COARSE COMPUTABILITY, THE DENSITY METRIC, HAUSDORFF DISTANCES BETWEEN TURING DEGREES, PERFECT TREES, AND REVERSE MATHEMATICS

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Abstract. For $A \subseteq \omega$, the *coarse similarity class* of A, denoted by $[A]$, is the set of all $B \subseteq \omega$ such that the symmetric difference of A and B has asymptotic density 0. There is a natural metric δ on the space $\mathcal{S}$ of coarse similarity classes defined by letting $\delta([A],[B])$ be the upper density of the symmetric difference of A and B. We study the metric space of coarse similarity classes under this metric, and show in particular that between any two distinct points in this space there are continuum many geodesic paths. We also study subspaces of the form $\{[A] : A \in \mathcal{U}\}$ where $\mathcal{U}$ is closed under Turing equivalence, and show that there is a tight connection between topological properties of such a space and computability-theoretic properties of $\mathcal{U}$.

We then define a distance between Turing degrees based on Hausdorff distance in the metric space $(\mathcal{S}, \delta)$. We adapt a proof of Monin to show that the Hausdorff distances between Turing degrees that occur are exactly 0, 1/2, and 1, and study which of these values occur most frequently in the senses of Lebesgue measure and Baire category. We define a degree $\mathbf{a}$ to be *attractive* if the class of all degrees at distance 1/2 from $\mathbf{a}$ has measure 1, and *dispersive* otherwise. In particular, we study the distribution of attractive and dispersive degrees. We also study some properties of the metric space of Turing degrees under this Hausdorff distance, in particular the question of which countable metric spaces are isometrically embeddable in it, giving a graph-theoretic sufficient condition for embeddability.

Motivated by a couple of issues arising in the above work, we also study the computability-theoretic and reverse-mathematical aspects of a Ramsey-theoretic theorem due to Mycielski, which in particular implies that there is a perfect set whose elements are mutually 1-random, as well as a perfect set whose elements are mutually 1-generic.

Finally, we study the completeness of $(\mathcal{S}, \delta)$ from the perspectives of computability theory and reverse mathematics.

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Contents

1. Introduction

We call two sets $A, B \subseteq \omega$ *coarsely similar* if their symmetric difference $A \triangle B$ has asymptotic density 0, i.e., $\lim_n \frac{|(A \triangle B) \cap [0,n)|}{n} = 0$. This is an equivalence relation on subsets of ω that arises naturally in studying coarse computability (see e.g. [37]), as well as in other areas of computability theory, for example in the work of Greenberg, Miller, Shen, and Westrick [26], who showed that a real has effective Hausdorff dimension 1 if and only if it is coarsely similar to a 1-random real (i.e., one that is random in the sense of Martin-Löf). A set A is called *coarsely computable* if it is coarsely similar to some computable set. Let $\mathcal{S}$ be the set of equivalence classes of the above equivalence relation. We consider a natural metric δ on $\mathcal{S}$ given by $\delta([A], [B]) = \overline{\rho}(A \triangle B)$, where $[A]$ is the equivalence class of A, and $\overline{\rho}$ is upper density, and explore several ways in which it can interact with Turing reducibility.

We first study the metric space $(\mathcal{S}, \delta)$. This space is non-separable and non-compact but is complete and contractible. We show that any two distinct points in $\mathcal{S}$ are joined by continuum many geodesic paths. We also study the topological properties of subspaces of this metric space generated by collections of sets that are closed under Turing equivalence. We establish close connections between topological properties of these subspaces and computability-theoretic properties of their generating collections of sets.

The second main topic of this paper is the application of the metric δ to the study of Turing degrees. Since δ is bounded, the notion of Hausdorff distance between subsets of $\mathcal{S}$ yields a metric on the closed subsets of $(\mathcal{S}, \delta)$. We define the *closure* $\overline{\mathbf{d}}$ of a Turing degree $\mathbf{d}$ to be the closure of $\{[A] : A \in \mathbf{d}\}$ in $(\mathcal{S}, \delta)$. We show that the Hausdorff distance between closures of Turing degrees yields a metric H on the set of Turing degrees $\mathcal{D}$. We show how to calculate Hausdorff distances between closures of Turing degrees using a relativized version of the function Γ (see [1]), which is based on a generalized version of coarse computability. We adapt a proof of B. Monin [47] to show that the distances that occur are exactly 0, 1/2, and 1, and determine which distances between Turing degrees occur most frequently in terms of Lebesgue measure and Baire category. We define a degree $\mathbf{a}$ to be *attractive* if the class of all degrees at distance $1/2$ from $\mathbf{a}$ has measure 1, and *dispersive* otherwise. In particular, we study the distribution of attractive and dispersive degrees.

We also study which countable metric spaces are isometrically embeddable in $(\mathcal{D}, H)$. If $\mathcal{M}$ is a countable metric space with all distances equal to 0, $1/2$, or 1, let $G_{\mathcal{M}}$ be the graph whose vertices are the points of $\mathcal{M}$, such that two points are joined by an edge if and only if the distance between them is 1. We show that if $G_{\mathcal{M}}$ is a comparability graph (i.e., there is a partial order $\prec$ on the set of vertices of $G_{\mathcal{M}}$ such that there is an edge between distinct vertices x and y if and only if they are $\prec$-comparable), then $\mathcal{M}$ is isometrically embeddable in $(\mathcal{D}, H)$. We also show that the complement of the graph $G_{(\mathcal{D},H)}$ (where pairs of degrees at distance $1/2$ are joined by an edge) is a connected graph with diameter at least 3 and at most 4.

The interplay between randomness and genericity (i.e., between being typical in the sense of measure and being typical in the sense of category) is a recurring theme in this paper. As we will see, having distance $1/2$ in the Hausdorff metric is related to mutual randomness, while having distance 1 is related to relative genericity. Thus, for instance, both our discussion of attractive and dispersive degrees in Section 5 and that of isometric embeddings into $(\mathcal{D}, H)$ in Section 6 rely heavily on randomness and genericity. For example, one of the results we obtain in the former section is that if A is weakly 2-generic and B is 2-random, then B computes a set that is weakly 1-generic relative to A, which, as we will see, implies that $H(A, B) = 1$.

At the end of Section 2, we discuss a Ramsey-theoretic theorem due to Mycielski [48], which in particular implies that there is a perfect set whose elements are mutually random (say in the sense of Martin-Löf randomness), as well as a perfect set whose elements are mutually generic. This theorem has several interesting computability-theoretic and reverse-mathematical aspects, which we discuss in Section 7. In particular, we show that there is a $\emptyset'$-computable perfect tree such that the join of any nonempty finite collection of pairwise distinct paths is 1-random.

In Section 8, we discuss the computability-theoretic and reverse-mathematical strength of the fact that $(\mathcal{S}, \delta)$ is complete. We finish with a section containing several open questions.

After the first part of Section 2, we assume familiarity with computability theory, as in [53], for instance. We will also use basic concepts and results from the theory of algorithmic randomness, and refer to [20] for details. This book also includes sections on genericity (Section 2.24) and on interactions between randomness and genericity (Section 8.21). Particularly useful is van Lambalgen's Theorem, which implies that if A and B are 1-random and A is 1-random relative to B, then B is 1-random relative to A. This fact is one of the reasons that we work with 1-randomness below, even though for some results, weaker notions of algorithmic randomness might suffice. We will also use the fact that the analog of van Lambalgen's Theorem for 1-genericity in place of 1-randomness holds, as shown by Yu [57]. In Sections 7 and 8 we will also assume some knowledge of reverse mathematics, as in [52], for instance.

We need to note several basic definitions. First we recall our definition of when two sets are "essentially the same". Since we are working with subsets of ω, we use classical asymptotic density from number theory. For $A \subseteq \omega$ and $n \in \omega$, let

$A \upharpoonright n = \{m < n : m \in A\}$ (which we also identify with the binary string σ of length n such that $\sigma(m) = 1$ if and only if $m \in A$).

Definition 1.1. If $A \subseteq \omega$, then, for $n \geqslant 1$, the *density of A at n* is

$$\rho_n(A) = \frac{|A \upharpoonright n|}{n}.$$

The *asymptotic density*, $\rho(A)$, of A is $\lim_n \rho_n(A)$ if this limit exists.

While the limit for density does not exist in general, the *upper density* $\overline{\rho}(A) = \limsup_n \rho_n(A)$ and the *lower density* $\underline{\rho}(A) = \liminf_n \rho_n(A)$ always exist.

Although easy, the following lemma is basic. Here we write $\neg A$ for the complement of A since we will use overlines for closures.

Lemma 1.2. *If $A \subseteq \omega$ then $\underline{\rho}(A) = 1 - \overline{\rho}(\neg A)$.*

Proof. Note that $\rho_n(A) = 1 - \rho_n(\neg A)$ for all $n \geqslant 1$. The lemma follows by taking the lim inf of both sides of this equation. □

We identify sets and their characteristic functions. The symmetric difference $A \triangle B = \{n : A(n) \neq B(n)\}$ is the subset of ω where A and B disagree. There does not seem to be a standard notation for the complement of $A \triangle B$, which is $\{n : A(n) = B(n)\}$, the *symmetric agreement* of A and B. We find it useful to use $A \triangledown B$ to denote $\{n : A(n) = B(n)\}$.

Definition 1.3. If $A, B \subseteq \omega$, then A and B are *coarsely similar*, written $A \sim_c B$, if the density of the symmetric difference of A and B is 0, that is, $\rho(A \triangle B) = 0$. Equivalently, $\rho(A \triangledown B) = 1$. Given A, any set B such that $B \sim_c A$ is called a *coarse description* of A.

It is easy to check that coarse similarity is indeed an equivalence relation on $\mathcal{P}(\omega)$. We write $[A]$ for the coarse similarity class of the set A. Let $\mathcal{S}$ denote the set of all coarse similarity classes. There is a natural pseudo-metric on $\mathcal{P}(\omega)$.

Definition 1.4. If $A, B \subseteq \omega$, let $\delta(A, B) = \overline{\rho}(A \triangle B)$.

A Venn diagram argument shows that δ satisfies the triangle inequality and is therefore a pseudo-metric on subsets of ω. Since $\delta(A, B) = 0$ exactly when A and B are coarsely similar, δ is actually a metric on the space $\mathcal{S}$ of coarse similarity classes, and we now work in the metric space $(\mathcal{S}, \delta)$.

Recall that we apply Lebesgue measure in computability theory by regarding the set $A \subseteq \omega$ as corresponding to the binary expansion defined by its characteristic function. For any set A, the Strong Law of Large Numbers implies that $\mathcal{M} = \{B : \delta(B, A) = 1/2\}$ has measure 1 in $\mathcal{P}(\omega)$. It follows that the complement of $\mathcal{M}$ has measure 0, which implies that any coarse similarity class has measure 0.

This metric has probably been rediscovered many times. It has recently been used on subsets of $\mathbb{Z}$ to study cellular automata on the line as dynamical systems. See [13] and [28]. The automata theory literature ascribes this metric to Besicovitch and cites his well-known book [10] on almost periodic functions as a reference. At least in his book, however, Besicovitch does not consider arbitrary subsets of ω or any metric on them. We will refer to this metric as the *density metric*.

Of course, $\mathcal{P}(\omega)$ can be made into an abelian group of exponent 2 by defining $A + B = A \triangle B$. The operation $+$ is well-defined as a map (also denoted $+$) from $\mathcal{S}^2$ to $\mathcal{S}$ and it is clear that $+$ is continuous in the metric on $\mathcal{S}$. Therefore $\mathcal{S}$ has the structure of a topological group by defining $[A] + [B] = [A \triangle B]$. The important property of this topological group is the following.

Observation 1.5. Let $C \subseteq \omega$. Define the translation map $\tau_C : \mathcal{S} \to \mathcal{S}$ by $\tau_C([A]) = [A] + [C]$. Since the operation $+$ is $\triangle$,

$$(A + C) + (B + C) = A + B$$

is the statement that

$$(A \triangle C) \triangle (B \triangle C) = A \triangle B,$$

which holds. Thus τ_C is an isometry, and $\mathcal{S}$ acts on itself by isometries.

Considering this topological group and its actions is of interest to computability theory. For instance, Kuyper and Miller [42] studied set stabilizers for these actions for the classes of 1-random and 1-generic sets.

2. Properties of the metric space $(\mathcal{S}, \delta)$

In this section, we explore the properties of the metric space $(\mathcal{S}, \delta)$. No knowledge of computability theory is needed to read most of this section, although our goal later will be to explore connections between $(\mathcal{S}, \delta)$ and computability theory.

We first show that the space $\mathcal{S}$ is non-separable and non-compact in a very strong sense.

The following sequence of intervals is basic to studying coarse computability.

Definition 2.1. Let $I_n = [n!, (n+1)!)$, and let $\mathcal{I}(A) = \bigcup_{n \in A} I_n$.

Theorem 2.2. (1) *If $\mathcal{A} = \{[A_i]\}$ is any countable subset of $\mathcal{S}$, then there is a class $[B]$ such that $\delta([B], [A_i]) = 1$ for all i.*

(2) *There is a subset $\mathcal{U}$ of $\mathcal{S}$ of size continuum such that the members of $\mathcal{U}$ are pairwise at distance 1 from each other.*

Proof. (1) Let $\langle i, m \rangle$ denote the pairing function from $\omega \times \omega \to \omega$. Define B as follows. If $n = \langle i, m \rangle$ then B agrees with the complement $\neg A_i$ of A_i on I_n. So $\rho_{(n+1)!-1}(B \bigtriangledown A_i) \leqslant 1/(n+1)$. Thus $\underline{\rho}(B \bigtriangledown A_i) = 0 = 1 - \overline{\rho}(B \triangle A_i)$, and $\delta([B], [A_i]) = 1$.

(2) Let $\mathcal{C}$ be a collection of continuum many infinite subsets of ω such that any two distinct sets in $\mathcal{C}$ have finite intersection. For example, identify the nodes of the infinite perfect binary tree T with natural numbers, and let $\mathcal{C}$ be the set of paths through T. Then let $\mathcal{U} = \{[\mathcal{I}(A)] : A \in \mathcal{C}\}$. Argue as in the proof of (1) that any two distinct elements of $\mathcal{U}$ are at distance 1 from each other. Clearly $\mathcal{U}$ has size continuum. □

The first part of the following corollary was shown by Blanchard, Formenti, and Kůrka [13].

Corollary 2.3. *The space $(\mathcal{S}, \delta)$ is not compact. Indeed, open covers need not have countable subcovers, so $(\mathcal{S}, \delta)$ is not Lindelöf.*

Proof. Consider any cover of $\mathcal{S}$ by open balls, all of whose radii are less than 1. Let $\mathcal{C} = \{\mathcal{B}_i\}$ be any countable subset of the cover. By the previous theorem, there is a point $[P]$ at distance 1 from all the centers of the $\mathcal{B}_i$ and thus $[P] \notin \bigcup_{\mathcal{B}_i \in \mathcal{C}} \mathcal{B}_i$. □

We now show that $(\mathcal{S},\delta)$ has several "good" properties, namely it is complete, contractible to a point, and geodesic in a strong sense. Completeness and pathwise connectedness (using $\mathbb{Z}$) were shown by completely different arguments by Blanchard, Formenti, and Kůrka [13], in a paper on automata theory (see also [28]). We give "computability-theoretic" proofs.

Definition 2.4. Let J_k be the interval $[2^k-1, 2^{k+1}-1)$. For any set C, let $d_k(C)$ be the density of C on J_k, that is, $d_k(C) = \frac{|C\cap J_k|}{2^k}$.

The following lemma, Lemma 5.10 of [31], relates $\overline{\rho}(C)$ to $\overline{d}(C) = \limsup_k d_k(C)$. We include the proof for the sake of self-containment.

Lemma 2.5 (Hirschfeldt, Jockusch, McNicholl, and Schupp [31]). *For every set C,*

$$\frac{\overline{d}(C)}{2} \leqslant \overline{\rho}(C) \leqslant 2\overline{d}(C).$$

Proof. For all k,

$$d_k(C) = \frac{|C\cap J_k|}{2^k} \leqslant \frac{|C \upharpoonright 2^{k+1}|}{2^k} = 2\rho_{2^{k+1}}(C).$$

Dividing both sides of this inequality by 2 and then taking the lim sup of both sides yields the fact that $\frac{\overline{d}(C)}{2} \leqslant \overline{\rho}(C)$.

To prove that $\overline{\rho}(C) \leqslant 2\overline{d}(C)$, assume that $k-1 \in J_n$, so $2^n \leqslant k < 2^{n+1}$. Then

$$\rho_k(C) = \frac{|C\upharpoonright k|}{k} \leqslant \frac{|C\upharpoonright (2^{n+1}-1)|}{2^n} = \frac{\sum_{0\leqslant i\leqslant n}|C\upharpoonright J_i|}{2^n} = \frac{\sum_{0\leqslant i\leqslant n} 2^i d_i(C)}{2^n} < 2\max_{i\leqslant n} d_i(C).$$

Let $\varepsilon > 0$ be given. Then $d_i(C) < \overline{d}(C) + \varepsilon$ for all sufficiently large i. Hence there is a finite set F such that $d_i(C\setminus F) < \overline{d}(C\setminus F) + \varepsilon$ for *all* i. Then, by the above inequality applied to $C \setminus F$, we have $\rho_k(C\setminus F) < 2(\overline{d}(C\setminus F)+\varepsilon)$ for all k, so $\overline{\rho}(C\setminus F) \leqslant 2\overline{d}(C\setminus F)$. As $\overline{\rho}$ and $\overline{d}$ are invariant under finite changes of their arguments and $\varepsilon > 0$ is arbitrary, it follows that $\overline{\rho}(C) \leqslant 2\overline{d}(C)$. □

Theorem 2.6 (Blanchard, Formenti, and Kůrka [13]). *The space $(\mathcal{S},\delta)$ is complete.*

Proof. Let $[C_0], [C_1], \dots$ be a Cauchy sequence of similarity classes with respect to the density metric δ. By passing to a subsequence, we can assume that if $m < n$ then $\delta(C_m, C_n) < 2^{-m-1}$, so that by the above lemma, $\overline{d}(C_m \triangle C_n) < 2^{-m}$. Then there is a sequence $0 = k_0 < k_1 < \cdots$ such that for all $m < n$, if $i \geqslant k_n$ then $d_i(C_m \triangle C_n) < 2^{-m}$. Let C be the unique set such that C and C_n agree on the interval J_i for each $i \in [k_n, k_{n+1})$.

Fix m. For every $n > m$ and $i \in [k_n, k_{n+1})$, we have $d_i(C_m \triangle C) = d_i(C_m \triangle C_n) < 2^{-m}$, since C and C_n agree on the interval J_i. Hence $\overline{d}(C_m \triangle C) = \limsup_i d_i(C_m \triangle C) \leqslant 2^{-m}$. So by the above lemma we have

$$\delta(C_m, C) = \overline{\rho}(C_m \triangle C) \leqslant 2\overline{d}(C_m \triangle C) \leqslant 2^{-m+1}.$$

Hence $\lim_m \delta(C_m, C) = 0$ and the sequence $\{[C_m]\}$ converges to $[C]$. □

The completeness of $(\mathcal{S},\delta)$ raises the question of how difficult it is to obtain the limit of a Cauchy sequence from the sequence, in the senses of computability theory and reverse mathematics. As this issue is not directly related to what we will pursue in the next few sections, we leave it to Section 8 below.

The first part of the following theorem was shown by Blanchard, Formenti, and Kůrka [13].

Theorem 2.7. *The space $(\mathcal{S},\delta)$ is pathwise connected. Indeed, it is contractible.*

Proof. First consider how we would make a path from $[\emptyset]$ to $[\omega]$.

For every real $r\in[0,1]$ in the unit interval, we will define a set C_r. We will have $C_s\subseteq C_r$ if $s\leqslant r$, and C_r will have density r. From this property it follows that if $0\leqslant s\leqslant r\leqslant 1$ then

$$\delta(C_r,C_s)=\overline{\rho}(C_r\triangle C_s)=\overline{\rho}(C_r\setminus C_s)=\rho(C_r)-\rho(C_s)=r-s.$$

Define $p:[0,1]\to\mathcal{S}$ by $p(r)=[C_r]$. Then p preserves the metric and hence is continuous. Furthermore, we will have $C_0=\emptyset$ and $C_1=\omega$.

