

# The Exact Computational Complexity of Evolutionarily Stable Strategies\*

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## Abstract

While the computational complexity of many game-theoretic solution concepts, notably Nash equilibrium, has now been settled, the question of determining the exact complexity of computing an evolutionarily stable strategy has resisted solution since attention was drawn to it in 2004. In this paper, I settle this question by proving that deciding the existence of an evolutionarily stable strategy is  $\Sigma_2^P$ -complete.

**Keywords:** Algorithmic game theory, equilibrium computation, evolutionarily stable strategies.

## 1 Introduction

Game theory provides ways of formally representing strategic interactions between multiple players, as well as a variety of *solution concepts* for the resulting games. The best-known solution concept is that of Nash equilibrium (Nash, 1950), where each player plays a best response to all the other players' strategies. The computational complexity of, given a game in normal form, computing a (any) Nash equilibrium, remained open for a long time and was accorded significant importance (Papadimitriou, 2001). (I will give a brief introduction to / review of computational complexity in Section 2; the reader unfamiliar with it may prefer to read this section first.) An elegant algorithm for the two-player case, the Lemke-Howson algorithm (Lemke and Howson, 1964), was proved to require exponential time on some game families by Savani and von Stengel (2006). Finally, in a breakthrough series of papers, the problem was established to be PPAD-complete, even in the two-player case (Daskalakis *et al.*, 2009; Chen *et al.*, 2009).<sup>1</sup>

Not all Nash equilibria are created equal; for example, one can Pareto-dominate another. Moreover, generally, the set of Nash equilibria does not satisfy *interchangeability*. That is, if player 1 plays her strategy from one Nash equilibrium, and player 2 plays his strategy from another Nash equilibrium, the result is not guaranteed to be a Nash equilibrium. This leads to the dreaded *equilibrium selection problem*: if one plays a game for the first time, how is one to know according to which equilibrium to play? This problem is arguably exacerbated by the fact that determining whether equilibria with particular properties, such as placing probability on a particular pure strategy or having at least a certain level of social welfare, exist is NP-complete in two-player games (and associated optimization

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<sup>1</sup>Depending on the precise formulation, the problem can actually be FIXP-complete for more than 2 players (Etessami and Yannakakis, 2010).

problems are inapproximable unless P=NP) (Gilboa and Zemel, 1989; Conitzer and Sandholm, 2008). In any case, equilibria are often seen as a state to which play could reasonably converge, rather than an outcome that can necessarily be arrived at immediately by deduction.

In this paper, we consider the concept of *evolutionarily stable strategies*, a solution concept for symmetric games with two players.  $s$  will denote a pure strategy and  $\sigma$  a mixed strategy, where  $\sigma(s)$  denotes the probability that mixed strategy  $\sigma$  places on pure strategy  $s$ .  $u(s, s')$  is the utility that a player playing  $s$  obtains when playing against a player playing  $s'$ , and

$$u(\sigma, \sigma') = \sum_{s, s'} \sigma(s)\sigma'(s')u(s, s')$$

is the natural extension to mixed strategies.

**Definition 1 (Price and Smith (1973))** *Given a symmetric two-player game, a mixed strategy  $\sigma$  is said to be an evolutionarily stable strategy (ESS) if both of the following properties hold.*

1. (*Symmetric Nash equilibrium property*) For any mixed strategy  $\sigma'$ , we have  $u(\sigma, \sigma) \geq u(\sigma', \sigma)$ .
2. For any mixed strategy  $\sigma'$  ( $\sigma' \neq \sigma$ ) for which  $u(\sigma, \sigma) = u(\sigma', \sigma)$ , we have  $u(\sigma, \sigma') > u(\sigma', \sigma')$ .

The intuition behind this definition is that a population of players playing  $\sigma$  cannot be successfully “invaded” by a small population of players playing some  $\sigma' \neq \sigma$ , because they will perform *strictly* worse than the players playing  $\sigma$  and therefore they will shrink as a fraction of the population. They perform strictly worse either because (1)  $u(\sigma, \sigma) > u(\sigma', \sigma)$ , and because  $\sigma$  has dominant presence in the population this outweighs performance against  $\sigma'$ ; or because (2)  $u(\sigma, \sigma) = u(\sigma', \sigma)$  so the second-order effect of performance against  $\sigma'$  becomes significant, but in fact  $\sigma'$  performs worse against itself than  $\sigma$  performs against it, that is,  $u(\sigma, \sigma') > u(\sigma', \sigma')$ .

**Example (Hawk-Dove game (Price and Smith, 1973)).** Consider the following symmetric two-player game:

	Dove	Hawk
Dove	1,1	0,2
Hawk	2,0	-1,-1

The unique symmetric Nash equilibrium  $\sigma$  of this game is 50% Dove, 50% Hawk. For any  $\sigma'$ , we have  $u(\sigma, \sigma) = u(\sigma', \sigma) = 1/2$ . That is, everything is a best response to  $\sigma$ . We also have  $u(\sigma, \sigma') = 1.5\sigma'(\text{Dove}) - 0.5\sigma'(\text{Hawk}) = 2\sigma'(\text{Dove}) - 0.5$ , and  $u(\sigma', \sigma') = 1\sigma'(\text{Dove})^2 + 2\sigma'(\text{Hawk})\sigma'(\text{Dove}) + 0\sigma'(\text{Dove})\sigma'(\text{Hawk}) - 1\sigma'(\text{Hawk})^2 = -2\sigma'(\text{Dove})^2 + 4\sigma'(\text{Dove}) - 1$ . The difference between the former and the latter expression is  $2\sigma'(\text{Dove})^2 - 2\sigma'(\text{Dove}) + 0.5 = 2(\sigma'(\text{Dove}) - 0.5)^2$ . The latter is clearly positive for all  $\sigma' \neq \sigma$ , implying that  $\sigma$  is an ESS.

Intuitively, the problem of computing an ESS appears significantly harder than that of computing a Nash equilibrium, or even a Nash equilibrium with a simple additional property such as those described earlier. In the latter type of problem, while it may be difficult to find the solution, once found, it is straightforward to verify that it is in fact a Nash equilibrium (with the desired simple property). This is not so for the notion of ESS: given a candidate strategy, it does not appear straightforward to figure out whether there exists a strategy that successfully invades it. However, appearances can be deceiving; perhaps there is a not entirely obvious, but nevertheless fast and elegant way of checking whether such an invading strategy exists. Even if not, it is not immediately clear whether this makes the problem of *finding* an ESS genuinely harder. Computational complexity provides the natural toolkit for answering these questions.

