R_\circ$ are as in Definition 6.13, and $[\boldsymbol{F}), [\boldsymbol{t})$ are the principal cones generated by $\{\boldsymbol{F}\}$ and $\{\boldsymbol{t}\}$.

We assume that the notions of admissible assignment and valuation from the previous section are adapted (with the obviously needed modifications) to the present frame.

Lemma 7.10 *Let s be an admissible assignment and h_s its extension to a valuation. Then*

(i) $(\forall \alpha \in h_s(A).\, a \in \alpha \wedge \forall \beta \in h_s(B).\, b \in \beta) \Rightarrow \forall \gamma\, (\gamma \in h_s(A \circ B) \Rightarrow a \circ b \in \gamma)$;
(ii) $\forall \alpha(\alpha \in h_s(A) \Rightarrow a \in \alpha) \Rightarrow \forall \beta(\beta \in h_s(A^n) \Rightarrow a^n \in \beta)$;
(iii) $\forall \alpha(\alpha \in h_s(A) \Rightarrow a \in \alpha) \Rightarrow \forall \beta(\beta \in h_s(A^*) \Rightarrow a^* \in \beta)$.

Proof

1 Let us assume the antecedent together with the negation of the consequent. Then $\exists \alpha\beta(R_\circ \alpha\beta\gamma \wedge \alpha \in h_s(A) \wedge \beta \in h_s(B))$. It follows that $a \in \alpha$, and $b \in \beta$; hence by the definition of $R_\circ$ also $a \circ b \in \gamma$—which is a contradiction.

2 h_s is a valuation, and so $\forall \alpha \in h_s(\boldsymbol{t}).\, \boldsymbol{t} \in \alpha$. For $n = 1$ the claim is a propositional tautology. From n we move to $n + 1$ using the previous step.

3 Using the definition of * together with Lemma 7.3 and the previous step in this proof, indeed, $a^* \in \beta$. □

Lemma 7.11 *The canonical structure $\mathfrak{F}$ with a valuation h_s, which is an extension of an admissible assignment, is a $*$-structure.*

Proof

1 Let us assume that $\gamma \in h_s(A^* \circ A^*)$, that is, $\exists \alpha\beta(R_\circ \alpha\beta\gamma \wedge \alpha \in h_s(A^*) \wedge \beta \in h_s(A^*))$. Since $h_s(A^*) = [\delta)$ for some δ, we also have $R_\circ \delta\delta\gamma \wedge \delta \in h_s(A^*)$ by $R_\circ{\downarrow\downarrow\uparrow}$. From Lemma 7.10 we know that if $a \in \vartheta$ for all $\vartheta \in h_s(A)$, then $a^* \in \delta$. By the definition of $R_\circ$, $a^* \in \gamma$. The definition of $h_s(A^*)$ means that it is generated by a cone of the form $[\{a_1, \ldots, a_n, \ldots\})$. Therefore, $a^* \in \gamma$ (for any a that is an element of the cone that generates $h_s(A)$) is sufficient for $\gamma \in h_s(A^*)$.

2 Let us assume that $h_s(\boldsymbol{t} \vee A \circ B) \subseteq h_s(B)$. Of course, for any valuation $h_s(\boldsymbol{t}) = [[\boldsymbol{t}))$; hence $\boldsymbol{t} \in \vartheta$ for all $\vartheta \in h_s(\boldsymbol{t})$. Assuming $a \in \alpha \in h_s(A)$ and $b \in \beta \in h_s(B)$, by Lemma 7.10, $a \circ b \in \gamma \in h_s(A \circ B)$. The inclusion in the first assumption guarantees that both $\boldsymbol{t} \leq b$ and $a \circ b \leq b$. In the Lindenbaum algebra then $a^* \leq b$.[22] However, by the definition of $h_s(A^*)$, a^* must be in $\delta \in h_s(A^*)$ and so $h_s(A^*) \subseteq h_s(B)$ as we wanted to show. □

Definition 7.12 The *canonical valuation* h is defined as

$$h(a) = \{ C : C \in \mathcal{C} \wedge a \in C \}.$$

Lemma 7.13 *$h(a^n) = (ha)^n$ (for all $n \in \mathbb{N}$), that is, h is a homomorphism for the power operation.*

Proof Let us assume that $\gamma \in h(a^n)$. Then $a^n \in \gamma$ by the definition of h.[23]

