# Unfolding Partiality and Disjunctions in Stable Model Semantics<sup>\*</sup>

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#### Abstract

The paper studies an implementation methodology for partial and disjunctive stable models where partiality and disjunctions are unfolded from a logic program so that an implementation of stable models for normal (disjunction-free) programs can be used as the core inference engine. The unfolding is done in two separate steps. Firstly, it is shown that partial stable models can be captured by total stable models using a simple linear and modular program transformation. Hence, reasoning tasks concerning partial stable models can be solved using an implementation of total stable models. Disjunctive partial stable models have been lacking implementations which now become available as the translation handles also the disjunctive case. Secondly, it is shown how total stable models of disjunctive programs can be determined by computing stable models for normal programs. Hence, an implementation of stable models of normal programs can be used as a core engine for implementing disjunctive programs. The feasibility of the approach is demonstrated by constructing a system for computing stable models of disjunctive programs using the SMODELS system as the core engine. The performance of the resulting system is compared to that of DLV which is a state-of-the-art system for disjunctive programs.

## **1** INTRODUCTION

Implementation techniques for declarative semantics of logic programs have advanced considerably during the last years. For example, the XSB system [40] is a WAM-based full logic programming system supporting the wellfounded semantics. In addition to this kind of a skeptical approach that is based on query evaluation also a credulous approach focusing on computing models of logic programs is gaining popularity. This work has been centered around the stable model semantics [17, 18]. There are reasonably efficient implementations available for computing stable models for disjunctive and normal (disjunction-free) programs, e.g., DLV [24], SMODELS [46, 45], CMOD-ELS [1], and ASSAT [29]. The implementations have provided a basis for a new paradigm for logic programming called *answer set programming* (a term coined by Vladimir Lifschitz). The basic idea is that a problem is solved by devising a logic program such that the stable models of the program provide the answers to the problem, i.e., solving the problem is reduced to a stable model computation task [27, 32, 34, 13, 5]. This approach has led to interesting applications in areas such as planning [8, 11, 2], model checking [30, 20], and software configuration [49].

This paper addresses two issues in the stable model semantics: partiality and disjunctions. The idea is to develop methodology such that efficient procedures for computing (total) stable models that are emerging can be exploited when dealing with partial stable models and disjunctive programs. Sometimes it is natural to use partial stable models to represent a domain. Even when working with total stable models, partial stable models could be useful, e.g., for debugging purposes to show what is wrong in a program without any total stable models. However, little has been done on implementing the computation of partial stable models and most of the work has focused on query evaluation w.r.t. the well-founded semantics. In the paper we show that total stable models can capture partial stable models using a simple linear program transformation. This transformation works also in the disjunctive case showing that implementations of total stable models, e.g. DLV, can be used for computing partial stable models. Using a suitable transformation of queries, a mechanism for query answering can be realized as well.

Our translation is interesting in many respects. First, it should be noted that the translation does not follow directly from the complexity results already available. It has been shown, e.g., that the problem of deciding whether a query is contained in some model (possibility inference) is  $\Sigma_2^p$ -complete for both partial and total stable models [12, 15]. This implies that there exists a polynomial time reduction from possibility inference w.r.t. partial models to possibility inference w.r.t. total models. However, this kind of a translation is guaranteed to preserve only the yes/no answer to the possibility inference problem. Second, not all translations are satisfactory from a computational point of view. In practice, when a program is compiled into another form to be executed, certain computational properties of the translation play an important role:

- efficiency of the compilation (in which order of polynomial),
- modularity (are independent, separate compilations of parts of a program possible), and
- structural preservation (are the composition and intuition of the original program preserved so that debugging and understanding of runtime

#### behavior are made possible).

All this points to the importance of finding good translation methods to enable the use of an existing inference engine to solve other interesting problems.

The efficiency of procedures for computing stable models of normal programs has increased substantially in recent years. An interesting possibility to exploit the computational power of such a procedure is to use it as a *core engine* for implementing other reasoning systems. In this paper, we follow this approach and develop a method for reducing stable model computation of disjunctive programs to the problem of determining stable models for normal programs. This is non-trivial as deciding whether a disjunctive program has a stable model is  $\Sigma_2^p$ -complete [12] whereas the problem is NP-complete in the non-disjunctive case [31]. The method has been implemented using the SMODELS system [46, 45] as the core engine. The performance of the implementation is compared to that of DLV, which is a state-of-the-art system for computing stable models for disjunctive programs.

There are a number of novelties in the work. Maximal partial stable models for normal programs are known as regular models, M-stable models, and preferred extensions [10, 39, 50]. Although this semantics has a sound and complete top-down query answering procedure [10, 16, 28], so far very little effort has been given to a serious implementation. For disjunctive programs, to our knowledge, no implementation has ever been attempted. As a result, we obtain (perhaps) the first scalable implementation of the regular model/preferred extension semantics, and the first implementation ever for partial stable model semantics for disjunctive programs. Our technical work on the relationship between stable and partial stable models via a translational approach provides a compelling argument for the naturalness of partial stable models: stable models and partial stable models share the same notion of unfoundedness, carefully studied earlier in [14, 26]. Finally, we demonstrate how key tasks in computing disjunctive stable models can be reduced to stable model computation for normal programs by suitable program transformations. In particular, we develop techniques for mapping a disjunctive program into a normal one such that the set of stable models of the normal program covers the set of stable models of the disjunctive one and in many case even coincides with it. Moreover, we devise a method where the stability of a model candidate for a disjunctive program can be determined by transforming the disjunctive program into a normal one and checking the existence of a stable model for it. Finally, in the experimental part of this paper, we present a new way of encoding quantified Boolean formulas as disjunctive logic programs. This transformation is more economical in the number of propositional atoms and disjunctive rules than earlier transformations presented in the literature [12, 25].

The rest of the paper is structured as follows. We first review the basic definitions and concepts in Section 2. It is then shown in Section 3 that partial stable models can be captured with total stable models using a simple program transformation. In Section 4, we describe the method for computing disjunctive stable models using an implementation of non-disjunctive programs as a core engine. After this, we present some experimental results in Section 5 and finish with concluding remarks in Section 6.

As a comment on the historical development of the translation given in Section 3, the characterization of partial stable models as stable models of the transformed program was first sketched for normal programs in a proof by Schlipf [42, Theorem 3.2]. For disjunctive programs, it was discovered and proven in [43], and independently in [21]. In the current paper we present a proof based on unfounded sets, which was given in [21], as this proof reveals some of the properties of unfounded sets which are of interest in their own right. Yet another approach to computing the partial stable models of a disjunctive program based on a program transformation has been developed by Ruiz and Minker [37]: a disjunctive program P is translated into a positive disjunctive program  $P^{3S}$  with constraints, the 3S-transformation of P, such that the total minimal models of  $P^{3S}$  that additionally fulfill the constraints coincide with the partial stable models of P.

## 2 DEFINITIONS AND NOTATIONS

A disjunctive logic program P (or, just disjunctive program P) is a set of rules of the form

$$a_1 \lor \cdots \lor a_k \leftarrow b_1, \dots, b_m, \sim c_1, \dots, \sim c_n$$
 (1)

where  $k \ge 1$ ,  $m, n \ge 0$  and  $a_i$ 's,  $b_i$ 's and  $c_i$ 's are atoms from the Herbrand base  $Hb(P)^1$  of P. Let us also distinguish subclasses of disjunctive programs.

<sup>&</sup>lt;sup>1</sup>For the sake of convenience, we assume that a given program P is already instantiated by the underlying Herbrand universe, and is thus ground.

If k = 1 for each rule of P, then P is a disjunction-free or normal program. If n = 0 for each rule of P, then P is called *positive*.

Literals are either atoms from Hb(P) or expressions of the form  $\sim a$  where  $a \in \text{Hb}(P)$ . For a set of atoms  $A \subseteq \text{Hb}(P)$ , we define  $\sim A$  as  $\{\sim a \mid a \in A\}$ . Let us introduce a shorthand  $A \leftarrow B, \sim C$  for rules where  $A \neq \emptyset$ , B and C are subsets of Hb(P). In harmony with (1), the set of atoms A in the head of the rule is interpreted disjunctively while the set of literals  $B \cup \sim C$  in the body of the rule is interpreted conjunctively. We wish to further simplify the notation  $A \leftarrow B, \sim C$  in some particular cases. When A, B or C is a singleton  $\{a\}$ , we write a instead of  $\{a\}$ . If  $B = \emptyset$  or  $C = \emptyset$  we omit B and  $\sim C$  (respectively) as well as the separating comma in the body of the rule.

#### 2.1 PARTIAL AND TOTAL MODELS

We review the basic model-theoretic concepts by following the presentation in [15]. Let P be any disjunctive program. A partial interpretation I for P is a pair  $\langle T, F \rangle$  of subsets of Hb(P) such that  $T \cap F = \emptyset$ . The atoms in the sets  $I^{\mathbf{t}} = T$ ,  $I^{\mathbf{f}} = F$  and  $I^{\mathbf{u}} = \text{Hb}(P) - (T \cup F)$  are considered to be true, false, and undefined, respectively. We introduce constants  $\mathbf{t}$ ,  $\mathbf{f}$ , and  $\mathbf{u}$ , to denote the respective three truth values. A partial interpretation I for P is a total interpretation for P whenever  $I^{\mathbf{u}} = \emptyset$ , i.e., if every atom of Hb(P) is either true or false. When no confusion arises, we use  $I^{\mathbf{t}}$  alone to specify a total interpretation I for P (then  $I^{\mathbf{f}} = \text{Hb}(P) - I^{\mathbf{t}}$  and  $I^{\mathbf{u}} = \emptyset$  hold).

Given a partial interpretation for P, the truth values of atoms are determined by  $I^{\mathbf{t}}$ ,  $I^{\mathbf{f}}$  and  $I^{\mathbf{u}}$  as explained above while  $\mathbf{t}$ ,  $\mathbf{f}$  and  $\mathbf{u}$  have their fixed truth values. For more complex logical expressions E, we use I(E) to denote the truth value of E in I. The value  $I(\sim a)$  is defined to be  $\mathbf{t}$ ,  $\mathbf{f}$ , or  $\mathbf{u}$  whenever I(a) is  $\mathbf{f}$ ,  $\mathbf{t}$ , or  $\mathbf{u}$ , respectively. To handle conjunctions and disjunctions, we introduce an ordering on the three truth values by setting  $\mathbf{f} < \mathbf{u} < \mathbf{t}$ . By default, a set of literals  $L = \{l_1, \ldots, l_n\}$  denotes the conjunction  $l_1 \wedge \cdots \wedge l_n$  while  $\bigvee L$  denotes the corresponding disjunction  $l_1 \vee \cdots \vee l_n$ . The truth values I(L) and  $I(\bigvee L)$  are defined as the respective minimum and maximum among the truth values  $I(l_1), \ldots, I(l_n)$ . A rule  $A \leftarrow B, \sim C$  is satisfied in I if and only if  $I(\bigvee A) \geq I(B \cup \sim C)$ . A partial interpretation M for P is a partial model of P if all rules of P are satisfied in M, and for a total model, also  $M^{\mathbf{u}} = \emptyset$  holds. Let us then introduce an ordering among partial models of a disjunctive program:  $M_1 \leq M_2$  if and only if  $M_1^{\mathbf{t}} \subseteq M_2^{\mathbf{t}}$  and  $M_1^{\mathbf{f}} \supseteq M_2^{\mathbf{f}}$ . A partial model M of P is a minimal one if there is no

partial model M' of P such that M' < M (i.e.,  $M' \leq M$  and  $M' \neq M$ ). In case of total models, we have  $N_1 \leq N_2$  if and only if  $N_1 \subseteq N_2$ . Moreover, a total model N of P is considered to be a *minimal* one if there is no total model N' of P such that  $N' \subset N$ .

#### 2.2 STABLE MODELS

Given a partial interpretation I for a disjunctive program P, we define a reduction of P as follows:

$$P^{I} = \{ A \leftarrow B \mid A \leftarrow B, \sim C \in P \text{ and } C \subseteq I^{\mathbf{f}} \}.$$

Note that this transformation coincides with the *Gelfond-Lifschitz reduction* of P (the GL-reduction of P) when I is a total interpretation.

**Definition 2.1 (Total stable model)** A total interpretation N for a disjunctive program P is a stable model if and only if N is a minimal total model of  $P^N$ .

The original definition of partial stable models [35, 36] is based on a weaker reduction. Given a disjunctive program P and an interpretation I, the reduction  $P_I$  is the set of rules obtained from P by replacing any  $\sim c$  in the body of a rule by  $I(\sim c)$ . As noted in [35], the only practical difference between  $P^I$  and  $P_I$  is that  $P_I$  has rules that correspond to rules of  $A \leftarrow B, \sim C \in P$  satisfying  $I(\sim C) = \mathbf{u}$ . Note that if  $I(\sim C) = \mathbf{t}$ , then  $A \leftarrow B \in P^I$ , and if  $I(\sim C) = \mathbf{f}$ , then the partial models of  $P_I$  are not constrained by the rule included in  $P_I$ .

**Definition 2.2 (Partial stable model)** A partial interpretation M for a disjunctive program P is a partial stable model of P if and only if M is a minimal partial model of  $P_M$ .

In the above definition, the relation between M and  $P_M$  is similar to the one for the total stable model, both for the purpose of preserving the stability condition. While maximizing falsity and minimizing true atoms, a partial stable model does not insist that every atom must be either true or false. (Partial) stable models are intimately related to unfounded sets [14, 26].

**Definition 2.3 (Unfounded sets)** Let I be a partial interpretation for a disjunctive program P. A set  $U \subseteq Hb(P)$  of ground atoms is an unfounded

set for P w.r.t. I, if at least one of the following conditions holds for each rule  $A \leftarrow B, \sim C \in P$  such that  $A \cap U \neq \emptyset$ :

UF1:  $B \cap I^{\mathbf{f}} \neq \emptyset$  or  $C \cap I^{\mathbf{t}} \neq \emptyset$ ,

UF2:  $B \cap U \neq \emptyset$ , or

UF3:  $(A - U) \cap (I^{\mathbf{t}} \cup I^{\mathbf{u}}) \neq \emptyset$ .

