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A MODIFICATION OF PARRY'S ANALYTIC IMPLICATION

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Parry [6] sets forth a system of sentential logic based upon the contention that for A to *analytically* imply B, every sentential variable that occurs in B must also occur in A. Parry's system is intended to be in step with Kant's notion of *analyticity*, and succeeds insofar as Parry is able to prove that $A \rightarrow B$ is a theorem of his system only if the above mentioned variable sharing criterion holds.

Parry's system might be better called a system of analytic *strict* implication since it is easily seen that every theorem of his system is a theorem of the Lewis modal logic S4. In this paper* we present a modification of Parry's system, the principal feature of which is a "demodalization" of Parry's original system which still preserves Parry's variable sharing criterion. We then give algebraic completeness results for this modified system, and show it decidable. These results parallel our work in [3] on RM and in [4] on LC.

1. Let us begin by presenting Parry's system for the sake of easy reference. Parry takes as primitive the connectives of negation, conjunction, and analytic implication, in our symbols -, &, and $\rightarrow$, respectively. Formation rules are as usual, and the connectives of analytic equivalence, disjunction, and material implication, in symbols, $\leftrightarrow$, $\vee$, and $\supset$, respectively, are introduced by definition in the usual manner. We take as axioms all instances of the following schemata, omitting parentheses according to the conventions of Church (as we do throughout the paper).

A1. $A \,\&\, B \rightarrow B \,\&\, A$
A2. $A \rightarrow A \,\&\, A$
A3. $A \rightarrow --A$
A4. $--A \rightarrow A$

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A5. $A \,\&\, (B \vee C) \rightarrow (A \,\&\, B) \vee (A \,\&\, C)$
A6. $A \vee (B \,\&\, -B) \rightarrow A$
A7. $(A \rightarrow B) \,\&\, (B \rightarrow C) \rightarrow .A \rightarrow C$
A8. $A \rightarrow B \,\&\, C \rightarrow .A \rightarrow C$
A9. $(A \rightarrow B) \,\&\, (C \rightarrow D) \rightarrow .A \,\&\, C \rightarrow B \,\&\, D$
A10. $(A \rightarrow B) \,\&\, (C \rightarrow D) \rightarrow .A \vee C \rightarrow B \vee D$
A11. $(A \rightarrow B) \rightarrow (A \supset B)$
A12. $(A \leftrightarrow B) \,\&\, f(A) \rightarrow f(B)$
A13. $f(A) \rightarrow .A \rightarrow A$

In axioms A12 and A13, we understand $f(A)$ to be any sentence in which A occurs as a subsentence, and in A12 we understand $f(B)$ to be the result of replacing A in one or more occurrences by B in the given sentence $f(A)$.

Parry [6] has a rule of substitution, which we dispense with by the well-known means of axiom schemata. The only other rule that Parry has is the rule of detachment for analytic implication, i.e., from A and $A \rightarrow B$ to infer B, but, as Anderson and Belnap [2] observe, the rule of adjunction, from A and B to infer $A \,\&\, B$, is obviously required. We hence take the rule of adjunction as a primitive rule of Parry's system, thereby correcting what we take to be a mere oversight on Parry's part.

Parry's system is still too weak for some of the purposes that we have in mind, since the following sentence is not a theorem of Parry's system, as we shall show in section 4.

A14. $A \,\&\, -B \rightarrow -(A \rightarrow B)$

The addition of A14 to Parry's list of axioms is equivalent to the addition of

T1. $(A \rightarrow A) \,\&\, (B \rightarrow B) \rightarrow .A \rightarrow B \rightarrow .A \rightarrow B.$

We sketch the derivation from A14 to T1, leaving the converse derivation to the reader. We take a lot of the obvious properties of conjunction and disjunction for granted and list only the most relevant axioms used, as shall be our practise throughout this paper.