Here is the construction of C_r. Partition ω into consecutive intervals $L_1,L_2,\dots$ with $|L_i|=i$ for all i. Given $r\in[0,1]$, let C_r be such that $C_r\cap L_i=[m_i,m_i+\lfloor ri\rfloor)$, where m_i is the least element of L_i. That is, $C_r\cap L_i$ is the longest initial segment of L_i whose density within L_i does not exceed r.

We now verify that C_r has the properties claimed above. It is obvious that $C_s\subseteq C_r$ if $s\leqslant r$. It remains to be shown that $\rho(C_r)=r$ for all $r\in[0,1]$. Fix r and let d_i be the density of C_r on L_i, that is, $d_i=|C_r\cap L_i|/i$. Since $|C_r\cap L_i|=\lfloor ri\rfloor$, we have $r-\frac{1}{i}\leqslant d_i\leqslant r$. It follows that $\lim_i d_i=r$.

For every i, we have $|C_r\upharpoonright m_i|=\sum_{j<i}|C_r\cap L_j|=\sum_{j<i}jd_j$. Hence $\rho_{m_i}(C_r)=\sum_{j<i}jd_j/m_i$ is the weighted average of $d_1,d_2,\dots,d_{i-1}$, where d_j has weight j. It follows that $\lim_i\rho_{m_i}(C_r)=r$.

If $n\in L_i$ then

$$\frac{m_i\rho_{m_i}(C_r)}{m_i+i}\quad\leqslant\quad\rho_{n+1}(C_r)\quad\leqslant\quad\frac{m_i\rho_{m_i}(C_r)+i}{m_i}.$$

Now let n approach infinity, so that i approaches infinity also. Since $\lim_i\rho_{m_i}(C_r)=r$ and $\lim_i i/m_i=0$, we have $\rho(C_r)=\lim_n\rho_{n+1}(C_r)=r$. This completes the proof that there is a path from $[\emptyset]$ to $[\omega]$.

For arbitrary $A\subseteq\omega$ define $A_r=A\cap C_r$ where C_r is as above. To obtain a path from $[\emptyset]$ to $[A]$, define $p_A:[0,1]\to\mathcal{S}$ by $p_A(r)=A_r$. Then $p_A(0)=[\emptyset]$ and $p_A(1)=[A]$. To show that p_A is continuous, it suffices to show that $\delta(p(s),p(r))\leqslant|r-s|$ for all $s,r\in[0,1]$. But this holds since

$$\begin{aligned}\delta(p(s),p(r))=\overline{\rho}((A\cap C_s)\triangle(A\cap C_r))&=\overline{\rho}(A\cap(C_s\triangle C_r))\\&\leqslant\overline{\rho}(C_s\triangle C_r)=\delta(C_s,C_r)=|r-s|.\end{aligned}$$

For a path between two arbitrary classes $[A]$ and $[B]$, use the reverse of the path from $[\emptyset]$ to $[A]$ followed by the path from $[\emptyset]$ to $[B]$.

To check contractibility, define $\Phi:\mathcal{S}\times[0,1]\to\mathcal{S}$ by $\Phi([A],r)=[A_r]$. By definition, for all $[A]$, we have $\Phi([A],1)=[A]$ and $\Phi([A],0)=[\emptyset]$. To verify that Φ is continuous check that

$$A_r\triangle B_r\subseteq(A\triangle B)\cup(C_r\triangle C_s).$$

□

The path just constructed from $[\emptyset]$ to an arbitrary point $[A]$ is called the *uniform path* between $[\emptyset]$ and $[A]$. In general, these uniform paths need not be geodesics, although the path between $[\emptyset]$ and $[\omega]$ is a geodesic. It turns out that $(\mathcal{S}, \delta)$ is actually a geodesic metric space.

Theorem 2.8. *The space $(\mathcal{S}, \delta)$ is a geodesic metric space.*

Proof. As before, we first want to construct a geodesic path between $[\emptyset]$ and an arbitrary point $[A]$. The idea is to relativize the previous construction by working within the set A. We suppose that $[A] \neq [\emptyset]$. Let $L_0^A, L_1^A, \cdots$ partition A into successive disjoint intervals with $|L_n^A| = n$ and $\max(L_n^A) < \min(L_{n+1}^A)$ for all n.

Define $f : [0,1] \to \mathcal{S}$ as follows:

$$f(s) = \left[\bigcup_n F_n^{s,A}\right],$$

where $F_n^{s,A}$ consists of the first $\lfloor sn \rfloor$ many elements of L_n^A.

It is clear that $f(0) = [\emptyset]$ and $f(1) = [A]$. We want to show that the length of the constructed path is indeed $\delta([\emptyset], [A])$. To do that, we show that if $0 \leqslant s \leqslant t \leqslant 1$ then

$$\delta([f(s)], [f(t)]) = (t - s)\delta([\emptyset], [A]). \tag{2.1}$$

Let $k_n^A = \max L_n^A + 1$. The idea is to show first that

$$\frac{|f(t) \upharpoonright k_n^A| - |f(s) \upharpoonright k_n^A|}{k_n^A} \approx \frac{t|A \upharpoonright k_n^A| - s|A \upharpoonright k_n^A|}{k_n^A},$$

and the error in this approximation approaches 0 as $n \to \infty$.

By the definition of k_n^A, we have $A \upharpoonright k_n^A = \bigcup_{i \leqslant n}(A \upharpoonright L_i^A)$. Since the L_i^A are pairwise disjoint we have

$$|A \upharpoonright k_n^A| = \sum_{i \leqslant n} |A \upharpoonright L_i^A| = \sum_{i \leqslant n} i = \frac{n(n+1)}{2}.$$

Then

$$|f(t) \upharpoonright k_n^A| = \sum_{i \leqslant n} |f(t) \upharpoonright L_i^A| = \sum_{i \leqslant n} \lfloor it \rfloor.$$

So

$$\sum_{i \leqslant n}(it - 1) \leqslant |f(t) \upharpoonright k_n^A| \leqslant \sum_{i \leqslant n} it,$$

yielding

$$t\left[\frac{n(n+1)}{2}\right] - n - 1 \leqslant |f(t) \upharpoonright k_n^A| \leqslant t\left[\frac{n(n+1)}{2}\right].$$

Since $k_n^A = \max L_n^A \geqslant \frac{n(n+1)}{2}$, if we divide by k_n^A we obtain

$$\lim_{n \to \infty}\left(\frac{t|A \upharpoonright k_n^A|}{k_n^A} - \frac{|f(t) \upharpoonright k_n^A|}{k_n^A}\right) = 0,$$

and the same holds with t replaced by s.

To complete the proof of Equation (2.1), we need to show that

(1) $\limsup_n \rho_{k_n^A}(f(t) \triangle f(s)) = \overline{\rho}(f(t) \triangle f(s))$ and
(2) $\limsup_n \rho_{k_n^A}(A) = \overline{\rho}(A)$.

Clearly, $\leqslant$ holds in both lines. We prove $\geqslant$ for part (2), and the proof for part (1) is essentially the same. Let x_n be such that $k_n^A \leqslant x_n \leqslant k_{n+1}^A$. By definition,

$$\rho_{x_n}(A) = \frac{|A \upharpoonright x_n|}{x_n} \leqslant \frac{|A \upharpoonright x_n| + (n+1)}{x_n}.$$

Since $x_n \geqslant k_n^A \geqslant \frac{n(n+1)}{2}$ we have $\rho_{x_n}(A) \leqslant \rho_{k_n^A}(A) + \mathrm{o}(n)$ and the equation follows.

We have established that there is a geodesic from $[\emptyset]$ to an arbitrary point $[C]$. Given arbitrary $[A]$ and $[B]$ there is a geodesic from $[\emptyset]$ to $[A+B]$. Applying the isometry τ_A from Observation 1.5 to this geodesic yields a geodesic from $[\emptyset]+[A] = [A]$ to $[A+B]+[A] = [B]$. □

Observation 2.9. It is interesting to note that $(\mathcal{S}, \delta)$ is a rather rich space. Consider the sets C_r in the proof of Theorem 2.7. The argument above with $A = \omega$ shows that the subspace $\{C_r : r \in [0,1]\}$ of $(\mathcal{S}, \delta)$ is homeomorphic to $[0,1]$. Let $R_k = \{m : 2^k \mid m \;\&\; 2^{(k+1)} \nmid m\}$. Let $i_0^k < i_1^k < \cdots$ be the elements of R_k. For sets $X_0, X_1, \ldots$, let $\bigoplus_k^{\mathcal{R}} X_k = \{i_n^k : n \in X_k\}$. It is not difficult to adapt this argument to show that the subspace $\{\bigoplus_i^{\mathcal{R}} C_{r_i} : r_0, r_1, \ldots \in [0,1]\}$ of $(\mathcal{S}, \delta)$ is homeomorphic to the Hilbert cube $[0,1]^\omega$. Thus, for instance, every second countable normal space embeds into $(\mathcal{S}, \delta)$.

Theorem 2.10. *Let $[A]$ and $[B]$ be distinct points in $\mathcal{S}$. Then there are continuum many geodesics from $[A]$ to $[B]$.*

Proof. It is enough to show that

$$M = \left\{ [Y] : \delta([A],[Y]) = \delta([Y],[B]) = \frac{1}{2}\delta([A],[B]) \right\}$$

is uncountable, because for each $[Y] \in M$ there is a geodesic path from $[A]$ to $[Y]$ and a geodesic path from $[Y]$ to B, by Theorem 2.8. Joining these two paths together yields a geodesic path from $[A]$ to $[B]$ with midpoint $[Y]$, and obviously paths with distinct midpoints are distinct.

Since $[A] \neq [B]$, the symmetric difference $A \triangle B$ is infinite, so let $d_0, d_1, \ldots$ list the elements of $A \triangle B$ in increasing order. We now define a function $F : 2^\omega \to 2^\omega$, which will have the property that $F(X) \in M$ whenever $\rho(X) = 1/2$. The idea is that $F(X)$ copies the common value of A and B at points where A and B agree, and at other points d_i, it copies A if $i \in X$ and otherwise copies B. Thus we define

$$F(X)(n) = \begin{cases} A(n) & \text{if } A(n) = B(n) \text{ or } (n = d_i \;\&\; i \in X) \\ B(n) & \text{if } n = d_i \;\&\; i \notin X. \end{cases}$$

Lemma 2.11. *If $\rho(X) = \frac{1}{2}$, then $[F(X)] \in M$.*

Proof. Recall that we denote the complement of X by $\neg X$. Suppose that $d_k < n \leqslant d_{k+1}$. By the definition of F,

$$|(A \triangle F(X)) \upharpoonright n| = |\neg X \upharpoonright k|.$$

Dividing both sides by n yields

$$\frac{|(A \triangle F(X)) \upharpoonright n|}{n} = \frac{|\neg X \upharpoonright k|}{k} \cdot \frac{k}{n}.$$

Then, by the definition of ρ, it follows that

$$\rho_n(A \triangle F(X)) = \rho_k(\neg X) \cdot \rho_n(A \triangle B).$$

Taking the lim sup of both sides as n goes to infinity, and hence also k goes to infinity, we obtain

$$\overline{\rho}(A \triangle F(X)) = \frac{1}{2}\overline{\rho}(A \triangle B).$$

Hence, by the definition of δ,

$$\delta([A],[F(X)]) = \frac{1}{2}\delta([A],[B]).$$

A similar argument shows that $\delta([B],[F(X)]) = \frac{1}{2}\delta([A],[B])$, completing the proof of the lemma. □

Lemma 2.12. *If* $\underline{\rho}(X_0 \triangle X_1) > 0$*, then* $[F(X_0)] \neq [F(X_1)]$.

Proof. One can argue as in the proof of the previous theorem that if $d_k < n \leqslant d_{k+1}$ then

$$\rho_n(F(X_0) \triangle F(X_1)) = \rho_k(X_0 \triangle X_1) \cdot \rho_n(A \triangle B).$$

Choose a real $\varepsilon > 0$ and a number n_0 such that $(\forall n \geqslant n_0)[\rho_n(A \triangle B) \geqslant \varepsilon]$. Then for all $n \geqslant n_0$ we have

$$\rho_n(F(X) \triangle F(X_1)) \geqslant \varepsilon \cdot \rho_n(A \triangle B).$$

Taking the lim sup of both sides we obtain

$$\delta([F(X_0)],[F(X_1)]) \geqslant \varepsilon\delta([A],[B]) > 0.$$

It follows that $[F(X_0)] \neq [F(X_1)]$. □

To complete the proof of the theorem, it suffices to show that there is a family $\mathcal{C} \subseteq 2^\omega$ of size continuum such that every element of $\mathcal{C}$ has density $1/2$, and the symmetric difference of any two distinct elements of $\mathcal{C}$ has positive lower density, since then by the lemmas $\{[F(X)] : X \in \mathcal{C}\}$ is a subset of M of size continuum.

Let C_r be as in the proof of Theorem 2.7, let $X_r = C_r \oplus \neg C_r$, and let $\mathcal{C} = \{X_r : r \in [0,1]\}$. Then $\mathcal{C}$ has size continuum, and every element X_r of $\mathcal{C}$ has density $1/2$, since for each n, exactly one of $2n$ and $2n+1$ is in X_r. Let $r \neq s$. As shown in the proof of Theorem 2.7, $\rho(C_r) = r$ and $\rho(C_s) = s$, so $\underline{\rho}(C_r \triangle C_s) > 0$. Furthermore, for each n, we have $\rho_{2n}(X_r \triangle X_s) = \rho_n(C_r \triangle C_s)$, so $\underline{\rho}(X_r \triangle X_s) > 0$. □

The properties of the family $\mathcal{C}$ in the proof of Theorem 2.10 suggest the question of whether we can build a family of size continuum such that each element has density $1/2$, and the symmetric difference of any two distinct elements also has density $1/2$. Such a family would behave as we expect pairwise mutually random sets to behave, and indeed it suffices to build a family of size continuum of pairwise mutually 1-random sets. (Of course, 1-randomness is considerably more than what is needed here. Computable randomness would suffice, as would even the fairly weak notion of Church stochasticity (see [20, Definition 7.4.1]).)

The existence of such a family of 1-randoms is a natural question, but it appears not to have been directly addressed in the literature, except in an unpublished manuscript of Miller and Yu [46]. They gave a direct construction of such a family, but noted [personal communication] that its existence also follows from a general Ramsey-theoretic theorem due to Jan Mycielski [48, Theorem 1]. (A simpler version of this theorem, which is sufficient for this purpose, is cited as Theorem 10.3.15 in [20].) We are grateful to Alekos Kechris for bringing [48] to our attention. We also thank Anush Tserunyan very much for formulating the result in a form that is much easier for us to apply and showing us how it can be proved using extensions

of some exercises in Kechris's book [38]. (This occurred before we were even aware of the possibility of applying [48] here.) We will give Tserunyan's formulation and sketch her proof in this section, and will discuss a different proof in Section 7.

For any set A, let $(A)^n$ denote the set of all ordered n-tuples of distinct elements of A, and let A^n denote the set of all ordered n-tuples of elements of A. We consider the usual Lebesgue measure on $[0,1]^n$, where $[0,1]$ is the unit interval.

Theorem 2.13 (Mycielski [48]). *For any sequence of relations $\{R_k\}_{k\in\omega}$ with $R_k \subseteq [0,1]^{n_k}$ such that R_k is of measure 1 for all k, there is a homeomorphic copy $C \subseteq [0,1]$ of the Cantor space 2^ω such that $(C)^{n_k} \subseteq R_k$ for all $k \in \omega$.*

Proof sketch (Tserunyan [private communication]). We first turn this measure-theoretic statement into a topological one by considering the (Lebesgue) density topology on $[0,1]$, as defined in [38, Exercise (17.47.ii)]. Note that the density topology on $[0,1]$ can be similarly defined on $[0,1]^n$ by using open balls in $[0,1]^n$ in place of open intervals in $(0,1)$ in the definition of $\varphi(A)$ in [38, Exercise (17.47.i)]. The remaining parts of [38, Exercise (17.47)] continue to hold for $[0,1]^n$ in place of $[0,1]$. The subsets of $[0,1]^n$ of Lebesgue measure 0 are exactly the meager sets in the density topology by [38, Exercise (17.47.iii)], extended to $[0,1]^n$, and this topological space is Choquet by [38, Exercise (17.47.vi)] and is clearly perfect, since nonempty open sets have positive Lebesgue measure. Thus, the Ramsey Theorem for perfect Choquet spaces with a weaker metric [38, Exercise (19.5)] applies for each fixed n_k. The latter result is easily extended to cover a sequence of relations of varying arity instead of a single relation, so the result follows. □

The above theorem also holds for 2^ω (equipped with its usual fair coin toss measure) in place of $[0,1]$ because the binary expansion map from 2^ω to $[0,1]$ is measure-preserving and almost a homeomorphism. More precisely, this map induces a homeomorphism from a co-countable subset of 2^ω (the set of binary sequences with infinitely many 1's) to a co-countable subset of $[0,1]$ (the set of non-dyadic reals in $[0,1]$).

The relation that holds of (X,Y) if and only if X and Y are mutually 1-random has measure 1, so we have the following.

Corollary 2.14. *There is a nonempty perfect set whose elements are pairwise mutually 1-random.*

Of course, there is nothing special about 1-randomness here. Mycielski's Theorem applies just as well to other notions of algorithmic randomness.

The following corollary was first proved by a direct construction, which we will return to in Section 7.

Corollary 2.15. *There is a perfect $\mathcal{C} \subseteq 2^\omega$ such that every element of $\mathcal{C}$ has density 1/2 and the symmetric difference of any two distinct elements of $\mathcal{C}$ also has density 1/2.*

We can strengthen Corollary 2.14 by using the fact that each of the relations $\{(X_0,\dots,X_n) : \text{each } X_i \text{ is 1-random relative to } \bigoplus_{j\leqslant n,\ j\neq i} X_j\}$ has measure 1.

Corollary 2.16. *There is a nonempty perfect set $\mathcal{P}$ such that if $\mathcal{F} \subset \mathcal{P}$ is a nonempty finite set, then the join of the elements of $\mathcal{F}$ is 1-random.*

It is not possible to extend this corollary to countable sets $\mathcal{F}$, because if $\mathcal{P}$ is perfect and $X \in \mathcal{P}$, then there are initial segments $\sigma_0 \prec \sigma_1 \prec \cdots$ of X such that for each i there is an $X_i \in \mathcal{P}$ for which σ_i is the longest common initial segment of X_i and X, and then $X \leqslant_{\mathrm{T}} \bigoplus_i X_i$.

It is not possible for $\mathcal{P}$ to have positive measure, even in Corollary 2.14, because such a $\mathcal{P}$ would be an antichain with respect to Turing reducibility, and Yu [58] has shown that such an antichain cannot have positive measure.

In Section 7, we will give a proof of Corollary 2.16 as an effectivization of a proof of Mycielski's Theorem along the lines of the one outlined in [56], and will also discuss the computability-theoretic and reverse-mathematical content of the above corollaries. It is worth noting that the relativized form of Corollary 2.16 is in fact equivalent to Mycielski's Theorem, in the sense that the latter can be proved easily from it. We will give the argument in Section 7, a version of which was first given by Miller and Yu [46].

Mycielski also proved an analog of Theorem 2.13 for category, which implies that there is a nonempty perfect set $\mathcal{P} \subseteq 2^\omega$ such that if $\mathcal{F} \subset \mathcal{P}$ is a nonempty finite set, then the join of the elements of $\mathcal{F}$ is 1-generic, and hence, in particular, any two distinct elements of $\mathcal{P}$ are mutually 1-generic.

Theorem 2.17 (Mycielski [48])**.** *For any sequence of relations $\{R_k\}_{k\in\omega}$ with $R_k \subseteq [0,1]^{n_k}$ such that R_k is comeager for all k, there is a homeomorphic copy $C \subseteq [0,1]$ of the Cantor space 2^ω such that $(C)^{n_k} \subseteq R_k$ for all $k \in \omega$.*

As with Theorem 2.13, this result also holds for 2^ω, for the same reason. We will prove it (in the 2^ω version) as Theorem 7.22 below, and discuss its computability-theoretic and reverse-mathematical content in Section 7.3.