An unfounded set U for P w.r.t. I is I-consistent if and only if  $U \cap I^{\mathbf{t}} = \emptyset$ .

The conditions UF1 and UF3 above coincide with the conditions

 $I(B \cup \sim C) = \mathbf{f}$  and  $I(\bigvee (A - U)) \neq \mathbf{f}$ ,

respectively. The intuition is that the atoms of an unfounded set U can be assumed to be false without violating the satisfiability of any rule  $A \leftarrow B, \sim C$ of the program whose head contains some atoms of U. For any such rule, either the rule body is false in I (UF1), or the rule body can be falsified by falsifying the atoms in U (UF2), or the head of the rule is not false in I(UF3). In particular, unfounded sets w.r.t. partial/total models can be used for constructing smaller partial/total models (recall the definition of minimal partial and total models) in a way that is made precise by what follows.

**Lemma 2.4** Let  $M = \langle T, F \rangle$  be a partial model of a positive disjunctive program P and U an unfounded set for P w.r.t. M. Then, if M is total or U is M-consistent,  $M' = \langle T - U, F \cup U \rangle$  is a partial model of P.

**PROOF.** Let M, P, U and M' be defined as above. Additionally, we assume that (a) M is total, or (b) U is M-consistent. Let us then assume that some rule  $A \leftarrow B$  of P is not satisfied in M' which means that  $M'(\bigvee A) < M'(B)$ . Thus (i)  $M'(\bigvee A) < \mathbf{t}$  and  $M'(B) = \mathbf{t}$ , or (ii)  $M'(\bigvee A) = \mathbf{f}$  and  $M'(B) = \mathbf{u}$ . Our proof splits in two separate threads.

**I.** Assume that  $A \cap U = \emptyset$  holds. Consider the case (i). Now  $M'(\bigvee A) < \mathbf{t}$ implies  $A \cap (T - U) = \emptyset$ . Since  $A \cap U = \emptyset$  holds, too, we obtain  $A \cap T = \emptyset$  so that  $M(\bigvee A) < \mathbf{t}$ . On the other hand,  $M'(B) = \mathbf{t}$ implies  $B \subseteq T - U$ . Thus  $B \subseteq T$  and  $M(B) = \mathbf{t}$  holds as well. But then  $M(\bigvee A) < M(B)$ , a contradiction. The case (ii) is analyzed next. Now  $M'(\bigvee A) = \mathbf{f}$  implies  $A \subseteq F \cup U$  as well as  $A \subseteq F$ , since  $A \cap U = \emptyset$ . Thus  $M(\bigvee A) = \mathbf{f}$ . Moreover, from  $M'(B) = \mathbf{u}$  we obtain  $B \cap (F \cup U) = \emptyset$ . Thus we obtain  $B \cap F \neq \emptyset$  so that  $M(B) > \mathbf{f}$  holds. To conclude, we have established that  $M(\bigvee A) < M(B)$ , a contradiction. **II.** Otherwise  $A \cap U \neq \emptyset$  holds. Then at least one of the unfoundedness conditions is applicable to  $A \leftarrow B$ . If UF1 is,  $M(B) = \mathbf{f}$  holds. It follows that  $B \subseteq F$  and  $B \subseteq F \cup U$ . Thus  $M'(B) = \mathbf{f}$  contradicting both (i) and (ii). If UF2 is applicable, we have  $B \cap U \neq \emptyset$ . It follows that  $B \cap (F \cup U) \neq \emptyset$  so that  $M'(B) = \mathbf{f}$ , a contradiction.

Thus UF3 must apply, i.e.,  $M(\bigvee (A - U)) > \mathbf{f}$  holds. Let us then consider cases (a) and (b) separately.

- (a) If M is total, we have necessarily  $M(\bigvee (A U)) = \mathbf{t}$ . This implies that some atom  $a \in A - U$  belongs to T. Thus also  $a \in T - U$ and  $M'(\bigvee A) = \mathbf{t}$ , a contradiction with both (i) and (ii).
- (b) If U is M-consistent, we have  $U \cap T = \emptyset$ . By  $M(\bigvee (A U)) = \mathbf{t}$ there is an atom  $a \in A - U$  such that  $a \notin F$ . Then  $a \notin F \cup U$ which implies  $M'(\bigvee A) > \mathbf{f}$ . Thus (ii) is impossible and (i) implies  $M'(\bigvee A) = \mathbf{u}$  and  $M'(B) = \mathbf{t}$ . It follows that  $A \cap (T - U) = \emptyset$ and  $B \subseteq T - U$ . Since  $U \cap T = \emptyset$ , the former implies  $A \cap T = \emptyset$ while the latter implies that  $B \subseteq T$ . Consequently,  $M(\bigvee A) < \mathbf{t}$ and  $M(B) = \mathbf{t}$ , i.e.,  $M(\bigvee A) < M(B)$ , a contradiction.

Let us yet emphasize the content of Lemma 2.4 when M is total (and U need not be M-consistent). Then M' = M - U is also a total model of P. A couple of examples on unfounded sets follow.

**Example 2.5** Consider a disjunctive program

$$P = \{a \lor b \leftarrow c, \sim a\}$$

and an interpretation  $I = \langle \emptyset, \{a\} \rangle$ . The only rule in P has its body undefined in I, hence UF1 is not applicable. The set  $\{a\}$  is unfounded w.r.t. I since bis undefined in I and not in the set, hence UF3 is applicable. On the other hand, the set  $\{b\}$  is not unfounded w.r.t. I whereas  $\{c\}$  is unfounded w.r.t. I. Once c belongs to an unfounded set, the atoms a and b can both get in due to UF2. Hence, we have  $U = \{a, b, c\}$  as an unfounded set w.r.t. I.

Comparing U with  $I^{\mathbf{f}}$ , we find that  $I^{\mathbf{f}}$  does not maximize the atoms that should be false. This program P has exactly one stable model (which is also a partial stable model) in which all three atoms are false.

Unlike the case for normal programs, the union of unfounded sets may not be an unfounded set.

**Example 2.6** Consider a program *P* containing only one rule

 $a \lor b \leftarrow$ 

and an interpretation  $I = \langle \{a, b\}, \emptyset \rangle$ . The program has two non-empty unfounded sets w.r.t. I,  $\{a\}$  and  $\{b\}$ . Either a or b depends on the other one not in the set for UF3 to be applicable. However, UF3 becomes not applicable when both a and b are in, thus the union  $\{a, b\}$  is not an unfounded set.  $\Box$ 

An interpretation I for P becomes particularly interesting when the union of all unfounded sets U for P w.r.t. I is also an unfounded set for P w.r.t. I. In this case, the program P possesses the greatest unfounded set U for Pw.r.t. I.

**Definition 2.7** A total interpretation I is said to be unfounded free for a program P if and only if there is no unfounded set U for P w.r.t. I such that  $U \cap I^{t} \neq \emptyset$ .

The notion of unfounded freeness captures the stable model beautifully.

**Theorem 2.8 [26]** Let M be a total interpretation for a disjunctive program P. Then, the following are equivalent

- M is a stable model of P.
- $M^{\mathbf{f}}$  is the greatest unfounded set for P w.r.t. M.
- *M* is unfounded free for *P*.

On the other hand, Eiter et al. [14] show that partial stable models can be defined essentially without reference to three-valued logic.

**Theorem 2.9 [14]** If M is a partial interpretation for a disjunctive program P, then M is a partial stable model of P if and only if

- $M^{\mathbf{t}}$  is a minimal total model of  $P^{M}$  and
- $M^{\mathbf{f}}$  is a maximal M-consistent unfounded set for P w.r.t. M.

The first condition in the theorem is called *foundedness* in [14]. The differences between these two theorems are quite subtle. The strictness of stable models enforces a simpler relationship between stable models and unfounded sets. Therefore, neither maximality nor consistency nor foundedness need be explicitly stated. The characterization of partial stable models in Theorem 2.9 accounts for a more reflexible situation: since M may not be a total model, maximality should extend the set of false atoms as much as possible without causing inconsistency. However, maximality and consistency are still not strong enough.

**Example 2.10** Consider a disjunctive program

$$P = \{a \lor b \leftarrow \sim a\}$$

and an interpretation  $I = \langle \{a\}, \{b\} \rangle$  which is total so that the definition of unfoundedness makes no difference in valuation under I. Since the body of the rule is false in  $I, U_1 = \{a\}, U_2 = \{b\}$ , and  $U_3 = \{a, b\}$  are all nonempty unfounded sets in this case. It follows immediately by Theorem 2.9 that Iis not a stable model. However,  $U_2$  is maximally I-consistent yet it is not a partial stable model because  $I^t$  is not a minimal model of  $P^I$ .

Note that P has a unique (partial) stable model, which is  $\langle \{b\}, \{a\} \rangle$ .  $\Box$ 

The characterizations for partial and total stable models in terms of unfounded sets provide a powerful tool for establishing relationships between stable models and partial stable models.

## **3 UNFOLDING PARTIALITY**

In this section, we first show a translation for a disjunctive program into another disjunctive program. We then prove that the translation preserves the semantics of partial stable models. This result allows us to compute the partial stable models of a program by computing the stable models of the translated program. Finally we address the problem of query answering under the translation.

#### 3.1 TRANSLATION

Let P be a disjunctive program. In the following, we describe a translation of P into another disjunctive program Tr(P) such that the stable models of Tr(P) correspond to the partial stable models of P.

Let us introduce a new atom  $a^{\bullet}$  for each  $a \in \operatorname{Hb}(P)$ . An atom  $a^{\bullet}$  is said to be *marked*, and an ordinary atom a is then said to be *unmarked*. The intuitive reading of  $a^{\bullet}$  is that a is *potentially* true. For a set of literals  $L \subseteq \operatorname{Hb}(P) \cup \sim \operatorname{Hb}(P)$ , we define  $L^{\bullet} = \{a^{\bullet} | a \in L\} \cup \{\sim a^{\bullet} | \sim a \in L\}$ . The translation  $\operatorname{Tr}(P)$  of a disjunctive program P is as follows:

$$\operatorname{Tr}(P) = \{A \leftarrow B, \sim C^{\bullet}; A^{\bullet} \leftarrow B^{\bullet}, \sim C \mid A \leftarrow B, \sim C \in P\} \cup \{a^{\bullet} \leftarrow a \mid a \in \operatorname{Hb}(P)\} (2)$$

where semicolons are used to separate program rules. Note that the Herbrand base of  $\operatorname{Hb}(\operatorname{Tr}(P))$  is  $\operatorname{Hb}(P) \cup \operatorname{Hb}(P)^{\bullet}$ . The rules  $a^{\bullet} \leftarrow a$  introduced for each  $a \in \operatorname{Hb}(P)$  enforce consistency in the sense that if a is true, then a must also be potentially true.

**Remark 3.1** Although for presentational purposes the translation is defined for ground programs, exactly the same translation applies to non-ground programs as well: for each predicate p we introduce a new predicate  $p^{\bullet}$ , hence for a (ground or non-ground) atom  $\phi = p(t_1, \ldots, t_n)$ , the new atom is  $\phi^{\bullet} = p^{\bullet}(t_1, \ldots, t_n)$  (cf. Example 3.5). Since our proofs do not depend on the assumption that a given program is finite, the conclusions reached cover also any non-ground program with function symbols whose semantics is determined by treating the program as a shorthand for its (possibly infinite) Herbrand instantation.

A partial stable model of a given program will be interpreted by a corresponding stable model of the transformed program. The extra symbol  $a^{\bullet}$  for each atom a provides an opportunity to represent undefined (in three-valued logic) in terms of truth values of  $a^{\bullet}$  and a in two-valued logic. For each pair aand  $a^{\bullet}$ , either of which can be true or false, there are four possibilities: when  $a^{\bullet}$  and a are in agreement, that is when they are both true or both false, the truth value of a is their commonly agreed truth value; the combination where a is false and  $a^{\bullet}$  is true then represents that a is undefined; and the fourth possibility where a is true and  $a^{\bullet}$  is false is ruled out by any models due to the consistency rules. This intended representation of a partial stable model is given by the following equations.

**Definition 3.2** Let M be a partial interpretation of a program P and N a total interpretation of Tr(P). The interpretations M and N are said to

satisfy the correspondence equations if and only if the following equations hold.

$$M^{\mathbf{t}} = \{ a \in \operatorname{Hb}(P) \, | \, a \in N^{\mathbf{t}} \text{ and } a^{\bullet} \in N^{\mathbf{t}} \}$$
(CE1)

$$M^{\mathbf{f}} = \{ a \in \operatorname{Hb}(P) \, | \, a \in N^{\mathbf{f}} \text{ and } a^{\bullet} \in N^{\mathbf{f}} \}$$
(CE2)

$$M^{\mathbf{u}} = \{ a \in \operatorname{Hb}(P) \mid a \in N^{\mathbf{f}} and a^{\bullet} \in N^{\mathbf{t}} \}$$
(CE3)

$$\emptyset = \{ a \in \operatorname{Hb}(P) \mid a \in N^{\mathsf{t}} \text{ and } a^{\bullet} \in N^{\mathsf{f}} \}$$
(CE4)

Note that total interpretations that are models of  $\operatorname{Tr}(P)$  satisfy CE4 immediately, since the set of rules  $\{a^{\bullet} \leftarrow a \mid a \in \operatorname{Hb}(P)\}$  is included in  $\operatorname{Tr}(P)$ . Consequently, the "fourth truth value" is ruled out. The following example demonstrates how the representation given in Definition 3.2 allows us to capture the partial stable models of a disjunctive program P with the total stable models of  $\operatorname{Tr}(P)$ .