(i) If $n = 0$ then $a^n = \boldsymbol{t}$, that is, $\boldsymbol{t} \in \gamma$. Since $\imath$ canonically is $[\boldsymbol{t})$ and $h(a^0) = [[\boldsymbol{t})), \gamma \in (ha)^0$.
(ii) If $n = 1$, then we have $h(a^1) = ha = (ha)^1$.
(iii) Let the hypothesis of induction be that the claim is true for n. Since $a^{n+1} \in \gamma$, surely $R_\circ[a)[a^n)\gamma$. By the hypothesis of induction $[a^n) \in (ha)^n$ and $[a) \in ha$. Therefore, $\exists\alpha\beta(R_\circ\alpha\beta\gamma \wedge \alpha \in ha \wedge \beta \in (ha)^n)$. This means exactly that $\gamma \in (ha)^{n+1}$.

For the converse, let us assume that $\gamma \in (ha)^n$.

(i) Let $n = 0$ to start with. Then $(ha)^n = [[\boldsymbol{t}))$, and since $a^0 = \boldsymbol{t}$, γ is indeed an element of $h(a^0)$.
(ii) Let $n = 1$. Again, $ha = ha$.
(iii) Lastly, if the claim is true for n, then we have by inductive hypothesis that $\exists\alpha\beta(R_\circ\alpha\beta\gamma \wedge \alpha \in ha \wedge \beta \in (ha)^n)$, and so $\beta \in h(a^n)$. The definition of $R_\circ$ implies that $a \circ a^n \in \gamma$, since $a \in \alpha$ and $a^n \in \beta$. Thus, $\gamma \in h(a^{n+1})$ as we wanted to show. □

Theorem 7.14 *The canonical valuation h is an isomorphism.*

Proof In view of the proof of the previous completeness theorem we only have to prove that h preserves *. (Note that we proved in Lemmas 6.17 and 6.18 that the current canonical structure is a distinguished, complete, inverted $R_\circ^{\leftarrow,\circ,\rightarrow}$-fit structure. In Lemma 7.11 we proved that the canonical structure with a valuation is a $*$-structure.)

Let us assume that $\delta \in h(a^*)$. Then $a^* \in \delta$. Since $a^n \leq a^*$ for all $n \in \mathbb{N}$, and by the preceding lemma h preserves the power operation, a^* is in the intersection of the $h(a^n)$s. Thus, $\delta \in (ha)^*$. For the converse let us assume the latter. Since $\delta \supseteq \bigcap_{n\in\mathbb{N}}[[a^n))$, any x such that $a^n \leq x$ for all ns is an element of δ. Therefore, $a^* \in \delta$, and so $\delta \in h(a^*)$. □

Corollary 7.15 *With a valuation the canonical structure yields a model of* **ACT**.

We conclude this section noting that to obtain an adequate semantics for $\mathbf{KL}_{10}$ only Definition 7.4 and the second step in Lemma 7.11 need a slight adjustment. Namely, the conditions on the structure have to be modified as before making $\circ$ additive and replacing (9) and (10) with $[\imath) \vee A \circ A^* \subseteq A^*$, $[\imath) \vee A^* \circ A \subseteq A^*$ and $A \circ B \subseteq B \;\Rightarrow\; A^* \circ B \subseteq B$, $B \circ A \subseteq B \;\Rightarrow\; B \circ A^* \subseteq B$, respectively. Then in Lemma 7.11 the new clauses have to be proven to hold. (We leave the few omitted details to be filled in by the reader.)

8 Extensions of Gaggle Theory

This paper may be seen to continue a program begun in [12] and [13] by Dunn—the results of which were applied to particular logics in further papers, for instance, Dunn [16] and Dunn [17].[24] The first paper [16] showed how to represent relation algebras using Routley-Meyer frames. The second paper [17] gave a philosophical interpretation that can be put on these kinds of representations, pointing out that a ternary relation on "states" can be viewed as an indexed binary relation that is a transition between states. Thus a set of states (a "proposition") can be simultaneously viewed as static and dynamic, and one can model von Neumann's concept of a "stored program." This paper additionally posed the problem of modeling action logic (representing action algebras) as well as modeling Pratt's dynamic logic and Hoare's logic using Routley-Meyer frames. The present paper solves the first of these problems.

In conclusion we would like to emphasize once again some of the ways in which we extended—on the examples of action logic and Kleene logic—gaggle theory. Our first semantics showed how to *dualize gaggles* to obtain a semantics for a join semi-lattice ordered monoid—[13] considered (only) meet semi-lattices with fusion. Our second semantics showed how to *add to a partial gaggle* only one of the lattice operations.