1. $(A \rightarrow A) \,\&\, (B \rightarrow B) \rightarrow (-A \vee A) \,\&\, (-B \vee B)$ A11.
2. $(-A \vee A) \,\&\, (-B \vee B) \rightarrow (-A \,\&\, -B) \vee (-A \,\&\, B) \vee (A \,\&\, -B) \vee (A \,\&\, B)$ A5.
3. $(-A \,\&\, -B) \vee (-A \,\&\, B) \vee (A \,\&\, -B) \vee (A \,\&\, B) \rightarrow .(-A \,\&\, -B) \vee (-A \,\&\, B) \vee -(A \rightarrow B) \vee (A \,\&\, B)$ A14.
4. $(-A \,\&\, -B) \vee (-A \,\&\, B) \vee -(A \rightarrow B) \vee (A \,\&\, B) \rightarrow .A \rightarrow B \rightarrow .A \rightarrow B$ A13.
5. $(A \rightarrow A) \,\&\, (B \rightarrow B) \rightarrow .A \rightarrow B \rightarrow .A \rightarrow B$ From 1-4, repeated uses of A7.

In what follows, we shall mean by the system of *analytic strict implication* (ASI) Parry's system with axiom A14 added. By the system of *analytic implication* (AI), we shall mean the system obtained from ASI by demodalizing it by the addition of the following axiom.

A15. $A \rightarrow .-A \rightarrow A$

It is well-known that when axiom A15 is added to the Lewis modal logic S4 the resulting system reduces to classical logic. That the same thing does not happen when A15 is added to ASI may be easily verified by referring to the matrix that Parry [6] used to show that $A \rightarrow B$ is a theorem of his system only when every propositional variable that occurs in B also occurs in A. This matrix not only satisfies the axioms A1-A13 of Parry's original system as well as axiom A14 and the rules of detachment and adjunction, but it also satisfies axiom A15.

Another theorem is derivable in AI which when added as an axiom to S4 gives classical logic, namely,

T3. $A \rightarrow .A \rightarrow A \rightarrow A.$

Indeed, A15 could be equivalently replaced with T3. This makes the relation of AI to ASI somewhat like the relation of Anderson and Belnap's system of R of Relevant Implication to their system E of Entailment since the former is obtainable from the latter by the addition of T3 (though not from the addition of A15). We content ourselves with a derivation of T3, first deriving T2*. (We star this theorem, as well as all others whose derivations hold in ASI.)

T2.* $(A \rightarrow A) \leftrightarrow -A \vee A$

1.	$A \rightarrow A \rightarrow -A \vee A$	A11.
2.	$-A \rightarrow .A \rightarrow A$	A13.
3.	$A \rightarrow .A \rightarrow A$	A13.
4.	$-A \vee A \rightarrow .A \rightarrow A$	From 2-3, by A10.
5.	$(A \rightarrow A) \leftrightarrow -A \vee A$	From 1, 4, by adjunction.

We now derive T3.

1.	$A \rightarrow .-A \rightarrow A$	A15.
2.	$A \rightarrow .A \rightarrow A$	A13.
3.	$A \rightarrow (-A \rightarrow A)\ \&\ (A \rightarrow A)$	From 1-2, by A9.
4.	$(-A \rightarrow A)\ \&\ (A \rightarrow A) \rightarrow .-A \vee A \rightarrow A$	A10.
5.	$A \rightarrow .-A \vee A \rightarrow A$	From 3-4, by A7.
6.	$A \rightarrow .A \rightarrow A \rightarrow A$	From 5, by T2 and A12.

2. Parry's axiom A13 seems to be a nod in the direction of classical implication. One can imagine motivating it as follows: "Since $A \rightarrow A$ is always true, then any sentence should analytically imply it as long as the sentence contains every variable which occurs in A." But this motivation would not require that A itself occurs as a subsentence of the sentence; it would suffice that the variables which occur in A occur in the sentence. Let us introduce the notation $\varphi(A)$ to mean any sentence in which occur all of the variables which occur in A. Then all instances of the following schema are derivable in ASI (note in the proof the importance of A15, which as we have noted above is independent from Parry's original formulation):

T4.* $\varphi(A) \rightarrow .A \rightarrow A$

Proof: We employ a strategy used by Anderson and Belnap in [1] in deriving the following lemma.