**Example 3.3** Consider a disjunctive program

$$P = \{ a \lor b \leftarrow \sim c; \ b \leftarrow \sim b; \ c \leftarrow \sim c \}.$$

Now a becomes false by the minimization of partial models, since the falsity of a does not affect the satisfiability of any rule. Thus the unique partial stable model of P is  $M = \langle \emptyset, \{a\} \rangle$ . Note that the reduction  $P_M = \{a \lor b \leftarrow \mathbf{u}; b \leftarrow \mathbf{u}; c \leftarrow \mathbf{u}\}$ . Then consider the translation

$$\operatorname{Tr}(P) = \{ a \lor b \leftarrow \sim c^{\bullet}; \quad b \leftarrow \sim b^{\bullet}; \quad c \leftarrow \sim c^{\bullet}; \\ a^{\bullet} \lor b^{\bullet} \leftarrow \sim c; \quad b^{\bullet} \leftarrow \sim b; \quad c^{\bullet} \leftarrow \sim c; \\ a^{\bullet} \leftarrow a; \quad b^{\bullet} \leftarrow b; \quad c^{\bullet} \leftarrow c \} \}.$$

The unique stable model of Tr(P) is  $N = \{b^{\bullet}, c^{\bullet}\}$  which represents (by CE2 and CE3) the setting that b and c are undefined and a is false in M.  $\Box$ 

It is well-known that a disjunctive program P may not have any partial stable models. In such cases, the translation Tr(P) should not have stable models either, if the translation Tr(P) is to be faithful.

Example 3.4 Consider a disjunctive program

$$P = \{a \lor b \lor c \leftarrow; a \leftarrow \sim b; b \leftarrow \sim c; c \leftarrow \sim a\}$$

and its translation

$$\operatorname{Tr}(P) = \{ a \lor b \lor c \leftarrow; \qquad a \leftarrow \sim b^{\bullet}; \quad b \leftarrow \sim c^{\bullet}; \quad c \leftarrow \sim a^{\bullet}; \\ a^{\bullet} \lor b^{\bullet} \lor c^{\bullet} \leftarrow; \quad a^{\bullet} \leftarrow \sim b; \quad b^{\bullet} \leftarrow \sim c; \quad c^{\bullet} \leftarrow \sim a \quad \} \cup C$$

where  $C = \{a^{\bullet} \leftarrow a; b^{\bullet} \leftarrow b; c^{\bullet} \leftarrow c\}$  is the set of consistency rules.

Consider a partial model  $M = \langle \{a, b\}, \emptyset \rangle$  of P and a total model  $N = \{a, a^{\bullet}, b, b^{\bullet}, c^{\bullet}\}$  of Tr(P) that satisfy the equations CE1–CE4 in Definition 3.2. Now the reduced program  $P_M$  is

$$\{a \lor b \lor c \leftarrow; a \leftarrow \mathbf{f}; b \leftarrow \mathbf{u}; c \leftarrow \mathbf{f}\}$$

and since  $M' = \langle \{a, b\}, \{c\} \rangle < M$  is a partial model of  $P_M$ , M is not a partial stable model of P. On the other hand, the reduct

$$\operatorname{Tr}(P)^{N} = \{ a \lor b \lor c \leftarrow; a^{\bullet} \lor b^{\bullet} \lor c^{\bullet} \leftarrow; b^{\bullet} \leftarrow \} \cup C.$$

But  $N' = \{a, a^{\bullet}, b, b^{\bullet}\} \subset N$  is a model of  $\operatorname{Tr}(P)^N$  so N is not a stable model of  $\operatorname{Tr}(P)$ . The reader may analyze the other candidates in a similar fashion. It turns out that P does not have partial stable models. Nor does  $\operatorname{Tr}(P)$  have stable models.  $\Box$ 

Partial stable models can be viewed as a logic programming account of the solution of semantic paradoxes due to Kripke [23]. In this account, undefined means *unknown* for some individuals which will not lose semantic interpretations for other individuals.

**Example 3.5** Consider the following program with variables:

$$P = \{ shave(bob, x) \leftarrow \sim shave(x, x); \\ pay_by_cash(y, x) \lor pay_by_credit(y, x) \leftarrow shave(x, y); \\ accepted(x, y) \leftarrow pay_by_cash(x, y); \\ accepted(x, y) \leftarrow pay_by_credit(x, y) \}.$$

This program intuitively says that Bob shaves those who do not shave themselves; if x shaves y then y pays x by cash or by credit; either way is accepted. The predicate **accepted** is used here to demonstrate disjunctive reasoning.

Assume there is another person, called Greg. Then clearly, we should conclude Bob shaves Greg, and Greg pays Bob by cash or by credit, either way is accepted. However, the program has no stable models in this case due to the paradox whether Bob shaves himself or not. But it has two partial stable models, in both of which shave(greg, greg) is false and shave(bob, bob) is undefined (unknown). By translating the first two rules of P we obtain

```
shaves(bob, x) \leftarrow \simshaves<sup>•</sup>(x, x);
shaves<sup>•</sup>(bob, x) \leftarrow \simshaves(x, x);
pay_by_cash(y, x) \lor pay_by_credit(y, x) \leftarrow shaves(x, y); and
pay_by_cash<sup>•</sup>(y, x) \lor pay_by_credit<sup>•</sup>(y, x) \leftarrow shaves<sup>•</sup>(x, y).
```

The full translation Tr(P) yields a Herbrand instantiation over the universe  $\{bob, greg\}$  which has four total stable models. One of them is

```
N = \{ shaves^{\bullet}(bob, bob), \\ shaves(bob, greg), shaves^{\bullet}(bob, greg), \\ pay_by_cash(greg, bob), pay_by_cash^{\bullet}(greg, bob), \\ pay_by_credit^{\bullet}(bob, bob), accepted^{\bullet}(bob, bob), \\ accepted(greg, bob), accepted^{\bullet}(greg, bob) \}.
```

Hence the fact that shaves(bob, bob) is undefined in the corresponding partial stable model M (recall the equations in Definition 3.2) is represented by shaves<sup>•</sup>(bob, bob) being true and shaves(bob, bob) being false in N.

### 3.2 CORRECTNESS OF THE TRANSLATION

The goal of this section is to establish a one-to-one correspondence between the partial stable models of a disjunctive program P and the (total) stable models of the translation Tr(P). It is first shown that the correspondence equations CE1–CE4 given in Definition 3.2 provide a syntactic way to transform a partial stable model M of P into a total stable model N of Tr(P)and back. More formally, we have the following theorem in mind.

**Theorem 3.6** Let M be a partial interpretation of a disjunctive program Pand N a total interpretation of the translation Tr(P) such that CE1–CE4 are satisfied. Then M is a partial stable model of P if and only if N is a (total) stable model of Tr(P).

Our strategy to prove Theorem 3.6 is as follows: first, in two separate lemmas, we show the correspondence, in each direction, between unfounded sets for P and Tr(P) under M and N, respectively. These two lemmas are interesting in their own right as they show very tight conditions under which the two previously studied notions of unfoundedness [14, 26] are related. These results will then be used in the proof of the theorem. We first state two relatively simple facts. The first says that that the GL-transform has no effect on unfoundedness, and the second states that the translation preserves models via CE1–CE4 in Definition 3.2.

**Proposition 3.7** Let P be a disjunctive program and N a total interpretation for P. Then,  $X \subseteq Hb(P)$  is an unfounded set for P w.r.t. N if and only if X is an unfounded set for  $P^N$  w.r.t. N.

**PROOF.** Note that  $A \leftarrow B \in P^N$  if and only if there is a rule  $A \leftarrow B, \sim C \in P$  such that  $C \subseteq N^{\mathbf{f}}$ , i.e.,  $C \cap N^{\mathbf{t}} = \emptyset$ . Then it holds for any  $X \subseteq \operatorname{Hb}(P)$  that

X is not an unfounded set for $P$ w.r.t. $N$
$\exists A \leftarrow B, \sim C \in P \text{ such that } (1) \ A \cap X \neq \emptyset, (2) \ B \cap N^{\mathbf{f}} = \emptyset,$
(3) $C \cap N^{\mathbf{t}} = \emptyset$ , (4) $B \cap X = \emptyset$ , and (5) $(A - X) \cap N^{\mathbf{t}} = \emptyset$
$\exists A \leftarrow B \in P^N$ such that (6) $A \cap X \neq \emptyset$ , (7) $B \cap N^{\mathbf{f}} = \emptyset$ ,
(8) $B \cap X = \emptyset$ , and (9) $(A - X) \cap N^{\mathbf{t}} = \emptyset$
X is not an unfounded set for $P^N$ w.r.t. N.

**Proposition 3.8** Let M be a partial interpretation for a disjunctive program P and N a total interpretation for the translation Tr(P). Assume M and N satisfy the CEs. Then, M is a partial model of P if and only if N is a total model of Tr(P).

**PROOF.** It follows by the correspondence equations CE1-CE4 in Definition 3.2 that M is not a partial model of P if and only if

$$\exists A \leftarrow B, \sim C \in P: M(\bigvee A) < M(B \cup \sim C)$$

$$\exists A \leftarrow B, \sim C \in P: M(\bigvee A) < \mathbf{t} \text{ and } M(B \cup \sim C) = \mathbf{t}, \text{ or }$$

$$\exists A \leftarrow B, \sim C \in P: M(\bigvee A) = \mathbf{f} \text{ and } M(B \cup \sim C) = \mathbf{u}$$

$$\Leftrightarrow \quad \exists A \leftarrow B, \sim C^{\bullet} \in \operatorname{Tr}(P): N(\bigvee A) = \mathbf{f} \text{ and } N(B \cup \sim C^{\bullet}) = \mathbf{t}, \text{ or }$$

$$\exists A^{\bullet} \leftarrow B^{\bullet}, \sim C \in \operatorname{Tr}(P): N(\bigvee A^{\bullet}) = \mathbf{f} \text{ and } N(B^{\bullet} \cup \sim C) = \mathbf{t}$$

$$\Leftrightarrow \quad \exists A \leftarrow B, \sim C^{\bullet} \in \operatorname{Tr}(P): N(\bigvee A^{\bullet}) = \mathbf{f} \text{ and } N(B^{\bullet} \cup \sim C) = \mathbf{t}$$

$$\Leftrightarrow \quad \exists A \leftarrow B, \sim C^{\bullet} \in \operatorname{Tr}(P): N(\bigvee A) < N(B \cup \sim C^{\bullet}), \text{ or }$$

$$\exists A^{\bullet} \leftarrow B^{\bullet}, \sim C \in \operatorname{Tr}(P): N(\bigvee A^{\bullet}) < N(B^{\bullet} \cup \sim C)$$

which is equivalent to stating that N is not a total model of Tr(P), since the consistency rules in Tr(P) are automatically satisfied by CE4.

Still assuming the setting determined by CEs, the following lemma gives a condition under which the unfounded sets w.r.t. N for Tr(P) can be converted into unfounded sets w.r.t. M for P.

**Lemma 3.9** Let P be a program, M a partial interpretation of P and N a total interpretation of the program Tr(P) such that CE1-CE4 are satisfied. Then, for any unfounded set X for Tr(P) w.r.t. N, the set of atoms  $Y = \{a \in \text{Hb}(P) \mid a^{\bullet} \in X\}$  is an unfounded set for P w.r.t. M. In addition, if X is N-consistent, then Y is M-consistent.

**PROOF.** Consider any rule  $A \leftarrow B, \sim C \in P$  such that  $A \cap Y \neq \emptyset$ . It is proven in the sequel that one of the unfoundedness conditions UF1–UF3 applies to  $A \leftarrow B, \sim C$ . Two cases arise depending on the value of  $M(B \cup \sim C)$ .

- **I.** If  $M(B \cup \sim C) = \mathbf{f}$ , then UF1 is directly applicable.
- **II.** Suppose that  $M(B \cup \sim C) \neq \mathbf{f}$  which implies  $N(B^{\bullet} \cup \sim C) = \mathbf{t}$  by the CEs. Now  $A \cap Y \neq \emptyset$  and the definition of Y imply  $A^{\bullet} \cap X \neq \emptyset$ . Since  $A^{\bullet} \leftarrow B^{\bullet}, \sim C \in \mathrm{Tr}(P), X$  is an unfounded set for  $\mathrm{Tr}(P)$  w.r.t. N and UF1 is not applicable to  $A^{\bullet} \leftarrow B^{\bullet}, \sim C$ , we know that either UF2 or UF3 applies to  $A^{\bullet} \leftarrow B^{\bullet}, \sim C$ .
  - (i) If UF2 applies to  $A^{\bullet} \leftarrow B^{\bullet}, \sim C$ , then  $B^{\bullet} \cap X \neq \emptyset$ . It follows by the definition of Y that  $B \cap Y \neq \emptyset$ , i.e., UF2 applies to  $A \leftarrow B, \sim C$ .
  - (ii) If UF3 applies to  $A^{\bullet} \leftarrow B^{\bullet}, \sim C$ , then  $N(\bigvee (A^{\bullet} X)) = \mathbf{t}$ . Since  $A^{\bullet} X = (A Y)^{\bullet}$  by the definition of Y, we obtain by the CEs that  $M(\bigvee (A Y)) \neq \mathbf{f}$ . Thus UF3 applies to  $A \leftarrow B, \sim C$ .

The proof of the consistency claim follows. To establish the contrapositive of the claim, suppose that Y is not M-consistent. Then  $Y \cap M^{\mathbf{t}} \neq \emptyset$ , i.e., there exists an atom  $a \in \operatorname{Hb}(P)$  such that  $a \in Y$  and  $a \in M^{\mathbf{t}}$ . The former implies  $a^{\bullet} \in X$  by the definition of Y while the latter gives us  $a^{\bullet} \in N^{\mathbf{t}}$  by the CEs. Thus  $X \cap N^{\mathbf{t}} \neq \emptyset$  and X is not N-consistent.  $\Box$ 

The next lemma shows that, under the specified conditions, an unfounded set for a given disjunctive program P corresponds to a collection of unfounded sets for the translation Tr(P).