3. Turing invariant subspaces of $(\mathcal{S}, \delta)$

In this section, we begin to examine some interactions between $(\mathcal{S}, \delta)$ and computability theory by considering subspaces of this metric space arising from sets of Turing degrees. In particular, we discuss conditions under which such subspaces can have some of the properties of $(\mathcal{S}, \delta)$ discussed in the previous section. Say that a subset of $\mathcal{P}(\omega)$ is *Turing invariant* if it closed under Turing equivalence (i.e., it is a union of Turing degrees). A particular example of a Turing invariant set is a *Turing ideal*, i.e., a subset of $\mathcal{P}(\omega)$ that is closed downward under Turing reducibility and closed under finite joins. Say that $U \subseteq \mathcal{P}(\omega)$ *generates* $\mathcal{U} \subseteq \mathcal{S}$ if $\mathcal{U} = \{[A] : A \in U\}$. We will be interested in subspaces of $(\mathcal{S}, \delta)$ that are generated by Turing invariant sets.

We first note that it does not matter whether we consider sets closed under Turing equivalence or sets that are downward closed under Turing reducibility.

Lemma 3.1. *If U is Turing invariant then*

$$\{[A] : A \in U\} = \{[B] : (\exists A \in U)[B \leqslant_{\mathrm{T}} A]\}.$$

Proof. If $A \in U$ and $B \leqslant_{\mathrm{T}} A$, then we can take a coinfinite computable set C of density 1, list the elements of its complement as $d_0 < d_1 < \cdots$, and define $D = (B \cap C) \cup \{d_n : n \in A\}$. Then $D \equiv_{\mathrm{T}} A$, so $D \in U$, and $[D] = [B]$. □

We now note a couple of other interesting facts about subsets of $\mathcal{S}$ generated by Turing invariant sets. Recall that $J_k = [2^k - 1, 2^{k+1} - 1)$.

Definition 3.2. Let $\mathcal{J}(A) = \bigcup_{k\in A} J_k$.

Lemma 3.3. *If $\delta(\mathcal{J}(A), C) < 1/4$ then $A \leqslant_{\mathrm{T}} C$.*

Proof. If $\delta(\mathcal{J}(A), C) < 1/4$ then, by Lemma 2.5, for all but finitely many n, more than half of the bits in $C \upharpoonright J_n$ are equal to $A(n)$. □

Recall that $\mathcal{S}$ has the structure of a topological group under the operation defined by $[A] + [B] = [A \triangle B]$.

Proposition 3.4. *Let U be downward closed under Turing reducibility. Then $\{[A] : A \in U\}$ is a subgroup of $\mathcal{S}$ if and only if U is a Turing ideal.*

Proof. Suppose that U is a Turing ideal. If $B, C \in U$ then $B \triangle C \leqslant_{\mathrm{T}} B \oplus C$ is also in U. Since each element of $\mathcal{S}$ is its own inverse, it follows that $\{[A] : A \in U\}$ is a subgroup of $\mathcal{S}$.

Now suppose that $\mathcal{U} = \{[A] : A \in U\}$ is a subgroup of $\mathcal{S}$ and let $B, C \in U$. Then $[\mathcal{J}(B) \oplus \emptyset], [\emptyset \oplus \mathcal{J}(C)] \in \mathcal{U}$, so $[\mathcal{J}(B) \oplus \mathcal{J}(C)] = [(\mathcal{J}(B) \oplus \emptyset) \triangle (\emptyset \oplus \mathcal{J}(C))] \in \mathcal{U}$. Thus U contains a coarse description of $\mathcal{J}(B) \oplus \mathcal{J}(C)$. This description computes coarse descriptions of $\mathcal{J}(B)$ and $\mathcal{J}(C)$, and hence computes both B and C. It follows that $B \oplus C \in U$. □

We say that a collection U of sets is *cofinal in the Turing degrees* if for every B, there is an $A \in U$ such that $B \leqslant_{\mathrm{T}} A$. The following fact follows from Lemmas 3.1 and 3.3.

Proposition 3.5. *Let U be Turing invariant. Then U is cofinal in the Turing degrees if and only if $\{[A] : A \in U\} = \mathcal{S}$.*

The space $(\mathcal{S}, \delta)$ has many compact subspaces. (For example, the subspace consisting of all the $[C_r]$ defined in the proof of Theorem 2.7 is homeomorphic to the interval $[0, 1]$.) However, none of them can be generated by a Turing invariant set.

Theorem 3.6. *If $\mathcal{U} \subseteq \mathcal{S}$ is nonempty and generated by a Turing invariant set, then $\mathcal{U}$ is not compact.*

Proof. Let U be a Turing invariant set that generates $\mathcal{U}$. The construction in part (1) of the proof of Theorem 2.2 shows that for sets $A_0, \ldots, A_n$, there is a set $A_{n+1} \leqslant_{\mathrm{T}} A_0 \oplus \cdots \oplus A_n$ such that $\delta([A_{n+1}], [A_i]) = 1$ for all $i \leqslant n$. So we can take any $A_0 \in U$ and build an infinite sequence $A_0, A_1, \ldots \leqslant_{\mathrm{T}} A_0$ such that $\delta([A_i], [A_j])) = 1$ for all $i \neq j$. Each $[A_i]$ is in $\mathcal{U}$.

Now take any open cover of $\mathcal{U}$ by open balls, all of whose radii are less than $1/2$. None of these balls can contain both $[A_i]$ and $[A_j]$ for $i \neq j$, so this cover has no finite subcover. □

We now turn to completeness. Since $(\mathcal{S}, \delta)$ is itself complete, a subspace is complete if and only if it is closed. The space $(\mathcal{S}, \delta)$ has many kinds of closed subspaces (for example, the closure $\overline{\mathbf{d}}$ of a degree $\mathbf{d}$ in Definition 4.5 below), but if a nonempty $\mathcal{U} \subsetneq \mathcal{S}$ is generated by a Turing invariant set, then it is difficult for $\mathcal{U}$ to be closed, as we now show.

The following sets defined in [36] are very useful in studying asymptotic density and computability.

Definition 3.7. Let $R_k = \{m : \, 2^k \mid m \;\&\; 2^{(k+1)} \nmid m\}$, and let $\mathcal{R}(A) = \bigcup_{k \in A} R_k$.

Lemma 3.8 (Asher Kach [personal communication])**.** *The set $\mathcal{R}(A)$ is a limit of computable sets for every $A \subseteq \omega$.*

Proof. Let $A_k = A \cap [0,k]$. Each $\mathcal{R}(A_k)$ is computable, and $\mathcal{R}(A) \triangle \mathcal{R}(A_k) \subseteq \bigcup_{j \geqslant k} R_j$. Since the density of the latter set goes to 0 as k goes to infinity, so does $\delta(\mathcal{R}(A), \mathcal{R}(A_k))$. □

The following lemma is a relativized version of one direction of Theorem 2.19 of [36].

Lemma 3.9. *If C is a coarse description of $\mathcal{R}(A)$ then $A \leqslant_{\mathrm{T}} C'$.*

Proof. For each k, the density of C within R_k must be 1 if $k \in A$, and 0 if $k \notin A$. Let

$$\rho_n^k(C) = \frac{|(C \cap R_k) \upharpoonright n|}{|R_k \upharpoonright n|}.$$

Given k, we can use C' to search for an m such that either $\rho_n^k(C) > \frac{1}{2}$ for all $n \geqslant m$, or $\rho_n^k(C) < \frac{1}{2}$ for all $n \geqslant m$. In the first case $k \in A$, while in the latter $k \notin A$. □

Recall the notation $\bigoplus_k^{\mathcal{R}} X_k$ from Observation 2.9. If C is a coarse description of $\bigoplus_k^{\mathcal{R}} X_k$ then $\{n : i_n^k \in C\}$ is a coarse description of X_k for each k, since R_k has positive density, so we have the following fact.

Lemma 3.10. *Every coarse description of $\bigoplus_k^{\mathcal{R}} X_k$ computes a coarse description of X_k for each k.*

We can now give necessary conditions for a subspace of $(\mathcal{S}, \delta)$ generated by a Turing invariant set to be closed, and hence complete.

Theorem 3.11. *If $\mathcal{U} \subseteq \mathcal{S}$ is generated by a Turing invariant set U and is closed, then the following hold.*

(1) *For every X and every $A \in U$, there is a $C \in U$ such that $A \leqslant_{\mathrm{T}} C$ and $X \leqslant_{\mathrm{T}} C'$.*
(2) *Every countable Turing ideal contained in U has an upper bound in U.*

Proof. (1) Fix X and $A \in U$. We have $[\mathcal{J}(A) \oplus B] \in \mathcal{U}$ for all computable B, so an easy adaptation of the proof of Lemma 3.8 shows that $[\mathcal{J}(A) \oplus \mathcal{R}(X)] \in \mathcal{U}$, i.e., there is a $C \in U$ that is a coarse description of $\mathcal{J}(A) \oplus \mathcal{R}(X)$. This C computes coarse descriptions of both $\mathcal{J}(A)$ and $\mathcal{R}(X)$, so by Lemmas 3.3 and 3.9, $A \leqslant_{\mathrm{T}} C$ and $X \leqslant_{\mathrm{T}} C'$.

(2) Suppose that $\mathcal{I}$ is a countable Turing ideal contained in U. Let $A_0, A_1, \ldots$ be the elements of $\mathcal{I}$. Let $i_0^k < i_1^k < \cdots$ be the elements of R_k. Let $Z_n = \bigoplus_k^{\mathcal{R}} X_k$ where $X_k = \mathcal{J}(A_k)$ for $k < n$ and $X_k = \emptyset$ for $k \geqslant n$, and let $Z = \bigoplus_k^{\mathcal{R}} \mathcal{J}(A_k)$. Then each Z_n is computable from finitely many elements of $\mathcal{I}$, and hence is in $\mathcal{I}$. It is also clear that $[Z]$ is the limit of the $[Z_n]$ in $(\mathcal{S}, \delta)$, so $[Z] \in \mathcal{U}$. That is, there is a $Y \in U$ that is a coarse description of Z. By Lemma 3.10, Y computes a coarse description of $\mathcal{J}(A_k)$ for each k, and hence Y computes each A_k, i.e., Y is an upper bound for $\mathcal{I}$. □

Corollary 3.12. *If $\mathcal{U} \subseteq \mathcal{S}$ is nonempty, generated by a Turing invariant set, and closed, then it has size continuum.*

Proof. Let U be a Turing invariant set generating $\mathcal{U}$. By the theorem, the set of jumps of elements of U is cofinal in the Turing degrees, and hence has size continuum, so U itself has size continuum. Thus the set of degrees of elements of

U has size continuum. If A and B have different degrees, then $[\mathcal{J}(A)] \neq [\mathcal{J}(B)]$, by Lemma 3.3, so $\mathcal{U}$ has size continuum. □

It is of course natural to ask whether the conditions in the above theorem can hold for any nonempty $\mathcal{U} \subsetneq \mathcal{S}$ generated by a Turing invariant set. We will discuss this question at the end of this section. We do not know whether these necessary conditions are also sufficient, but we will show that if we strengthen condition (2) from countable Turing ideals to all countable subsets of U, then we do obtain sufficient conditions for a subspace of $(\mathcal{S}, \delta)$ generated by a Turing invariant set to be closed. Of course, in that case, the downward closure of U under Turing reducibility is a Turing ideal, so we might as well state our result for Turing ideals, which also allows us to simplify condition (1). We say that a collection U of sets is *jump-cofinal in the Turing degrees* if for every X, there is a $C \in U$ such that $X \leqslant_{\mathrm{T}} C'$. It is easy to see that if U is a nonempty Turing ideal, then U satisfies condition (1) if and only if it is jump-cofinal in the Turing degrees.

We will use the relativized form of the following result, which follows from a theorem due to J. Miller (see [31]).

Lemma 3.13 (Hirschfeldt, Jockusch, McNicholl, and Schupp [31, Corollary 5.11])**.** *Suppose there is a $\emptyset'$-computable function f such that, for all e, we have that $\Phi_{f(e)}$ is total and $\{0,1\}$-valued, and $\overline{\rho}(B \triangle \Phi_{f(e)}) \leqslant 2^{-e}$. Then B is coarsely computable.*

Theorem 3.14. *Let $\mathcal{U} \subseteq \mathcal{S}$ be nonempty and generated by a Turing ideal U. Then $\mathcal{U}$ is closed if and only if U is jump-cofinal in the Turing degrees and every countable subset of U has an upper bound in U.*

Proof. The "only if" direction follows from Theorem 3.11 and the fact that every countable subset of U is contained in a countable subideal of U. For the "if" direction, let $B_0, B_1, \ldots \in U$ be such that $[B_0], [B_1], \ldots$ is a Cauchy sequence, and let $[B]$ be its limit. By passing to a subsequence, we can assume that $\delta([B_m], [B_n]) < 2^{-m}$ for all $m < n$. Let $D \in U$ be such that $B_0, B_1, \ldots \leqslant_{\mathrm{T}} D$ (which exists because the subideal of U generated by the B_i has an upper bound in U). Let e_i be such that $\Phi^D_{e_i} = B_i$ for all i. There is a $C \in U$ such that $D \leqslant_{\mathrm{T}} C$ and the function $i \mapsto e_i$ is computable in C'. Then there is a function $f \leqslant_{\mathrm{T}} C'$ such that $\Phi^C_{f(i)} = B_i$ for all i. By the relativized form of Lemma 3.13, B is coarsely C-computable, so $[B] \in \mathcal{U}$. □

Again, we will discuss the question of whether there are any $\mathcal{U} \subsetneq \mathcal{S}$ that satisfy the conditions in this theorem at the end of this section.

We now turn to the properties in Theorems 2.7 and 2.8, for which we have the following exact characterization, which again involves condition (1) above. A real is *right-c.e.* if the set of rationals greater than it is c.e., and *left-c.e.* if the set of rationals less than it is c.e. If a real is both right-c.e. and left-c.e., then it is computable.

Lemma 3.15. *For any sets A and B, we have that $\delta(A, B)$ is right-c.e. relative to $(A \oplus B)'$.*

Proof. We can assume $\delta(A, B)$ is irrational, since every rational is a computable real. Then for a rational q, we have $q > \delta(A, B)$ if and only if $(\exists m)(\forall n > m)[\rho_n(A \triangle B) < q]$, which is a c.e. condition relative to $(A \oplus B)'$. □

Theorem 3.16. *Let $\mathcal{U} \subseteq \mathcal{S}$ be nonempty and generated by a Turing invariant set U. The following are equivalent.*

(1) *$\mathcal{U}$ is connected.*
(2) *$\mathcal{U}$ is pathwise connected.*
(3) *$\mathcal{U}$ is contractible.*
(4) *$\mathcal{U}$ is geodesic.*
(5) *For every real r and every $A \in U$, there is a $B \in U$ such that $A \leqslant_{\mathrm{T}} B$ and r is right-c.e. relative to B'.*
(6) *For every real r and every $A \in U$, there is a $C \in U$ such that $A \leqslant_{\mathrm{T}} C$ and $r \leqslant_{\mathrm{T}} C'$.*

Proof. By Lemma 3.1, we can assume that U is downward closed under Turing reducibility.

It is well-known that (3) and (4) both imply (2), which in turn implies (1).

Suppose that (5) holds and fix a real r and a set $A \in U$. Let $B \in U$ be such that $A \leqslant_{\mathrm{T}} B$ and r is right-c.e. relative to B'. Now let $C \in U$ be such that $B \leqslant_{\mathrm{T}} C$ and $1-r$ is right-c.e. relative to C', which implies that r is left-c.e. relative to C'. Since $B' \leqslant_{\mathrm{T}} C'$, we have that r is both right-c.e. and left-c.e. relative to C', so $r \leqslant_{\mathrm{T}} C'$. Thus (5) implies (6). Clearly (6) implies (5), so (5) and (6) are equivalent.

Now suppose that (5) fails. We show that $\mathcal{U}$ is not connected. Let r and $A \in U$ be such that if $B \in U$ and $A \leqslant_{\mathrm{T}} B$, then r is not right-c.e. relative to B'. If A is computable, then Lemma 3.15 implies that there is no $B \in U$ with $\delta(A, B) = r$, which allows us to separate $\mathcal{U}$ into the disjoint nonempty open sets $\{[Y] \in \mathcal{U} : \delta(Y, A) < r\}$ and $\{[Y] \in \mathcal{U} : \delta(Y, A) > r\}$. Since we cannot assume that A is computable, we replace A by $\mathcal{J}(A)$ (which is in $\mathcal{U}$), and replace r by a real s that differs from r by a rational and is small enough to ensure that every B such that $\delta(B, \mathcal{J}(A)) < s$ computes A.

In detail: Let $s < 1/4$ differ from r by a rational. Note that s cannot be right-c.e. relative to B' for any $B \geqslant_{\mathrm{T}} A$ in U. Let $\mathcal{D}$ consist of all $[Y] \in \mathcal{U}$ such that $\delta(Y, \mathcal{J}(A)) < s$, and let $\mathcal{E}$ consist of all $[Y] \in \mathcal{U}$ such that $\delta(Y, \mathcal{J}(A)) > s$. Then $\mathcal{D}$ and $\mathcal{E}$ are open in $\mathcal{U}$, disjoint, and nonempty. (We have $[\mathcal{J}(A)] \in \mathcal{D}$, and U contains sets of density 0 and 1, at least one of which is in $\mathcal{E}$.)

If $B \in U$ and $\delta(B, \mathcal{J}(A)) = s$, then $\delta(B, \mathcal{J}(A)) < 1/4$, so that by Lemma 3.3, $A \leqslant_{\mathrm{T}} B$, and hence s is not right-c.e. relative to B'. But $B \oplus \mathcal{J}(A) \equiv_{\mathrm{T}} B \oplus A \equiv_{\mathrm{T}} B$, so this fact contradicts Lemma 3.15. Thus there is no such B, and hence $\mathcal{U} = \mathcal{D} \cup \mathcal{E}$ is not connected.

Finally, suppose that (6) holds. We first show that $\mathcal{U}$ is contractible. It is enough to show that for $A \in U$ and each of the sets A_r in the proof of Theorem 2.7, $[A_r] \in \mathcal{U}$. We use the notation of that proof. The basic idea is that C_r has density very close to r within each L_i, but we can instead take a sequence $q_0, q_1, \dots$ of rationals with limit r, and build a set that is defined like C_r but has density very close to q_i within each L_i. We can then show that this set is coarsely equivalent to C_r, and thus its intersection with A is coarsely equivalent to A_r.

In detail: Given r, let $C \in U$ be such that $A \leqslant_{\mathrm{T}} C$ and there is a C-computable sequence of rationals $q_0, q_1, \dots$ with limit r. Define D_r by letting $D_r \cap L_i = [m_i, m_i + \lfloor q_i i \rfloor]$. Let $B_r = A \cap D_r$. Then $B_r \leqslant_{\mathrm{T}} A \oplus C \equiv_{\mathrm{T}} C$, so $B_r \in U$. Furthermore, $\delta(A_r, B_r) = \overline{\rho}(A_r \bigtriangleup B_r) \leqslant \overline{\rho}(C_r \bigtriangleup D_r) = \delta(C_r, D_r)$. Fix $\varepsilon > 0$. Let $s < t$ be such that $r \in (s, t)$ and $t - s < \varepsilon$. Then $C_r \bigtriangleup D_r \subseteq^* C_s \bigtriangleup C_t$, so $\delta(C_r, D_r) \leqslant$

$\delta(C_s, C_t) = t - s < \varepsilon$. (Here $\subseteq^*$ is containment up to finitely many elements.) Since ε is arbitrary, $\delta(A_r, B_r) \leqslant \delta(C_r, D_r) = 0$, that is, $[A_r] = [B_r] \in \mathcal{U}$.

Essentially the same argument, working with L_i^A instead of L_i, shows that $\mathcal{U}$ is geodesic. □

By Proposition 3.5, the following examples of sets satisfying the conditions of Theorem 3.16 are nontrivial.