L1. *Let $p_1, \ldots, p_n$ be the sentential variables occurring in A. Then if $t_n = (p_1 \rightarrow p_1) \& \ldots \& (p_n \rightarrow p_n)$, $t_n \rightarrow .A \rightarrow A$ is a theorem.*

The proof of L1 is by induction on the length of A.

(i) If $A = p_1$, then $t_1 = A \rightarrow A$, and we have just an instance of $B \rightarrow B$, which is derivable from A3 and A4 by A7.

(ii) If $A = -B$, then by inductive hypothesis, $t_n \rightarrow .B \rightarrow B$ is a theorem. But by T2, both $B \rightarrow B$ and $-B \rightarrow -B$ are provably equivalent to $-B \vee B$ and hence to each other. So by A12, $t_n \rightarrow .-B \rightarrow -B$ is a theorem.

(iii) Suppose $A = B \& C$. Let t_j be appropriate for B and t_k for C. Then by hypothesis, both $t_j \rightarrow .B \rightarrow B$ and $t_k \rightarrow .C \rightarrow C$ are theorems. But then by A9, $t_j \& t_k \rightarrow .B \& C \rightarrow B \& C$ is a theorem. But $t_n \rightarrow t_j \& t_k$ is then a theorem by obvious properties of conjunction, and so is $t_n \rightarrow .A \rightarrow A$ by A7.

(iv) Suppose $A = B \rightarrow C$. Let t_j and t_k be as in (iii). Then again both $t_j \rightarrow .B \rightarrow B$ and $t_k \rightarrow .C \rightarrow C$ are theorems. But then by A15, $t_j \& t_k \rightarrow .B \rightarrow C \rightarrow .B \rightarrow C$ is a theorem, and like in (ii), by A7, so is $t_n \rightarrow .A \rightarrow A$, which completes the lemma.

That T4 is a theorem of ASI now follows easily from the lemma. For let $p_1, \ldots, p_n$ be all the propositional variables occurring in A. Then $\varphi(A)$ may be looked at as $f(p_i)$, so as instances of A13 we have as theorems, for all $i \leq n$ $\varphi(A) \rightarrow .p_i \rightarrow p_i$. Hence by repeated uses of A9, we have as a theorem, $\varphi(A) \rightarrow t_n$. But by the lemma, we have $t_n \rightarrow .A \rightarrow A$, so by A7, we obtain T4 from these two.

We shall use T4 in showing an appropriate deduction theorem for analytic implication for the system AI. But first we observe that the standard deduction theorem for the horseshoe holds for the system AI (indeed for the system ASI and for Parry's original formulation). This is easily seen by modifying the proof of the deduction theorem for the classical logic so as to take care of the case where the rule of adjunction is used. All of the theorems of classical logic are available since Anderson and Belnap [2] show that Parry's original formulation contains the classical propositional calculus. So using $\vdash$ for deducibility, we have for Parry's original formulation (and hence for ASI and AI) the

Classical Deduction Theorem (CDT).* *If Γ is a set of sentences, and $\Gamma, A \vdash B$, then $\Gamma \vdash A \supset B$.*

Using CDT, it is easy to establish the following as a theorem.

T5. $A \supset .\varphi(A) \rightarrow A$

Proof:	1. A	Assumption.
	2. $A \rightarrow .A \rightarrow A \rightarrow A$	T3.
	3. $A \rightarrow A \rightarrow A$	1, 2 by detachment for $\rightarrow$.
	4. $\varphi(A) \rightarrow .A \rightarrow A$	T4.
	5. $\varphi(A) \rightarrow A$	From 3, 4 by A7.

Hence T5 is a theorem by CDT.

We use the Classical Deduction Theorem together with T5 to prove for AI the following.

Analytic Deduction Theorem (ADT). *If* Γ *is a set of sentences,* $\Gamma, A \vdash B$, *and every variable which occurs in* B *also occurs in* A, *then* $\Gamma \vdash A \rightarrow B$.