**Lemma 3.10** Let M be a partial model of a disjunctive program P and N a total interpretation of Tr(P) satisfying the CEs. If X is an M-consistent

unfounded set for P w.r.t. M, then  $Y = F \cup U$  where  $F = \{a, a^{\bullet} | a \in X\}$ and  $U \subseteq \{a | a \in N^{\mathbf{f}}, a^{\bullet} \in N^{\mathbf{t}}\}$  is an unfounded set for  $\operatorname{Tr}(P)$  w.r.t. N.

**PROOF.** Let X be an M-consistent unfounded set for P w.r.t. M and let  $Y = F \cup U$  satisfy the requirements above. Since any atom in Y is either marked or unmarked, two cases arise.

- **I.** Suppose that  $a^{\bullet} \in Y$  which implies by the definition of Y that  $a \in Y$ . Then it is clear that that UF2 applies to the consistency rule  $a^{\bullet} \leftarrow a \in \operatorname{Tr}(P)$ . Let us then prove that one of the unfoundedness conditions applies to any rule  $A^{\bullet} \leftarrow B^{\bullet}, \sim C \in \operatorname{Tr}(P)$  satisfying  $a^{\bullet} \in A^{\bullet}$ . Since N is a total interpretation, we have  $N(B^{\bullet} \cup \sim C) = \mathbf{f}$  (in which case UF1 applies to  $A^{\bullet} \leftarrow B^{\bullet}, \sim C$ ) or  $N(B^{\bullet} \cup \sim C) = \mathbf{t}$  in which case  $M(B \cup \sim C) > \mathbf{f}$ . Since  $a \in X$  and  $a \in A$ , and UF1 does not apply to  $A \leftarrow B, \sim C$ , we only need to consider UF2 and UF3. If UF2 applies to  $A \leftarrow B, \sim C, \exists b \in B$  such that  $b \in X$ . It follows by the definition of Y that  $b^{\bullet} \in Y$ . Hence UF2 applies to  $A^{\bullet} \leftarrow B^{\bullet}, \sim C$ . If UF3 applies to  $A \leftarrow B, \sim C, \exists b \in A$  such that  $M(b) > \mathbf{f}$  and  $b \notin X$ . Then we know that  $N(b^{\bullet}) = \mathbf{t}$  by the CEs. Further, by the definition of Y,  $b \notin X$  implies  $b^{\bullet} \notin Y$ . Hence UF3 applies to  $A^{\bullet} \leftarrow B^{\bullet}, \sim C$ .
- **II.** Suppose that  $a \in Y$ . Then consider any rule  $A \leftarrow B, \sim C^{\bullet} \in \operatorname{Tr}(P)$ such that  $a \in A$ . Since N is a total interpretation,  $N(B \cup \sim C^{\bullet}) = \mathbf{f}$ (in which case UF1 applies to  $A \leftarrow B, \sim C^{\bullet}$ ) or  $N(B \cup \sim C^{\bullet}) = \mathbf{t}$ . In the latter case, we know that  $\exists b \in A$  such that  $N(b) = \mathbf{t}$ , since N is a model of  $\operatorname{Tr}(P)$  by Proposition 3.8 (recall that M is a partial model of P). Then suppose that  $b \in Y$ , i.e.,  $b \in F$  or  $b \in U$  by the definition of Y. If  $b \in F$ , then  $b \in X$  by the definition of F. On the other hand,  $N(b) = \mathbf{t}$  implies  $M(b) = \mathbf{t}$ . Thus  $M^{\mathbf{t}} \cap X \neq \emptyset$ , contradicting the M-consistency of X. If  $b \in U$ , the the definition of U implies  $N(b) = \mathbf{f}$ , a contradiction. Hence,  $b \notin Y$  and UF3 applies to  $A \leftarrow B, \sim C^{\bullet}$ .

We note that the M-consistency of X is also a necessary condition for the correspondence to hold.

**Example 3.11** Consider a disjunctive program  $P = \{a \lor b \leftarrow; a \leftarrow \neg a\}$ and a partial model  $M = \langle \{b\}, \emptyset \rangle$  of P. It can be checked easily that  $X = \{b\}$  is an unfounded set for P w.r.t. M: UF3 applies to the only rule in which b appears in the head. But X is not M-consistent. Now consider

$$\operatorname{Tr}(P) = \{ a \lor b \leftarrow; \quad a^{\bullet} \lor b^{\bullet} \leftarrow; \quad a \leftarrow \sim a^{\bullet}; \\ a^{\bullet} \leftarrow \sim a; \quad a^{\bullet} \leftarrow a; \quad b^{\bullet} \leftarrow b \}.$$

The total interpretation corresponding to M above is  $N = \{a^{\bullet}, b, b^{\bullet}\}$ . However,  $Y = \{a, b, b^{\bullet}\}$  is not unfounded for Tr(P) w.r.t. N, since for  $b \in Y$  and the first rule in Tr(P), none of the unfoundedness conditions applies.  $\Box$ 

Let us establish Theorem 3.6 in two separate theorems.

**Theorem 3.12** Let P be a disjunctive program. If N is a stable model of the translation Tr(P), then the partial interpretation M of P satisfying the correspondence equations CE1-CE4 is a partial stable model of P.

**PROOF.** Let N be a stable model of  $\operatorname{Tr}(P)$ . Then it follows by the presence of consistency rules  $\{a^{\bullet} \leftarrow a \mid a \in \operatorname{Hb}(P)\}$  in  $\operatorname{Tr}(P)$  that there is no  $a \in \operatorname{Hb}(P)$  such that  $a \in N$  and  $a^{\bullet} \notin N$ , since N is a total model of  $\operatorname{Tr}(P)$ . Thus it makes sense to define M as the partial interpretation satisfying CE1-CE4. We prove that  $M^{\mathsf{t}}$  is a minimal total model of  $P^{M}$ , and  $M^{\mathsf{f}}$ is a maximal M-consistent unfounded set for P w.r.t. M.

- **I.** Let us first establish that for any rule  $A \leftarrow B, \sim C \in P, A \leftarrow B \in P^M$  $\iff A \leftarrow B \in \operatorname{Tr}(P)^N$ . So consider any  $A \leftarrow B, \sim C \in P$ . It follows by the CEs and the definitions of  $P^M$ ,  $\operatorname{Tr}(P)$  and  $\operatorname{Tr}(P)^N$  that  $A \leftarrow B \in P^M \iff$  there is a rule  $A \leftarrow B, \sim D \in P$  such that  $D \subseteq M^{\mathbf{f}}$  $\iff$  there is a rule  $A \leftarrow B, \sim D^{\bullet} \in \operatorname{Tr}(P)$  such that  $D^{\bullet} \subseteq N^{\mathbf{f}} \iff$  $A \leftarrow B \in \operatorname{Tr}(P)^N$ . Note that within these equivalences  $A \leftarrow B, \sim C$ and  $A \leftarrow B, \sim D$  need not be the same rules of P.
- **II.** Let us then prove that  $M^{\mathbf{t}}$  is a minimal total model of  $P^{M}$ . If we assume the contrary, two cases arise.
  - $M^{\mathbf{t}}$  is not a total model of  $P^M$ , i.e., there is a rule  $A \leftarrow B \in P^M$ such that  $M^{\mathbf{t}}(B) = \mathbf{t}$ , but  $M^{\mathbf{t}}(A) = \mathbf{f}$ . It follows by the CEs that  $N(B) = \mathbf{t}$  and  $N(A) = \mathbf{f}$ . Thus  $A \leftarrow B$  is not satisfied in N and thus N is not a model of  $\operatorname{Tr}(P)^N$ , as  $P^M \subset \operatorname{Tr}(P)^N$  holds by (I) above. A contradiction, since N is a stable model of  $\operatorname{Tr}(P)$ .

- There is a total model M' of  $P^M$  such that  $M' \subset M^t$ . Then define a total interpretation  $N' = M' \cup \{a^{\bullet} \mid a^{\bullet} \in N\}$ . By  $M' \subset M^t$  and the CEs, we obtain  $N' \subset N$  (only some unmarked atoms of Nare not in N'). Since M' is a total model of  $P^M$ , and M' and N'coincide on the atoms of Hb(P), every rule in  $P^M$  is satisfied by N'. By (I), the difference  $\operatorname{Tr}(P)^N - P^M$  contains only consistency rules  $a^{\bullet} \leftarrow a$  (for every  $a \in \operatorname{Hb}(P)$ ) and rules of the form  $A^{\bullet} \leftarrow B^{\bullet}$ (for some  $A \leftarrow B, \sim C \in P$ ). These rules are all satisfied by N', since N is a total model of  $\operatorname{Tr}(P)^N$ ,  $N' \subset N$ , and N' and Ncoincide on the marked atoms in  $\operatorname{Hb}(P)^{\bullet}$ . Thus N' is a total model of  $\operatorname{Tr}(P)^N$  nor a total stable model of P. A contradiction.
- **III.** Since N is a total stable model of  $\operatorname{Tr}(P)$ , it holds by Theorem 2.8 that  $N^{\mathbf{f}}$  is the greatest unfounded set for  $\operatorname{Tr}(P)$  w.r.t. N. Moreover,  $N^{\mathbf{f}}$  is N-consistent, since  $N^{\mathbf{f}} \cap N^{\mathbf{t}} = \emptyset$ . Note that  $a^{\bullet} \in N^{\mathbf{f}}$  implies  $a \in N^{\mathbf{f}}$ , since N satisfies  $a^{\bullet} \leftarrow a \in \operatorname{Tr}(P)$ . Thus  $M^{\mathbf{f}} = \{a \in \operatorname{Hb}(P) \mid a \in N^{\mathbf{f}} \text{ and } a^{\bullet} \in N^{\mathbf{f}}\} = \{a \in \operatorname{Hb}(P) \mid a^{\bullet} \in N^{\mathbf{f}}\}$ . It follows by Lemma 3.9 that  $M^{\mathbf{f}}$  is an M-consistent unfounded set for P w.r.t. M.

Then assume that  $M^{\mathbf{f}}$  is not maximal, i.e., there is an *M*-consistent unfounded set *X* for *P* w.r.t. *M* such that  $X \supset M^{\mathbf{f}}$ . So there is an atom  $a \in X$  such that  $a \notin M^{\mathbf{f}}$ . Then  $a \notin M^{\mathbf{f}}$  implies  $a \in M^{\mathbf{t}}$  or  $a \in M^{\mathbf{u}}$ . In both cases, by the CEs,  $a^{\bullet} \in N^{\mathbf{t}}$ , i.e.,  $a^{\bullet} \notin N^{\mathbf{f}}$ . Then construct  $Y = \{a, a^{\bullet} | a \in X\}$ . According to Lemma 3.10, that *X* is an *M*-consistent unfounded set for *P* w.r.t. *M* implies that *Y* is an unfounded set for  $\operatorname{Tr}(P)$  w.r.t *N*. However,  $a \in X$  implies  $a^{\bullet} \in Y$ but  $a^{\bullet} \notin N^{\mathbf{f}}$ . Thus  $a^{\bullet} \in N^{\mathbf{t}}$  indicating that *N* is not unfounded free for  $\operatorname{Tr}(P)$ . Consequently, *N* is not a stable model of  $\operatorname{Tr}(P)$  by the characterization of stable models in Theorem 2.8, a contradiction.

**Theorem 3.13** Let P be a disjunctive program. If M is a partial stable model of P, then the total interpretation N satisfying the correspondence equations CE1-CE4 is a stable model of the translation Tr(P).

**PROOF.** Suppose that M is a partial stable model of P. Then we know by Theorem 2.9 that (i)  $M^{\mathbf{t}}$  is a minimal total model of  $P^{M}$  and (ii)  $M^{\mathbf{f}}$  is a

maximal *M*-consistent unfounded set for *P* w.r.t. *M*. Then define *N* as the total interpretation of Tr(P) satisfying the CEs. It follows by Lemma 3.10 that  $N^{\mathbf{f}} = \{a, a^{\bullet} \mid a \in M^{\mathbf{f}}\} \cup M^{\mathbf{u}}$  is an unfounded set for Tr(P) w.r.t. *N*.

Let us then assume that N is not a stable model of  $\operatorname{Tr}(P)$ . Equivalently, it holds by Theorem 2.8 that  $N^{\mathbf{f}}$  is not the greatest unfounded set for  $\operatorname{Tr}(P)$ w.r.t. N. So there is an unfounded set X for  $\operatorname{Tr}(P)$  w.r.t. N such that  $N^{\mathbf{f}} \subset X$  and  $N^{\mathbf{t}} \cap X \neq \emptyset$  hold. It follows by Proposition 3.7 that X is also an unfounded set for  $\operatorname{Tr}(P)^N$  w.r.t. N.

Then consider any  $A \leftarrow B \in P^M$  for which there is a rule  $A \leftarrow B, \sim C \in P$ such that  $C \subseteq M^{\mathbf{f}}$ . It follows by the CEs that  $C^{\bullet} \subseteq N^{\mathbf{f}}$ . Since  $A \leftarrow B, \sim C^{\bullet} \in$  $\operatorname{Tr}(P)$ , it follows that  $A \leftarrow B \in \operatorname{Tr}(P)^N$ . Thus  $P^M \subset \operatorname{Tr}(P)^N$  holds, as  $\operatorname{Tr}(P)^N$  contains among others the consistency rules  $\{a^{\bullet} \leftarrow a \mid a \in \operatorname{Hb}(P)\}$ .

Recall that  $M^{\mathbf{t}} = N^{\mathbf{t}} \cap \operatorname{Hb}(P)$  is a minimal total model of  $P^{M}$ . We also distinguish a set of atoms  $X' = X \cap \operatorname{Hb}(P)$ . Let us then establish that X' is an unfounded set for  $P^{M}$  with respect to  $M^{\mathbf{t}}$  in the two-valued sense.