The complexity of computing whether a game has an evolutionarily stable strategy (for an overview, see Chapter 29 of the Algorithmic Game Theory book (Suri, 2007)) was first studied by Etessami and Lochbihler (2008), who proved that the problem is both NP-hard and coNP-hard, as well as that the problem is contained in  $\Sigma_2^P$  (the class of decision problems that can be solved in nondeterministic polynomial time when given access to an NP oracle; see also Section 2). Nisan (2006) subsequently<sup>2</sup> proved the stronger hardness result that the problem is  $\text{co}D^P$ -hard. He also observed that it follows from his reduction that the problem of determining whether a given strategy is an ESS is coNP-hard (and Etessami and Lochbihler (2008) then pointed out that this also follows from their reduction). Etessami and Lochbihler (2008) also showed that the problem of determining the existence of a *regular* ESS is NP-complete. As was pointed out in both papers, all of this still leaves the main question of the exact complexity of the general ESS problem open. In this paper, this is settled: the problem is in fact  $\Sigma_2^P$ -complete. After the review of computational complexity (Section 2), I will briefly discuss the significance of this result (Section 3).

The remainder of the paper—to which the reader not interested in a review of computational complexity or a discussion of the significance of the result is welcome to jump—contains the proof, which is structured as follows. In Section 4, Lemma 1 states that the slightly more general problem of determining whether an ESS exists whose support is restricted to a subset of the strategies is  $\Sigma_2^P$ -hard. This is the main part of the proof. Then, in Section 5, Lemma 2 points out that if two pure strategies are exact duplicates, neither of them can occur in the support of any ESS. By this, we can disallow selected strategies from taking part in any ESS simply by duplicating them. Combining this with the first result, we arrive at the main result, Theorem 1.

One may well complain that Lemma 2 is a bit of a cheat; perhaps we should just consider duplicate strategies to be “the same” strategy and merge them back into one. As the reader probably suspects, such a hasty and limited patch will not avoid the hardness result. Even something a little more thorough, such as iterated elimination of very weakly dominated strategies (in some order), will not suffice: in Appendix A I show, with additional analysis and modifications, that the result holds even in games where each pure strategy is the unique best response to some mixed strategy.

## 2 Brief Background on Computational Complexity

Much of theoretical computer science is concerned with designing algorithms that solve computational problems *fast* (as well as, of course, correctly). For example, one computational problem is the following: given a two-player game in normal form, determine whether there exists a Nash equilibrium in which player 1 obtains utility at least 1. A specific two-player normal-form game would be an *instance* of that problem. What does it mean to solve a problem fast? This is fundamentally about how the runtime scales with the size of the input (e.g., the size of the game). The focus is generally primarily on whether the runtime scales as a *polynomial* function of the input, which is considered fast (or *efficient*)—as opposed to, say, an exponential function.

For many problems, including the one described in the previous paragraph, we do not have any efficient algorithm, nor do we have a proof that no such algorithm exists. However, in these situations, we can often prove that the problem is at least as hard as any other problem in a large class. That is, we can prove that if the problem under consideration admits an efficient algorithm, then so do all other problems in a large class. The most famous such class is NP, which consists of *decision problems*, i.e., problems for which every instance has a “yes” or “no” answer. Specifically, it consists of decision problems that are such that for every “yes” instance, there is a succinct proof (that can be efficiently checked) that the answer is “yes.” A problem that is at least as hard as any problem in NP is said to be NP-hard. If an NP-hard problem is also in the class NP, it is said to be NP-complete; thus, in a

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<sup>2</sup>An early version of Etessami and Lochbihler (2008) appeared in 2004.

sense, all NP-complete problems are equally hard.

Many problems of interest are NP-complete. The paradigmatic NP-complete problem is the *satisfiability* problem, which asks, given a propositional logic formula, whether there is a way to set the variables in this formula to *true* or *false* in such a way that the formula as a whole evaluates to *true*. For example, the formula  $(x_1 \vee x_2) \wedge (\neg x_2)$  is a “yes” instance, because setting  $x_1$  to *true* and  $x_2$  to *false* results in the formula evaluating to *true*. The succinct proof that an instance is a “yes” instance consists simply of values that the variables can take to make the formula evaluate to *true*. As it turns out, the problem introduced at the beginning of this section is NP-complete. It is in NP because given the supports of the strategies in a Nash equilibrium with high utility for player 1, we can easily reconstruct such an equilibrium; therefore, the supports serve as the proof that it is a “yes” instance. Many similar problems are also NP-complete (Gilboa and Zemel, 1989; Conitzer and Sandholm, 2008).

A standard way to prove that a problem  $A$  is NP-hard is to take another problem  $B$  that is already known to be NP-hard, and *reduce* it to problem  $A$ . A reduction here is an efficiently computable function that maps every instance of  $B$  to some instance of  $A$  with the same truth value (“yes” or “no”). Given such a reduction, an efficient algorithm for  $A$  could be used to solve  $B$  as well, proving that in the relevant sense,  $A$  is at least as hard as  $B$ .

There are other classes of interest besides NP, with hardness and completeness defined similarly. For example, coNP consists of problems where there is a succinct proof of an instance being a “no” instance. The class  $\Sigma_2^P$  is most easily illustrated by a standard complete problem for it. As in the satisfiability problem, we are given a propositional logic formula, but this time, the variables are split into two sets,  $X_1$  and  $X_2$ . We are asked whether there exists a way to set the variables in  $X_1$  such that *no matter how* the variables in  $X_2$  are set, the formula evaluates to *true*. (Note here the similarity to the ESS problem, where we are asked whether there exists a strategy  $\sigma$  such that *no matter which*  $\sigma'$  invades, the invasion is repelled.) Similarly, a complete problem for the class  $\Pi_2^P$  (which equals  $\text{co}\Sigma_2^P$ ) asks whether no matter how the variables in  $X_1$  are set, there is a way to set the variables in  $X_2$  so that the formula evaluates to *true*. These classes are said to be at the *second level of the polynomial hierarchy*, and the generalization to higher levels is straightforward.

### 3 Significance of the Result

What is the significance of establishing the  $\Sigma_2^P$ -completeness of deciding whether an evolutionarily stable strategy exists? When the computational problem of determining the existence of an ESS comes up, it is surely more satisfying to be able to simply state the exact complexity of the problem than to have to state that it is hard for some classes, included in another, and the exact complexity is unknown. Moreover, the latter situation also left open the possibility that the ESS problem exposed a fundamental gap in our understanding of computational complexity theory. It could even have been the case that the ESS problem required the definition of an entirely new complexity class for which the problem was complete.<sup>3</sup> The result presented here implies that this is not the case; while  $\Sigma_2^P$  is not as well known as NP, it is a well-established complexity class.