Corollary 3.17. *Let X be noncomputable. Then $\{[A] : X \nleqslant_{\mathrm{T}} A\}$, thought of as a subspace of $(\mathcal{S}, \delta)$, is pathwise connected (and hence connected), and is in fact contractible and geodesic.*

Proof. By the theorem, it is enough to show that if $X \nleqslant_{\mathrm{T}} A$ then for any real r there is a $C \geqslant_{\mathrm{T}} A$ such that $X \nleqslant_{\mathrm{T}} C$ and $r \leqslant_{\mathrm{T}} C'$. The Friedberg Jump Inversion Theorem can be combined with cone avoidance, as noted in [43, Exercise 4.18]. Relativizing this fact to A produces the desired C. □

Combining Theorems 3.11 and 3.16 yields the following consequence.

Corollary 3.18. *Let $\mathcal{U} \subseteq \mathcal{S}$ be nonempty, generated by a Turing invariant set, and closed. Then it is pathwise connected (and hence connected), and is in fact contractible and geodesic.*

The following corollary has the same proof as Corollary 3.12.

Corollary 3.19. *If $\mathcal{U} \subseteq \mathcal{S}$ is nonempty, generated by a Turing invariant set, and connected, then it has size continuum.*

As an example of the negative application of the above theorems, we can take U to be the set of hyperimmune-free degrees. This is an uncountable set, downward closed under Turing reducibility, and every degree above $\mathbf{0}''$ is the double jump of a hyperimmune-free degree, so there might seem to be some hope that $\mathcal{U} = \{[A] : A \in U\}$ is connected. However no hyperimmune-free degree can be high, so by Theorems 3.11 and 3.16, $\mathcal{U}$ is neither connected nor closed.

If U is a Turing ideal, then conditions 5 and 6 in Theorem 3.16 can be replaced by the following simpler equivalent ones:

(5′) For every real r there is a $B \in U$ such that r is right-c.e. relative to B'.
(6′) For every real r there is a $C \in U$ such that $r \leqslant_{\mathrm{T}} C'$, i.e., U is jump-cofinal in the Turing degrees.

We now show that it is possible for a Turing ideal to satisfy these conditions without being cofinal in the Turing degrees, using the following definition.

Definition 3.20. A *perfect tree* is a map $T : 2^{<\omega} \to 2^{<\omega}$ such that for each σ, the strings $T(\sigma^\frown 0)$ and $T(\sigma^\frown 1)$ are incompatible and both properly extend $T(\sigma)$. Let $T(A) = \bigcup_{\sigma \prec A} T(\sigma)$. We say that X is a *path through* T if there is an A such that $X = T(A)$.

Proposition 3.21. *There exists a $\emptyset'$-computable perfect tree T such that if $\mathcal{F}$ is a nonempty finite collection of paths through T, then $\emptyset' \nleqslant_{\mathrm{T}} \bigoplus_{Y \in \mathcal{F}} Y$.*

Proof. This proposition can be proved directly, and Yu Liang [personal communication] has noted that it also follows easily from the result of Binns and Simpson [12] that there is a nonempty Π_1^0 class $\mathcal{C}$ such that if $\mathcal{F}$ is a nonempty finite collection of elements of $\mathcal{C}$ and X is an element of $\mathcal{C} \setminus \mathcal{F}$, then $X \nleqslant_{\mathrm{T}} \bigoplus_{Y \in \mathcal{F}} Y$.

The proposition also follows from results in Section 7. In Theorem 7.5, we will show that there is a $\emptyset'$-computable perfect tree T such for any nonempty finite collection $\mathcal{F}$ of paths through T, the set $\bigoplus_{Y \in \mathcal{F}} Y$ is 1-random. We will do the same for 1-genericity in place of 1-randomness in the proof of Theorem 7.22, as noted in Theorem 7.24. Either of these trees can be used to prove the theorem. Let us do it using 1-randomness.

Let T be as above, and suppose there is a nonempty finite collection $\mathcal{F}$ of paths through T such that $\emptyset' \leqslant_{\mathrm{T}} \bigoplus_{Y \in \mathcal{F}} Y$. If Z is computable then $T(Z)$ is $\emptyset'$-computable, so there is a $\emptyset'$-computable path X through T such that $X \notin \mathcal{F}$. Then, by van Lambalgen's Theorem, X is 1-random relative to $\bigoplus_{Y \in \mathcal{F}} Y$, and hence relative to $\emptyset'$, contradicting the fact that $X \leqslant_{\mathrm{T}} \emptyset'$. □

Proposition 3.22. *Let T be as in Proposition 3.21 and let U be the downward closure under Turing reducibility of the class of all $\bigoplus_{Y \in \mathcal{F}} Y$ such that $\mathcal{F}$ is a nonempty finite collection of paths through T. Then U is a Turing ideal and is not cofinal in the Turing degrees. Furthermore, U is jump-cofinal in the Turing degrees, and hence $\{[A] : A \in U\}$, thought of as a subspace of $(\mathcal{S}, \delta)$, is pathwise connected (and hence connected), and is in fact contractible and geodesic.*

Proof. That U is a Turing ideal and is not cofinal in the Turing degrees is clear from its definition. To see that U is jump-cofinal in the Turing degrees, fix a set Z. We can compute Z from $T \oplus T(Z) \leqslant_{\mathrm{T}} \emptyset' \oplus T(Z) \leqslant_{\mathrm{T}} T(Z)'$, and $T(Z) \in U$. □

In general, we cannot expect an exact computability-theoretic criterion for connectedness for arbitrary subspaces of $(\mathcal{S}, \delta)$. For example, if we let $\mathcal{U}$ consist of $[C_r]$ for all the sets C_r in the proof of Theorem 2.7, then $\mathcal{U}$ is pathwise connected, but if we add a single point $[A]$ to $\mathcal{U}$ for a set A that does not have density, then the resulting set is no longer connected. Adding $[A]$ has no effect as far as Turing degrees go, however, since the C_r already have all possible degrees. Nevertheless, one direction of Theorem 3.16 can be adapted as follows.

Theorem 3.23. *Let the subspace $\mathcal{U} \subseteq \mathcal{S}$ generated by U be connected and have at least two points. Then for every real r and every $A \in U$, there is a $C \in U$ such that $r \leqslant_{\mathrm{T}} (A \oplus C)'$.*

Proof. Assume that $\mathcal{U}$ has at least two points but the condition in the second sentence of the theorem fails. The same argument as in the proof of Theorem 3.16 shows that there exist a real r and an $A \in U$ such that for all $B \in U$, we have that r is not right-c.e. relative to $(A \oplus B)'$. Now repeat the argument that not (5) implies not (1) in the proof of Theorem 3.16, but with A itself in place of $\mathcal{J}(A)$. The only other significant difference is that we now choose s small enough so that there is a point in $\mathcal{U}$ at distance greater than s from $[A]$, to ensure that $\mathcal{E}$ is nonempty. □

Let us return to the question of the existence of sets $\mathcal{U} \subsetneq \mathcal{S}$ satisfying the conditions in Theorem 3.11, or the stronger ones in Theorem 3.14, i.e. nonempty closed sets $\mathcal{U} \subsetneq \mathcal{S}$ generated by Turing invariant sets (or ideals). We do not know whether such sets exist in an absolute sense, but they do exist if we assume the Continuum Hypothesis (CH).

Theorem 3.24 (Richard Shore [personal communication])**.** *Assuming CH, there is a Turing ideal $U \subsetneq 2^\omega$ that is jump-cofinal and such that every countable subset of U has an upper bound in U.*

Proof. Assuming CH, let $(X_\alpha)_{\alpha<\omega_1}$ list 2^ω. Let D be noncomputable. We define a sequence $(C_\alpha)_{\alpha<\omega_1}$ with $C_\alpha \leqslant_{\mathrm{T}} C_\beta$ for $\alpha < \beta$, such that $X_\alpha \leqslant_{\mathrm{T}} C'_{\alpha+1}$ for all $\alpha < \omega_1$ but $D \nleqslant_{\mathrm{T}} C_\alpha$ for all $\alpha < \omega_1$. Then $\{Y : (\exists \alpha < \omega_1)[Y \leqslant_{\mathrm{T}} C_\alpha]\}$ is our desired ideal.

Let $C_0 = \emptyset$. Given C_α such that $D \nleqslant_{\mathrm{T}} C_\alpha$, as in the proof of Corollary 3.17, relativizing the combination of the Friedberg Jump Inversion Theorem with cone-avoidance, we see that there is a $Z \geqslant_{\mathrm{T}} C_\alpha$ such that $X_\alpha \leqslant_{\mathrm{T}} Z'$ and $D \nleqslant_{\mathrm{T}} Z$. Let $C_{\alpha+1} = Z$. For a limit ordinal $\gamma < \omega_1$, given $C_\alpha \ngeqslant_{\mathrm{T}} D$ for all $\alpha < \gamma$, let Z_0, Z_1 be an exact pair for the ideal $\mathcal{I} = \{Y : (\exists \alpha < \gamma)[Y \leqslant_{\mathrm{T}} C_\alpha]\}$, as constructed by Spector [54]. That is, $Y \in \mathcal{I}$ if and only if $Y \leqslant_{\mathrm{T}} X_0$ and $Y \leqslant_{\mathrm{T}} X_1$. Since $D \notin \mathcal{I}$, there is an $i < 2$ such that $D \nleqslant_{\mathrm{T}} X_i$. Let $C_\gamma = X_i$. □

4. Computability theory and Hausdorff distance

As mentioned in the introduction, a *coarse description* of A is a set C such that $\delta(A, C) = 0$, and a set is *coarsely computable* if it has a computable coarse description. Even if A is not coarsely computable, one can measure how closely A can be approximated by computable sets. Let r be a real number such that $0 \leqslant r \leqslant 1$. A set A is *coarsely computable at density* r if there is a computable set C such that the symmetric agreement between A and C has lower density at least r, that is, $\underline{\rho}(A \bigtriangledown C) \geqslant r$. A set C such that $\underline{\rho}(A \bigtriangledown C) \geqslant r$ is called an r*-description* of A. Then, as in [31], define the *coarse computability bound* $\gamma(A)$ of A by

$$\gamma(A) = \sup\{r : A \text{ is coarsely computable at density } r\}.$$

We can relativize the definition of coarsely computable sets in [36] and the coarse computability bound γ in [31] to any Turing degree $\mathbf{d}$.

Definition 4.1. The set A is *coarsely* $\mathbf{d}$*-computable* if it has a coarse description computable from $\mathbf{d}$.

The set A is *coarsely* $\mathbf{d}$*-computable at density* r if there is an r-description B of A such that $B \leqslant_{\mathrm{T}} \mathbf{d}$.

The *coarse* $\mathbf{d}$*-computability bound* of a set A is

$$\gamma_{\mathbf{d}}(A) = \sup\{r : A \text{ is coarsely } \mathbf{d}\text{-computable at density } r\}.$$

If D is a set whose degree is $\mathbf{d}$, we also write $\gamma_D(A)$ for $\gamma_{\mathbf{d}}(A)$.

Note that these definitions depend only on similarity classes, so we can consider them to be defined on such classes, and in particular let $\gamma_{\mathbf{d}}([A]) = \gamma_{\mathbf{d}}(A)$ for any $[A] \in \mathcal{S}$. Observe that $\gamma_{\mathbf{d}}([A]) = 1$ if and only if $[A]$ is a limit of coarse similarity classes of $\mathbf{d}$-computable sets.

Our goal is to use the topology of $\mathcal{S}$ to investigate coarse computability and Turing degrees. For any degree $\mathbf{d}$, there are sets A with $\gamma_{\mathbf{d}}(A) = 0$. (For example, if A is weakly 1-generic relative to $\mathbf{d}$, by the relativization of a result in [31] that we will revisit in Theorem 5.1.) The following result follows from relativizing the theorem for γ given in [31, Theorem 3.4]. The proof in that paper is a slightly messy computability-theoretic construction, but the result is obvious in the present context.

Theorem 4.2. *For a degree* $\mathbf{d}$*, if* $0 \leqslant r \leqslant 1$ *then there is a set* B *with* $\gamma_{\mathbf{d}}(B) = r$.

Proof. Let α be a path from $[\emptyset]$ to a set $[A]$ with $\gamma_{\mathbf{d}}(A) = 0$. The function $\gamma_{\mathbf{d}}$ is continuous along α, so there is a point $[B]$ on α with $\gamma_{\mathbf{d}}(B) = r$ by the Intermediate Value Theorem. □

If $[A]$ is coarsely $\mathbf{d}$-computable, then $\gamma_{\mathbf{d}}([A]) = 1$, but it follows from Lemmas 3.8 and 3.9 that the converse fails for all degrees $\mathbf{d}$.

Definition 4.3. The *core*, $\kappa(\mathbf{d})$, of a degree $\mathbf{d}$ is the collection of all coarse similarity classes $[A]$ such that A is computable from $\mathbf{d}$. So $[A] \in \kappa(\mathbf{d})$ if and only if there is a set $D \leqslant_{\mathrm{T}} \mathbf{d}$ such that $A \sim_{\mathrm{c}} D$.

By Lemma 3.1, $\kappa(\mathbf{d})$ is also the collection of all coarse similarity classes $[A]$ such that $A \in \mathbf{d}$. It is clear that cores are countable since $\mathbf{d}$ computes only countably many sets.

Lemma 4.4. *If* $\mathbf{d}$ *and* $\mathbf{e}$ *are degrees, then* $\mathbf{d} \leqslant \mathbf{e}$ *if and only if* $\kappa(\mathbf{d}) \subseteq \kappa(\mathbf{e})$. *Thus* $\mathbf{d} = \mathbf{e}$ *if and only if* $\kappa(\mathbf{d}) = \kappa(\mathbf{e})$.

Proof. Suppose that $\kappa(\mathbf{d}) \subseteq \kappa(\mathbf{e})$ and let $D \in \mathbf{d}$. Then $[\mathcal{J}(D)] \in \kappa(\mathbf{d})$, where $\mathcal{J}(D)$ is as in Definition 3.2, so there is an $\mathbf{e}$-computable coarse description of $\mathcal{J}(D)$. By Lemma 3.3, $\mathbf{d} \leqslant \mathbf{e}$. The other direction is obvious. □

Definition 4.5. The *closure* $\overline{\mathbf{d}}$ of the degree $\mathbf{d}$ is the closure of $\kappa(\mathbf{d})$ in the metric space $(\mathcal{S}, \delta)$.

So $\overline{\mathbf{d}}$ is exactly the set of those classes that are limits of points of $\kappa(\mathbf{d})$ in the sense of the metric topology induced by δ. Thus $\overline{\mathbf{d}} = \{[A] : \gamma_{\mathbf{d}}(A) = 1\}$.

Observation 4.6. Recall that $\mathcal{S}$ has a group structure as described at the end of Section 1. Note that $\kappa(\mathbf{d})$ is a subgroup of $\mathcal{S}$ since if we can compute A and B from $\mathbf{d}$, then we can compute their symmetric difference from $\mathbf{d}$. Therefore, $\overline{\mathbf{d}}$ is also a subgroup since the closure of a subgroup of a topological group is again a subgroup. Indeed, since $\overline{\mathbf{d}}$ is closed in $(\mathcal{S}, \delta)$ and $\kappa(\mathbf{d})$ is countable, $\overline{\mathbf{d}}$ is also a Polish space, and hence is a Polish group.

In order to prove that closures determine degrees, that is, $\mathbf{d} \leqslant \mathbf{e}$ if and only if $\overline{\mathbf{d}} \subseteq \overline{\mathbf{e}}$, we use a relativized version of the sets $\mathcal{R}(C)$, defined using the notation in Observation 2.9. Recall that we denote the complement of A by $\neg A$.

Definition 4.7. For sets A and C, let $\mathcal{R}^A(C) = \bigoplus_n^{\mathcal{R}} X_n$, where $X_n = A$ if $n \in C$, and $X_n = \neg A$ if $n \notin C$. In particular, $\mathcal{R}^{\omega}(C) = \mathcal{R}(C)$ for all C.

We note some properties of this definition.

Lemma 4.8.
(i) *For all* A *and* C, *the set* $\mathcal{R}^A(C)$ *is a limit of* A*-computable sets.*
(ii) *For all sets* A, C, *and* G, *if* $\gamma_G(\mathcal{R}^A(C)) = 1$ *then* $\gamma_G(A) = 1$.
(iii) *For all sets* A, C_1, *and* C_2, *if* $C_1 \neq C_2$, *then* $[\mathcal{R}^A(C_1)] \neq [\mathcal{R}^A(C_2)]$.

Proof. The proof of part (i) is essentially the same as for $A = \omega$.

For part (ii), note first that if $\gamma_G(X \oplus Y) = 1$ then $\gamma_G(X) = \gamma_G(Y) = 1$. This fact is pointed out for $G = \emptyset$ in Lemma 2.4 of [2], and the relativization to arbitrary G is routine. Now assume that $\gamma_G(\mathcal{R}^A(C)) = 1$. First consider the case where $0 \in C$. Then $\mathcal{R}^A(C) = Y \oplus A$ for some Y, since A is coded into the odds in $\mathcal{R}^A(C)$. It follows that $\gamma_G(A) = 1$. If $0 \notin C$, then $\mathcal{R}^A(C) = Y \oplus \neg A$ for some Y, so $\gamma_G(\neg A) = 1$, and again it follows that $\gamma_G(A) = 1$.

Part (iii) follows from the fact that every $R_n = \{m : 2^k \mid m \ \& \ 2^{(k+1)} \nmid m\}$ has positive density, and if C_1 and C_2 differ at n, then $\mathcal{R}^A(C_1)$ and $\mathcal{R}^A(C_2)$ differ at every point of R_n. □

The following theorem shows that closures of distinct degrees are indeed distinct.

Theorem 4.9. *If* $\mathbf{d}$ *and* $\mathbf{e}$ *are degrees and* $\mathbf{d} \nleqslant \mathbf{e}$*, then there are continuum many similarity classes that belong to the closure of* $\mathbf{d}$ *but not to the closure of* $\mathbf{e}$.

Proof. Suppose that $\mathbf{d} \nleqslant \mathbf{e}$. We first construct a single set F whose similarity class belongs to the closure of $\mathbf{d}$ but not the closure of $\mathbf{e}$. Let D and E be sets of degree $\mathbf{d}$ and $\mathbf{e}$ respectively. Let $A = \mathcal{J}(D)$ (as defined in Definition 3.2), which is Turing equivalent to D. Choose any set C and let $F = \mathcal{R}^A(C)$. We must show that $[F]$ belongs to the closure of $\mathbf{d}$ but not the closure of $\mathbf{e}$.

To show that $[F]$ belongs to the closure of $\mathbf{d}$, apply part (i) of the above lemma, using the fact that A is Turing equivalent to D.

To show that $[F]$ is not in the closure of $\mathbf{e}$ it suffices to prove that $\gamma_{\mathbf{e}}(F) < 1$. Suppose for a contradiction that $\gamma_E(F) = 1$, so $\gamma_E(\mathcal{R}^A(C)) = 1$. It follows from part (ii) of the lemma that $\gamma_E(A) = 1$, so $\gamma_E(\mathcal{J}(D)) = 1$. But then D would be computable from E by Lemma 3.3, contradicting the hypothesis that $\mathbf{d} \nleqslant \mathbf{e}$.

To complete the proof of the theorem, just note that there are continuum many choices for C, and apply part (iii) of the lemma. □

We have noted that closures of degrees are subgroups. Any group G of exponent 2 has a well-defined dimension, $\dim(G)$, as a vector space over the field of two elements.

Observation 4.10. If $\mathbf{d}$ and $\mathbf{e}$ are degrees then

$$\dim\left(\frac{\overline{\mathbf{e}}}{\overline{\mathbf{d}} \cap \overline{\mathbf{e}}}\right)$$

is either 0 (exactly when $\mathbf{e} \leqslant \mathbf{d}$) or the cardinality of the continuum.

We can consider the distance from a single point to a subset of $\mathcal{S}$ in the usual way.