**I.** If X' is not such a set, it follows by Definition 2.3 that there is  $A \leftarrow B \in P^M$  with  $A \cap X' \neq \emptyset$  such that  $B \subseteq M^t$ ,  $B \cap X' = \emptyset$  and  $(A - X') \cap M^t = \emptyset$ . It follows that  $A \leftarrow B \in \operatorname{Tr}(P)^N$ , as  $P^M \subset \operatorname{Tr}(P)^N$ . Since A and B are subsets of Hb(P),  $M^t = N^t \cap \operatorname{Hb}(P)$  and  $X' = X \cap \operatorname{Hb}(P)$ , we obtain  $A \cap X \neq \emptyset$ ,  $B \subseteq N^t$ ,  $B \cap X = \emptyset$  and  $(A - X) \cap N^t = \emptyset$ . Then X is not an unfounded set for  $\operatorname{Tr}(P)^N$  w.r.t. N, a contradiction.

It follows by Lemma 2.4 that  $M^{\mathbf{t}} - X'$  is a total model of  $P^{M}$ . It follows by the minimality of  $M^{\mathbf{t}}$  that  $M^{\mathbf{t}} \cap X' = \emptyset$  and  $M^{\mathbf{t}} \cap X = \emptyset$ . Moreover, it follows by Lemma 3.9 that  $Y = \{a \in \operatorname{Hb}(P) \mid a^{\bullet} \in X\}$  is an unfounded set for P w.r.t. M. It remains to establish that Y is M-consistent and  $M^{\mathbf{f}} \subset Y$ .

- **II.** Suppose that Y is not M-consistent, i.e., it holds for some  $\mathbf{a} \in \operatorname{Hb}(P)$  that (a)  $a \in Y$  and (b)  $a \in M^{\mathfrak{t}}$ . Now (b) implies by the CEs that  $a \in N^{\mathfrak{t}}$  and  $a^{\bullet} \in N^{\mathfrak{t}}$ . On the other hand, it follows by (a) and the definition of Y that  $a^{\bullet} \in X$ . Thus one of the unfoundedness conditions applies to the rule  $a^{\bullet} \leftarrow a \in \operatorname{Tr}(P)$ , as X is an unfounded set for  $\operatorname{Tr}(P)$  w.r.t. N. Now UF1 is not applicable, as  $\mathbf{a} \notin N^{\mathfrak{f}}$ , and UF3 is not applicable, as  $a^{\bullet} \in X$ . Thus UF2 must be applicable to  $a^{\bullet} \leftarrow a$ . It follows that  $\mathbf{a} \in X$ , too. Then there is  $a \in \operatorname{Hb}(P)$  such that  $a \in M^{\mathfrak{t}}$  and  $a \in X$ . A contradiction with  $M^{\mathfrak{t}} \cap X = \emptyset$  established above.
- **III.** Consider any  $a \in M^{\mathbf{f}}$ . Thus  $a \in \operatorname{Hb}(P)$  and  $a^{\bullet} \in N^{\mathbf{f}}$  follows by the CEs. Then  $N^{\mathbf{f}} \subset X$  implies  $a^{\bullet} \in X$  as well as  $a \in Y$  by the definition

of Y. Thus  $M^{\mathbf{f}} \subseteq Y$ . On the other hand, recall that  $N^{\mathbf{t}} \cap X \neq \emptyset$  and  $M^{\mathbf{t}} \cap X = \emptyset$ . Then  $a^{\bullet} \in N^{\mathbf{t}} \cap X$  holds for some  $a \in \operatorname{Hb}(P)$ . It follows that  $a^{\bullet} \in N^{\mathbf{t}}$  and  $a^{\bullet} \in X$ . The former implies  $\mathbf{a} \notin M^{\mathbf{f}}$  by the CEs. The latter implies  $a \notin Y$  by the definition of Y. Hence  $M^{\mathbf{f}} \subset Y$ .

Thus  $M^{\mathbf{f}}$  is not a maximal *M*-consistent unfounded set for *P* w.r.t. *M*, a contradiction. Hence *N* must be a stable model of  $\operatorname{Tr}(P)$ .

It is worthwhile at this point to briefly comment on the proof of Theorem 3.6 as given in [43], which proceeds in several steps. Given two partial interpretations M and M' of a disjunctive program P, let N and N', respectively, be the corresponding total interpretations of Tr(P) such that CE1–CE4 are satisfied. Firstly, it can be shown that M is a partial model of  $P_{M'}$  if and only if N is a total model of  $\text{Tr}(P)^{N'}$ . Secondly, since the truth-ordering for partial interpretations corresponds to the subset ordering for total interpretations, it can be shown that the minimal partial models of  $P_{M'}$  correspond to the minimal total models of  $\text{Tr}(P)^{N'}$ . Thirdly, based on a characterization of partial models in general [43], we conclude that M is a partial stable model of P if and only if N is a total stable model of Tr(P).

Looking back to results established so far, we know by Theorem 3.13 that any partial stable model M of P can be mapped to a stable model

$$f(M) = M^{\mathbf{t}} \cup (M^{\mathbf{t}} \cup M^{\mathbf{u}})^{\bullet}$$
(3)

of  $\operatorname{Tr}(P)$ . Similarly, any stable model N of  $\operatorname{Tr}(P)$  can be projected to a partial stable model

$$g(N) = \langle \{ a \in \operatorname{Hb}(P) \, | \, a \in N \}, \{ a \in \operatorname{Hb}(P) \, | \, a \notin N \text{ and } a^{\bullet} \notin N \} \rangle \quad (4)$$

of P by Theorem 3.12. These equations and the corresponding theorems indicate that f and g are functions between the set of partial stable models of P and the set of stable models of Tr(P). In the sequel, it is established that these functions are bijections, which means that our translation technique does not yield spurious models for programs although new atoms are used. This is highly desirable from the knowledge representation perspective.

**Theorem 3.14** The partial stable models of a disjunctive program P and the total stable models of the translation Tr(P) are in a one-to-one correspondence.

**PROOF.** Let f and g be defined by the equations (3) and (4), respectively. It is straightforward to see that f is injective, i.e.,  $f(M_1) = f(M_2)$  implies  $M_1 = M_2$ . Then assume that  $g(N_1) = g(N_2)$  holds for some stable models  $N_1$  and  $N_2$  of  $\operatorname{Tr}(P)$ . It follows by the definition of g for any  $a \in \operatorname{Hb}(P)$  that (i)  $a \in N_1 \iff a \in N_2$  and (ii)  $a \notin N_1$  and  $a^{\bullet} \notin N_1 \iff a \notin N_2$  and  $a^{\bullet} \notin N_2$ . Then consider any  $a \in \operatorname{Hb}(P)$  such that  $a^{\bullet} \in N_1$ . Two cases arise. If  $a \in N_1$ , it follows by (i) that  $a \in N_2$ . Since  $N_2$  satisfies the rule  $a^{\bullet} \leftarrow a \in \operatorname{Tr}(P)^{N_2}$ , we obtain  $a^{\bullet} \in N_2$ . On the other hand, if  $a \notin N_1$  it follows by (i) that  $a \notin N_2$ . Assuming that  $a^{\bullet} \notin N_2$  implies by (ii) that  $a^{\bullet} \notin N_1$ , a contradiction. Hence  $a^{\bullet} \in N_2$  also in this case. By symmetry,  $a^{\bullet} \in N_2$  implies  $a^{\bullet} \in N_1$ .

Thus it holds for any  $a \in \text{Hb}(P)$  that (iii)  $a^{\bullet} \in N_1 \iff a^{\bullet} \in N_2$ . It follows by (i) and (iii) that  $N_1 = N_2$  so that g is injective, too. Thus f and g are bijections and inverses of each other, as g(f(M)) = M and f(g(N)) = N hold for any (partial) stable models M and N. Hence the claim.  $\Box$ 

None of the preceding proofs relies on the assumption that the given program is finite. Therefore, all of these results presented in this section apply in the non-ground case as well.

#### 3.3 QUERY ANSWERING

Let us yet address the possibility of using an inference engine for computing total stable models to answer queries concerning partial stable models. This is highly interesting, because there are already systems available for computing total stable models [1, 29, 24, 45] while partial stable models lack implementations. Here we must remind the reader that partial stable models can be used in different ways in order to evaluate queries. Typically two modes of reasoning are used: certainty inference and possibility inference. In the former approach, a query Q should be true in all (intended) models of P while Q should be true in some (intended) model of P in the latter approach. Moreover, maximal partial stable models (under set inclusion) are sometimes distinguished; this is how regular models and preferred extensions are obtained for normal programs [10, 39, 50]. We are particularly interested in possibility inference where the maximality condition makes no difference (see [15, 38] for certainty inference):  $M(Q) = \mathbf{t}$  for some partial stable model M of P if and only if  $M'(Q) = \mathbf{t}$  for some maximal partial stable model M' of P.

We consider queries Q that are sets of literals over Hb(P) and queries are translated in harmony with the CEs:  $Tr(Q) = Q \cup Q^{\bullet}$ . As a direct consequence of Theorem 3.13 and CE1, we obtain the following.

**Corollary 3.15** A query Q is true in a (maximal) partial stable model of P if and only if Tr(Q) is true in a stable model of Tr(P).

What about using a query answering procedure for partial stable models to answer queries concerning stable models? A slight extension of the translation  $\operatorname{Tr}(P)$  is needed for this purpose: let  $\operatorname{Tr}_2(P)$  be  $\operatorname{Tr}(P)$  augmented with a set of rules  $\{f \leftarrow a^{\bullet}, \sim a \mid a \in \operatorname{Hb}(P)\}$  where  $f \notin \operatorname{Hb}(P)$  is a new atom. The purpose of these additional rules is to detect partial stable models with remaining undefined atoms. A query Q is translated into  $\operatorname{Tr}_2(Q) = Q \cup \{\sim f\}$ .

**Corollary 3.16** A query Q is true in a stable model of P if and only if  $\operatorname{Tr}_2(Q)$  is true in a partial stable model of  $\operatorname{Tr}_2(P)$ .

This result allows query answering for stable models to be conducted by a procedure for partial stable models, e.g., by the abductive procedure of Eshghi and Kowalski [16].

### 4 UNFOLDING DISJUNCTIONS

In this section we develop a method for reducing the task of computing a (total) stable model of a disjunctive program to computing stable models for normal (disjunction-free) programs. This objective demands us to  $unfold^2$  disjunctions from programs in a way or another. Since the problem of deciding whether a disjunctive program has a stable model is  $\Sigma_2^p$ -complete [12] whereas the problem is NP-complete in the non-disjunctive case [31], the reduction cannot be computable in polynomial time unless the polynomial hierarchy collapses. This is why our reduction is based on a generate and test approach.

The basic idea is that given a disjunctive program P we compute its stable models in two phases: (i) we generate model candidates and (ii) test candidates for stability until we find a suitable model. For generating model

<sup>&</sup>lt;sup>2</sup>The idea of unfolding disjunction generally refers to performing some transformations on disjunctions in order to remove them [4, 9, 41]. However, such transformations do not necessarily remove all disjunctions or do not preserve stable semantics.

candidates we construct a normal program Gen(P) such that the stable models of Gen(P) give the candidate models. For testing a candidate model Mwe build another normal program Test(P, M) such that Test(P, M) has no stable models if and only if M is a stable model of the original disjunctive program P. Hence, given a procedure for computing stable models for normal programs all stable models of a disjunctive program P can be generated as follows: for each stable model M of Gen(P), decide whether Test(P, M)has a stable model and if this is not the case, output M as a stable model of P. This kind of a generate and test approach is used also in DLV [24] which is a state-of-the-art system for disjunctive programs. The difference is that we reduce the test and generate subtasks directly to problems of computing stable models of normal programs whereas in DLV special techniques for the two subtasks have been developed based on the notion of unfounded sets for disjunctive programs.

It is easy to construct a normal program for generating candidate models for a disjunctive program P. Consider, e.g., a program G0(P) which contains for each atom  $a \in Hb(P)$ , two rules  $a \leftarrow \sim \hat{a}$ ;  $\hat{a} \leftarrow \sim a$  where  $\hat{a}$  is a new atom denoting the complement of the atom a, i.e., a is in a stable model exactly when  $\hat{a}$  is not. These rules generate stable models corresponding to every subset of Hb(P). In order to prune this set of models to those with all rules in P satisfied, it is sufficient to include a rule

$$f \leftarrow \sim f, \sim a_1, \dots, \sim a_k, b_1, \dots, b_m, \sim c_1, \dots, \sim c_n \tag{5}$$

for each rule of the form (1) in P where f is a new atom. As f cannot be in any stable model, the rule functions as an integrity constraint eliminating the models where each  $b_i$  is included, every  $c_j$  is excluded but no  $a_l$  is included.

In order to guarantee completeness, it is sufficient that for each stable model M of P there is a corresponding model candidate which agrees with M w.r.t. Hb(P). It is clear that G0(P) satisfies this condition. However, for efficiency it is important to devise a generating program that has as few as possible (candidate) stable models provided that completeness is not lost. An obvious shortcoming of G0(P) is that it generates many candidates even if the program P is disjunction-free. In order to solve this problem we construct for given a disjunctive program P a generating program G1(P) as follows:

$$G1(P) = \{a \leftarrow \sim \hat{a}, B, \sim C \mid A \leftarrow B, \sim C \in P_{D}, a \in A\} \cup \\ \{\hat{a} \leftarrow \sim a \mid a \in Heads(P_{D})\} \cup \\ \{f \leftarrow \sim f, \sim A, B, \sim C \mid A \leftarrow B, \sim C \in P_{D}\} \cup P_{N} \end{cases}$$

where  $P_{\rm N}$  is the set of the normal rules in P and  $P_{\rm D}$  are the other (proper disjunctive) rules in P, i.e.  $P = P_{\rm N} \cup P_{\rm D}$ , and  $\text{Heads}(P_{\rm D})$  is the set of atoms appearing in the heads of the rules in  $P_{\rm D}$ .