In both semantics where we used a separate relation to model * we relied on distribution patterns in a way that was not predicted by gaggle theory itself. More precisely, we observed that the only distribution pattern that could hold for * on a join semi-lattice—namely, $\vee \longrightarrow \vee$—does not hold. Therefore, we based the choice of the modeling on the *preservation* (or the lack of preservation) of the *least* (and in the case of action logic also on the *greatest*) element of the Lindenbaum algebra. (Incidentally, we also modeled the least element differently than usual. We contend that our modeling is the general one, since it does not use special properties of the empty set—which are customarily exploited in definitions of normal operations on lattices with a bottom element.)

Lastly, in the star semantics we showed that a partial gaggle may be extended with an operation of *different arity* when the new operation is *inductively definable*—as the *power* operation—or the new operation is the *closure* of an inductively definable operation—as the *Kleene star* operation.

Notes

1. A comprehensible introduction to FSAs, regular languages, and regular expressions (and more) is, for instance, Sipser [26].

2. We use the names $\mathbf{KL}_{10}$ and $\mathbf{KA}_{10}$ to honor Kozen's claim that there were nine other inequivalent definitions of Kleene algebras before his—a claim we did not attempt to disprove or verify.

3. We use $p, q, r, p_0, \ldots$ for propositional variables of the object language and φ, ψ, χ, $\varphi_0 \ldots$ as metavariables for well-formed formulas (wffs). We assume familiarity with basic logical notions and Hilbert-style axiom systems, in particular. (See, e.g., Curry [9] and Enderton [21].) Accidentally, $\mathbf{KL}_{10}$ is neither of Kleene's two well-known three-valued logics, the weak nor the strong one.

4. Kozen and Pratt—by and large—use different notation than we do. We treat $\mathbf{KL}_{10}$ and **ACT** as nonclassical logics and use a notation that (probably) originated in relevance logics and by now is widely accepted in substructural logics. In particular, in naming the

residuals we follow Anderson and Belnap [2], §18, not Pratt. (However, note that * here is very different from * in the syntax (or in the semantics) of some relevance logics.)

5. A monoid has a binary associative operation and an identity element. For standard definitions of algebraic notions see, for instance, Grätzer [22] and Dunn and Hardegree [18].

6. We use the same symbol for an operation in the Lindenbaum algebra as we used for the connective from which it originated. However, instead of $[\varphi]$, $[\psi]$, ... we denote equivalence classes of formulas simply by $a, b, c \ldots$.

7. For various semantics that involve ideals see Birkhoff and Frink [8], Urquhart [28], Allwein and Dunn [1], Hartonas and Dunn [23], [6], and Bimbó [3]. We note that ideals may not be necessary to define a closure operation. For example, in one of the semantics in [6] the closure operation is defined from a proto-order (which is canonically exemplified by set inclusion).

8. * *has no distribution pattern* since it does not distribute over $\vee$ in the model on regular sets, and, of course, there is no conjunction in the language. (Actually * does not distribute over the intersection of regular languages either.) * *does not preserve* $\boldsymbol{F}$, because $\boldsymbol{F}^* = \boldsymbol{t}$.

9. For a complete exposition of *generalized Galois logics* (gaggle theory) and some of its applications we refer the interested reader to Dunn's papers [12], [13], [14], and [15].

10. For different formulations of Zorn's lemma see, for example, Stoll [27], p. 116, or Dunn and Hardegree [18], p. 30. Zorn's lemma is commonly used in semantics to obtain a suitable ultra, prime, or join irreducible filter, or a maximally disjoint pair of filters and ideals. We use the lemma to define appropriate ideals because there seems to be no straightforward way to define an α and a β from the information that $a \circ b \notin \gamma$. (The problem with a Lindenbaum-style construction here is that it is not clear what ordering of the wffs to use to ensure that both (12) is satisfied and the minimally necessary additions are made to α and β.)

11. Of course, from the point of view of "size" these ideals are minimal, but from the point of view of the ordering $\breve{\subseteq}$ they are maximal elements.

12. Since we have to distinguish between two sorts of objects, we will use the terms 'positive states' and 'negative states', mostly for mnemonic purposes. Lowercase Greek letters range over positive states, whereas lowercase Roman letters are variables for negative states. $A, B, C, \ldots$ are subsets of U^+.

13. Notice that $\varnothing$ does not play a role in the representation; $\varnothing$ is not stable due to the second half of condition (2).

14. We excluded $\varnothing$ only because it has no role to play in the representation.

15. The definition of h differs from that in [6], however, very much like the valuation h in the first representation there—in its spirit.