Definition 4.11. If $[A] \in \mathcal{S}$ and $\mathcal{B} \subseteq \mathcal{S}$, then

$$\delta([A], \mathcal{B}) = \inf\{\delta([A], [B]) : [B] \in \mathcal{B}\}.$$

There is also a natural definition of the "computational distance" between a set and the closure $\overline{\mathbf{d}}$ of a degree $\mathbf{d}$, namely:

Definition 4.12. $c([A], \overline{\mathbf{d}}) = 1 - \gamma_{\mathbf{d}}(A)$.

Lemma 1.2 shows that

$$\gamma_{\mathbf{d}}([A]) = 1 - \delta([A], \overline{\mathbf{d}}),$$

so $c([A], \overline{\mathbf{d}}) = \delta([A], \overline{\mathbf{d}})$, so the computational distance equals the metric distance. This fact shows that the density metric is the correct metric for our situation.

If $\mathcal{A}, \mathcal{B}$ are subsets of a metric space, the *Hausdorff distance* between them is, roughly speaking, the greatest distance from a point in either set to the other set. More precisely, for the metric space $(\mathcal{S}, \delta)$ the definition is as follows: If $\mathcal{A}, \mathcal{B} \subseteq \mathcal{S}$ then the Hausdorff distance between them is given by

$$H(\mathcal{A}, \mathcal{B}) = \max\{\sup_{x \in \mathcal{A}} \delta(x, \mathcal{B}), \sup_{y \in \mathcal{B}} \delta(y, \mathcal{A})\}. \tag{4.1}$$

The Hausdorff distance is always a pseudo-metric on the subsets of a metric space, and is a metric on its closed bounded subsets. Since δ is bounded, it is a metric on the closed subsets of $(\mathcal{S}, \delta)$.

In any metric space, the Hausdorff distance between two subsets of the space is the same as the Hausdorff distance between their closures. Thus, for any degrees $\mathbf{d}, \mathbf{e}$, the Hausdorff distance between their cores is the same as the Hausdorff distance between their closures, and it seems that this distance is a reasonable measure of the distance between the degrees. We will therefore use the following definition.

Definition 4.13. The Hausdorff distance, $H(\mathbf{d}, \mathbf{e})$, between the degrees $\mathbf{d}$ and $\mathbf{e}$ is the Hausdorff distance $H(\overline{\mathbf{d}}, \overline{\mathbf{e}})$ between their closures in the space $(\mathcal{S}, \delta)$. If $D \in \mathbf{d}$ and $E \in \mathbf{e}$, we also write $H(D, E)$ for $H(\mathbf{d}, \mathbf{e})$.

Using Equation (4.1) we have

$$H(\mathbf{d}, \mathbf{e}) = \max\{ \sup_{[A] \in \overline{\mathbf{d}}} \{1 - \gamma_{\mathbf{e}}([A])\}, \sup_{[B] \in \overline{\mathbf{e}}} \{1 - \gamma_{\mathbf{d}}([B])\}\}. \tag{4.2}$$

Let $\mathcal{D}$ denote the set of all Turing degrees. In order to calculate these Hausdorff distances between degrees, we use a relativized version of the function $\Gamma : \mathcal{D} \to [0, 1]$ defined in [1]. We first review the definition of Γ and some of its known properties.

For $\mathbf{a} \in \mathcal{D}$, define

$$\Gamma(\mathbf{a}) = \inf\{\gamma(A) : A \leqslant_{\mathrm{T}} \mathbf{a}\}.$$

Obviously, $\Gamma(\mathbf{0}) = 1$. We define Γ on sets by $\Gamma(A) = \Gamma(\mathbf{a})$, where $\mathbf{a}$ is the degree of A. Let $\mathcal{I}(A)$ be as in Definition 2.1. It was proved in [31] that for all A, if $\gamma(\mathcal{I}(A)) > 1/2$ then A is computable. It follows that if $\Gamma(\mathbf{a}) > 1/2$ then $\mathbf{a} = \mathbf{0}$, so $\Gamma(\mathbf{a}) = 1$. It was also proved in [31] that if the degree $\mathbf{a}$ is hyperimmune or PA, then $\Gamma(\mathbf{a}) = 0$. Furthermore, it was proved in [1] that if the degree $\mathbf{a}$ is either nonzero and computably traceable or both hyperimmune-free and 1-random, then $\Gamma(\mathbf{a}) = 1/2$. These results showed that the range R of Γ satisfies $\{0,\ 1/2,\ 1\} \subseteq R \subseteq [0, 1/2] \cup \{1\}$, and in fact there are continuum many degrees $\mathbf{a_1}$ such that $\Gamma(\mathbf{a_1}) = 1/2$ and also there are continuum many degrees $\mathbf{a_2}$ such that $\Gamma(\mathbf{a_2}) = 0$. Several years later, these results were capped off in [47] by B. Monin, who proved that, for all degrees $\mathbf{a}$, if $\Gamma(\mathbf{a}) < 1/2$ then $\Gamma(\mathbf{a}) = 0$, thus establishing that $R = \{0,\ 1/2,\ 1\}$.

We can use Γ to calculate the Hausdorff distance between $\mathbf{0}$ and an arbitrary degree $\mathbf{a}$. Namely, it is easy to see that $H(\mathbf{0}, \mathbf{a}) = 1 - \Gamma(\mathbf{a})$. In order to find Hausdorff distances between arbitrary pairs of degrees we need to relativize Γ. This is done in the obvious way, by defining

$$\Gamma_{\mathbf{c}}(\mathbf{a}) = \inf\{\gamma_{\mathbf{c}}(A) : A \leqslant_{\mathrm{T}} \mathbf{a}\}. \tag{4.3}$$

If $C \in \mathbf{c}$ and $A \in \mathbf{a}$ then we also write $\Gamma_C(A)$ for $\Gamma_{\mathbf{c}}(\mathbf{a})$.

Many of the above results on Γ relativize routinely to $\Gamma_{\mathbf{c}}$ for an arbitrary degree $\mathbf{c}$ as noted in the following proposition.

Proposition 4.14. *Let* $\mathbf{a}, \mathbf{c}$ *be degrees.*

(1) $\Gamma_{\mathbf{c}}(\mathbf{a}) = 1$ *if and only if* $\mathbf{a} \leqslant \mathbf{c}$.
(2) *If* $\Gamma_{\mathbf{c}}(\mathbf{a}) > 1/2$*, then* $\mathbf{a} \leqslant \mathbf{c}$*, so* $\Gamma_{\mathbf{c}}(\mathbf{a}) = 1$.
(3) *There exist continuum many degrees* $\mathbf{a_1} > \mathbf{c}$ *such that* $\Gamma_{\mathbf{c}}(\mathbf{a_1}) = 1/2$.
(4) *There exist continuum many degrees* $\mathbf{a_2} > \mathbf{c}$ *such that* $\Gamma_{\mathbf{c}}(\mathbf{a_2}) = 0$.

We omit the routine proof. The result of Monin also relativizes, but more care is needed for that, and we will deal with it later in Theorem 4.20. The next proposition shows how the relativized version of Γ can be used to compute Hausdorff distances between degrees.

Proposition 4.15. *For any two degrees* $\mathbf{d}$ *and* $\mathbf{e}$*, we have*

$$H(\mathbf{d},\mathbf{e}) = 1 - \min\{\Gamma_{\mathbf{d}}(\mathbf{e}), \Gamma_{\mathbf{e}}(\mathbf{d})\}.$$

Proof. The proposition follows directly from Equations (4.2) and (4.3). To express the right-hand side of (4.2) in terms of H, first note by (4.3) that

$$\sup_{[A]\in\overline{\mathbf{d}}}\{1 - \gamma_{\mathbf{e}}([A])\} = 1 - \inf_{[A]\in\overline{\mathbf{d}}} \gamma_{\mathbf{e}}([A]) = 1 - \Gamma_{\mathbf{e}}(\mathbf{d}).$$

Of course, the same result holds if $\mathbf{d}$ and $\mathbf{e}$ are interchanged.

Then by equation (4.2),

$$H(\mathbf{d},\mathbf{e}) = \max\{1 - \Gamma_{\mathbf{e}}(\mathbf{d}), 1 - \Gamma_{\mathbf{d}}(\mathbf{e})\} = 1 - \min\{\Gamma_{\mathbf{e}}(\mathbf{d}), \Gamma_{\mathbf{d}}(\mathbf{e})\}.$$

□

The next corollary follows immediately from Propositions 4.14 and 4.15.

Corollary 4.16. *If* $\mathbf{a} \leqslant \mathbf{b}$*, then* $H(\mathbf{a},\mathbf{b}) = 1 - \Gamma_{\mathbf{a}}(\mathbf{b})$*.*

Note that H respects the ordering of degrees in the sense that if $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ are degrees and $\mathbf{a} \leqslant \mathbf{b} \leqslant \mathbf{c} \leqslant \mathbf{d}$, then $H(\mathbf{b},\mathbf{c}) \leqslant H(\mathbf{a},\mathbf{d})$. To prove that this is the case, observe that

$$H(\mathbf{b},\mathbf{c}) = 1 - \Gamma_{\mathbf{b}}(\mathbf{c}) \leqslant 1 - \Gamma_{\mathbf{a}}(\mathbf{d}) = H(\mathbf{a},\mathbf{d}),$$

since $\Gamma_{\mathbf{b}}(\mathbf{c}) \geqslant \Gamma_{\mathbf{a}}(\mathbf{c}) \geqslant \Gamma_{\mathbf{a}}(\mathbf{d})$.

The next corollary also follows immediately from Propositions 4.14 and 4.15.

Corollary 4.17. *Let* $\mathbf{d}$ *and* $\mathbf{e}$ *be degrees.*

(1) $H(\mathbf{d},\mathbf{e}) = 0$ *if and only if* $\mathbf{d} = \mathbf{e}$*.*
(2) $H(\mathbf{d},\mathbf{e}) = 1/2$ *if and only if one of the following holds:*

$$\Gamma_{\mathbf{d}}(\mathbf{e}) = \Gamma_{\mathbf{e}}(\mathbf{d}) = 1/2$$
$$\mathbf{d} \leqslant \mathbf{e} \ \&\ \Gamma_{\mathbf{d}}(\mathbf{e}) = 1/2$$
$$\mathbf{e} \leqslant \mathbf{d} \ \&\ \Gamma_{\mathbf{e}}(\mathbf{d}) = 1/2.$$

(3) $H(\mathbf{d},\mathbf{e}) = 1$ *if and only if either* $\Gamma_{\mathbf{d}}(\mathbf{e}) = 0$ *or* $\Gamma_{\mathbf{e}}(\mathbf{d}) = 0$*.*

The following corollary follows immediately from the above corollary together with Proposition 4.14.

Corollary 4.18. (1) *For every degree* $\mathbf{c}$ *there are continuum many degrees* $\mathbf{d} > \mathbf{c}$ *such that* $H(\mathbf{c},\mathbf{d}) = 1/2$*.*
(2) *For every degree* $\mathbf{c}$ *there are continuum many degrees* $\mathbf{d} > \mathbf{c}$ *such that* $H(\mathbf{c},\mathbf{d}) = 1$*.*

Corollary 4.19. $(\mathcal{D}, H)$ *is a metric space.*

Proof. As mentioned above, H is a pseudo-metric on the subsets of any metric space. To show that it is a metric on $\mathcal{D}$, assume that $H(\mathbf{d},\mathbf{e}) = 0$. It then follows from part (1) of the previous proposition that $\mathbf{d} = \mathbf{e}$. □

The following result is a relativized form of Monin's theorem in [47] that if $\Gamma(A) < 1/2$ then $\Gamma(A) = 0$. However, some care is needed to prove it, as explained below.

Theorem 4.20 (Monin [47], relativized)**.** *For all sets* A *and* D*, if* $\Gamma_D(A) < 1/2$ *then* $\Gamma_D(A) = 0$*.*

Proof. The proof of this result is a straightforward modification of Monin's proof for the case where D is computable, but it is somewhat lengthy because Monin's proof involves so many steps. First, if we state Monin's theorem in terms of γ rather than Γ, it says the following:

$$(\forall \varepsilon > 0)(\forall X)[\text{if } \gamma(X) < 1/2 \text{ then } (\exists Y \leqslant_{\mathrm{T}} X)[\gamma(Y) < \varepsilon]].$$

Any proof of Monin's theorem should easily relativize to show the following:

$$(\forall D)(\forall \varepsilon > 0)(\forall X)[\text{if } \gamma_D(X) < 1/2 \text{ then } (\exists Y \leqslant_{\mathrm{T}} X \oplus D)[\gamma_D(Y) < \varepsilon]].$$

However, this relativization is not good enough for us. We need to ensure that $Y \leqslant_{\mathrm{T}} X$, not merely that $Y \leqslant_{\mathrm{T}} X \oplus D$. In fact, this stronger result comes right out of Monin's proof: The set Y is constructed from X by a series of intermediate steps in his argument. If one relativizes this argument to D, one never uses a D-oracle to construct a set or function, although the constructed objects have certain properties pertaining to D-computable functions.

We follow the proof of Theorem 3.8 of Monin's paper [47] and use his notation. We assume the reader is familiar with Monin's proof and has access to his paper. We start with a set X, an oracle D such that $\gamma_D(X) < 1/2$, and a real $\varepsilon > 0$. Our goal is to construct a set $B \leqslant_{\mathrm{T}} X$ with $\gamma_D(B) < \varepsilon$. We do so via the following steps, where all references are to [47].

(1) Define the notion of infinitely often equal as in Definition 3.3, but with "computable" replaced by "D-computable".
(2) From X compute a sequence of strings $\{\sigma_n\}_{n\in\omega}$ as in Corollary 3.9. This corollary uses Theorem 3.8. The proof of that theorem allows us to choose this sequence of strings to be X-computable, rather than merely $(X \oplus D)$-computable.
(3) Using Theorem 2.4, construct for each n a set C_n of strings as described in the proof of Theorem 3.11. The sets C_n are uniformly computable, and relativization to D has no effect on this step. Let C_n be effectively listed as $\tau_0^n, \tau_1^n, \dots$.
(4) As in the proof of Theorem 3.11, define $T_n = \{i : \delta(\sigma_n, \tau_i^n) \leqslant 1/2 - \varepsilon'\}$, where ε' is such that $0 < \varepsilon' < \varepsilon$ and δ is normalized Hamming distance as defined in Definition 2.1. Then, as shown in the proof of Theorem 3.11, there is an L such that $|T_n| \leqslant L$ for all n. Also. the sets T_n are uniformly computable from the sequence $\{\sigma_n\}_{n\in\omega}$ in (2), again without using a D-oracle.
(5) Show that for every suitably bounded D-computable function g, we have $g(n) \in T_n$ for infinitely many n, as in the proof of Theorem 3.11. (This step does not involve a construction, so we do not worry that it involves D-computable functions.)
(6) Again as in the proof of Theorem 3.11, show that there is a sequence of sets $\{T'_n\}_{n\in\omega}$, uniformly computable from the sequence $\{T_n\}_{n\in\omega}$ (not using D), with each T'_n of size at most L and $\max T'_n \leqslant 2^{L2^n}$ for all n, such that for every D-computable function g bounded by 2^{L2^n}, we have $g(n) \in T'_n$ for infinitely many n.
(7) Continuing to follow the proof of Theorem 3.11, show that there is a function h computable from the sequence $\{T'_n\}_{n\in\omega}$ (not using D) such that for every D-computable function g bounded by 2^{2^n} there exist infinitely many n for which $h(n) = g(n)$.

(8) Use the proof of Theorem 3.6 to show that there is a set $B \leqslant_{\mathrm{T}} h$ (not $h \oplus D$) such that $\gamma_D(B) < \varepsilon$. We have $B \leqslant_{\mathrm{T}} X$ since
$$B \leqslant_{\mathrm{T}} h \leqslant_{\mathrm{T}} \{T'_n\}_{n\in\omega} \leqslant_{\mathrm{T}} \{T_n\}_{n\in\omega} \leqslant_{\mathrm{T}} \{\sigma_n\}_{n\in\omega} \leqslant_{\mathrm{T}} X.$$
□

The following corollary follows at once from the theorem and Proposition 4.14.

Corollary 4.21. *For every degree* $\mathbf{c}$ *the range of* $\Gamma_{\mathbf{c}}$ *is* $\{0, 1/2, 1\}$.

This corollary and Proposition 4.15 yield the following.

Corollary 4.22. *The possible Hausdorff distances between degrees are exactly* 0*,* $1/2$*, and* 1*.*

Although we do not pursue this idea further here, it is worth noting that H can be extended from degrees to downward-closed sets of degrees in a straightforward way. Let $\mathcal{E}$ be the collection of all downward-closed sets of degrees. Then $(\mathcal{E}, H)$ is still a $0, 1/2, 1$-valued metric space, and we can think of $(\mathcal{D}, H)$ as a subspace by identifying a degree with its lower cone. An intermediate space that could also be worth studying is that of Turing ideals.

5. Hausdorff distance, Lebesgue measure, and Baire category

It is natural to ask which Hausdorff distance occurs "most frequently" between pairs of degrees. The answer depends on whether the question is formalized using Lebesgue measure or Baire category. Indeed, as we will see in this and the next section, the interplay between typicality in the sense of measure (as captured computability-theoretically in notions such as 1-randomness) and typicality in the sense of category (as captured computability-theoretically in notions such as (weak) 1-genericity) seems central to understanding the structure of $(\mathcal{D}, H)$. Randomness leads to constructions of degrees at distance $1/2$, while genericity leads to constructions of degrees at distance 1. We use (c. m. A) to abbreviate "for comeager many A" (in the usual topology on 2^ω) and (a. e. A) to abbreviate "for almost every A" (in the usual coin-toss measure on 2^ω).

We first consider Baire category, in the usual topology on 2^ω. Recall that we write Γ_A for $\Gamma_{\mathbf{a}}$ where $\mathbf{a}$ is the degree of A. In [31] it is shown that if B is weakly 1-generic then $\gamma(B) = 0$. The proof of this result relativizes to establish the following fact.

Theorem 5.1 (Hirschfeldt, Jockusch, McNicholl, and Schupp [31, proof of Theorem 2.2, relativized])**.** *If* B *is weakly* 1*-generic relative to* A*, then* $\gamma_A(B) = 0$*, so* $\Gamma_A(B) = 0$*, and hence* $H(A, B) = 1$*. Therefore, for all* $C \geqslant_{\mathrm{T}} B$*, we have* $\Gamma_A(C) = 0$*, and hence* $H(A, C) = 1$*.*

Corollary 5.2. $(\forall A)(\text{c. m. } B)[H(A, B) = 1]$.

It also follows that there is an uncountable family $\mathcal{C}$ of degrees such that any two distinct degrees $\mathbf{a}, \mathbf{b} \in \mathcal{C}$ satisfy $H(\mathbf{a}, \mathbf{b}) = 1$: Consider a family $\mathcal{C}$ with the property that any two distinct degrees $\mathbf{a}, \mathbf{b}$ in $\mathcal{C}$ satisfy $H(\mathbf{a}, \mathbf{b}) = 1$. If $\mathcal{C}$ is countable, then it follows from the previous corollary that there is a $\mathbf{c} \notin \mathcal{C}$ such that $\mathcal{C} \cup \{\mathbf{c}\}$ also has this property. Thus $\mathcal{C}$ is not maximal. So any maximal family with this property is uncountable. We can improve this result using Mycielski's Theorem for category, which we stated as Theorem 2.17 and will prove as Theorem 7.22 below.

Corollary 5.3. *There is a family $\mathcal{C}$ of degrees of size continuum such that any two distinct degrees $\mathbf{a}, \mathbf{b} \in \mathcal{C}$ satisfy $H(\mathbf{a}, \mathbf{b}) = 1$.*

We now consider analogous results for measure. We first examine Hausdorff distances from the least degree $\mathbf{0}$.

Proposition 5.4. $(\text{a.e. } B)[H(\emptyset, B) = 1]$.