The program G1(P) has typically far fewer stable models than G0(P) and the number of "extra" candidate models which do not match stable models of P is related to the number of disjunctions in P. For example, if P is disjunction-free, the stable models of G1(P) correspond exactly to the stable models of P. However, for a disjunctive program P, G1(P) can easily have "extra" stable models. Consider, e.g.,

$$P = \{a \lor b \leftarrow\} \tag{6}$$

for which  $G1(P) = \{a \leftarrow \sim \hat{a}; \hat{a} \leftarrow \sim a; b \leftarrow \sim \hat{b}; \hat{b} \leftarrow \sim b; f \leftarrow \sim f, \sim a, \sim b\}$  has a stable model,  $\{a, b\}$ , not corresponding to a stable model of P. In fact, G1(P) only requires for each proper disjunctive rule in P that some non-empty subset of the head atoms of the rule is included in the model candidate when the body of the rule holds. Hence, for such a rule with d disjuncts in the head there are  $2^d - 1$  possible subsets and in the worst case  $2^d - 2$  of these could lead to "extra" model candidates. This means that in the worst case G1(P) can have an exponential number of "extra" model candidates w.r.t. the number of disjunctions in P.

In order to decrease the number of "extra" models we introduce a technique exploiting a key property of supported models [3]: each atom a true in a model M of P must have a rule supporting it, i.e., there is a rule  $A \leftarrow B, \sim C \in P$  such that  $a \in A, M(B \cup \sim C) = \mathbf{t}$ , and  $M(\bigvee (A - \{a\})) = \mathbf{f}$ . Since every stable model of P is also a supported model of P, it makes perfect sense to require supportedness from the candidate stable models. For this, we introduce a new atom  $a^{s}$ , which denotes the fact that atom a has a supporting rule, for each atom a appearing in the head of a disjunctive rule. The intuition behind the set of rules  $\operatorname{Supp}(P)$  below is that a rule can support exactly one of its head atoms and we may exclude every model that has an atom without a supporting rule:

$$Supp(P) = \{a^{s} \leftarrow \sim (A - \{a\}), B, \sim C \mid A \leftarrow B, \sim C \in P, a \in A \cap Heads(P_{D})\} \cup \{f \leftarrow \sim f, a, \sim a^{s} \mid a \in Heads(P_{D})\}$$
(7)

where  $\text{Heads}(P_{\text{D}})$  is the set of atoms appearing in the heads of the proper disjunctive rules in P. For example, for P in (6),

 $\operatorname{Supp}(P) = \{ a^{\operatorname{s}} \leftarrow \sim b; \ b^{\operatorname{s}} \leftarrow \sim a; \ f \leftarrow \sim f, a, \sim a^{\operatorname{s}}; \ f \leftarrow \sim f, b, \sim b^{\operatorname{s}} \}.$ 

Now  $G1(P) \cup Supp(P)$  has exactly two stable models  $\{a, a^s, \hat{b}\}$  and  $\{b, b^s, \hat{a}\}$  corresponding to the two stable models  $\{a\}$  and  $\{b\}$  of P.

Combining this idea with G1(P) gives a promising generating program

$$\operatorname{Gen}(P) = \operatorname{G1}(P) \cup \operatorname{Supp}(P) \tag{8}$$

which still preserves completeness.

**Proposition 4.1** Let P be a disjunctive program. Then if M is a stable model of P, there is a stable model N of  $\text{Gen}(P) = \text{G1}(P) \cup \text{Supp}(P)$  with  $M = N \cap \text{Hb}(P)$ .

**PROOF.** Let M be a stable model of P and

$$N = M \cup \{\hat{a} \mid a \in \text{Heads}(P_{D}) - M\} \cup \{a^{s} \mid a \in M \cap \text{Heads}(P_{D})\}.$$

Now clearly  $M = N \cap \text{Hb}(P)$ . We show first that (i) N is a model of  $\text{Gen}(P)^N$ and then that (ii) if there is a model N' of  $\text{Gen}(P)^N$  such that  $N' \subseteq N$  then  $N \subseteq N'$  holds. These together imply that N is a stable model of Gen(P).

For property (i) consider rules in  $\operatorname{Gen}(P)^N = \operatorname{G1}(P)^N \cup \operatorname{Supp}(P)^N$  starting with those in  $\operatorname{G1}(P)^N$ . Suppose  $a \leftarrow B \in \operatorname{G1}(P)^N$ . If  $a \in \operatorname{Heads}(P_D)$ , then  $\hat{a} \notin N$  and, hence,  $a \in M \subseteq N$ . Otherwise if  $a \notin \operatorname{Heads}(P_D)$ , then  $a \leftarrow B \in P^M$  implying that  $N(\supseteq M)$  satisfies  $a \leftarrow B$ . If  $\hat{a} \leftarrow \operatorname{G1}(P)^N$ , then  $a \notin M$  and  $\hat{a} \in N$ . If  $f \leftarrow B \in \operatorname{G1}(P)^N$ , then there is a rule  $A \leftarrow B, \sim C \in P_D$  such that  $A \cap M = \emptyset$  and  $A \leftarrow B \in P^M$ . As M is a model of  $P^M$ ,  $B \nsubseteq M$  and, consequently,  $B \nsubseteq N$ . Thus, N is a model of  $\operatorname{G1}(P)^N$ . Next consider rules in  $\operatorname{Supp}(P)^N$ . If  $a^s \leftarrow B \in \operatorname{Supp}(P)^N$ , then there is a rule  $A \leftarrow B, \sim C \in P$  such that  $A \leftarrow B \in P^M$  and  $(A - \{a\}) \cap M = \emptyset$ . If  $B \subseteq N$ , then  $B \subseteq M$  and, hence,  $A \cap M \neq \emptyset$  as M is a model of  $P^M$ . Thus,  $a \in M$ . If  $f \leftarrow a \in \operatorname{Supp}(P)^N$ , then  $a^s \notin N$ , and  $a \notin N$ . This implies that N is a model of  $\operatorname{Supp}(P)^N$  and that (i) holds.

For property (ii) consider a model N' of  $\operatorname{Gen}(P)^N$  such that  $N' \subseteq N$ . First we show that  $N' \cap \operatorname{Hb}(P)$  is a model of  $P^M$  implying that  $M \subseteq N' \cap \operatorname{Hb}(P)$ . Consider  $A \leftarrow B \in P^M$ . If the body B is true in  $N' \cap \operatorname{Hb}(P) \subseteq N \cap \operatorname{Hb}(P) =$ M, then at least one  $a \in A \cap M$ . Then  $a \leftarrow B \in \operatorname{Gen}(P)^N$  and  $a \in N'$ . Hence,  $M \subseteq N' \cap \operatorname{Hb}(P)$ . If  $\hat{a} \in N$ , then  $a \notin N$  and, hence,  $\hat{a} \leftarrow \operatorname{G1}(P)^N$ implying  $\hat{a} \in N'$ . If  $a^{\mathrm{s}} \in N$ , then  $a \in M \cap \operatorname{Heads}(P_D)$ . Then there is a rule  $A \leftarrow B \in P^M$  such that  $B \subseteq M$  but  $(A - \{a\}) \cap M = \emptyset$ . This is because otherwise  $M - \{a\}$  would be a model of  $P^M$  contradicting the minimality of M. Hence,  $a^{s} \leftarrow B \in \text{Supp}(P)^N$  and  $B \subseteq M \subseteq N'$  implying  $a^{s} \in N'$ . Thus,  $N \subseteq N'$  and (ii) holds.  $\Box$ 

A (total) model candidate  $M \subseteq \text{Hb}(P)$  is a stable model of a program P if it is a minimal model of the GL-transform  $P^M$  of the program. This test can be reduced to an unsatisfiability problem in propositional logic using techniques presented in [33]: M is a minimal model of  $P^M$  if and only if

$$P^{M} \cup \{\neg a \mid a \in \operatorname{Hb}(P) - M\} \cup \{\neg b_{1} \lor \cdots \lor \neg b_{m}\}$$

$$(9)$$

is unsatisfiable where  $M = \{b_1, \ldots, b_m\}$  and the rules in  $P^M$  are seen as clauses. This can be determined by testing non-existence of stable models for a normal program Test(P, M) which is constructed for a disjunctive program P and a total interpretation  $M \subseteq \text{Hb}(P)$  as follows:

$$\begin{aligned} \operatorname{Test}(P,M) &= & \{ a \leftarrow \sim \hat{a}, B \mid A \leftarrow B \in P_{\mathrm{D}}^{M}, a \in A \cap M, B \subseteq M \} \cup \\ & \{ \hat{a} \leftarrow \sim a \mid a \in \operatorname{Heads}(P_{\mathrm{D}}) \} \cup \\ & \{ f \leftarrow \sim f, \sim A, B \mid A \leftarrow B \in P_{\mathrm{D}}^{M}, B \subseteq M \} \cup \\ & \{ a \leftarrow B \in P_{\mathrm{N}}^{M} \mid a \in M, B \subseteq M \} \cup \\ & \{ f \leftarrow \sim f, M \} \end{aligned}$$

where  $P_{\rm N}$  is the set of the normal rules in P and  $P_{\rm D}$  are the proper disjunctive rules in P and Heads $(P_{\rm D})$  is the set of atoms appearing in the heads of the rules in  $P_{\rm D}$ . The idea is that stable models of Test(P, M) capture models of the reduct  $P^M$  that are properly included in M.

**Proposition 4.2** Let P be a disjunctive program and M a (total) model of P. Then M is a minimal model of  $P^M$  if and only if Test(P, M) has no stable model.

**PROOF.** Let  $M \subseteq Hb(P)$  be a total model of P.

 $(\Rightarrow)$  Let N be a stable model of Test(P, M). If  $a \in \operatorname{Hb}(P) - M$ , then there is no rule with a in the head in Test(P, M) and  $a \notin N$ . Hence,  $N \cap \operatorname{Hb}(P) \subseteq M$ . As  $f \notin N$  and  $f \leftarrow M \in \operatorname{Test}(P, M)^N$ , there is some  $a \in M$  such that  $a \notin N \cap \operatorname{Hb}(P)$ . Consider  $A \leftarrow B \in P^M$ . Let  $B \subseteq N \cap \operatorname{Hb}(P) \subseteq M$  but suppose  $A \cap N \cap \operatorname{Hb}(P) = \emptyset$ . If  $A = \{a\}, a \leftarrow B \in \operatorname{Test}(P, M)^N$  and  $a \in N$ , a contradiction. Otherwise  $f \leftarrow B \in \operatorname{Test}(P, M)^N$  and  $f \in N$ , a contradiction. Hence,  $N \cap \operatorname{Hb}(P)$  is a model of  $P^M$  but  $N \cap \operatorname{Hb}(P) \subset M$  implying that Mis not a minimal model of  $P^M$ . ( $\Leftarrow$ ) Assume that M is not a minimal model of  $P^M$ . As M is a model of  $P^M$ , there is a minimal model  $M' \subset M$  of  $P^M$ . We show that  $N = M' \cup \{\hat{a} \mid a \in \text{Heads}(P_D) - M'\}$  is a minimal model of  $\text{Test}(P, M)^N$ , i.e., N a stable model of Test(P, M). Now  $\text{Test}(P, M)^N =$ 

$$\begin{split} &\{a \leftarrow B \mid A \leftarrow B \in P_{\mathrm{D}}^{M}, a \in A \cap M \cap M', B \subseteq M\} \cup \\ &\{\hat{a} \leftarrow \mid a \in \mathrm{Heads}(P_{\mathrm{D}}) - M'\} \cup \\ &\{f \leftarrow B \mid A \leftarrow B \in P_{\mathrm{D}}^{M}, A \cap M' = \emptyset, B \subseteq M\} \cup \\ &\{a \leftarrow B \in P_{\mathrm{N}}^{M} \mid a \in M, B \subseteq M\} \cup \\ &\{f \leftarrow M\}. \end{split}$$

It is easy to check that (i) N is a model of  $\operatorname{Test}(P, M)^N$ . Assume there is a model N' of  $\operatorname{Test}(P, M)^N$  such that  $N' \subseteq N$  holds. We show that then (ii)  $N \subseteq N'$  holds as follows. We notice that for all  $a \in \operatorname{Hb}(P)$ ,  $a \in N'$ implies  $a \in M'$ . Consider  $A \leftarrow B \in P^M$ . If B is true in  $N' \cap \operatorname{Hb}(P)$ , then Bis true in M' and, thus,  $B \subseteq M$  and some  $a \in A \cap M \cap M'$ . Then  $a \leftarrow B \in$  $\operatorname{Test}(P, M)^N$  and  $a \in N'$ . Hence,  $N' \cap \operatorname{Hb}(P)$  is a model of  $P^M$  which implies  $M' \subseteq N' \cap \operatorname{Hb}(P)$ . If  $\hat{a} \in N$ , then  $a \notin M'$  and  $\hat{a} \leftarrow \in \operatorname{Test}(P, M)^N$  implying  $\hat{a} \in N'$ . Then  $N \subseteq N'$  holds. Now (i) and (ii) imply that N is a minimal model of  $\operatorname{Test}(P, M)^N$  and, hence, a stable model of  $\operatorname{Test}(P, M)$ .  $\Box$ 

**Example 4.3** Consider a disjunctive program P and its generator Gen(P):

$$Gen(P) = \{ a \leftarrow \sim \hat{a}, \sim c; b \leftarrow \sim \hat{b}, \sim c; \\ \hat{a} \leftarrow \sim a; \hat{b} \leftarrow \sim b; \\ f \leftarrow \sim f, \sim a, \sim b, \sim c; \\ a^{s} \leftarrow \sim b, \sim c; b^{s} \leftarrow \sim a, \sim c; \\ f \leftarrow \sim f, a, \sim a^{s}; f \leftarrow \sim f, b, \sim b^{s} \}$$

 $P = \{a \lor b \leftarrow \sim c\}$ 

For a stable model  $\{b, b^s, \hat{a}\}$  of Gen(P) the corresponding model candidate is  $M_1 = \{b, b^s, \hat{a}\} \cap \text{Hb}(P) = \{b\}$  and the test program:

$$\operatorname{Test}(P, M_1) = \{ b \leftarrow \sim \hat{b}; \\ \hat{a} \leftarrow \sim a; \quad \hat{b} \leftarrow \sim b; \\ f \leftarrow \sim f, \sim a, \sim b; \quad f \leftarrow \sim f, b \}$$

Test $(P, M_1)$  has no stable models and, hence,  $M_1$  is a stable model of P.  $\Box$ 

The simple generate and test paradigm can be optimized by building model candidates gradually. This means that we start from the empty partial interpretation and extend the interpretation step by step. An interesting observation is that the technique for testing minimality can be used to rule out a partial model candidate of Gen(P) at any stage of the search and not just when a total model of the program P has been found. This can be done by treating a partial interpretation M as a total interpretation where undefined atoms are taken to be false and using the Test(P, M) program.