Additionally, some of the significance of the result is in the irony that a key solution concept in evolutionary game theory, which is often taken to be a model of how equilibria might actually be reached in practice by a simple process, is actually computationally significantly less tractable (as far as our current understanding of computational complexity goes) than the concept of Nash equilibrium. This was already implied by the earlier hardness results referenced in the introduction, but the result obtained here shows the gap to be even wider. This perhaps suggests that modified solution concepts are called for, and more generally that the computational complexity of solution concepts should be

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<sup>3</sup>In the case of computing one Nash equilibrium, the class PPAD had previously been defined (Papadimitriou, 1994), but it did not have much in the way of known complete problems before the Nash equilibrium result—and the standing of the class was quite diminished by this lack of natural problems known to be complete for it.

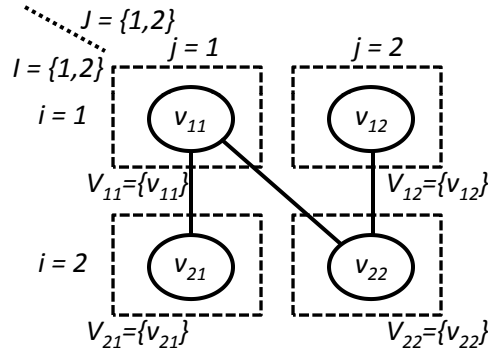


Figure 1: An example MINMAX-CLIQUE instance (with  $k = 2$ ), for which the answer is “no.”

taken into account in assessing their reasonableness for the purpose at hand. On the other hand, it is important to note that it may yet be possible to find evolutionarily stable strategies fast for most games actually encountered in practice. Games encountered in practice may have additional structure that puts the problem in a lower complexity class, possibly even P. If so, this would clearly reduce the force of the call for new solution concepts.

## 4 Hardness with Restricted Support

Having completed a review of the relevant computational complexity theory and a discussion of the significance of the result, we now begin the technical part of the paper. As outlined earlier, we first introduce a slightly different problem, which we will then show is  $\Sigma_2^P$ -hard. From this, it will be fairly easy to show, in Section 5, that the main problem is  $\Sigma_2^P$ -hard.

**Definition 2** *In ESS-RESTRICTED-SUPPORT, we are given a symmetric two-player normal-form game  $G$  with strategies  $S$ , and a subset  $T \subseteq S$ . We are asked whether there exists an evolutionarily stable strategy of  $G$  that places positive probability only on strategies in  $T$  (but not necessarily on all strategies in  $T$ ).*

We will establish  $\Sigma_2^P$ -hardness by reduction from (the complement of) the following problem.

**Definition 3 (MINMAX-CLIQUE)** *We are given a graph  $G = (V, E)$ , sets  $I$  and  $J$ , a partition of  $V$  into subsets  $V_{ij}$  for  $i \in I$  and  $j \in J$ , and a number  $k$ . We are asked whether it is the case that for every function  $t : I \rightarrow J$ , there is a clique of size (at least)  $k$  in the subgraph induced on  $\bigcup_{i \in I} V_{i,t(i)}$ . (Without loss of generality, we will require  $k > 1$ .)*

**Example.** Figure 1 shows a tiny MINMAX-CLIQUE instance (let  $k = 2$ ). The answer to this instance is “no” because for  $t(1) = 2, t(2) = 1$ , the graph induced on  $\bigcup_{i \in I} V_{i,t(i)} = V_{12} \cup V_{21} = \{v_{12}, v_{21}\}$  has no clique of size at least 2.

We have the following known hardness result for this problem. (Recall that  $\Pi_2^P = \text{co}\Sigma_2^P$ .)

**Known Theorem 1 ((Ko and Lin, 1995))** *MINMAX-CLIQUE is  $\Pi_2^P$ -complete.*

We are now ready to present the main part of the proof.

**Lemma 1** *ESS-RESTRICTED-SUPPORT* is  $\Sigma_2^P$ -hard.

**Proof:** We reduce from the complement of MINMAX-CLIQUE. That is, we show how to transform any instance of MINMAX-CLIQUE into a symmetric two-player normal-form game with a distinguished subset  $T$  of its strategies, so that this game has an ESS with support in  $T$  if and only if the answer to the MINMAX-CLIQUE instance is “no.”

**The Reduction.** For every  $i \in I$  and every  $j \in J$ , create a strategy  $s_{ij}$ . For every  $v \in V$ , create a strategy  $s_v$ . Finally, create a single additional strategy  $s_0$ .

- For all  $i \in I$  and  $j \in J$ ,  $u(s_{ij}, s_{ij}) = 1$ .
- For all  $i \in I$  and  $j, j' \in J$  with  $j \neq j'$ ,  $u(s_{ij}, s_{ij'}) = 0$ .
- For all  $i, i' \in I$  with  $i \neq i'$  and  $j, j' \in J$ ,  $u(s_{ij}, s_{i'j'}) = 2$ .
- For all  $i \in I$ ,  $j \in J$ , and  $v \in V$ ,  $u(s_{ij}, s_v) = 2 - 1/|I|$ .
- For all  $i \in I$  and  $j \in J$ ,  $u(s_{ij}, s_0) = 2 - 1/|I|$ .
- For all  $i \in I$ ,  $j \in J$ , and  $v \in V_{ij}$ ,  $u(s_v, s_{ij}) = 2 - 1/|I|$ .
- For all  $i \in I$ ,  $j, j' \in J$  with  $j \neq j'$ , and  $v \in V_{ij}$ ,  $u(s_v, s_{ij'}) = 0$ .
- For all  $i, i' \in I$  with  $i \neq i'$ ,  $j, j' \in J$ , and  $v \in V_{ij}$ ,  $u(s_v, s_{i'j'}) = 2 - 1/|I|$ .
- For all  $v \in V$ ,  $u(s_v, s_v) = 0$ .
- For all  $v, v' \in V$  with  $v \neq v'$  where  $(v, v') \notin E$ ,  $u(s_v, s_{v'}) = 0$ .
- For all  $v, v' \in V$  with  $v \neq v'$  where  $(v, v') \in E$ ,  $u(s_v, s_{v'}) = (k/(k-1))(2 - 1/|I|)$ .
- For all  $v \in V$ ,  $u(s_v, s_0) = 0$ .
- For all  $i \in I$  and  $j \in J$ ,  $u(s_0, s_{ij}) = 2 - 1/|I|$ .
- For all  $v \in V$ ,  $u(s_0, s_v) = 0$ .
- $u(s_0, s_0) = 0$ .

We are asked whether there exists an ESS that places positive probability only on strategies  $s_{ij}$  with  $i \in I$  and  $j \in J$ . That is,  $T = \{s_{ij} : i \in I, j \in J\}$ .