Proof: Let Γ, A, and B be as in the hypothesis of the theorem. Then by CDT, $\Gamma \vdash A \supset B$. But since every variable of B is a variable of A, we may regard A as $\varphi(A \supset B)$, and so as an instance of T5 we have $(A \supset B) \supset (A \rightarrow (A \supset B))$, and hence $\Gamma \vdash A \rightarrow (A \supset B)$. But $A \rightarrow (A \supset B) \vdash A \rightarrow B$ as we see by the following deduction.

1. $A \rightarrow (-A \vee B)$	Assumption.
2. $A \rightarrow (-A \vee B) \rightarrow .A \rightarrow A$	A13.
3. $A \rightarrow A$	From 1-2, by detachment.
4. $(A \rightarrow A) \& (A \rightarrow (-A \vee B))$	From 1, 3, by adjunction.
5. $A \rightarrow A \& (-A \vee B)$	From 4, by A9.
6. $A \rightarrow ((A \& -A) \vee B)$	From 5, by A5.
7. $A \rightarrow B$	From 6, by A6.

So $\Gamma \vdash A \rightarrow B$.

The Analytic Deduction Theorem provides a kind of affirmative answer to a question that Gödel raised about Parry's original system in the discussion following its presentation in [6], namely, can we interpret "A analytically implies B" to mean that B is derivable from A and the logical axioms and that the content of B is contained in the content of A. This adds additional plausibility to axiom A14 and its alternative T1, because both of these can be established using the Analytic Deduction Theorem, and yet they may be shown independent of Parry's original formulation as we shall see in section 4.

3. We turn now to the business of a completeness proof. But first we must establish some additional facts about the system AI from the syntactic side.

T6.* $A \leftrightarrow B \supset .f(A) \leftrightarrow f(B)$

Proof: 1. $(A \leftrightarrow B) \& (f(A) \leftrightarrow f(A)) \rightarrow .f(A) \leftrightarrow f(B)$	A12.
2. $f(A) \leftrightarrow f(A) \supset .A \leftrightarrow B \supset .f(A) \leftrightarrow f(B)$	A11.
3. $f(A) \leftrightarrow f(A)$	By A3 and A4.
4. $A \leftrightarrow B \supset .f(A) \leftrightarrow f(B)$	From 2, 3, by A6.

We next see that the "normal form" theorems that follow all come easily from the axioms and definitions of the connectives.

T7.* $--A \leftrightarrow A$	
T8.* $A \& A \leftrightarrow A$	T9.* $A \vee A \leftrightarrow A$
T10.* $A \& B \leftrightarrow B \& A$	T11.* $A \vee B \leftrightarrow B \vee A$
T12.* $A \& (B \& C) \leftrightarrow (A \& B) \& C$	T13.* $A \vee (B \vee C) \leftrightarrow (A \vee B) \vee C$
T14.* $A \& (B \vee C) \leftrightarrow (A \& B) \vee (A \& C)$	T15.* $A \vee (B \& C) \leftrightarrow (A \vee B) \& (A \vee C)$
T16.* $-(A \& B) \leftrightarrow (-A \vee -B)$	T17.* $-(A \vee B) \leftrightarrow (-A \& -B)$

By means of T6-T17 it is possible to put every sentence of AI which does not contain an $\rightarrow$ into disjunctive normal form. If we allow no repetition of conjuncts or disjuncts in the disjunctive normal form, then there will be at most a finite number of these disjunctive normal forms made up of a finite number of sentential variables. We list this fact as

T18.* *Let W^n be the set of all sentences which contain no $\rightarrow$ and in which occur only the sentential variables $p_1, \ldots, p_n$. Then there are sentences $A_1, \ldots, A_m \in W^n$ such that for every sentence $B \in W^n$, one of $A_1 \leftrightarrow B, \ldots, A_m \leftrightarrow B$ is a theorem.*

We next note as a theorem,

T19.* $A \,\&\, (A \supset B) \rightarrow B$.

The proof is immediate, mostly from axioms A6 and A5.