**Proposition 4.4** Let P be a disjunctive program and M a total interpretation. If Test(P, M) has a stable model, then there is no (total) stable model M' of P such that  $M \subseteq M'$ .

**PROOF.** Let Test(P, M) have a stable model. As shown in the proof of Proposition 4.2, then there is a model M'' of  $P^M$  with  $M'' \subset M$ . Consider any total interpretation M' such that  $M \subseteq M'$  and M' is a model of  $P^{M'}$ . Now M' is not a minimal model of  $P^{M'}$  as  $P^{M'} \subseteq P^M$  and, hence, M'' is a model of  $P^{M'}$  but  $M'' \subset M \subseteq M'$ .

Notice that for a total interpretation M, Proposition 4.4 can only be used for eliminating stable models of P extending M. For guaranteeing the existence of a stable model of P, a total model of P needs to be found making Proposition 4.2 applicable.

Our approach to testing minimality of model candidates differs from that used in DLV [22]. We check minimality by directly searching for models of the reduct strictly contained in the candidate model. In DLV a dual approach is used based on the notion of unfounded sets for disjunctive programs [26] and minimality testing is done using a SAT solver. Our approach could be implemented straightforwardly using a SAT solver, too, but we have chosen to use the same logic program core engine for generating and testing subtasks in order to keep the implementation as simple as possible. A basic difference is that in our approach the set of clauses (9) used for minimality testing follow the structure of the original program whereas in the DLV approach dual clauses (with each literal complemented) are employed. Moreover, DLV employs a couple of optimizations which have not been exploited in our approach. First, DLV adopts specialized algorithms for some syntactically recognizable classes of rules like head cycle free programs. Second, DLV employs modular evaluation techniques for minimality testing where the program is divided into components based on its dependency graph and the

minimality of a candidate model is tested for each component separately by exploiting specialized algorithms for components with corresponding restricted form whenever possible. For a more detailed comparison, see [22].

## 5 EXPERIMENTS

In this section, we compare DLV [24], a state-of-the-art implementation of the stable model semantics for disjunctive logic programs, with an implementation of the generate and test approach of the previous section which we call GNT. In Section 5.1 we explain briefly implementation techniques employed in GNT and explain the setup for the experiments. For comparisons we use three families of test problems related to reasoning about *minimal models* [12], evaluating *quantified Boolean formulas* [47], and *planning* [34] for which encodings of the problem instances as logic programs and test results are presented in Sections 5.2-5.4, respectively. All benchmarks used in the experiments are available at http://www.tcs.hut.fi/Software/gnt/benchmarks/jnssy-tests-20

### 5.1 IMPLEMENTATION

The implementation of GNT [44] is based on SMODELS [46, 45], a program that computes stable models of normal logic programs. The basic idea behind GNT is to use two instances of the SMODELS engine, one that generates the model candidates and one that checks if they are minimal. To implement the idea it is enough to extend the SMODELS engine only slightly. Figure 1 shows the pseudo-code for GNT modified from the original *smodels* function presented in [45]. The function gnt(G, P, A) takes as input a normal (generator) program G, a disjunctive program P and a partial model (a set of literals) A and performs a backtracking search for stable models of G. It returns a stable model M of G which agrees with A and for which minimal(P, M)returns true if such a stable model exists and otherwise it returns false. It uses functions expand(G, A), extend(G, A), conflict(G, A), heuristic(G, A), and minimal(P, A). The first four are as in the original smodels procedure:

• expand(G, A) returns a partial model which expands the given partial model A by literals satisfied by all (total) stable models of G agreeing with A (obtained using a generalized well-founded computation);

- extend(G, A) returns a partial model extending the partial model A by literals obtained by *expand* enhanced with lookahead techniques.
- conflict(G, A) checks whether there is an immediate conflict, i.e., if the partial model A contains a complementary pair of literals and
- heuristic(G, A) returns an atom undefined in A to be used as the next choice point in the backtracking search for stable models.

For further details on these functions see [45]. The function minimal(P, A)performs the minimality test for a disjunctive program P and a partial model A given in Proposition 4.4 using a call to SMODELS, i.e., it views A as a total model A' where all atoms undefined in A are taken to be false, builds the program Test(P, A'), calls SMODELS and returns false if Test(P, A') has a stable model and otherwise returns true. To compute a stable model for a disjunctive program P, the procedure  $qnt(\text{Gen}(P), P, \emptyset)$  is called. First qntextends the given partial model and checks for conflicts. If all atoms are covered by the extended partial model, then a (total) model candidate has been found and it is checked for minimality. Otherwise the heuristic function selects a new undefined atom x and qnt searches recursively first for models where x is false. If no such model is found, the partial model is expanded by making x true. If there is a conflict or the expanded model does not pass an "early" minimality test, the procedure backtracks and otherwise it continues the search recursively using the expanded model. As the "early" minimality tests are computationally quite expensive, some optimization has been employed so that such tests are performed only when backtracking from a model candidate. For this there is a global variable 'WasCovered' which is initially set to false and which is set to true when a model candidate is found. However, it should be noticed that when backtracking from a model candidate, the test could be repeated at each backtracking level until it succeeds. The implementation of the qnt procedure shown in Figure 1 consists of a few hundred lines of code [44] on top of the SMODELS system.

In the sequel, we report several experiments which we carry out in order to compare DLV (version 2003-05-16) with GNT which is based on SMOD-ELS (version 2.27) and uses LPARSE (version 1.0.13) as an instantiator. We consider two versions of our approach, GNT1 and GNT2, which are similar except that in GNT1 generating program G1(P) is used and in GNT2,  $Gen(P) = G1(P) \cup Supp(P)$ . All of our tests are run under Linux 2.4.20 operating system on a 1.7 GHz AMD Athlon XP 2000+ computer with 1

```
function gnt(G, P, A)
A := extend(G, A)
if conflict(G, A) then
  return false
else if A covers Hb(G) then
  WasCovered := true
  if minimal(P, A) then
    return A
  else
    return false
  end if
else
  x := heuristic(G, A)
  A' := gnt(G, P, A \cup \{\sim x\})
  if A' \neq false then
    return A'
  else
    A' := expand(G, A \cup \{x\})
    if conflict(G, A') then
       return false
    else if WasCovered then
       if not minimal(P, A') then
         return false
       end if
    end if
    WasCovered := false
    return gnt(G, P, A')
  end if
end if.
```

Figure 1: GNT Procedure

GB of memory. Execution times are measured using the customary Unix /usr/bin/time command.

### 5.2 MINIMAL MODELS

Our first test problem is the  $\Sigma_2^p$ -complete problem of deciding the existence of a minimal model of a set of clauses in which some specified atoms are true [12]. This problem is mapped to a stable model computation problem as follows. For a problem instance consisting of a set of clauses and some specified atoms, a program P is constructed where each clause  $a_1 \vee \cdots \vee a_n \vee \neg b_1 \vee \cdots \vee \neg b_m$ is translated into a rule  $a_1 \vee \cdots \vee a_n \leftarrow b_1, \ldots, b_m$  and for each specified atom  $c_i$ , a rule

$$f \leftarrow \sim f, \sim c_i \tag{10}$$

is included. Now P has a stable model if and only if there is a minimal model of the clauses containing all specified atoms  $c_i$ .

The test cases (random disjunctive 3-SAT programs) are based on random 3-SAT problems having a fixed clauses/atoms ratio c and they are constructed as follows. Given a number of atoms n, a random 3-SAT problem is generated, i.e.  $c \times n$  clauses are generated each by picking randomly three distinct atoms from the n available and selecting their polarity uniformly. This is done using a program MAKEWFF developed by Bart Selman. Then the clauses are translated into rules as described above and for  $i = 1, \ldots, \lfloor 2n/100 \rfloor$  and for random atoms  $c_i$ , the extra rules (10) are added. The problem size is controlled by the number of atoms n which is increased by increments of 10. For each n, we test 100 random 3-SAT programs and measure the maximum, average, and minimum time it takes to decide whether a stable model exists.

In the first set of tests we study the effect of different generating programs on the performance of our approach, i.e., we compare GNT1 and GNT2, which are similar except that in GNT1 generating program G1(P) is used and in GNT2,  $Gen(P) = G1(P) \cup Supp(P)$ . We test at two clauses/atoms ratios. The first test is at 4.258 which is in the phase transition region [7] where roughly 50% of the generated 3-SAT clause sets are satisfiable. The second test is at clauses/atoms ratio 3.750 where practically all generated 3-SAT clause sets are satisfiable.

The test results are shown in Figure 2. In the first test set the key problem seems to be finding at least one model candidate. The simpler generator (GNT1) appears to perform relatively well except for a few outliers,

i.e. instances with significantly higher running time than the average. The outliers occur when the generator program G1(P) allows a high number of candidate models. At clauses/atoms ratio 3.750 the frequency of outliers for GNT1 increases and outliers occur already in smaller problem sizes. The more involved generating program Gen(P) behaves in a much more robust way and the average running time of GNT2 is significantly lower than that of GNT1. Next we use the same two test sets for comparing GNT2 and DLV. The results are shown in Figure 3. The systems scale very similarly in both test sets but DLV seems to be roughly a constant factor faster than GNT2. This is probably due to the overhead caused by the more complicated generating program in GNT2 and by the two level architecture of GNT2 where two instances of SMODELS are cooperating.

### 5.3 QUANTIFIED BOOLEAN FORMULAS

We continue the comparison of GNT2 and DLV using instances of quantified Boolean formulas (QBFs) and develop a new way to encode such formulas as disjunctive logic programs. In our experiments, we consider a specific subclass of QBFs, namely 2,  $\exists$ -QBFs. Such formulas are of the form  $\exists X \forall Y \phi$ where X and Y are sets of existentially and universally quantified propositional variables, respectively, and  $\phi$  is a Boolean formula based on  $X \cup Y$ . Deciding the validity of such a formula forms a  $\Sigma_2^p$ -complete decision problem [47] even if  $\phi$  is assumed to be a Boolean formula in 3DNF [48]. Recall that checking the existence of a stable model for a disjunctive logic program is of equal computational complexity [12], which implies the existence of polynomial time transformations between the decision problems mentioned above. In fact, Eiter and Gottlob [12] show how a QBF of the form  $\exists X \forall Y \phi$  with  $\phi$  in 3DNF can be translated into a disjunctive logic program P such that  $\exists X \forall Y \phi$  is valid if and only if P has a stable model. This translation is used by Leone et al. [25] to compare DLV and GNT2.

However, we present an alternative transformation in order to obtain a better performance for the two systems under comparison. Our transformation is based on the following ideas. The first observation is that we can rewrite  $\exists X \forall Y \phi$  with  $\phi$  in DNF as  $\exists X \neg \exists Y \neg \phi$  where  $\neg \phi$  can be understood as a Boolean formula in CNF or as a *set of clauses* S so that each clause  $c \in S$  can represented as a disjunction of the form

$$X_1 \lor \neg X_2 \lor Y_1 \lor \neg Y_2 \tag{11}$$



Disjunctive 3-SAT: clauses/atoms = 4.258

Figure 2: Minimal Models: GNT1 vs. GNT2





Figure 3: Minimal Models: DLV vs. GNT2

where  $X_1$  and  $\neg X_2$  are the sets of positive and negative literals, respectively, which appear in c and involve variables from X while  $Y_1$  and  $\neg Y_2$  are similarly related to Y. It follows that  $\exists X \forall Y \phi$  is valid if and only if we can find an interpretation<sup>3</sup>  $I : X \cup Y \to \{\mathbf{t}, \mathbf{f}\}$  such that  $\neg \phi_{I(X)}$  is unsatisfiable where  $\neg \phi_{I(X)}$  denotes the set of clauses  $Y_1 \lor \neg Y_2$  for which (11) belongs to S and  $I \not\models X_1 \lor \neg X_2$ . The second idea behind our transformation is to choose the truth value of the condition  $X_1 \lor \neg X_2$  for each clause (11) in S rather than the truth values of the variables in  $X \cup Y$ . This line of thinking leads to the following translation of S.