**Example.** Consider again the MINMAX-CLIQUE instance from Figure 1. The game to which the reduction maps this instance is:

	$s_{11}$	$s_{12}$	$s_{21}$	$s_{22}$	$s_{v_{11}}$	$s_{v_{12}}$	$s_{v_{21}}$	$s_{v_{22}}$	$s_0$
$s_{11}$	1	0	2	2	3/2	3/2	3/2	3/2	3/2
$s_{12}$	0	1	2	2	3/2	3/2	3/2	3/2	3/2
$s_{21}$	2	2	1	0	3/2	3/2	3/2	3/2	3/2
$s_{22}$	2	2	0	1	3/2	3/2	3/2	3/2	3/2
$s_{v_{11}}$	3/2	0	3/2	3/2	0	0	3	3	0
$s_{v_{12}}$	0	3/2	3/2	3/2	0	0	0	3	0
$s_{v_{21}}$	3/2	3/2	3/2	0	3	0	0	0	0
$s_{v_{22}}$	3/2	3/2	0	3/2	3	3	0	0	0
$s_0$	3/2	3/2	3/2	3/2	0	0	0	0	0

It has an ESS  $\sigma$  with weight  $1/2$  on each of  $s_{12}$  and  $s_{21}$ . In contrast, (for example)  $\sigma'$  with weight  $1/2$  on each of  $s_{11}$  and  $s_{21}$  is invaded by the strategy  $\sigma''$  with weight  $1/2$  on each of  $s_{v_{11}}$  and  $s_{v_{21}}$ , because  $u(\sigma'', \sigma') = u(\sigma', \sigma') = 3/2$  and  $u(\sigma'', \sigma'') = u(\sigma', \sigma'') = 3/2$ .

**Proof of equivalence.** Suppose there exists a function  $t : I \rightarrow J$  such that every clique in the subgraph induced on  $\bigcup_{i \in I} V_{i,t(i)}$  has size strictly less than  $k$ . We will show that the mixed strategy  $\sigma$  that places probability  $1/|I|$  on  $s_{i,t(i)}$  for each  $i \in I$  (and 0 everywhere else) is an ESS.

First, we show that  $\sigma$  is a best response against itself. For any  $s_{ij}$  in the support of  $\sigma$ , we have  $u(s_{ij}, \sigma) = (1/|I|) \cdot 1 + (1 - 1/|I|) \cdot 2 = 2 - 1/|I|$ , and hence we also have  $u(\sigma, \sigma) = 2 - 1/|I|$ . For  $s_{ij}$  not in the support of  $\sigma$ , we have  $u(s_{ij}, \sigma) = (1/|I|) \cdot 0 + (1 - 1/|I|) \cdot 2 = 2 - 2/|I| < 2 - 1/|I|$ . For all  $i \in I$ , for all  $v \in V_{i,t(i)}$ , we have  $u(s_v, \sigma) = (1/|I|) \cdot (2 - 1/|I|) + (1 - 1/|I|) \cdot (2 - 1/|I|) = 2 - 1/|I|$ . For all  $i \in I$ ,  $j \in J$  with  $j \neq t(i)$ , and  $v \in V_{ij}$ , we have  $u(s_v, \sigma) = (1/|I|) \cdot 0 + (1 - 1/|I|) \cdot (2 - 1/|I|) = (1 - 1/|I|)(2 - 1/|I|) < 2 - 1/|I|$ . Finally,  $u(s_0, \sigma) = 2 - 1/|I|$ . So  $\sigma$  is a best response to itself.

It follows that if there were a strategy  $\sigma' \neq \sigma$  that could successfully invade  $\sigma$ , then  $\sigma'$  must put probability only on best responses to  $\sigma$ . Based on the calculations in the previous paragraph, these best responses are  $s_0$ , and, for any  $i$ ,  $s_{i,t(i)}$  and, for all  $v \in V_{i,t(i)}$ ,  $s_v$ . The expected utility of  $\sigma$  against any of these is  $2 - 1/|I|$  (in particular, for any  $i$ , we have  $u(\sigma, s_{i,t(i)}) = (1/|I|) \cdot 1 + (1 - 1/|I|) \cdot 2 = 2 - 1/|I|$ ). Hence,  $u(\sigma, \sigma') = 2 - 1/|I|$ , and to successfully invade,  $\sigma'$  must attain  $u(\sigma', \sigma') \geq 2 - 1/|I|$ .

We can write  $\sigma' = p_0 s_0 + p_1 \sigma'_1 + p_2 \sigma'_2$ , where  $p_0 + p_1 + p_2 = 1$ ,  $\sigma'_1$  only puts positive probability on the  $s_{i,t(i)}$  strategies, and  $\sigma'_2$  only puts positive probability on the  $s_v$  strategies with  $v \in V_{i,t(i)}$ . The strategy that results from conditioning  $\sigma'$  on  $\sigma'_1$  not being played may be written as

$$(p_0/(p_0 + p_2))s_0 + (p_2/(p_0 + p_2))\sigma'_2$$

and thus we may write

$$\begin{aligned} u(\sigma', \sigma') &= p_1^2 u(\sigma'_1, \sigma'_1) + p_1(p_0 + p_2)u(\sigma'_1, (p_0/(p_0 + p_2))s_0 + (p_2/(p_0 + p_2))\sigma'_2) \\ &\quad + (p_0 + p_2)p_1 u((p_0/(p_0 + p_2))s_0 + (p_2/(p_0 + p_2))\sigma'_2, \sigma'_1) \\ &\quad + (p_0 + p_2)^2 u((p_0/(p_0 + p_2))s_0 + (p_2/(p_0 + p_2))\sigma'_2, (p_0/(p_0 + p_2))s_0 + (p_2/(p_0 + p_2))\sigma'_2) \end{aligned}$$

Now, if we shift probability mass from  $s_0$  to  $\sigma'_2$ , i.e., we decrease  $p_0$  and increase  $p_2$  by the same amount, this will not affect any of the coefficients in the previous expression; it will not affect any of

$$u(\sigma'_1, \sigma'_1),$$

$$u(\sigma'_1, (p_0/(p_0 + p_2))s_0 + (p_2/(p_0 + p_2))\sigma'_2)$$

(because  $u(s_{ij}, s_v) = u(s_{ij}, s_0) = 2 - 1/|I|$ ), and

$$u((p_0/(p_0 + p_2))s_0 + (p_2/(p_0 + p_2))\sigma'_2, \sigma'_1)$$

(because  $u(s_0, s_{ij}) = u(s_v, s_{ij}) = 2 - 1/|I|$  when  $v \in V_{ij}$  or  $v \in V_{i'j'}$  with  $i' \neq i$ );

and it will not decrease

$$u((p_0/(p_0 + p_2))s_0 + (p_2/(p_0 + p_2))\sigma'_2, (p_0/(p_0 + p_2))s_0 + (p_2/(p_0 + p_2))\sigma'_2)$$

(because for any  $v \in V$ ,  $u(s_0, s_0) = u(s_0, s_v) = u(s_v, s_0) = 0$ ).