16. Of course, in modal logics (e.g., in a frame for any of the unimodal normal modal logics) there is only one binary relation, whereas here there are two relations; moreover, one of them is ternary. However, in **ACT** there are binary intensional operations in addition to a unary * operation.

17. We pointed out that $\varnothing \notin \mathcal{P}(U^+)^\curlyvee$. It is easy to see that $\varnothing \notin \mathcal{P}(\mathfrak{C})^\curlyvee$ because the Lindenbaum algebra has a least element. Indeed, $(\overset{\rightarrow}{\ulcorner}\varnothing)^{\overset{\leftarrow}{\urcorner}} = \mathfrak{C}^{\overset{\leftarrow}{\urcorner}} = \{\,[\boldsymbol{F})\,\}$.

18. Also, the present proof is somewhat different from the other one, although the proofs, perhaps, merely reflect different ways of looking at algebras that are similar to an extent.

19. A cocone is the dual of a cone, that is, a downward closed subset.

20. We use here $\circ$, $\vee$, and * as abbreviations—to enhance readability—which are meant to be understood in the sense of the previous definition for $\circ$ and $\vee$, and in the sense of the following definition (Definition 7.5) for *.

21. A reader who scrutinized the two semantics probably noticed that in the second we postulated a single set of information states only. However, nothing in the proof of Theorem 6.11 depended on U^- and U^+ being distinct—indeed the second canonical construction identified the positive and negative states.

22. The quasi equation $\boldsymbol{t} \vee a \circ b \leq b \Rightarrow a^* \leq b$ holds in the Lindenbaum algebra. (Cf. [25].)

23. The definition of a^n was inductive; therefore, our proof is inductive too—a situation somewhat unusual in the context of the semantics of nonclassical logics only because ordinarily operations are not inductively defined.

24. We singled out the two papers that are motivationally most closely related to our treatment of $\mathbf{KL}_{10}$ and **ACT**. However, we could mention a whole series of papers which define semantics for various logics by generalizing in one way or other the Jónsson-Tarski-Kripke-Routley-Meyer-Dunn approach. (From among the references we could list here also [1], [3], [4], [5], [6], [14], [15], [16], [17], [19], [23].)

References

[1] Allwein, G., and J. M. Dunn, "Kripke models for linear logic," *The Journal of Symbolic Logic*, vol. 58 (1993), pp. 514–45. Zbl 0795.03013. MR 1233922. 487, 488

[2] Anderson, A. R., and N. D. Belnap, Jr., *Entailment. The Logic of Relevance and Necessity*, vol. 1, Princeton University Press, Princeton, 1975. Zbl 0323.02030. MR 0406756. 487

[3] Bimbó, K., "Semantics for structurally free logics LC+," *Logic Journal of the IGPL*, vol. 9 (2001), pp. 525–39. Zbl 0986.03019. MR 1844688. 487, 488

[4] Bimbó, K., "Semantics for dual and symmetric combinatory calculi," *Journal of Philosophical Logic*, vol. 33 (2004), pp. 125–53. Zbl 02081492. MR 2052250. 488

[5] Bimbó, K., and J. M. Dunn, "Two extensions of the structurally free logic LC," *Logic Journal of the IGPL*, vol. 6 (1998), pp. 403–24. Zbl 0904.03006. MR 1625125. 488

[6] Bimbó, K., and J. M. Dunn, "Four-valued logic," *Notre Dame Journal of Formal Logic*, vol. 42 (2001), pp. 171–92 (2003). Zbl 1034.03021. MR 2010180. 475, 477, 487, 488

[7] Birkhoff, G., *Lattice Theory*, American Mathematical Society Colloquium Publications, Vol. XXV. American Mathematical Society, Providence, 1967. Zbl 0153.02501. MR 0227053. 474

[8] Birkhoff, G., and O. Frink, Jr., "Representations of lattices by sets," *Transactions of the American Mathematical Society*, vol. 64 (1948), pp. 299–316. Zbl 0032.00504. MR 0027263. 487

[9] Curry, H. B., *Foundations of Mathematical Logic*, McGraw-Hill Book Co., New York, 1963. Zbl 0163.24209. MR 0148529. 486

[10] Dunn, J. M., "Intuitive semantics for first-degree entailments and 'coupled trees'," *Philosophical Studies*, vol. 29 (1976), pp. 149–68. MR 0491026. 477