For our next result, we define a set Γ of sentences to be *consistent* if for no sentence A is it the case that $\Gamma \vdash A \,\&\, -A$. A set of sentences M is *maximally* consistent if M is consistent, and no consistent set of sentences properly includes M.

T20.* *Let Γ be a consistent set of sentences. Then Γ is included in a maximally consistent set of sentences.*

Proof: The proof will be like that of Henkin [5], and indeed our whole completeness proof will be reminiscent of Henkin's completeness proof for classical logic. Let Γ be a consistent set. Let $A_1, A_2, \ldots$ be an enumeration of all the sentences. Let $M_0 = \Gamma$, and inductively, let $M_{n+1} = M_n \cup \{A_{n+1}\}$ if this is a consistent set, and otherwise let $M_{n+1} = M_n$. Let M be the union of all the M_n.

We first show that M is consistent. Suppose for some B, $M \vdash B \,\&\, -B$. Then there are sentences $B_1, \ldots, B_n \in M$ such that $B_1, \ldots, B_n \vdash B \,\&\, -B$. But then by the construction of M, there would be some M_i such that $B_1, \ldots, B_n \in M_i$, and so M_i would be inconsistent. But M_i is consistent by construction, so M must be consistent after all.

We now show that M is *maximally* consistent. Suppose $M \subseteq N$ and $M \neq N$, where N is consistent. Then there exists $A_i \in N$ such that $A_i \notin M$. Consider M_i. $M_{i-1} \cup \{A_i\}$ is consistent, since $M_{i-1} \cup \{A_i\} \subseteq N$, hence $A_i \in M_i \subseteq M$, contrary to hypothesis.

We next show that maximally consistent sets have some of the properties expected of them.

T21.* *If M is a maximally consistent set, then*
(i) *$A \in M$ iff $M \vdash A$, for any sentence A,*
(ii) *exactly one of A and $-A$ is in M, for any sentence A,*
(iii) *$A \vee B \in M$ iff $A \in M$ or $B \in M$, for any sentences A and B,*
(iv) *$A \,\&\, B \in M$ iff $A \in M$ and $B \in M$, for any sentences A and B.*

The proof of T21 proceeds in a straightforward manner, mimicking Henkin's proof for classical logic.

4. We now turn to the completeness proof proper. Let us define a Parry matrix $\mathfrak{M}$ to be a quintuple $\langle D, U, ^-, A, \Rightarrow\rangle$ satisfying the following description: D and U are disjoint isomorphic semi-lattices under the relations $\leq_D$ and $\leq_U$, respectively, and D is the set of designated elements and U is the set of undesignated elements. For some isomorphism f of D onto U, $^-$ is the union of f and the inverse of f. We consider $D \cup U$ itself as a semi-lattice by defining for a, $b \in D \cup U$, $a \leq b$ iff either (i) a, $b \in D$ and $a \leq_D b$, or (ii) a, $b \in U$ and $a \leq_U b$, or (iii) $a \in U$, $b \in D$ and $a \leq_U \overline{b}$ (note that D and U are then sub-semi-lattices of $D \cup U$). The meet operation in this new semi-lattice is then $\wedge$. If $a \leq b$, then $a \Rightarrow b = max(a, \overline{a})$, and if $a \not\leq b$, then $a \Rightarrow b = a \wedge \overline{a} \wedge b \wedge \overline{b}$. We point out that the original matrix found in Parry [6] is that Parry matrix obtained by letting D and U be the two element Boolean algebra.

It is now convenient to establish the independence of A14, which we do through a matrix for which we are indebted to Professor Robert Meyer. Let D be a semi-lattice with three elements 1, 2, and 3, where $1 \leq 2 \leq 3$. Let U be a disjoint copy of D, having elements -1, -2, and -3. As in the definition of a Parry matrix, set $-i \leq i$, and define $-$ and $\wedge$ accordingly. Define $a \Rightarrow b$ the same way as in a Parry matrix if the absolute value of a is $\leq$ the absolute value of b, but otherwise let $a \Rightarrow b = -1$. It is easily verified that this matrix satisfies all the axioms and rules of AI except A14, which is rejected when A is assigned the value +3 and B the value -2. The sentence T1 is also rejected by this matrix, letting A be +3 and B be +2.