Proof. By a result of D. A. Martin (unpublished, see e.g. [20, Theorem 8.21.1]), almost every set is of hyperimmune degree. As noted in [31], since every hyperimmune degree computes a weakly 1-generic, it follows from Theorem 5.1 that $\Gamma(\mathbf{b}) = 0$ for every hyperimmune degree $\mathbf{b}$, so $H(\mathbf{0}, \mathbf{b}) = 1$ for every such $\mathbf{b}$. □

One might hope to prove by relativizing the above proposition that for every A, we have $H(A, B) = 1$ for almost every B, which would be in complete analogy with Corollary 5.2. However, relativization yields only the result that for every A, we have $H(A, A \oplus B) = 1$ for almost every B. In fact, we have the following result. We state it in terms of the notion of Church stochasticity (see [20, Definition 7.4.1]), which is implied by 1-randomness (and even by computable randomness), but all we need is the fact that if B is Church stochastic relative to A, then for any infinite A-computable set C, the density of B within C is $1/2$.

Theorem 5.5. *If B is Church stochastic relative to A then $\Gamma_B(A) \geqslant 1/2$, so if A and B are Church stochastic relative to each other then $H(A, B) = 1/2$.*

Proof. Suppose that B is Church stochastic relative to A. Let $C \leqslant_{\mathrm{T}} A$ be infinite and coinfinite. Then the density of B within C and the density of B within the complement of C must both be $1/2$. Thus $\rho(C \bigtriangleup B) = 1/2$. That is, B is a $1/2$-description of every infinite, coinfinite A-computable set, and hence $\Gamma_B(A) \geqslant 1/2$.

If A and B are Church stochastic relative to each other then $\Gamma_B(A) \geqslant 1/2$ and $\Gamma_A(B) \geqslant 1/2$, but A and B are also Turing incomparable, so in fact $\Gamma_B(A) = \Gamma_A(B) = 1/2$, and hence $H(A, B) = 1/2$. □

The following corollary holds for computable randomness as well, but we state it for 1-randomness as that is the version we will use below.

Corollary 5.6. *If B is 1-random relative to A then $\Gamma_B(A) \geqslant 1/2$, so if we also have $A \nleqslant_{\mathrm{T}} B$, then $\Gamma_B(A) = 1/2$. Thus, if A and B are relatively 1-random, then $H(A, B) = 1/2$.*

Corollary 5.7. *For every A there exists a B such that $H(A, B) = 1/2$ and B is 1-random relative to A.*

Proof. Let B be 1-random relative to A and such that every B-computable function is dominated by an A-computable function. Such a set exists by the relativized version of the hyperimmune-free basis theorem. Then $\Gamma_A(B) = 1/2$ by the relativized form of [1, Corollary 1.13], while $\Gamma_B(A) = 1/2$ by Corollary 5.6. Thus $H(A, B) = 1/2$. □

The class of 1-randoms has measure 1. Furthermore, if A is 1-random then the class of sets that are 1-random relative to A also has measure 1, and if B is in this class then A and B are relatively 1-random, by van Lambalgen's Theorem. Thus we have the following fact.

Corollary 5.8. $(\text{a.e. } A)(\text{a.e. } B)[H(A, B) = 1/2]$.

It also follows that there is an uncountable subset $\mathcal{C}$ of 2^ω such that if A and B are any two distinct element of $\mathcal{C}$, then $H(A,B) = 1/2$: Let $\mathcal{E}$ be the class of all A such that $H(A,B) = 1/2$ for almost every B. By the previous corollary, $\mathcal{E}$ has measure 1. By Zorn's Lemma there is a maximal family $\mathcal{C} \subseteq \mathcal{E}$ with the property in the statement of the corollary. It follows from the previous corollary and the countable additivity of Lebesgue measure that $\mathcal{C}$ cannot be countable. We can improve on this result by applying Corollary 2.14.

Corollary 5.9. *There is a subset $\mathcal{C}$ of 2^ω of size continuum such that if A and B are any two distinct element of $\mathcal{C}$, then $H(A,B) = 1/2$.*

Kolmogorov's 0-1 Law implies that any measurable collection of sets that is closed under Turing equivalence has measure 0 or 1. In particular, for every A, the class of all B such that $H(A,B) = 1$ always has measure 0 or 1. As we have seen, if A is 1-random then this class has measure 0.

Definition 5.10. A set A, and the degree of A, are *attractive* if the class of all B such that $H(A,B) = 1/2$ has measure 1, or equivalently, the class of all B such that $H(A,B) = 1$ has measure 0. Otherwise, A and its degree are *dispersive.*

It follows from Proposition 5.4 that $\emptyset$ is dispersive, and it follows from Corollary 5.6 and van Lambalgen's Theorem that every 1-random set is attractive.

Proposition 5.11. *The class of attractive degrees is closed upwards. Equivalently, the class of dispersive degrees is closed downwards.*

Proof. Suppose that A is dispersive and $C \leqslant_{\mathrm{T}} A$. By Corollary 4.17, we have (a. e. B)$[\Gamma_A(B) = 0$ or $\Gamma_B(A) = 0]$. By Corollary 5.6, we have $\Gamma_B(A) \geqslant 1/2$ for almost every B, so $\Gamma_A(B) = 0$ for almost every B. Since $C \leqslant_{\mathrm{T}} A$, we have $\Gamma_C(B) \leqslant \Gamma_A(B)$ for all B. Hence $\Gamma_C(B) = 0$ for almost every B. It follows from Corollary 4.17 that $H(C,B) = 1$ for almost every B, so C is dispersive. □

It follows from the above proposition and the remark just before it that if a set computes a 1-random then it is attractive. In particular, $\emptyset'$ is attractive. We will see in Observation 5.19 that not every attractive set computes a 1-random, but let us first discuss the dispersive sets. The following proposition follows at once from Theorem 5.1 and will be used frequently to show that sets are dispersive.

Proposition 5.12. *If almost every set computes a set that is weakly* 1*-generic relative to A, then A is dispersive.*

Recall from Proposition 5.4 that the empty set is dispersive. It follows from known results that there are noncomputable dispersive sets. A degree $\mathbf{a}$ is *low for weak* 1*-genericity* if every weakly 1-generic is still weakly 1-generic relative to $\mathbf{a}$. If $\mathbf{a}$ is low for weak 1-genericity, then $\mathbf{a}$ is dispersive by the above proposition, since almost every degree is hyperimmune, and thus computes a set that is weakly 1-generic relative to $\mathbf{a}$. Stephan and Yu [55] showed that a degree is low for weak 1-genericity if and only if it is hyperimmune-free and not diagonally noncomputable. It follows that all sets that are sufficiently generic for forcing with computable perfect trees are dispersive. The following theorems give further examples. In particular, we show that there is a high c.e. degree that is dispersive and hence, as we will remark, that there is a computably random set that is dispersive. We also show that every low c.e. degree is dispersive, and that every weakly 2-generic degree is dispersive, so that the class of dispersive sets is comeager. Indeed, we will show that if A is weakly 2-generic and B is 2-random, then $H(A,B)=1$.

We begin with the following result. Although we will strengthen it below, we include a proof because it will be helpful in explaining the proofs of these stronger versions.

Theorem 5.13. *There is a noncomputable c.e. set A such that almost every set computes a set that is weakly 1-generic relative to A, and hence A is dispersive.*

Proof. This proof is a finite-injury priority construction, based on Martin's proof mentioned above that the hyperimmune degrees have measure 1. That proof, as presented for instance in [20, Theorem 8.21.1], can easily be adapted to give a direct proof that almost every set computes a weakly 1-generic (which is essentially what we would get if we removed the R-requirements from the proof below).

By Kolmogorov's 0-1 Law, it is enough to build a Turing functional Ψ such that the set of X for which Ψ^X is weakly 1-generic relative to A has positive measure. We will ensure that the set of X such that Ψ^X is total has positive measure, while satisfying requirements

$$R_e : \Phi_e \neq A$$

and

$$Q_e : W_e^A \text{ dense and } \Psi^X \text{ total } \Rightarrow \ \Psi^X \text{ meets } W_e^A.$$

We arrange these in a priority ordering $Q_0, R_0, Q_1, R_1, \ldots$. We think of Ψ as being defined in stages, where at stage s we define Ψ^τ for the strings τ of length s. During the construction, certain strings will be *claimed* by Q-requirements. If τ extends such a string σ, then we do not allow Ψ^τ to converge on any new values (i.e., we define $\Psi^\tau = \Psi^{\tau\restriction(s-1)}$), unless the strategy for the requirement claiming σ defines it otherwise, as discussed below. Otherwise, we ensure that $\Psi^\tau(n)$ is defined for all $n \leqslant s$. (The actual values do not matter in this case, so if $\Psi^{\tau\restriction(s-1)}(n)$ is not defined for such an n, then we just let $\Psi^\tau(n)=0$.)

We satisfy R_e in the usual way by choosing a witness n, waiting until we see that $\Phi_e(n)=0$ (if ever), and then enumerating n into A. Each time R_e is initialized, it chooses a new witness larger than any number previously mentioned in the construction, which means that the value of A on this number cannot affect any currently existing computation.

To satisfy a single Q_e, we could proceed as follows. Let $\sigma_0, \ldots, \sigma_{2^{e+2}-1}$ be the strings of length $e+2$. We begin by claiming σ_0 at some stage $s > e+2$. Let $\tau_0, \ldots, \tau_m$ be the extensions of σ_0 of length $s-1$. For each such τ_i, we have defined $\Psi^{\tau_i} = \mu_i$ for some μ_i. We now wait until a stage $t \geqslant s$ such that for each $i \leqslant m$, there is an extension ν_i of μ_i in $W_e^A[t]$. If such a t is never found, then σ_0 is permanently claimed by Q_e, and Ψ^X is not total for $X \succ \sigma_0$, but the set of such X has measure only $2^{-(e+2)}$. If t is found then we try to ensure that each ν_i is in W_e^A by initializing weaker priority R-requirements. (Of course, stronger priority R-requirements might still act, but in that case we simply restart our strategy for Q_e.) We then ensure that $\nu_i \prec \Psi^X$ for each $i \leqslant m$ and each X extending τ_i, drop our claim on σ_0, claim σ_1, and repeat our procedure.

In this way, we move through the strings of length $e+2$, with one of two eventual outcomes. We might eventually permanently claim some σ_k. If so, then W_e^A is not dense, so Q_e is satisfied, and we have removed only $2^{-(e+2)}$ from the measure of

$\{X : \Psi^X \text{ total}\}$. Otherwise, we ensure that Ψ^X extends some element of W_e^A for every X, again satisfying Q_e.

When considering all our Q-requirements at once, the only difference is that we need to ensure that while Q_e is claiming a string σ, no Q_i with $i > e$ can claim an extension of σ. Notice that the total measure removed from $\{X : \Psi^X \text{ total}\}$ by permanently claimed strings over the whole construction is at most $\sum_e 2^{-(e+2)} = 1/2$.

We now proceed with the full construction. We think of Ψ as a computable function $2^{<\omega} \to 2^{<\omega}$, whose value at σ is denoted by Ψ^σ, such that if $\sigma \prec \tau$ then $\Psi^\sigma \preccurlyeq \Psi^\tau$. Then $\Psi^X = \bigcup_n \Psi^{X\restriction n}$, so Ψ^X is total if and only if $\lim_n |\Psi^{X\restriction n}| = \infty$. For two strings σ and τ of the same length, let $\sigma <_1 \tau$ if σ comes before τ in the lexicographic ordering.

When an R-strategy is initialized, its witness becomes undefined. When a Q-strategy is initialized, it gives up any current claims it might have, and is declared to be unsatisfied.

We begin with $\Psi^\lambda = \lambda$, where λ is the empty string. At each stage $s > 0$, we define Ψ^σ for all σ of length s, proceeding as follows.

First, for each R_e with $e \leqslant s$ that is not yet satisfied and does not currently have a witness, assign R_e a witness larger than any number appearing in the construction so far. Then, for each R_e with $e \leqslant s$ that has a witness n such that $\Phi_e(n)[s] = 0$ and n is not yet in A, put n into A, initialize all Q_i with $i > e$, and declare R_e to be satisfied.

Now say that Q_e with $e < s-2$ *requires attention* if Q_e is not currently declared to be satisfied, and either Q_e is not claiming any string or it is claiming a string σ, and for every $\tau \succcurlyeq \sigma$ of length $s-1$, there is an extension ν of Ψ^τ currently in W_e^A. For the least e such that Q_e requires attention (if any), we act as follows.

If Q_e is not claiming any string, then Q_e claims the $<_1$-least string τ of length $e+2$ such that no initial segment of τ is currently being claimed by any Q_i with $i < e$. (Since each Q_i can be claiming at most one string, and that string must have length $i+2$, such a τ must exist.)

Otherwise, proceed as follows. For each $\tau \succcurlyeq \sigma$ of length $s-1$, let ν be an extension of Ψ^τ currently in W_e^A and define $\Psi^{\tau^\frown 0} = \Psi^{\tau^\frown 1} = \nu$. Now Q_e drops its claim on σ and claims the next $<_1$-least string $\tau >_1 \sigma$ of length $e+2$ such that no initial segment of τ is currently being claimed by any Q_i with $i < e$. If there is no such τ, then declare Q_e to be satisfied.

In any case, initialize every Q_i with $i > e$ and every R_i with $i \geqslant e$.

Finally, for each μ of length s such that Ψ^μ is not yet defined, if some initial segment of μ is currently claimed by a Q-strategy then define $\Psi^\mu = \Psi^{\mu\restriction(s-1)}$, and otherwise define $\Psi^\mu = \Psi^{\mu\restriction(s-1)^\frown 0}$.

This completes the construction of A and Ψ. We now verify its correctness. Note that, for all s, no two compatible strings are claimed by different requirements at the end of stage s. If no initial segment of X is ever permanently claimed, then there are infinitely many stages s such that no initial segment of X is claimed at the end of stage s, and hence Ψ^X is total, so $\mu(\{X : \Psi^X \text{ total}\}) \geqslant 1 - \sum_e 2^{-(e+2)} = 1/2$.

Whenever an R-strategy puts a number into A, it is permanently satisfied. If Q_e is not initialized, then it goes through the strings of length $e+2$ in lexicographic order and thus eventually stops requiring attention. Thus, by induction, each requirement is initialized only finitely often.

Thus each R_e has a final witness n, and we ensure that $\Phi_e(n) \neq A(n)$.

If Q_e ever permanently claims a string, then W_e^A is not dense. Otherwise, for each string σ of length $e+2$ such that no initial segment of σ is eventually permanently claimed by some Q_i, the strategy for Q_e ensures that if X extends σ then Ψ^X has an initial segment in W_e^A. So if Ψ^X is total then it meets W_e^A. □

The above proof can be adapted to show that A can be made to be high, and also that it can be chosen to be any low set, as we show in the next two theorems. (See Observation 5.19 for a remark on how far these results could be extended.)

Theorem 5.14. *There is a high c.e. set A such that almost every set computes a set that is weakly 1-generic relative to A, and hence A is dispersive.*

Proof. For a set X, let $X^{[e]} = \{n : \langle e, n\rangle \in X\}$. To make A high, we use the fact that there is a c.e. set C such that each $C^{[e]}$ is either finite or equal to $\omega^{[e]}$, and if $A^{[e]} =^* C^{[e]}$ for all e then A is high. (See e.g. [20, Section 2.14.3]. Here $=^*$ is equality up to finitely many elements.)

The basic idea of this proof is that we have the same requirements

$$Q_e : W_e^A \text{ dense and } \Psi^X \text{ total } \Rightarrow \Psi^X \text{ meets } W_e^A$$

as in the previous proof, but the R-requirements in that proof are replaced by

$$R_e : A^{[e]} =^* C^{[e]}.$$

If $C^{[e]}$ is finite, then the action we need to take to satisfy R_e is finitary, so its effect is similar to that of the R-requirements in the previous proof. Otherwise, this action is infinitary, but it is computable. A weaker priority strategy for satisfying a Q-requirement can guess at an m such that $n \in A^{[e]}$ for all $n \geqslant m$. Then it does not believe an enumeration into $W_e^A[s]$ with use u unless every $n \in [m, u)$ is in $A^{[e]}[s]$. Of course, we cannot know whether $C^{[e]}$ is finite or not, so we proceed as usual and make this into an infinite-injury priority construction, using a tree of strategies.

For each e and $\alpha \in 2^e$, we have strategies R_α for R_e and Q_α for Q_e. The strategy Q_α works under the assumption that R_β is infinitary if $\beta^\frown 0 \preccurlyeq \alpha$ and finitary if $\beta^\frown 1 \preccurlyeq \alpha$.

For binary strings α and β, write $\alpha <_{\mathrm{L}} \beta$ if α is above or to the left of β, i.e., if $\alpha \prec \beta$ or there is a γ such that $\gamma^\frown 0 \preccurlyeq \alpha$ and $\gamma^\frown 1 \preccurlyeq \beta$.

At stage s, we define a string γ_s of length s such that R_α and Q_α are allowed to act at that stage if and only if $\alpha \preccurlyeq \gamma_s$. We say these strategies are *accessible* at stage s, and allow them to act in order (i.e., $Q_{\gamma_s \upharpoonright n}$ acts before $R_{\gamma_s \upharpoonright n}$, which acts before $Q_{\gamma_s \upharpoonright n+1}$). We will describe the details of these actions below, but we can already say how γ_s is defined. Suppose we have defined $\gamma_s \upharpoonright n$ for $n < s-1$. If $R_{\gamma_s \upharpoonright n}$ enumerates any numbers into A at stage s, then $\gamma_s(n) = 0$. Otherwise, $\gamma_s(n) = 1$. As usual, the *true path* of the construction is $\liminf_s \gamma_s$, i.e., the leftmost path visited infinitely often.

The strategy R_α works as follows at stages at which it is accessible. It chooses a number m_α, which is picked to be a fresh large number each time R_α is initialized. At any stage s at which R_α is accessible, it enumerates every $n \geqslant m_\alpha$ in $C^{[e]}[s] \setminus A^{[e]}[s]$ into $A^{[e]}[s]$. If there is at least one such n, then it initializes all Q_β with $\alpha <_{\mathrm{L}} \beta$.

The strategy Q_α acts similarly to the strategies for Q-requirements in the previous construction, but instead of working with strings of a length fixed ahead of time, it has a parameter $k_\alpha > 1$, and works with strings of that length. Every time Q_α is initialized, k_α is picked to be a fresh large number (which ensures that it is larger than the lengths of the strings currently being used by strategies above or to the left of Q_α in the tree of strategies). As before, for strings σ and τ of the same length, let $\sigma <_{\mathrm{l}} \tau$ if σ comes before τ in the lexicographic ordering.

Suppose that Q_α is accessible at stage s. If Q_α is not claiming any string, is not currently satisfied, and was not initialized at this stage, then Q_α claims the $<_{\mathrm{L}}$-least string τ of length k_α such that no initial segment of τ is currently being claimed by any Q_β with $\beta <_{\mathrm{L}} \alpha$. To see that such a τ exists, note that k_α is chosen to be larger than k_β for all $\beta <_L \alpha$, the k_β's are distinct, and each strategy can claim at most one string at a time. Hence there is at most one claimed string of each length less than k_α that Q_α must respect, so the total number of strings of length k_α that cannot be chosen to be τ is at most the sum of 2^n for $1 < n < 2^{k_\alpha}$. Thus not every τ of length 2^{k_α} is forbidden.

Now suppose that Q_α is claiming a string σ, and for every $\tau \succcurlyeq \sigma$ of length $s-1$, there is an extension ν of Ψ^τ currently in W_e^A such that, for the use u of this enumeration and every β such that $\beta^\frown 0 \prec \alpha$, every number in $[m_\beta, u)$ is currently in $A^{[|\beta|]}$. Then proceed as follows. For each $\tau \succcurlyeq \sigma$ of length $s-1$, choose a ν as above and define $\Psi^{\tau^\frown 0} = \Psi^{\tau^\frown 1} = \nu$. Now Q_α drops its claim on σ and claims the next $<_{\mathrm{l}}$-least string $\tau >_{\mathrm{l}} \sigma$ of length k_α such that no initial segment of τ is currently being claimed by any Q_β with $\beta <_{\mathrm{L}} \alpha$. If there is no such τ, then declare Q_α to be satisfied.

In either of the above cases, initialize every Q_β with $\alpha <_{\mathrm{L}} \beta$ and every R_β with $\alpha \leqslant_{\mathrm{L}} \beta$.

If neither of these cases holds, then Q_α does nothing at this stage.