**Definition 5.1** A clause c of the form (11) where  $X_1, X_2 \subseteq X$  and  $Y_1, Y_2 \subseteq Y$  is translated into following sets of rules:

$$\begin{aligned} \operatorname{Tr}_{\mathrm{V}}(c) &= \{ c \leftarrow \sim \hat{c}; \ \hat{c} \leftarrow \sim c \}, \\ \operatorname{Tr}_{\mathrm{E}}(c) &= \{ f \leftarrow x, \sim \hat{c}, \sim f \mid x \in X_1 \} \cup \{ x \leftarrow \sim \hat{c} \mid x \in X_2 \} \cup \\ \{ f \leftarrow X_2, \sim X_1, \sim c, \sim f \}, \ and \\ \operatorname{Tr}_{\mathrm{U}}(c) &= \{ y \leftarrow u \mid y \in Y_1 \cup Y_2 \} \cup \{ Y_1 \cup \{ u \} \leftarrow Y_2, \sim \hat{c} \} \end{aligned}$$

where c and  $\hat{c}$  are new atoms associated with the clause c, and f and u are new atoms. A set of clauses S is translated into

$$\bigcup_{c \in S} (\operatorname{Tr}_{\mathcal{V}}(c) \cup \operatorname{Tr}_{\mathcal{E}}(c) \cup \operatorname{Tr}_{\mathcal{U}}(c)) \cup \{u \leftarrow \sim u\}.$$

We use Boolean variables from  $X \cup Y$  as propositional atoms in the translation. Intuitively, the rules of  $\operatorname{Tr}_{V}(c)$  choose whether a clause c is *active*, i.e.  $X_1 \vee \neg X_2$  evaluates to false so that the satisfaction of the clause c depends on the values assigned to  $Y_1 \cup Y_2$ . The rules in  $\operatorname{Tr}_{E}(c)$  try to *explain* the preceding choice by checking that the values of the variables in X can be assigned accordingly. Finally, the rules in  $\operatorname{Tr}_{U}(c)$  implement the test for unsatisfiability together with the rule  $u \leftarrow \sim u$ . Basically, the same unsatisfiability check is used in the translation proposed by Eiter and Gottlob. However, the transformation given in Definition 5.1 is more economical as it uses far less new atoms and disjunctive rules. In particular, note that variables from  $X \cup Y$ not appearing in the clauses do not contribute any rules to the translation.

Next we address the correctness of our transformation and consider a 2,  $\exists$ -QBF  $\exists X \forall Y \phi$  where  $\phi$  is in DNF and the disjunctive logic program P obtained by translating  $\neg \phi$  (a set of clauses S) according to Definition 5.1.

<sup>&</sup>lt;sup>3</sup>The values assigned by I to the variables in Y are not important, but make I a proper interpretation over  $X \cup Y$ .

**Lemma 5.2** Let  $M \subseteq Hb(P)$  be a total propositional interpretation for the translation P such that for every clause  $c \in S$  of the form (11), (i)  $c \in M$  $\iff \hat{c} \notin M$  and (ii)  $c \in M \iff X_1 \cap M = \emptyset$  and  $X_2 \subseteq M$ .

Then the programs  $\operatorname{Tr}_{V}(c)$ ,  $\operatorname{Tr}_{E}(c)$ , and  $\operatorname{Tr}_{U}(c)$  associated with a clause  $c \in S$  of the form (11) satisfy the following.

- (R1) The fact  $c \leftarrow$  belongs to  $\operatorname{Tr}_{\mathcal{V}}(c)^M \iff c \in M$ .
- (R2) The fact  $\hat{c} \leftarrow$  belongs to  $\operatorname{Tr}_{\mathcal{V}}(c)^M \iff \hat{c} \in M$ .
- (R3) For  $x \in X_1$ , the rule  $f \leftarrow x$  belongs to  $\operatorname{Tr}_{\mathrm{E}}(c)^M \iff x \notin M$  and  $f \notin M$ .
- (R4) For  $x \in X_2$ , the fact  $x \leftarrow$  belongs to  $\operatorname{Tr}_{\mathrm{E}}(c)^M \iff x \in M$ .
- (R5) The rule  $f \leftarrow X_2$  belongs to  $\operatorname{Tr}_{\mathrm{E}}(c)^M \iff X_2 \not\subseteq M$  and  $f \notin M$ .
- (R6) For  $y \in Y_1 \cup Y_2$ , the rule  $y \leftarrow u$  belongs to  $\operatorname{Tr}_{U}(c)^M$  unconditionally.
- (R7) The rule  $Y_1 \cup \{u\} \leftarrow Y_2$  belongs to  $\operatorname{Tr}_{\mathrm{U}}(c)^M \iff c \in M$ .

**Theorem 5.3** The quantified Boolean formula  $\exists X \forall Y \phi$  is valid if and only if the translation P has a stable model.

**PROOF.** We may safely assume that all variables in  $X \cup Y$  actually appear in  $\phi$ , since redundant variables can be dropped without affecting the validity of the formula nor the structure of its translation.

 $(\Longrightarrow)$  Suppose that  $\exists X \forall Y \phi$  is valid. Then there is an interpretation  $I : X \cup Y \to \{\mathbf{t}, \mathbf{f}\}$  such that  $I \models \forall Y \phi$ . Then define  $X_I = \{x \in X \mid I(x) = \mathbf{t}\}$ . Without loss of generality we may assume that  $X_I$  is minimal, i.e. there is no interpretation J such that  $J \models \forall Y \phi$  and  $X_J \subset X_I$ . Then define a total propositional interpretation

$$M = X_I \cup Y \cup \{u\} \cup$$

$$\{c \mid c = X_1 \vee \neg X_2 \vee Y_1 \vee \neg Y_2 \in \neg \phi \text{ and } I \not\models X_1 \vee \neg X_2\} \cup$$

$$\{\hat{c} \mid c = X_1 \vee \neg X_2 \vee Y_1 \vee \neg Y_2 \in \neg \phi \text{ and } I \models X_1 \vee \neg X_2\}. (12)$$

It is verified next that M is a stable model of P. The definition of M implies that M satisfies the requirements of Lemma 5.2. Then (R1)–(R7) effectively describe the structure of  $P^M$  and it is easy to verify that M is a model of  $P^M$ on the basis of these relationships, as  $Y \cup \{u\} \subseteq M$  and  $f \notin M$  by definition. Next we assume that  $N \subseteq M$  is a model of  $P^M$  and show that  $M \subseteq N$ . (i) If  $c \in M$  for some clause c of the form (11), then  $c \in N$ , as  $c \leftarrow$  belongs to  $P^M$  by (R1). (ii) Similarly,  $\hat{c} \in M$  implies  $\hat{c} \in N$  by (R2). (iii) We have  $u \in N$  because otherwise N would form a model of  $\neg \phi_{I(X)}$  by satisfying the rules  $Y_1 \cup \{u\} \leftarrow Y_2$  included in  $P^M$  by (R7). (iv) Moreover,  $Y \subseteq N$ holds, as  $u \in N$  and N satisfies all the rules  $y \leftarrow u$  belonging to  $P^M$  by (R6). (v) Let us define an interpretation  $J : X \cup Y \to \{\mathbf{t}, \mathbf{f}\}$  such that for  $x' \in X, J(x') = \mathbf{t} \iff x' \in N$ , and for  $y \in Y, J(y) = I(y)$ . Using (R4), we can establish for any (11) that  $I \not\models X_1 \lor \neg X_2$  implies  $J \not\models X_1 \lor \neg X_2$ . Thus  $\neg \phi_{I(X)} \subseteq \neg \phi_{J(X)}$  where  $\neg \phi_{I(X)}$  is known to be unsatisfiable. The same follows for  $\neg \phi_{J(X)}$  so that J qualifies as an assignment for which  $J \models \forall Y \phi$ holds. But then the minimality of I implies  $J = I, X_J = X_I$ , and  $X_I \subseteq N$ . To conclude the preceding analysis,  $M \subseteq N$  and M is a stable model of P.

 $(\Leftarrow)$  Suppose that P has a stable model M. Then define an interpretation  $I: X \cup Y \to \{\mathbf{t}, \mathbf{f}\}$  by setting  $I(z) = \mathbf{t} \iff z \in M$  for any  $z \in X \cup Y$ . Let us then establish that M and I satisfy (12). (i) The definition of I implies that  $X_I = M \cap X$ . (ii) Now  $u \in M$ , because P contains  $u \leftarrow \sim u$  and M is a stable model of P. (iii) For the same reason,  $f \notin M$ , because all the rules having f as the head have  $\sim f$  among the negative body literals. (iv) Since  $u \in M$  and  $P^M$  contains the rule  $y \leftarrow u$  for every  $y \in Y$ , we obtain  $Y \subseteq M$ . (v) For any clause c of the form (11), the structure of  $\operatorname{Tr}_V(c) \subseteq P$ implies that  $c \in M \iff c \leftarrow$  belongs to  $\operatorname{Tr}_V(c)^M \subseteq P^M \iff \hat{c} \notin M$ . Using this property, we can establish that  $c \in M \iff I \not\models X_1 \vee \neg X_2$  holds for the interpretation I defined above. (vi) Thus  $\hat{c} \in M \iff I \not\models X_1 \vee \neg X_2$ is implied by the fact that  $c \in M \iff \hat{c} \notin M$ , as shown above in (v).

It remains to show that  $\neg \phi_{I(X)}$  is unsatisfiable. So let us assume the contrary, i.e. there is a model  $Y' \subseteq Y$  for  $\neg \phi_{I(X)}$ . Note that M meets the requirements of Lemma 5.2 by (v) and (vi) above, as the definition of I implies  $I \not\models X_1 \lor \neg X_2 \iff X_1 \cap M = \emptyset$  and  $X_2 \subseteq M$ . The relationships (R1)–(R7) imply that  $N = (M - (Y \cup \{u\})) \cup Y'$  is a model of  $P^M$ , too. Since  $u \notin N$ , we have  $N \subset M$  indicating that M is not stable, a contradiction. Thus  $\neg \phi_{I(x)}$  is unsatisfiable which implies  $I \models \forall Y \phi$  and the validity of  $\exists X \forall Y \phi$ .  $\Box$ 

Using an implementation of the translation given in Definition 5.1 we are able to transform 2,  $\exists$ -QBFs into disjunctive programs. The remaining question is how to generate 2,  $\exists$ -QBF instances. We use two different schemes based on random instances [6, 19]. In the first scheme, the sets of variables X and Y satisfy |X| = |Y|. Each random instance is based on v = |X| +|Y| variables and a Boolean formula  $\phi$  which is a disjunction of  $d = 2 \times v$  conjunctions of 5 random literals out of which at least two literals involve a variable from Y, as suggested by Gent and Walsh [19]. This scheme is slightly different from  $2\text{QBF}_{GW}$  in [25] based on 3 literal conjunctions just to obtain a more challenging benchmark. The constant factor 2 in the equation relating d and v has been determined as a phase transition point for the DLV system by keeping v = 50 fixed and varying the number of disjunctions in  $\phi$ . In the actual experiment, the number of v variables is varied from 5 to 50 by increments of 5. We generate 100 instances of 2,  $\exists$ -QBFs for each value of v and translate them into corresponding disjunctive logic programs. The running times for DLV and GNT2 are depicted in the upper graph of Figure 4. The systems scale very similarly, but DLV is on the average from one to two decades faster than GNT2.

In the second experiment with 2,  $\exists$ -QBFs, we use a different scheme for the number of disjunctions  $d = \lfloor \sqrt{v/2} \rfloor$  as well as the number of literals which is 3 in each conjunction. The resulting instances are much easier to solve, because d remains relatively low (e.g.  $d \approx 41$  for v = 3500) and many variables do not appear in  $\phi$  at all. We let v vary from 50 to 3550 by increments of 50 and generate 100 instances of 2,  $\exists$ -QBFs for each value of v. The resulting running times are shown in the lower graph of Figure 4. The shapes of the curves are basically the same, but the performance of GNT2 degrades faster than that of DLV. However, the benefits of the translation given in Definition 5.1 are clear, as GNT2 is able to solve much larger instances than reported in [25] where 40 variables turn out to be too much for GNT2. As far as we understand, this is due to the sizes of search spaces associated with the translated instances of 2,  $\exists$ -QBFs. For the translation given in Definition 5.1, the size of the search space examined by GNT2 is of order  $2^{\sqrt{v/2}}$  whereas it is of order  $2^v$  if the translation proposed by Eiter and Gottlob [12] is used.

### 5.4 PLANNING

In order to get an idea of the overhead of GNT2 when compared to SMOD-ELS, we study three blocks world planning problems encoded as normal programs [34]:

- large.c is a 15 blocks problem requiring a 8 step plan using the encoding given in [34] allowing parallel execution of operators,
- large.d is a 17 blocks problem with a 9 step plan and



Quantified Boolean Formulas: Scheme 1

Figure 4: Quantified Boolean Formulas: DLV vs. GNT2

Table 1: Planning: SMODELS vs. GN12					
$\operatorname{Problem}$	Number of	Number of	Time $(s)$	Time $(s)$	
	$\operatorname{steps}$	ground rules	SMODELS	GNT2	
large.c	8	81681	4.5	10.3	
	7	72527	0.6	2.1	
large.d	9	127999	10.1	21.2	
	8	115109	1.4	5.2	
large.e	10	191621	18.2	35.0	
	9	174099	2.2	8.7	

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• large.e is a 19 blocks problem with a 10 step plan.

Table 1 contains two entries for each problem: one reporting the time needed to find a valid plan with the "optimal" number of steps given as input and one reporting the time needed to show optimality, i.e., that no plan (no stable model) exists when the number of situations is decreased by one. The times reported for each test case are the execution times of SMODELS and GNT2 given a ground normal program (generated by LPARSE) as input. The results show that there is some overhead in the current implementation of GNT2 even for normal programs and GNT2 handles these examples 2-4 times slower than SMODELS.

#### CONCLUSIONS 6

The paper presents an approach to implementing partial and disjunctive stable models using an implementation of stable models for disjunction-free programs as the core inference engine. The approach is based on unfolding partiality and disjunctions from a logic program in two separate steps. In the first step partial stable models of disjunctive programs are captured by total stable models using a simple linear program transformation. Thus, reasoning tasks concerning partial models can be solved using an implementation of total models such as the DLV system. This also sheds new light on the relationship between partial and total stable models by establishing a close correspondence. In the second step a generate and test approach is developed for computing total stable models of disjunctive programs using a core engine capable of computing stable models of normal programs. We have developed an implementation of the approach using SMODELS as the core engine. The

extension is fairly simple consisting of a few hundred lines of code. The approach turns out to be competitive even against a state-of-the-art system for disjunctive programs. The efficiency of the approach comes partly from the fact that normal programs can capture essential properties of disjunctive stable models that help with decreasing the computational complexity of the generate and test phases in the approach. However, a major part of the success can be accounted for by the efficiency of the core engine. This suggests that more efforts should be spent in developing efficient core engines.

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