Therefore, we may assume without loss of generality that  $p_0 = 0$ , and hence  $\sigma' = p_1 \sigma'_1 + p_2 \sigma'_2$ . It follows that we can write

$$u(\sigma', \sigma') = p_1^2 u(\sigma'_1, \sigma'_1) + p_1 p_2 u(\sigma'_1, \sigma'_2) + p_2 p_1 u(\sigma'_2, \sigma'_1) + p_2^2 u(\sigma'_2, \sigma'_2)$$

We first note that  $u(\sigma'_1, \sigma'_1)$  can be at most  $2 - 1/|I|$ . Specifically,

$$u(\sigma'_1, \sigma'_1) = \left( \sum_i \sigma'_1(s_{i,t(i)})^2 \right) \cdot 1 + \left( 1 - \sum_i \sigma'_1(s_{i,t(i)})^2 \right) \cdot 2$$

and this expression is uniquely maximized by setting each  $\sigma'_1(s_{i,t(i)})$  to  $1/|I|$ .  $u(\sigma'_1, \sigma'_2)$  is easily seen to also be  $2 - 1/|I|$ , and  $u(\sigma'_2, \sigma'_1)$  is easily seen to be at most  $2 - 1/|I|$  (in fact, it is exactly that). Thus, to obtain  $u(\sigma', \sigma') \geq 2 - 1/|I|$ , we must have either  $p_1 = 1$  or  $u(\sigma'_2, \sigma'_2) \geq 2 - 1/|I|$ . However, in the former case, we would require  $u(\sigma'_1, \sigma'_1) = 2 - 1/|I|$ , which can only be attained by setting each  $\sigma'_1(s_{i,t(i)})$  to  $1/|I|$ —but this would result in  $\sigma' = \sigma$ . Thus, we can conclude  $u(\sigma'_2, \sigma'_2) \geq 2 - 1/|I|$ . But then  $\sigma'_2$  would also successfully invade  $\sigma$ . Hence, we can assume without loss of generality that  $\sigma' = \sigma'_2$ , i.e.,  $p_0 = p_1 = 0$  and  $p_2 = 1$ .

That is, we can assume that  $\sigma'$  only places positive probability on strategies  $s_v$  with  $v \in \bigcup_{i \in I} V_{i,t(i)}$ . For any  $v, v' \in V$ , we have  $u(s_v, s_{v'}) = u(s_{v'}, s_v)$ . Specifically,  $u(s_v, s_{v'}) = u(s_{v'}, s_v) = (k/(k-1))(2 - 1/|I|)$  if  $v \neq v'$  and  $(v, v') \in E$ , and  $u(s_v, s_{v'}) = u(s_{v'}, s_v) = 0$  otherwise. Now, suppose that  $\sigma'(s_v) > 0$  and  $\sigma'(s_{v'}) > 0$  for  $v \neq v'$  with  $(v, v') \notin E$ . We can write  $\sigma' = p_0\sigma'' + p_1s_v + p_2s_{v'}$ , where  $p_0, p_1 = \sigma'(s_v)$ , and  $p_2 = \sigma'(s_{v'})$  sum to 1. We have

$$u(\sigma', \sigma') = p_0^2 u(\sigma'', \sigma'') + 2p_0p_1 u(\sigma'', s_v) + 2p_0p_2 u(\sigma'', s_{v'})$$

(because  $u(s_v, s_v) = u(s_{v'}, s_{v'}) = u(s_v, s_{v'}) = 0$ ). Suppose, without loss of generality, that  $u(\sigma'', s_v) \geq u(\sigma'', s_{v'})$ . Then, if we shift all the mass from  $s_{v'}$  to  $s_v$  (so that the mass on the latter becomes  $p_1 + p_2$ ), this can only increase  $u(\sigma', \sigma')$ , and it reduces the size of the support of  $\sigma'$  by 1. By repeated application, we can assume without loss of generality that the support of  $\sigma'$  corresponds to a clique of the induced subgraph on  $\bigcup_{i \in I} V_{i,t(i)}$ . We know this clique has size  $c$  where  $c < k$ .  $u(\sigma', \sigma')$  is maximized if  $\sigma'$  randomizes uniformly over its support, in which case

$$u(\sigma', \sigma') = ((c-1)/c)(k/(k-1))(2 - 1/|I|) < ((k-1)/k)(k/(k-1))(2 - 1/|I|) = 2 - 1/|I|$$

But this contradicts that  $\sigma'$  would successfully invade  $\sigma$ . It follows that  $\sigma$  is indeed an ESS.

Conversely, suppose that there exists an ESS  $\sigma$  that places positive probability only on strategies  $s_{ij}$  with  $i \in I$  and  $j \in J$ . We must have  $u(\sigma, \sigma) \geq 2 - 1/|I|$ , because otherwise  $s_0$  would be a better response to  $\sigma$ . First suppose that for every  $i \in I$ , there is at most one  $j \in J$  such that  $\sigma$  places positive probability on  $s_{ij}$  (we will shortly show that this must be the case). Let  $t(i)$  denote the  $j \in J$  such that  $\sigma(s_{ij}) > 0$  (if there is no such  $j$  for some  $i$ , then choose an arbitrary  $j$  to equal  $t(i)$ ). Then,  $u(\sigma, \sigma)$  is uniquely maximized by setting  $\sigma(s_{i,t(i)}) = 1/|I|$  for all  $i \in I$ , resulting in

$$u(\sigma, \sigma) = (1/|I|) \cdot 1 + (1 - 1/|I|) \cdot 2 = 2 - 1/|I|$$

Hence, this is the only way to ensure that  $u(\sigma, \sigma) \geq 2 - 1/|I|$ , under the assumption that for every  $i \in I$ , there is at most one  $j \in J$  such that  $\sigma$  places positive probability on  $s_{ij}$ .