After all accessible strategies have acted at stage s, for each μ of length s such that Ψ^μ is not yet defined, if some initial segment of μ is currently claimed by a Q-strategy then define $\Psi^\mu = \Psi^{\mu \upharpoonright (s-1)}$, and otherwise define $\Psi^\mu = \Psi^{\mu \upharpoonright (s-1)^\frown 0}$.

Now, as before, only one string of each length greater than 1 can ever be permanently claimed by any strategy, and Ψ^X is total unless some initial segment of it is eventually permanently claimed, because whenever a strategy gives up its claim on a string, no other strategy can claim an extension of that string at that stage. Thus $\mu(\{X : \Psi^X \text{ total}\}) \geqslant 1 - \sum_n 2^{-(n+2)} = 1/2$.

An argument by induction, much as before, shows that any strategy on the true path is initialized only finitely often. Thus, if R_α with $|\alpha| = e$ is on the true path then it puts all sufficiently large elements of $C^{[e]}$ into $A^{[e]}$, and hence ensures that R_e is satisfied.

Now let Q_α with $|\alpha| = e$ be on the true path. Let $\beta_0, \dots, \beta_{k-1}$ be all of the strings such that $\beta_i{}^\frown 0 \preccurlyeq \alpha$, and let m_i be the final value of m_{β_i}. If Q_α ever permanently claims a string σ, then there is a $\tau \succcurlyeq \sigma$ such that for every sufficiently large stage s at which Q_α is accessible, if there is an extension ν of τ in $W_e^A[s]$, then for the use u of this enumeration, there are an $i < k$ and an $n \in [m_i, u)$ such that $n \notin A^{[|\beta_i|]}[s]$. This n will eventually be put into $A^{[|\beta_i|]}$ by R_{β_i}, so τ has no extension in W_e^A. Thus in this case W_e^A is not dense.

Otherwise, for the final value of k_α and each string σ of length k_α such that no initial segment of σ is eventually permanently claimed by some Q_β with $\beta <_{\mathrm{L}} \alpha$, the strategy Q_α ensures that if X extends σ then Ψ^X has an initial segment in W_e^A. So if Ψ^X is total then it meets W_e^A. □

An interesting consequence of the above theorem is that computable randomness is not enough to ensure that a set is attractive, because every high c.e. degree contains a computably random set, as shown by Nies, Stephan, and Terwijn [50]. (A computably random dispersive set gives an example of a set that is computably random, but not computably random relative to A for measure-1 many sets A, which is a strong form of the failure of van Lambalgen's Theorem for computable randomness, though there are easier means to obtain this kind of example.)

Theorem 5.15. *Let A be a low c.e. set. Then almost every set computes a set that is weakly 1-generic relative to A, and hence A is dispersive.*

Proof. This proof is again based on that of Theorem 5.13, but now we have no R-requirements, since A is given to us. Thus we have only the requirements

$$Q_e : W_e^A \text{ dense and } \Psi^X \text{ total } \Rightarrow \Psi^X \text{ meets } W_e^A.$$

The basic idea is the following. Suppose that at a stage s, the requirement Q_e is currently claiming a string σ and seems to require attention, i.e., for every $\tau \succcurlyeq \sigma$ of length $s-1$, there is an extension ν of Ψ^τ currently in W_e^A. Then we can test the enumerations of these strings ν into W_e^A using a computable approximation to A'. Roughly speaking, we can use this approximation to A' to guess at whether these ν are truly in W_e^A, by waiting until either the approximation says that they are, or at least one of them leaves W_e^A. If the former happens, then we think of A' as certifying the enumerations of the strings ν, and declare that Q_e does in fact require attention. The idea is that by using the approximation to A' carefully, certified enumerations will be incorrect only finitely often.

More precisely, during the construction we build uniformly c.e. sets of strings $D_{e,k,\sigma}$ for $e, k \in \omega$ and $\sigma \in 2^{<\omega}$. By the Recursion Theorem, we can assume that we have a computable function $f : \omega \times \omega \times 2^{<\omega} \to \omega$ such that A has an initial segment in $D_{e,k,\sigma}$ if and only if $A'(f(e,k,\sigma)) = 1$. Suppose that Q_e seems to require attention at stage s as above. For each $\tau \succcurlyeq \sigma$ of length $s-1$, pick the extension ν_τ of Ψ^τ currently in W_e^A that has been in that set the longest, let u_τ be the use of the current enumeration of ν_τ into W_e^A, and let u be the maximum of u_τ over all $\tau \succcurlyeq \sigma$ of length $s-1$. Let k be the number of times Q_e has been initialized. Put $A[s] \upharpoonright u$ into $D_{e,k,\sigma}$, and search for a $t \geqslant s$ such that either $A'(f(e,k,\sigma))[t] = 1$ or $A[t] \upharpoonright u \neq A[s] \upharpoonright u$. Such a t must exist. (Notice that, in the second case, $A \upharpoonright u \neq A[s] \upharpoonright u$, since A is c.e.) In the first case, Q_e actually requires attention at stage s. As before, we choose the least e such that Q_e requires attention at stage s and act for it as in the proof of Theorem 5.13, making sure that we use the strings ν_τ mentioned above. If later we find that we were mistaken, i.e., that $A'(f(e,k,\sigma))[u] = 0$ for some $u > t$, then we restart Q_e as if it had been initialized, though we do not count this as an initialization (so that we keep working with the same sets $D_{e,k,\sigma}$).

This restarting process can happen only finitely often between initializations, as each occurrence requires a change in the approximation to $A'(f(e,k,\sigma))$ for a fixed k and one of the finitely many strings σ of length $2^{-(e+2)}$. Thus the proof that each strategy is initialized only finitely often remains the same as before.

We can then argue that each Q_e is satisfied more or less as before: Let k be the total number of times that Q_e is initialized. If Q_e ever permanently claims a string σ, then it eventually stops requiring attention, so there is at least one extension τ of σ such that whenever an extension of τ is in $W_e^A[s]$ with use u, there is a

$t \geqslant s$ such that $A[t] \upharpoonright u \neq A[s] \upharpoonright u$. Since A is c.e., τ has no extension in W_e^A, so W_e^A is not dense. Otherwise, for each string σ of length $e+2$ such that no initial segment of σ is eventually permanently claimed by some Q_i, the strategy for Q_e eventually requires attention at a stage at which it is claiming σ, such that it never gets initialized or restarted after that stage. Thus this strategy ensures that if X extends σ then Ψ^X has an initial segment in W_e^A. So if Ψ^X is total then it meets W_e^A. $\square$

Moving away from c.e. sets, we have the following result. Note that it does not imply any of our previous results, because a weakly 2-generic set cannot compute any noncomputable c.e. sets.

Theorem 5.16. *Let A be weakly 2-generic. Then almost every set computes a set that is weakly 1-generic relative to A, and hence A is dispersive.*

In fact, almost every set computes a set that is weakly 1-generic relative to every weakly 2-generic set.

Proof. By Kolmogorov's 0-1 Law, it is enough to show that there are positive-measure many sets that compute a set that is weakly 1-generic relative to every weakly 2-generic set. We do this by defining a Turing functional Ψ such that Ψ^X is total for positive-measure many X, and such that if Ψ^X is total then it is weakly 1-generic relative to every weakly 2-generic set.

The construction of Ψ is once again based on the one in the proof of Theorem 5.13. We no longer have any R-requirements, but now have requirements

$$Q_e : A \text{ weakly 2-generic, } W_e^A \text{ dense, and } \Psi^X \text{ total } \Rightarrow \Psi^X \text{ meets } W_e^A.$$

For each Q_e, we will have infinitely many strategies Q_e^α, one for each binary string α. Associated with each Q_e^α will be a string $\beta_e^\alpha \succcurlyeq \alpha$, whose value might change during the construction. This string will be such that, for the final value of β_e^α, we will have ensured that Q_e holds as long as A extends β_e^α. As this value will be computably approximated, and the set of β_e^α will be dense, if A is weakly 2-generic then it will extend some β_e^α.

Let $g : \omega \times 2^{<\omega} \to \omega$ be a computable injective function. We arrange the Q_e^α's into a priority list using g, declaring that Q_e^α is stronger that Q_i^β if $g(e,\alpha) < g(i,\beta)$. The string β_e^α is initially equal to α, and is again set to α every time Q_e^α is initialized. At each stage, Q_e^α might be claiming a string. Initially, no Q_e^α claims a string. When Q_e^α is initialized, it gives up its current claim if it has one. As before, for two strings σ and τ of the same length, let $\sigma <_{\mathrm{l}} \tau$ if σ comes before τ in the lexicographic ordering.

At stage s, say that Q_e^α with $g(e,\alpha) < s-2$ *requires attention* if Q_e^α is not currently declared to be satisfied, and either Q_e^α is not claiming any string or it is claiming a string σ, and there is a $\gamma \succcurlyeq \beta_e^\alpha$ of length s such for every $\tau \succcurlyeq \sigma$ of length $s-1$, there is an extension ν of Ψ^τ in $W_e^\gamma[s]$. For the least value of $g(e,\alpha)$ such that Q_e^α requires attention (if any), we act as follows.

If Q_e^α is not claiming any string, then it claims the $<_{\mathrm{l}}$-least string τ of length $g(e,\alpha)+2$ such that no initial segment of τ is currently being claimed by any Q_i^δ with $g(i,\delta) < g(e,\alpha)$. Such a τ exists by the same argument used to show the existence of the analogous string τ in the proof of Theorem 5.14.

Otherwise, proceed as follows. For each $\tau \succcurlyeq \sigma$ of length $s-1$, let ν be an extension of Ψ^τ in $W_e^\gamma[s]$ and define $\Psi^{\tau^\frown 0} = \Psi^{\tau^\frown 1} = \nu$. Now Q_e^α drops its claim on σ and claims the next $<_l$-least string $\tau >_l \sigma$ of length $g(e,\alpha)+2$ such that no initial segment of τ is currently being claimed by any Q_i^δ with $g(i,\delta) < g(e,\alpha)$. If there is no such τ, then declare Q_e^α to be satisfied. In either case, redefine $\beta_\alpha = \gamma$.

In any case, initialize every Q_i^δ with $g(i,\delta) > g(e,\alpha)$.

Finally, for each μ of length s such that Ψ^μ is not yet defined, if some initial segment of μ is currently claimed by a strategy then define $\Psi^\mu = \Psi^{\mu\upharpoonright(s-1)}$, and otherwise define $\Psi^\mu = \Psi^{\mu\upharpoonright(s-1)}{}^\frown 0$.

As before, if no initial segment of X is ever permanently claimed, then there are infinitely many stages s such that no initial segment of X is claimed at the end of stage s, and hence Ψ^X is total, so $\mu(\{X : \Psi^X \text{ total}\}) \geqslant 1 - \sum_n 2^{-(n+2)} = 1/2$. Also as before, by induction, each strategy is initialized only finitely often.

Fix e and a weakly 2-generic A. Writing β_α for the final value of that string, $\{\beta_\alpha : \alpha \in 2^{<\omega}\}$ is dense and $\emptyset'$-c.e. (In fact, it is $\emptyset'$-computable.) Thus there is an α such that $\beta_\alpha \prec A$. If Q_e^α ever permanently claims a string, then W_e^X is not dense for any X extending β_α, so in particular W_e^A is not dense. Otherwise, the construction ensures that for each string σ of length $g(e,\alpha)+2$ such that no initial segment of σ is eventually permanently claimed by some strategy, if X extends σ then Ψ^X has an initial segment in $W_e^{\beta_\alpha}$, and hence in W_e^A. So if Ψ^X is total then it meets W_e^A. □

We do not know whether every 1-generic set is dispersive.

Corollary 5.17. *The class of attractive sets is meager, i.e.,*

$$(\text{c.m. } A)(\text{a.e. } B)[H(A,B)=1].$$

The proof of Theorem 5.16 yields the following more precise version of the above corollary.

Theorem 5.18. *If A is weakly 2-generic and B is 2-random, then B computes a set that is weakly 1-generic relative to A, and hence $H(A,B)=1$.*

Proof. The proof of Theorem 5.16 can easily be modified to show that for each i there is a Turing functional Ψ_i that meets all the requirements Q_e with Ψ_i replacing Ψ, and has the further property that $\mu(\{X : \Psi_i^X \text{ total}\}) \geqslant 1-2^{-i}$. Furthermore, an index for such a Ψ_i can be obtained effectively from i. Let $\mathcal{C}_i = \{X : \Psi_i^X \text{ not total}\}$. Then $\mathcal{C}_0, \mathcal{C}_1, \dots$ are uniformly Σ_2^0, and hence form a Σ_2^0 Martin-Löf test. Since B is 2-random, we can fix i with $B \notin \mathcal{C}_i$. Since all Q_e are satisfied for $\Psi = \Psi_i$, the conclusion of the theorem follows. □

Notice that we cannot improve the above theorem to all weakly 2-random B, because there are weakly 2-randoms that have hyperimmune-free degree (see e.g. [20, Theorem 8.11.12]), and hence do not compute any weakly 1-generics. It would be interesting to have an exact characterization of the attractive degrees that does not mention Hausdorff distance. It would also be interesting to know whether A can be dispersive without it being the case that almost every set computes a set that is weakly 1-generic relative to A. Finally, we do not know whether the above theorem can be improved to hold of all 1-generic A.

Observation 5.19. A related notion that has been completely characterized is almost everywhere domination. Dobrinen and Simpson [19] defined a set A to be *almost everywhere dominating* if for almost every B, every B-computable function is dominated by an A-computable function. Binns, Kjos-Hanssen, Lerman, and

Solomon [11] and Kjos-Hanssen, Miller, and Solomon [39] characterized the almost everywhere dominating sets as those sets A such that every set that is 1-random relative to A is 2-random. If almost every set computes a set that is weakly 1-generic relative to A, then A cannot be almost everywhere dominating (because if a set is weakly 1-generic relative to A, then it computes a function that is not dominated by any A-computable function).

We can strengthen this fact by noting that the proof in [1, Corollary 1.13(i)] that if B is 1-random and has hyperimmune-free degree then $\Gamma(B) = 1/2$ relativizes to show that if B is 1-random relative to A and every B-computable function is dominated by an A-computable function, then $\Gamma_A(B) = 1/2$. Of course, in this case we also have $\Gamma_B(A) = 1/2$, since B is 1-random relative to A, and thus $H(A, B) = 1/2$. It follows that if A is almost everywhere dominating, then it is attractive. The other direction does not hold, however, as there are 1-random sets that are not almost everywhere dominating, for instance any 1-random set of hyperimmune-free degree, since if A has hyperimmune-free degree and every B-computable function is dominated by an A-computable function, then B also has hyperimmune-free degree. It is possible, however, that for c.e. sets the two notions coincide. As explained below Corollary 11.2.7 in [20], there are incomplete c.e. sets that are almost everywhere dominating. It follows that there are attractive sets that do not compute any 1-randoms. We do not know whether every attractive set of hyperimmune-free degree computes a 1-random.

Much of the work in this section has been devoted to determining the truth of sentences of the form

$$(Q_1A)(Q_2B)[H(A, B) = r],$$

where each Q_i is c. m. or a. e. , and r is $1/2$ or 1. We pause to summarize the results of this form. For $r = 1/2$, the above statement is true if and only if both Q_1 and Q_2 are a. e. . For $r = 1$ the above statement is true if and only if either Q_1 or Q_2 is c. m. .

6. Isometric embeddings into the Turing Degrees

The motivation for the results in this section is the question of which finite metric spaces with every distance equal to 0, 1/2, or 1 are isometrically embeddable in $(\mathcal{D}, H)$. We begin with a partial answer to this question.

By a graph we will mean an undirected graph with no loops. For a metric space $\mathcal{M}$ with every distance equal to 0, 1/2, or 1, let $G_{\mathcal{M}}$ be the graph whose vertices are the points of $\mathcal{M}$, with an edge between x and y if and only if the distance between x and y is 1. We denote the complement of this graph, where there is an edge between x and y if and only if the distance between x and y is 1/2, by $G^{c}_{\mathcal{M}}$. We write $G_{\mathcal{D}}$ for $G_{(\mathcal{D},H)}$. Notice that every graph is $G_{\mathcal{M}}$ for some $0, 1/2, 1$-valued metric space $\mathcal{M}$.

A graph (V, E) is a *comparability graph* if there is a partial order $(V, \preccurlyeq)$ such that $E(x, y)$ if and only if $x \prec y$ or $y \prec x$.

Theorem 6.1. *Let $\mathcal{M}$ be a countable metric space with every distance equal to 0, 1/2, or 1, such that $G_{\mathcal{M}}$ is a comparability graph. Then $\mathcal{M}$ is isometrically embeddable in $(\mathcal{D}, H)$.*

Proof. There is a computable partial ordering $(\omega, \preccurlyeq)$ that is countably universal, i.e., every countable partial ordering is order-isomorphic to a subordering of $(\omega, \preccurlyeq)$.

(See e.g. [53, Exercise 6.15].) It is enough to show that there are pairwise distinct degrees $\mathbf{a}_i$ such that $H(\mathbf{a}_i, \mathbf{a}_j) = 1$ if and only if $i \neq j$ and i and j are $\preccurlyeq$-comparable.

Let $\bigoplus_{n\in\omega} B_n$ be a Δ^0_2 1-random. By van Lambalgen's Theorem, each B_n is 1-random relative to $\bigoplus_{m\neq n} B_m$. Let $A_n = \bigoplus_{i\preccurlyeq n} B_i$ and let $\mathbf{a}_n$ be the degree of A_n.

If $i \prec j$ then $A_i <_{\mathrm{T}} A_j$. Furthermore, A_j is Δ^0_2, so A_j has hyperimmune degree relative to A_i, by relativizing the result that nonzero Δ^0_2 degrees are hyperimmune. Since $A_i <_{\mathrm{T}} A_j$ and A_j has hyperimmune degree relative to A_i, it follows that A_j computes a weakly 1-generic relative to A_i, and hence, by Theorem 5.1, $H(\mathbf{a}_i, \mathbf{a}_j) = 1$.

Now suppose that i and j are $\preccurlyeq$-incomparable. Then B_i is 1-random relative to $\bigoplus_{k\preccurlyeq j} B_k$, since the latter set is computable from $\bigoplus_{k\neq i} B_k$. By Corollary 5.6, $\Gamma_{\mathbf{a}_i}(\mathbf{a}_j) \geqslant \Gamma_{B_i}(\mathbf{a}_j) = 1/2$, so in fact $\Gamma_{\mathbf{a}_i}(\mathbf{a}_j) = 1/2$. The symmetric argument shows that $\Gamma_{\mathbf{a}_j}(\mathbf{a}_i) = 1/2$. Thus $H(\mathbf{a}_i, \mathbf{a}_j) = 1/2$. □

We do not know what other countable $0, 1/2, 1$-valued metric spaces (if any) are isometrically embeddable in $(\mathcal{D}, H)$. An interesting test case is the finite $0, 1/2, 1$-valued metric space $\mathcal{M}$ such that $G_{\mathcal{M}}$ is a cycle of length 5, which is the simplest example of a graph that is not a comparability graph. Answering this question will likely require better knowledge of the possible distances between elements of sets of pairwise incomparable degrees. A first step in that direction is to show that there are pairwise incomparable degrees $\mathbf{a}, \mathbf{b}, \mathbf{c}$ such that $H(\mathbf{a}, \mathbf{b}) = H(\mathbf{b}, \mathbf{c}) = 1/2$ and $H(\mathbf{a}, \mathbf{c}) = 1$. The following result will do so, and indeed ensure that the degrees $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are 1-random, which might be useful in obtaining further embeddings of metric spaces into $(\mathcal{D}, H)$.

Theorem 6.2. *There are incomparable* 1*-random degrees* $\mathbf{a}, \mathbf{c}$ *such that* $H(\mathbf{a}, \mathbf{c}) = 1$.