It is routinely verified that any Parry matrix satisfies the axioms of AI, and it is easily seen that detachment and adjunction preserve designation. Hence we have

T22.* Strong Correctness Theorem. *Let* Γ *be a set of sentences. Then for any sentence* A, *if* $\Gamma \vdash A$, *then every Parry matrix which simultaneously satisfies* Γ, *also satisfies* A.

We shall eventually prove for AI the Strong Completeness Theorem, the converse of T22. We shall do so in a Henkin-style manner, first showing

T23. *Every consistent set is satisfied by some Parry matrix.*

T23 will follow as an immediate consequence of T20 and the following lemma.

L23. *Every maximally consistent set is satisfied by some Parry matrix.*

We now go about establishing the lemma by noting the following further theorems of AI.

T24. $A \,\&\, B \,\&\, (A \rightarrow B) \rightarrow .\text{-}A \rightarrow \text{-}B$

Proof: By ADT, the following deduction suffices.

1.	A	Assumption.
2.	B	Assumption.
3.	$A \rightarrow B$	Assumption.
4.	$\text{-}A \rightarrow A$	From 1, by A15.
5.	$\text{-}A \rightarrow B$	From 3, 4, by A7.
6.	$\text{-}A \rightarrow A \,\&\, B$	From 5, 4 by A9 and A2.
7.	$A \,\&\, B \rightarrow A \vee \text{-}B$	Classical propositional calculus and ADT.
8.	$\text{-}A \rightarrow A \vee \text{-}B$	From 6, 7 by A7.
9.	$\text{-}A \rightarrow \text{-}B$	From 8 by A5 and A6.

T25.* $-A \leftrightarrow -B \supset .A \leftrightarrow B$

Proof: 1. $-A \leftrightarrow -B \supset .--A \leftrightarrow --B$ T6.
2. $-A \leftrightarrow -B \supset .A \leftrightarrow B$
Two applications of A12, using $--A \leftrightarrow A$ and $--B \leftrightarrow B$.

T26. $-A \ \& \ B \ \& \ (A \rightarrow B) \rightarrow .A \rightarrow -B$

Proof: By ADT and the following deduction:
1. $-A \ \& \ B$ Assumption.
2. $A \rightarrow B$ Assumption.
3. $A \leftrightarrow A \ \& \ B$ From 2, by A8 and A9.
4. $-A \vee -B$ From 1, by classical propositional calculus.
5. $A \ \& \ B \rightarrow -A \vee -B$ From 4, by A15.
6. $A \rightarrow (-A \vee -B)$ From 3 and 6, by A12.
7. $A \rightarrow -B$ From 6 by A5 and A6.

T27.* $A \rightarrow B \supset .A \rightarrow B \rightarrow .A \rightarrow A$

Proof: 1. $A \rightarrow B \rightarrow .A \rightarrow A$ A13.
2. $A \rightarrow B \supset .A \rightarrow B \rightarrow .A \rightarrow A$
From 1, by classical propositional calculus.

T28.* $A \rightarrow B \supset .A \rightarrow A \rightarrow .A \rightarrow B$

Proof: By CDT and the following deduction:
1. $A \rightarrow B$ Assumption.
2. $A \leftrightarrow A \ \& \ B$ From 1, by A8 and A9.
3. $A \rightarrow A \ \& \ B \rightarrow .A \rightarrow B$ A8.
4. $A \rightarrow A \rightarrow .A \rightarrow B$ From 2 and 3, by A12.

T29.* $A \rightarrow B \supset .(A \rightarrow B) \leftrightarrow (-A \vee A)$

Proof: By adjunction from T27 and T28, using CDT, and then using T2 and A12.

T30. $-(A \rightarrow B) \supset .A \ \& \ -A \ \& \ B \ \& \ -B \rightarrow .A \rightarrow B$

Proof: By CDT and ADT, using the fact that in classical propositional calculus a contradiction implies anything.