[11] Dunn, J. M., "Relevance logic and entailment," pp. 117–229 in *Handbook of Philosophical Logic*, edited by D. Gabbay and F. Guenther, vol. 3, Reidel, Dordrecht, 1986. Zbl 0875.03051. 477

[12] Dunn, J. M., "Gaggle theory: An abstraction of Galois connections and residuation, with applications to negation, implication, and various logical operators," pp. 31–51 in *Logics in AI (Amsterdam, 1990)*, edited by J. van Eijck, vol. 478 of *Lecture Notes in Computer Science*, Springer, Berlin, 1991. Zbl 0814.03044. MR 1099620. 485, 487

[13] Dunn, J. M., "Partial gaggles applied to logics with restricted structural rules," pp. 63–108 in *Substructural Logics (Tübingen, 1990)*, edited by K. Došen and P. Schroeder-Heister, vol. 2 of *Studies in Logic and Computation*, Oxford University Press, New York, 1993. Zbl 0941.03521. MR 1283194. 485, 486, 487

[14] Dunn, J. M., "Gaggle theory applied to intuitionistic, modal and relevance logic," pp. 335–68 in *Logik und Mathematik. Frege-Kolloquium Jena*, W. de Gruyter, Berlin, 1995. 487, 488

[15] Dunn, J. M., "Generalized ortho-negation," pp. 3–26 in *Negation: A Notion in Focus*, edited by H. Wansing, W. de Gruyter, Berlin, 1996. Zbl 0979.03027. 487, 488

[16] Dunn, J. M., "A representation of relation algebras using Routley-Meyer frames," pp. 77–108 in *Logic, Meaning and Computation*, edited by C. A. Anderson and M. Zelëny, vol. 305 of *Synthese Library*, Kluwer Academic Publishers, Dordrecht, 2001. Zbl 1039.03051. MR 2067967. 486, 488

[17] Dunn, J. M., "Ternary relational semantics and beyond: Programs as data and programs as instructions," *Logical Studies*, no. 7, online journal, (2001). Special Issue: Proceedings of the International Conference *Third Smirnov Readings* (Moscow, May 24–27, 2001), Part 2, http://www.logic.ru/LogStud/. 486, 488

[18] Dunn, J. M., and G. M. Hardegree, *Algebraic Methods in Philosophical Logic*, vol. 41 of *Oxford Logic Guides*, The Clarendon Press, New York, 2001. Zbl 1014.03002. MR 1858927. 487

[19] Dunn, J. M., and R. K. Meyer, "Combinators and structurally free logic," *Logic Journal of the IGPL*, vol. 5 (1997), pp. 505–37 (electronic). Zbl 0878.03008. MR 1465611. 488

[20] Ebbinghaus, H.-D., and J. Flum, *Finite Model Theory*, 2d edition, Perspectives in Mathematical Logic. Springer-Verlag, Berlin, 1999. Zbl 0932.03032. MR 1716820. 471

[21] Enderton, H. B., *A Mathematical Introduction to Logic*, 2d edition, Harcourt/Academic Press, Burlington, 2001. Zbl 0992.03001. MR 1801397. 486

[22] Grätzer, G., *Universal Algebra*, 2d edition, Springer-Verlag, New York, 1979. Zbl 0412.08001. MR 538623. 487

[23] Hartonas, C., and J. M. Dunn, "Stone duality for lattices," *Algebra Universalis*, vol. 37 (1997), pp. 391–401. Zbl 0902.06008. MR 1452408. 487, 488

[24] Kozen, D., "A completeness theorem for Kleene algebras and the algebra of regular events," *Information and Computation*, vol. 110 (1994), pp. 366–90. 1991 IEEE Symposium on Logic in Computer Science (Amsterdam, 1991). Zbl 0806.68082. MR 1276741. 461

[25] Pratt, V., "Action logic and pure induction," pp. 97–120 in *Logics in AI (Amsterdam, 1990)*, edited by J. van Eijck, vol. 478 of *Lecture Notes in Computer Science*, Springer, Berlin, 1991. Zbl 0814.03024. MR 1099624. 462, 482, 488

[26] Sipser, M., *Introduction to the Theory of Computation*, PWS Publishing Company, New York, 1997. 486

[27] Stoll, R. R., *Set Theory and Logic*, Dover Publications Inc., New York, 1979. Corrected reprint of the 1963 edition. Zbl 0489.03002. MR 634799. 487

[28] Urquhart, A., "A topological representation theory for lattices," *Algebra Universalis*, vol. 8 (1978), pp. 45–58. Zbl 0382.06010. MR 0450150. 487

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