Now, let us consider the case where there exists an  $i \in I$  such that there exist  $j, j' \in J$  with  $j \neq j'$ ,  $\sigma(s_{ij}) > 0$ , and  $\sigma(s_{ij'}) > 0$ , to show that such a strategy cannot obtain a utility of  $2 - 1/|I|$  or more against itself. We can write  $\sigma = p_0\sigma' + p_1s_{ij} + p_2s_{ij'}$ , where  $\sigma'$  places probability zero on  $s_{ij}$  and  $s_{ij'}$ . We observe that  $u(\sigma', s_{ij}) = u(s_{ij}, \sigma')$  and  $u(\sigma', s_{ij'}) = u(s_{ij'}, \sigma')$ , because when the game is restricted to these strategies, each player always gets the same payoff as the other player. Moreover,  $u(\sigma', s_{ij}) = u(\sigma', s_{ij'})$ , because  $\sigma'$  does not place positive probability on either  $s_{ij}$  or  $s_{ij'}$ . Hence, we have that

$$u(\sigma, \sigma) = p_0^2 u(\sigma', \sigma') + 2p_0(p_1 + p_2)u(\sigma', s_{ij}) + p_1^2 + p_2^2$$

But then, if we shift all the mass from  $s_{ij'}$  to  $s_{ij}$  (so that the mass on the latter becomes  $p_1 + p_2$ ) to obtain strategy  $\sigma''$ , it follows that  $u(\sigma'', \sigma'') > u(\sigma, \sigma)$ . By repeated application, we can find a strategy  $\sigma'''$  such that  $u(\sigma''', \sigma''') > u(\sigma, \sigma)$  and for every  $i \in I$ , there is at most one  $j \in J$  such that  $\sigma'''$  places positive probability on  $s_{ij}$ . Because we showed previously that the latter type of strategy can obtain expected utility at most  $2 - 1/|I|$  against itself, it follows that it is in fact the *only* type of



strategy (among those that randomize only over the  $s_{ij}$  strategies) that can obtain expected utility  $2 - 1/|I|$  against itself. Hence, we can conclude that the ESS  $\sigma$  must have, for each  $i \in I$ , exactly one  $j \in J$  (to which we will refer as  $t(i)$ ) such that  $\sigma(s_{i,t(i)}) = 1/|I|$ , and that  $\sigma$  places probability 0 on every other strategy.

Finally, suppose, for the sake of contradiction, that there exists a clique of size  $k$  in the induced subgraph on  $\bigcup_{i \in I} V_{i,t(i)}$ . Consider the strategy  $\sigma'$  that places probability  $1/k$  on each of the corresponding strategies  $s_v$ . We have that  $u(\sigma, \sigma) = u(\sigma, \sigma') = u(\sigma', \sigma) = 2 - 1/|I|$ . Moreover,

$$u(\sigma', \sigma') = (1/k) \cdot 0 + ((k-1)/k) \cdot (k/(k-1))(2 - 1/|I|) = 2 - 1/|I|$$

It follows that  $\sigma'$  successfully invades  $\sigma$ —but this contradicts  $\sigma$  being an ESS. It follows, then, that  $t$  is such that every clique in the induced graph on  $\bigcup_{i \in I} V_{i,t(i)}$  has size strictly less than  $k$ . ■

## 5 Hardness without Restricted Support

All that remains is to reduce the modified problem to the main problem of determining whether a game has an ESS. The following lemma makes this fairly straightforward.

**Lemma 2 (No duplicates in ESS)** *Suppose that strategies  $s_1$  and  $s_2$  ( $s_1 \neq s_2$ ) are duplicates, i.e., for all  $s$ ,  $u(s_1, s) = u(s_2, s)$ .<sup>4</sup> Then no ESS places positive probability on  $s_1$  or  $s_2$ .*

**Proof:** For the sake of contradiction, suppose  $\sigma$  is an ESS that places positive probability on  $s_1$  or  $s_2$  (or both). Then, let  $\sigma' \neq \sigma$  be identical to  $\sigma$  with the exception that  $\sigma'(s_1) \neq \sigma(s_1)$  and  $\sigma'(s_2) \neq \sigma(s_2)$  (but it must be that  $\sigma'(s_1) + \sigma'(s_2) = \sigma(s_1) + \sigma(s_2)$ ). That is,  $\sigma'$  redistributes some mass between  $s_1$  and  $s_2$ . Then,  $\sigma$  cannot repel  $\sigma'$ , because  $u(\sigma, \sigma) = u(\sigma', \sigma)$  and  $u(\sigma, \sigma') = u(\sigma', \sigma')$ . ■

We now formally define the main problem:

**Definition 4** *In ESS, we are given a symmetric two-player normal-form game  $G$ . We are asked whether there exists an evolutionarily stable strategy of  $G$ .*

We now obtain the main result as follows.

**Theorem 1** *ESS is  $\Sigma_2^P$ -complete.*

**Proof:** Etessami and Lochbihler (2008) proved membership in  $\Sigma_2^P$ . We prove hardness by reduction from ESS-RESTRICTED-SUPPORT, which is hard by Lemma 1. Given the game  $G$  with strategies  $S$  and subset of strategies  $T \subseteq S$  that can receive positive probability, construct a modified game  $G'$  by duplicating all the strategies in  $S \setminus T$ . (At this point, for duplicate strategies  $s_1$  and  $s_2$ , we require  $u(s, s_1) = u(s, s_2)$  as well as  $u(s_1, s) = u(s_2, s)$ .) If  $G$  has an ESS  $\sigma$  that places positive probability only on strategies in  $T$ , this will still be an ESS in  $G'$ , because any strategy that uses the new duplicate strategies will still be repelled, just as its equivalent strategy that does not use the new duplicates was repelled in the original game. (Here, it should be noted that the equivalent strategy in the original game cannot turn out to be  $\sigma$ , because  $\sigma$  does not put any probability on a strategy that is duplicated.) On the other hand, if  $G'$  has an ESS, then by Lemma 2, this ESS can place positive probability only on strategies in  $T$ . This ESS will still be an ESS in  $G$  (all of whose strategies also exist in  $G'$ ), and naturally it will still place positive probability only on strategies in  $T$ . ■

<sup>4</sup>It is fine to require  $u(s, s_1) = u(s, s_2)$  as well, and we will do so in the proof of Theorem 1, but it is not necessary for this lemma to hold.

## A Hardness without duplication

In this appendix, it is shown that with some additional analysis and modifications, the result holds even in games where each pure strategy is the unique best response to some mixed strategy. That is, the hardness is not simply an artifact of the introduction of duplicate or otherwise redundant strategies.