T31. $-(A \rightarrow B) \supset .A \rightarrow B \rightarrow A \ \& \ -A \ \& \ B \ \& \ -B$

Proof: By CDT and ADT, as in T30.

T32. $-(A \rightarrow B) \supset .(A \rightarrow B) \leftrightarrow (A \ \& \ -A \ \& \ B \ \& \ -B)$

Proof: From T30 and T31, using CDT.

We now prove L23. Let M be a maximally consistent set of sentences of AI, and let W be the set of all sentences of AI. Then define W/M as follows. Let $|A|$ be the set of wffs B such that $A \leftrightarrow B \in M$. By obvious theorems of AI, this partitions W into equivalence classes. Let $D_M = \{|A|: A \in M\}$, and let $U_M = \{|A|: A \notin M\}$. Define operations on the $|A|$ so that $|\overline{A}| = |-A|$, $|A| \wedge |B| = |A \ \& \ B|$, and $|A| \Rightarrow |B| = |A \rightarrow B|$. Define

$|A| \leq |B|$ iff $A \rightarrow B \in M$. Then let $W/M = \langle D_M, U_M, \bar{\ }, \wedge, \Rightarrow \rangle$. I claim that

W/M is a Parry matrix satisfying M.

Proof: The replacement theorem T6 ensures that the operations are well defined (single-valued). Obvious theorems regarding & and $\rightarrow$ ensure that $\wedge$ is a semi-lattice operation under $\leq$. For the next remarks, keep in mind that M is closed under deducibility (T21, i). Adjunction guarantees that D_M is closed under $\wedge$, and so is a semi-lattice in its own right. Also U_M is closed under $\wedge$, for if $|A|, |B| \in U_M$, then $A, B \notin M$. But then by T21, (ii) $-A, -B \in M$. But then it is impossible for $A \& B \in M$, since $A \& B \rightarrow A$ is a theorem of AI, and so $A \in M$, contradicting T21, (ii). So $A \& B \notin M$, i.e., $|A| \wedge |B| \in U_M$. So both D_M, and U_M are semi-lattices, and their disjointness follows also from T21, (ii).

We can construct an isomorphism f from D_M onto U_M, letting $f(|A|) = \overline{|A|}$. For $|A|, |B| \in D_M$, $|A| \leq |B|$ implies $f(|A|) \leq f(|B|)$, by T24. f is onto by T21, (ii), and f is one-one by T25. Further $\leq$ is defined as on a Parry matrix by virtue of T26. And finally $\Rightarrow$ is defined as on a Parry matrix by virtue of T28 and T32.

Obviously W/M satisfies M under the natural valuation, v_M, such that $v_M(A) = |A|$. So we have L23 and hence T23. We next use T23 to establish

T33. Strong Completeness Theorem. *Let Γ be a set of sentences. Then for every sentence A, if every Parry matrix which simultaneously satisfies Γ also satisfies A, then $\Gamma \vdash A$.*

Proof: Still mimicking Henkin, we prove the contrapositive by supposing that A is not deducible from Γ, and showing that then $\Gamma \cup \{-A\}$ must be consistent. For suppose to the contrary that $\Gamma, -A \vdash B \& -B$. Then by CDT, $\Gamma \vdash -A \supset B \& -B$. But since AI contains the classical propositional calculus, $\Gamma \vdash A$, contrary to hypothesis. But since $\Gamma \cup \{-A\}$ is consistent, then by T23 there must be a Parry matrix which satisfies it, i.e., a Parry matrix which satisfies Γ, but rejects A, which completes the proof.

5. We now draw morals concerning the decidability of AI from our work above. Putting T22 and T33 together, we have

T34. *Let Γ be a set of sentences. Then for any sentence A, $\Gamma \vdash A$ iff every Parry matrix which satisfies Γ also satisfies A.*

The trouble for decidability is that the matrix W/M constructed in the proof of T33 may well be infinite. But suppose only the sentential variables $p_1, \ldots, p_n$ occur in A and in the wffs of Γ. Let W^n be the set of all wffs containing only the variables $p_1, \ldots, p_n$, and consider a formulation of AI with only these n variables. Let us call this AI^n. It is obvious that A is deducible from Γ in AI iff A is deducible from Γ in AI^n (replace any variable occurring in a deduction in AI which is not one of $p_1, \ldots, p_n$ uniformly by say p_1). A matrix W^n/M for AI^n can be constructed analogously to W/M, and it obviously will be finite by virtue of T18. The reason is that the definition of $\Rightarrow$ for W^n/M ultimately reduces $\Rightarrow$ to the operations $\wedge$ and $^-$. Indeed, a recursive bound may be set upon the m occurring in T18, namely

$$m \leq 2^{2^{2n}} = \alpha$$

So we have from the proof of T34, *via* the above remarks,

T35. For any wff A, A is a theorem of AI iff A is valid in every Parry matrix of at most α elements.

This gives us

T36. AI *is decidable.*

6. We now make some remarks which have both algebraic and informal interest. It is well-known that every semi-lattice is isomorphic to a semi-lattice of sets. It is usual, of course, to take the semi-lattice operation as intersection, but one can dually take it as union. This gives us a way of representing every Parry matrix in a way which is both algebraically and logically satisfying. Given a Parry matrix $\langle D, U, ^-, \wedge, \Rightarrow\rangle$, let S be a semi-lattice of sets, taking union as the semi-lattice operation, which is isomorphic to D. Then consider the direct product of S with the two-element Boolean algebra 2, $S \times 2$. The elements of $S \times 2$ are the pairs $\langle X, a\rangle$, where $X \in S$ and a is 0 or 1, and $S \times 2$ is a semi-lattice where for two pairs, $\langle X, a\rangle \leq \langle Y, b\rangle$ iff $Y \subseteq X$ and $a \leq b$ in 2. Indeed, $S \times 2$ is isomorphic to the original Parry matrix regarded as a semi-lattice. We can in fact represent the original Parry matrix entirely by $S \times 2$. Let D' be the set of pairs $\langle X, 1\rangle$ and U' be the set of pairs $\langle X, 0\rangle$, where $X \in S$. Define $\overline{\langle X, a\rangle} = \langle X, \overline{a}\rangle$, where $\overline{a}$ is the Boolean complement of a in 2, and define $\langle X, a\rangle \Rightarrow \langle Y, b\rangle = \langle X \cup Y, 1\rangle$, if $\langle X, a\rangle \leq \langle Y, b\rangle$, and otherwise let it be $\langle X \cup Y, 0\rangle$. The result is easily seen to be isomorphic to the original Parry matrix.

We can put a natural intuitive interpretation upon the above piece of algebra, as was called to our attention by Professor Meyer. Suppose we regard a proposition as having two important features, a content and a truth value. We might then choose to regard a proposition as a pair $\langle X, a\rangle$, where X is a set of objects that the proposition mentions, and a is the truth value of the proposition. This latter we can regard in the usual way as an element of 2, letting 1 stand for true and 0 stand for false. It would then be natural to say that a proposition $\langle X, a\rangle$ *analytically implies* a proposition $\langle Y, b\rangle$ iff the content of the second proposition is included in the content of the first, and furthermore the first proposition is not true while the second is false. But this is just $\langle X, a\rangle \leq \langle Y, b\rangle$.

To complete the story we suppose that combining propositions by logical connectives gives us a proposition that has exactly the total content of the propositions being combined, and that the truth value of the resultant proposition is computed in a classical sort of way. Thus a conjunction has as its content the union of the contents of its conjuncts, and is true iff both conjuncts are true. Similarly, a disjunction has as its content the union of the contents of its disjuncts, but is true iff at least one of its disjuncts is true. A negation has as its content the content of the original unnegated proposition, and is true iff the original proposition is false. An analytic implication is slightly more complicated than these "truth functions," for the contents of its antecedent and consequent can affect the truth value of the implication. Thus an analytic implication has as its content the union of the contents of its antecedent and consequent, and is true iff its antecedent analytically implies its consequent. This interpretation thus gives a complete and correct semantics for the system AI which is in accord with Parry's motivations.

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