**Definition 5** *In the MINMAX-CLIQUE problem, say vertex  $v$  dominates vertex  $v'$  if they are in the same partition element  $V_{ij}$ , there is no edge between them, and the set of neighbors of  $v$  is a superset (not necessarily strict) of the set of neighbors of  $v'$ .*

**Lemma 3** *Removing a dominated vertex does not change the answer to a MINMAX-CLIQUE instance.*

**Proof:** In any clique in which dominated vertex  $v'$  participates (and therefore its dominator  $v$  does not),  $v$  can participate in its stead. ■

**Modified Lemma 1** *ESS-RESTRICTED-SUPPORT is  $\Sigma_2^P$ -hard, even if every pure strategy is the unique best response to some mixed strategy.*

**Proof:** We use the same reduction as in the proof of Lemma 1. We restrict our attention to instances of the MINMAX-CLIQUE problem where  $|I| \geq 2$ ,  $|J| \geq 2$ , there are no dominated vertices, and every vertex is part of at least one edge. Clearly, the problem remains  $\Pi_2^P$ -complete when restricting attention to these instances. For the games resulting from these restricted instances, we show that every pure strategy is the unique best response to some mixed strategy. Specifically:

- $s_{ij}$  is the unique best response to the strategy that distributes  $1 - \epsilon$  mass uniformly over the  $s_{i'j'}$  with  $i' \neq i$ , and  $\epsilon$  mass uniformly over the  $s_{ij'}$  with  $j' \neq j$ . (This is because only pure strategies  $s_{ij'}$  will get a utility of 2 against the part with mass  $1 - \epsilon$ , and among these only  $s_{ij}$  will get a utility of 1 against the part with mass  $\epsilon$ .)
- $s_v$  (with  $v \in V_{ij}$ ) is the unique best response to the strategy that places  $(1/|I|)(1 - \epsilon)$  probability on  $s_{ij}$  and  $(1/(|I||J|))(1 - \epsilon)$  probability on every  $s_{i'j'}$  with  $i' \neq i$ , and that distributes the remaining  $\epsilon$  mass uniformly over the vertex strategies corresponding to neighbors of  $v$ . (This is because  $s_v$  obtains an expected utility of  $2 - 1/|I|$  against the part with mass  $1 - \epsilon$ , and an expected utility of  $(k/(k-1))(2 - 1/|I|)$  against the part with mass  $\epsilon$ ; strategies  $s_{v'}$  with  $v' \notin V_{ij}$  obtain utility strictly less than  $2 - 1/|I|$  against the part with mass  $1 - \epsilon$ ; and strategies  $s_{i''j''}$ ,  $s_0$ , and  $s_{v'}$  with  $v' \in V_{ij}$  obtain utility at most  $2 - 1/|I|$  against the part with mass  $1 - \epsilon$ , and an expected utility of strictly less than  $(k/(k-1))(2 - 1/|I|)$  against the part with mass  $\epsilon$ . (In the case of  $s_{v'}$  with  $v' \in V_{ij}$ , this is because by assumption,  $v'$  does not dominate  $v$ , so either  $v$  has a neighbor that  $v'$  does not have, which gets positive probability and against which  $s_{v'}$  gets a utility of 0; or, there is an edge between  $v$  and  $v'$ , so that  $s_{v'}$  gets positive probability and  $s_{v'}$  gets utility 0 against itself.))
- $s_0$  is the unique best response to the strategy that randomizes uniformly over all the  $s_{ij}$ . (This is because it obtains utility  $2 - 1/|I|$  against that strategy, and all the other pure strategies obtain utility strictly less against that strategy, due to getting utility 0 against at least one pure strategy in its support.)

■

The following lemma is a generalization of Lemma 2.

**Modified Lemma 2** *Suppose that subset  $S' \subseteq S$  satisfies:*

- *for all  $s \in S \setminus S'$  and  $s', s'' \in S'$ , we have  $u(s', s) = u(s'', s)$  (that is, strategies in  $S'$  are interchangeable when they face a strategy outside  $S'$ );<sup>5</sup> and*
- *the restricted game where players must choose from  $S'$  has no ESS.*

*Then no ESS of the full game places positive probability on any strategy in  $S'$ .*

**Proof:** Consider a strategy  $\sigma$  that places positive probability on  $S'$ . We can write  $\sigma = p_1\sigma_1 + p_2\sigma_2$ , where  $p_1 + p_2 = 1$ ,  $\sigma_1$  places positive probability only on  $S \setminus S'$ , and  $\sigma_2$  places positive probability only on  $S'$ . Because no ESS exists in the game restricted to  $S'$ , there must be a strategy  $\sigma'_2$  (with  $\sigma'_2 \neq \sigma_2$ ) whose support is contained in  $S'$  that successfully invades  $\sigma_2$ , so either (1)  $u(\sigma'_2, \sigma_2) > u(\sigma_2, \sigma_2)$  or (2)  $u(\sigma'_2, \sigma_2) = u(\sigma_2, \sigma_2)$  and  $u(\sigma'_2, \sigma'_2) \geq u(\sigma_2, \sigma'_2)$ . Now consider the strategy  $\sigma' = p_1\sigma_1 + p_2\sigma'_2$ ; we will show that it successfully invades  $\sigma$ . This is because

$$\begin{aligned} u(\sigma', \sigma) &= p_1^2 u(\sigma_1, \sigma_1) + p_1 p_2 u(\sigma_1, \sigma_2) + p_2 p_1 u(\sigma'_2, \sigma_1) + p_2^2 u(\sigma'_2, \sigma_2) \\ &= p_1^2 u(\sigma_1, \sigma_1) + p_1 p_2 u(\sigma_1, \sigma_2) + p_2 p_1 u(\sigma_2, \sigma_1) + p_2^2 u(\sigma'_2, \sigma_2) \\ &\geq p_1^2 u(\sigma_1, \sigma_1) + p_1 p_2 u(\sigma_1, \sigma_2) + p_2 p_1 u(\sigma_2, \sigma_1) + p_2^2 u(\sigma_2, \sigma_2) = u(\sigma, \sigma) \end{aligned}$$

where the second equality follows from the property assumed in the lemma. If case (1) above holds, then the inequality is strict and  $\sigma$  is not a best response against itself. If case (2) holds, then we have equality; moreover,

$$\begin{aligned} u(\sigma', \sigma') &= p_1^2 u(\sigma_1, \sigma_1) + p_1 p_2 u(\sigma_1, \sigma'_2) + p_2 p_1 u(\sigma'_2, \sigma_1) + p_2^2 u(\sigma'_2, \sigma'_2) \\ &= p_1^2 u(\sigma_1, \sigma_1) + p_1 p_2 u(\sigma_1, \sigma'_2) + p_2 p_1 u(\sigma_2, \sigma_1) + p_2^2 u(\sigma'_2, \sigma'_2) \\ &\geq p_1^2 u(\sigma_1, \sigma_1) + p_1 p_2 u(\sigma_1, \sigma'_2) + p_2 p_1 u(\sigma_2, \sigma_1) + p_2^2 u(\sigma_2, \sigma'_2) = u(\sigma, \sigma') \end{aligned}$$

where the second equality follows from the property assumed in the lemma. So in this case too,  $\sigma'$  successfully invades  $\sigma$ . ■

**Modified Theorem 1** *ESS is  $\Sigma_2^P$ -complete, even if every pure strategy is the unique best response to some mixed strategy.*

**Proof:** Again, Etessami and Lochbihler (2008) proved membership in  $\Sigma_2^P$ . For hardness, we use a similar proof strategy as in Theorem 1, again reducing from ESS-RESTRICTED-SUPPORT, which is hard even if every pure strategy is the unique best response to some mixed strategy, by Modified Lemma 1. Given the game  $G$  with strategies  $S$  and subset of strategies  $T \subseteq S$  that can receive positive probability, construct a modified game  $G'$  by replacing each pure strategy  $s \in S \setminus T$  by three new pure strategies,  $s^1, s^2, s^3$ . For each  $s' \notin \{s^1, s^2, s^3\}$ , we will have  $u(s^i, s') = u(s, s')$  (the utility of the original  $s$ ) and  $u(s', s^i) = u(s', s)$  for all  $i \in \{1, 2, 3\}$ ; for all  $i, j \in \{1, 2, 3\}$ , we will have  $u(s^i, s^j) = u(s, s) + \rho(i, j)$ , where  $\rho$  gives the payoffs of rock-paper-scissors (with  $-1$  for a loss,  $0$  for a tie, and  $1$  for a win).

If  $G$  has an ESS that places positive probabilities only on strategies in  $T$ , this will still be an ESS in  $G'$  because any strategy  $\sigma'$  that uses new strategies  $s^i$  will still be repelled, just as the corresponding strategy  $\sigma''$  that put the mass on the corresponding original strategies  $s$  (i.e.,  $\sigma''(s) = \sigma'(s^1) + \sigma'(s^2) + \sigma'(s^3)$ ) was repelled in the original game. (Unlike in the proof of the original

<sup>5</sup>Again, it is fine to require  $u(s, s') = u(s, s'')$  as well, and we will do so in the proof of Modified Theorem 1, but it is not necessary for the lemma to hold.

Theorem 1, here it is perhaps not immediately obvious that  $u(\sigma'', \sigma'') = u(\sigma', \sigma')$ , because the right-hand side involves additional terms involving  $\rho$ . But  $\rho$  is a symmetric zero-sum game, and any strategy results in an expected utility of 0 against itself in such a game.) On the other hand, if  $G'$  has an ESS, then by Modified Lemma 2 (letting  $S' = \{s^1, s^2, s^3\}$  and using the fact that rock-paper-scissors has no ESS), this ESS can place positive probability only on strategies in  $T$ . This ESS will still be an ESS in  $G$  (for any potentially invading strategy in  $G$  there would be an equivalent such strategy in  $G'$ , for example replacing  $s$  by  $s^1$  as needed), and naturally it will still place positive probability only on strategies in  $T$ .

Finally it remains to be shown that in  $G'$  each pure strategy is the unique best response to some mixed strategy, using the fact that this is the case for  $G$ . For a pure strategy in  $T$ , we can simply use the same mixed strategy as we use for that pure strategy in  $G$ , replacing mass placed on each  $s \notin T$  in  $G$  with a uniform mixture over  $s^1, s^2, s^3$  where needed. (By using a uniform mixture, we guarantee that each  $s^i$  obtains the same expected utility against the mixed strategy as the corresponding  $s$  strategy in  $G$ .) For a pure strategy  $s^i \notin T$ , we cannot simply use the same mixed strategy as we use for the corresponding  $s$  in  $G$  (with the same uniform mixture trick), because  $s^1, s^2, s^3$  would all be equally good responses. But because these three would be the *only* best responses, we can mix in a sufficiently small amount of  $s^{i+1 \pmod{3}}$  (where  $i$  beats  $i+1 \pmod{3}$  in  $\rho$ ) to make  $s^i$  the unique best response. ■

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## References

- Xi Chen, Xiaotie Deng, and Shang-Hua Teng. Settling the complexity of computing two-player Nash equilibria. *Journal of the ACM*, 56(3), 2009.
- Vincent Conitzer and Tuomas Sandholm. New complexity results about Nash equilibria. *Games and Economic Behavior*, 63(2):621–641, 2008.
- Constantinos Daskalakis, Paul Goldberg, and Christos H. Papadimitriou. The complexity of computing a Nash equilibrium. *SIAM Journal on Computing*, 39(1):195–259, 2009.
- Kousha Etessami and Andreas Lochbihler. The computational complexity of evolutionarily stable strategies. *International Journal of Game Theory*, 37(1):93–113, 2008. An earlier version was made available as ECCC tech report TR04-055, 2004.
- Kousha Etessami and Mihalis Yannakakis. On the complexity of Nash equilibria and other fixed points. *SIAM Journal on Computing*, 39(6):2531–2597, 2010.
- Itzhak Gilboa and Eitan Zemel. Nash and correlated equilibria: Some complexity considerations. *Games and Economic Behavior*, 1:80–93, 1989.
- Ker-I Ko and Chih-Long Lin. On the complexity of min-max optimization problems and their approximation. In D.-Z. Du and P. M. Pardalos, editors, *Minimax and Applications*, pages 219–239. Kluwer Academic Publishers, Boston, 1995.

- Carlton Lemke and Joseph Howson. Equilibrium points of bimatrix games. *Journal of the Society of Industrial and Applied Mathematics*, 12:413–423, 1964.
- John Nash. Equilibrium points in n-person games. *Proceedings of the National Academy of Sciences*, 36:48–49, 1950.
- Noam Nisan. A note on the computational hardness of evolutionary stable strategies. *Electronic Colloquium on Computational Complexity (ECCC)*, 13(076), 2006.
- Christos H. Papadimitriou. On the complexity of the parity argument and other inefficient proofs of existence. *Journal of Computer and System Sciences*, 48(3):498–532, 1994.
- Christos H. Papadimitriou. Algorithms, games and the Internet. In *Proceedings of the Annual Symposium on Theory of Computing (STOC)*, pages 749–753, 2001.
- George Price and John Maynard Smith. The logic of animal conflict. *Nature*, 246:15–18, 1973.
- Rahul Savani and Bernhard von Stengel. Hard-to-solve bimatrix games. *Econometrica*, 74:397–429, 2006.
- Marcus Schaefer and Christopher Umans. Completeness in the polynomial-time hierarchy: A compendium, 2008.
- Siddharth Suri. Computational evolutionary game theory. In Noam Nisan, Tim Roughgarden, Eva Tardos, and Vijay Vazirani, editors, *Algorithmic Game Theory*, chapter 29. Cambridge University Press, 2007.