Proof. By a theorem of Kučera [40], every degree $\geqslant \mathbf{0}'$ is 1-random. Hence it suffices to show that there are incomparable degrees $\mathbf{a}, \mathbf{c} \geqslant \mathbf{0}'$ such that $H(\mathbf{a}, \mathbf{c}) = 1$. This can be done via what is essentially a relativization of a proof that there exist incomparable degrees $\mathbf{a}, \mathbf{c}$ such that $H(\mathbf{a}, \mathbf{c}) = 1$. That is, let $G_0 \oplus G_1$ be 2-generic, let $\mathbf{a}$ be the degree of $\emptyset' \oplus G_0$, and let $\mathbf{c}$ be the degree of $\emptyset' \oplus G_1$. Then $\mathbf{a}$ and $\mathbf{c}$ are incomparable, since $G_0 \oplus G_1$ is 1-generic relative to $\emptyset'$. We also have $\Gamma_{\mathbf{a}}(\mathbf{c}) \leqslant \gamma_{\mathbf{a}}(G_1) = 0$, by the relativization of Theorem 5.1 to $\emptyset'$. It follows that $H(\mathbf{a}, \mathbf{c}) = 1$. □

Corollary 6.3. *There are pairwise incomparable* 1*-random degrees* $\mathbf{a}, \mathbf{b}, \mathbf{c}$ *such that* $H(\mathbf{a}, \mathbf{b}) = H(\mathbf{b}, \mathbf{c}) = 1/2$ *and* $H(\mathbf{a}, \mathbf{c}) = 1$.

Proof. Let $\mathbf{a}$ and $\mathbf{c}$ be as in the theorem, and let $\mathbf{b}$ be 1-random relative to $\mathbf{a} \vee \mathbf{c}$, so that by van Lambalgen's Theorem, $\mathbf{a}$ and $\mathbf{b}$ are relatively 1-random, as are $\mathbf{b}$ and $\mathbf{c}$. Then $H(\mathbf{a}, \mathbf{b}) = H(\mathbf{b}, \mathbf{c}) = 1/2$ by Corollary 5.6. □

The proof of Theorem 6.2 does not seem very flexible, relying as it does on 1-random degrees above $\mathbf{0}'$, which are atypical in several ways. It also does not relativize to show that $\mathbf{a}, \mathbf{c}$ can be chosen to have higher levels of algorithmic randomness. Thus we give the following alternate proof, which establishes the result in relativized form.

Theorem 6.4. *Let X be any oracle. There are incomparable degrees $\mathbf{a}, \mathbf{c}$ that are 1-random relative to X and such that $H(\mathbf{a}, \mathbf{c}) = 1$.*

Proof. Recall the notion of almost everywhere dominating sets from Observation 5.19. There are several ways to see that the class of such sets has measure 0, for instance because every such set is high (see [20, Section 10.6]), and the class of high sets has measure 0 (see the proof of [20, Lemma 11.8.7])). Thus there is a set D that is 1-random relative to X and is not almost everywhere dominating. Then there is a set E that is 1-random relative to $X \oplus D$ and computes a function that is not dominated by any D-computable function. Let $C = D \oplus E$. Then C is 1-random relative to X by van Lambalgen's Theorem relative to X. Furthermore, $D \leqslant_{\mathrm{T}} C$ and C has hyperimmune degree relative to D, so C computes a set G that is weakly 1-generic relative to D.

Let $\mathcal{P}$ be a nonempty $\Pi^{0,X}_1$ class of sets that are 1-random relative to X. We build a set A by forcing with nonempty $\Pi^{0,X}_1$ subclasses of $\mathcal{P}$. Since $C \not\leqslant_{\mathrm{T}} X$, any set that is sufficiently generic for this notion of forcing is Turing incomparable with C. (See for instance [18, Section 4].)

We claim that if A is sufficiently generic for this notion of forcing, then G is weakly 1-generic relative to A. Then we can take $\mathbf{a}$ and $\mathbf{c}$ to be the degrees of A and C, respectively, and these will be incomparable degrees that are 1-random relative to X and such that $H(\mathbf{a}, \mathbf{c}) = 1$, by Theorem 5.1. The key here will be the result of Kučera [40] (in relativized form) that if Y is 1-random relative to X and $\mathcal{Q}$ is a $\Pi^{0,X}_1$ class of positive measure, then $\mathcal{Q}$ contains an element of the same degree as Y.

Thinking of c.e. operators as enumerating sets of binary strings, it is enough to show that for each nonempty $\Pi^{0,X}_1$ subclass $\mathcal{Q}$ of $\mathcal{P}$ and each e, there is a nonempty $\Pi^{0,X}_1$ subclass $\mathcal{R}$ of $\mathcal{Q}$ such that either for each $Z \in \mathcal{R}$, the set W^Z_e is not dense, or for each $Z \in \mathcal{R}$, there is an initial segment of G in W^Z_e.

For each binary string σ, consider the $\Pi^{0,X}_1$ subclass of $\mathcal{Q}$ consisting of all $Z \in \mathcal{Q}$ such that W^Z_e does not contain an extension of σ. If any of these classes is nonempty, we can take it to be $\mathcal{R}$. Otherwise, W^Z_e is dense for all $Z \in \mathcal{Q}$. Since $\mathcal{Q}$ is a nonempty $\Pi^{0,X}_1$ class of sets that are 1-random relative to X, it has positive measure, and hence contains a set B of the same degree as D. Then W^B_e is dense and D-c.e., so it contains an initial segment ρ of G. Let τ be an initial segment of B such that $\rho \in W^\tau_e$. Then we can take $\mathcal{R}$ to be the restriction of $\mathcal{Q}$ to extensions of τ. □

Corollary 6.5. *For any n, there are pairwise incomparable n-random degrees $\mathbf{a}, \mathbf{b}, \mathbf{c}$ such that $H(\mathbf{a}, \mathbf{b}) = H(\mathbf{b}, \mathbf{c}) = 1/2$ and $H(\mathbf{a}, \mathbf{c}) = 1$.*

The graph $G_{\mathcal{D}}$ is connected and has diameter 2, since for any degrees $\mathbf{a}$ and $\mathbf{c}$, there is a degree $\mathbf{b}$ that is weakly 1-generic relative to $\mathbf{a} \vee \mathbf{c}$, and then $H(\mathbf{a}, \mathbf{b}) = H(\mathbf{b}, \mathbf{c}) = 1$. The graph $G^{\mathrm{c}}_{\mathcal{D}}$ is also connected, as we now show, but its diameter is more difficult to determine.

Theorem 6.6. *The graph $G^{\mathrm{c}}_{\mathcal{D}}$ is connected and has diameter at most 4.*

Proof. Let $\mathbf{a}$ and $\mathbf{b}$ be any two degrees. By Corollary 5.7, there are 1-random degrees $\mathbf{c}$ and $\mathbf{d}$ such that $H(\mathbf{a}, \mathbf{c}) = 1/2$ and $H(\mathbf{d}, \mathbf{b}) = 1/2$. Now let $\mathbf{e}$ be 1-random relative to $\mathbf{a} \vee \mathbf{b} \vee \mathbf{c} \vee \mathbf{d}$. By van Lambalgen's Theorem, $\mathbf{c}$ and $\mathbf{e}$ are relatively 1-random, as are $\mathbf{d}$ and $\mathbf{e}$. So by Corollary 5.6, $H(\mathbf{c}, \mathbf{e}) = H(\mathbf{e}, \mathbf{d}) = 1/2$. Thus we conclude that $H(\mathbf{a}, \mathbf{c}) = H(\mathbf{c}, \mathbf{e}) = H(\mathbf{e}, \mathbf{d}) = H(\mathbf{d}, \mathbf{b}) = 1/2$. □

We now show that the diameter of $G_{\mathcal{D}}$ is at least 3, via a couple of results of independent interest. To prove them we use a lemma due to Ng, Stephan, Yang, and Yu [49]. We include a proof since one does not appear in [49].

Lemma 6.7 (Ng, Stephan, Yang, and Yu [49])**.** *If $\mathcal{P}$ is a $\Pi^{0,\emptyset'}_1$ class with a member A of hyperimmune-free degree, then $\mathcal{P}$ has a Π^0_1 subclass containing A.*

Proof. Let T be a Δ^0_2 binary tree whose paths are exactly the elements of $\mathcal{P}$, and let $T[0], T[1], \ldots$ be uniformly computable binary trees approximating T. Let $f(n)$ be the least $k \geqslant n$ such that $A \upharpoonright n$ is in $T[k]$. Then $f \leqslant_{\mathrm{T}} A$, so f is majorized by some computable function g. Let Q consist of all binary strings σ such that every predecessor of σ is in Q and $\sigma \in T[k]$ for some $k \in [|\sigma|, g(|\sigma|)]$. Then Q is a computable tree. Furthermore, every $A \upharpoonright n$ is in Q, since $f(n) \in [n, g(n)]$, so A is a path on Q. Finally, if X is not a path on T, then there is an n such that $X \upharpoonright n \notin T[k]$ for all $k \geqslant n$. Then $X \upharpoonright n \notin Q$, so X is not a path on Q. Thus the class of paths on Q is our desired Π^0_1 subclass of $\mathcal{P}$. □

Theorem 6.8. *If $\mathbf{a}$ is hyperimmune-free and $\mathbf{b}$ is a Δ^0_2 PA degree, then $\Gamma_{\mathbf{a}}(\mathbf{b}) = 0$, and hence $H(\mathbf{a}, \mathbf{b}) = 1$.*

Proof. We will need two properties of PA degrees. One is the relativized version of the theorem due to Jockusch [35, Proposition 4] that if $\mathbf{b}$ is PA then there is a uniformly $\mathbf{b}$-computable sequence of sets that includes all computable sets. The second, due to Simpson [51, Theorem 6.5], is that if $\mathbf{b}$ is PA then there is another PA degree $\mathbf{c}$ such that $\mathbf{b}$ is PA relative to $\mathbf{c}$.

Let $\mathbf{a}$ be hyperimmune-free and let $\mathbf{b}$ be a Δ^0_2 PA degree. We begin with the following claim. Let g be a computable function. Then there are uniformly $\mathbf{b}$-computable sets of strings $S_0, S_1, \ldots$ such that $|S_n| = n$ for all n, every element of S_n has length $g(n)$ for all n, and for every $A \leqslant_{\mathrm{T}} \mathbf{a}$, we have $A \upharpoonright g(n) \in S_n$ for infinitely many n.

To establish the claim, suppose not. Let $\mathbf{c}$ be a PA degree such that $\mathbf{b}$ is PA relative to $\mathbf{c}$. Let $E_0, E_1, \ldots$ be uniformly $\mathbf{b}$-computable sets such that every $\mathbf{c}$-computable set is on this list. For each n, let $S_n = \{E_i \upharpoonright g(n) : i < n\}$. Then $S_0, S_1, \ldots$ are uniformly $\mathbf{b}$-computable, and $|S_n| = n$ for all n, so there is an $A \leqslant_{\mathrm{T}} \mathbf{a}$ and an m such that if $n > m$ then $A \upharpoonright g(n) \notin S_n$. Let $\mathcal{P} = \{X : (\forall n > m)[X \upharpoonright g(n) \notin S_n]\}$. Then, since $\mathbf{b} \leqslant \mathbf{0}'$, we have that $\mathcal{P}$ is a $\Pi^{0,\mathbf{0}'}_1$ class containing A. By Lemma 6.7, $\mathcal{P}$ has a nonempty Π^0_1 subclass $\mathcal{Q}$. Since $\mathbf{c}$ is PA, $\mathcal{Q}$ contains a $\mathbf{c}$-computable set, which is equal to E_i for some i. Let $n > i, m$. Then $E_i \upharpoonright g(n) \in S_n$, contradicting the fact that $E_i \in \mathcal{Q} \subseteq \mathcal{P}$. Thus we have established the claim.

The idea now is that by choosing g to be sufficiently fast growing, we can build a set $B \leqslant_{\mathrm{T}} \mathbf{b}$ to diagonalize against each element of each S_n on a large segment, and thus in particular to diagonalize against every $A \leqslant_{\mathrm{T}} \mathbf{a}$ on infinitely many large segments, ensuring that, for every such A, the upper density of $A \triangle B$ is equal to 1, so that $\gamma_{\mathbf{a}}(B) = 0$. It then follows that $\Gamma_{\mathbf{a}}(\mathbf{b}) = 0$, and hence $H(\mathbf{a}, \mathbf{b}) = 1$.

Let $F_0, F_1, \ldots$ be consecutive segments of ω with $|F_n| = n+1$. Let $I_k = [k!, (k+1)!)$. Let $g(n) = (k+1)!$ for the largest $k \in F_n$. Apply the claim to this g to obtain sets $S_0, S_1, \ldots$ as above. Note that these sets are pairwise disjoint since g is injective. Assign each $\sigma \in S_n$ to a $k_\sigma \in F_n$, so that $k_\sigma \neq k_\tau$ for $\sigma \neq \tau \in S_n$. This is possible since $|F_n| = |S_n|+1$ and the sets S_n are pairwise disjoint. Furthermore, this assignment can be made computably in $\mathbf{b}$. Define $B \leqslant_{\mathrm{T}} \mathbf{b}$ as follows. For $i \in I_{k_\sigma}$, let $B(i) = 1 - \sigma(i)$. Now for each of the infinitely many n such that $A \upharpoonright g(n) \in S_n$, letting $k = k_{A\upharpoonright g(n)}$, we have that $k \in F_n$, and by definition $B(i) = 1 - A(i)$ for all $i \in I_k$. Hence $\rho_{(k+1)!}(A \bigtriangledown B) \leqslant k!/(k+1)! = 1/(k+1)$. Since there are arbitrarily large such k, it follows that $\underline{\rho}(A \bigtriangledown B) = 0$. Since A was an arbitrary **a**-computable set, we have that $\gamma_{\mathbf{a}}(B) = 0$. It then follows as above that $H(\mathbf{a}, \mathbf{b}) = 1$. □

Notice that this theorem gives us yet another proof of Theorem 6.2, by considering a hyperimmune-free 1-random degree and $\mathbf{0}'$ (which is PA and 1-random).

It would be interesting to know how far the above theorem can be extended, and in particular whether it holds for all hyperimmune PA degrees.

Corollary 6.9. *There is a degree* $\mathbf{b}$ *such that for all degrees* $\mathbf{a}$*, if* $H(\mathbf{0}, \mathbf{a}) = 1/2$ *then* $H(\mathbf{a}, \mathbf{b}) = 1$.

Proof. Let $\mathbf{b}$ be a Δ^0_2 PA degree. By [31, Theorem 2.2], if $H(\mathbf{0}, \mathbf{a}) = 1/2$, then $\mathbf{a}$ is hyperimmune-free, so by Theorem 6.8, $H(\mathbf{a}, \mathbf{b}) = 1$. Thus there is no degree $\mathbf{a}$ such that $H(\mathbf{0}, \mathbf{a}) = H(\mathbf{a}, \mathbf{b}) = 1/2$. □

This result has implications for the issue of extensions of isometric embeddings of metric spaces into $(\mathcal{D}, H)$, which we will not pursue further here. Let $\mathcal{M}$ be the metric space with two points x and y at distance 1 from each other, and let $\mathcal{M}'$ be the extension of $\mathcal{M}$ obtained by adding a point z such that the distances between x and z and between y and z are both 1/2. Notice that $\mathcal{M}'$ is isometrically embeddable into $(\mathcal{D}, H)$. Let $\mathbf{b}$ be as in the proof of Corollary 6.9. Then the isometric embedding of $\mathcal{M}$ into $(\mathcal{D}, H)$ obtained by mapping x to $\mathbf{0}$ and y to $\mathbf{b}$ cannot be extended to an isometric embedding of $\mathcal{M}'$ into $(\mathcal{D}, H)$.

Corollary 6.10. *The diameter of* $G^{\mathrm{c}}_{\mathcal{D}}$ *is at least* 3.

We do not know whether the diameter of $G^{\mathrm{c}}_{\mathcal{D}}$ is 3 or 4.

7. Mycielski's Theorem, computability, and reverse mathematics

In this section, we analyze Mycielski's Theorems 2.13 and 2.17 and their consequences from the points of view of computability theory and reverse mathematics. To talk about perfect sets in this context, we use perfect trees, as defined in Definition 3.20. We can think of a perfect tree T as a 1-1 function from 2^ω to 2^ω, so there are continuum many paths through T. Indeed, the paths through T form a perfect set. Notice that for every A we have $A \oplus T \equiv_{\mathrm{T}} T(A) \oplus T$. An equivalent way to think of a perfect tree is as a binary tree (in the usual sense) that has no dead ends and no isolated paths.

7.1. Mycielski's Theorem for measure and computability theory.

We begin by effectivizing Corollary 2.16. The following notions will be useful.

Definition 7.1. For a binary string σ, let $[\sigma] = \{X \in 2^\omega : \sigma \prec X\}$. For a measurable class $\mathcal{C}$ and $X \in \mathcal{C}$, let

$$d(X \mid \mathcal{C}) = \liminf_n 2^n \mu([X \upharpoonright n] \cap \mathcal{C})$$

be the *density of* $\mathcal{C}$ *near* X.

A set X is a *density-one point* if $d(X \mid \mathcal{P}) = 1$ for all Π^0_1 classes $\mathcal{P}$ containing X.

If σ is a string, the *relative measure of* $\mathcal{C}$ *above* σ is $2^{|\sigma|}\mu([\sigma] \cap \mathcal{C})$.

For example, every 1-generic set is a density-one point, since it lies in the interior of every Π^0_1 class it belongs to.

The following lemma will be useful below.

Lemma 7.2. *Suppose that $d(X \mid \mathcal{C}) = 1$. Let $c \in \omega$ and $\delta > 0$. For all sufficiently large t, if $\tau \succ X \restriction t$ has length $t + c$, then $\mu([\tau] \cap \mathcal{C}) > (1 - \delta)2^{-|\tau|}$.*

Proof. Let $\mathcal{D} = 2^\omega \setminus \mathcal{C}$. Since $[\tau] \cap \mathcal{D} \subseteq [X \restriction t] \cap \mathcal{D}$, the relative measure of $\mathcal{D}$ above τ is at most 2^c times the relative measure of $\mathcal{D}$ above $X \restriction t$, and the latter relative measure goes to 0 as t increases. □

We write $\bigoplus_{i \leqslant n} X_i$ for the set given by the sequence

$$X_0(0) \cdots X_n(0) X_0(1) \cdots X_n(1) X_0(2) \cdots X_n(2) \cdots ,$$

and for strings $\sigma_0, \dots, \sigma_n$ of the same length k, we write $\bigoplus_{i \leqslant n} \sigma_i$ for the string

$$\sigma_0(0) \cdots \sigma_n(0) \sigma_0(1) \cdots \sigma_n(1) \cdots \sigma_0(k-1) \cdots \sigma_n(k-1).$$

The following basic properties are easy to check.

Lemma 7.3. *Let $\bigoplus_{i<m} X_i$ be a density-one point.*

1. *For any pairwise distinct $i_0, \dots, i_k < m$, the set $\bigoplus_{j<k} X_{i_j}$ is a density-one point.*
2. *Let $n > m$. If $\mathcal{P}$ is a Π^0_1 class containing $\bigoplus_{i<m} X_i$ and $\mathcal{C} = \{\bigoplus_{i<n} Y_i : \bigoplus_{i<m} Y_i \in \mathcal{P}\}$ then $d(\bigoplus_{i<n} X_i \mid \mathcal{C}) = 1$ for all $X_{m+1}, \dots, X_n$.*

By the Lebesgue Density Theorem (see e.g. [20, Theorem 1.2.3]), for each measurable class $\mathcal{C}$, for almost every X, if X is in $\mathcal{C}$ then $d(X \mid \mathcal{C}) = 1$. Since there are only countably many Π^0_1 classes, we have the following.

Lemma 7.4. *There are measure-1 many density-one points, so there are measure-1 many density-one 1-random points.*

We can also relativize this notion by saying that X is a *density-one point relative to A* if $d(X \mid \mathcal{P}) = 1$ for all $\Pi^{0,A}_1$ classes $\mathcal{P}$ containing X. The analogs of the above properties continue to hold in this case.

Theorem 7.5. *For any A there is an A'-computable perfect tree T such that for any nonempty finite collection $\mathcal{F}$ of paths through T, the set $\bigoplus_{Y \in \mathcal{F}} Y$ is 1-random relative to A, and hence the joins of any two finite, disjoint, nonempty collections of paths through T are mutually 1-random relative to A.*

Proof. We do the proof for $A = \emptyset$, as the full proof is a straightforward relativization. We begin by discussing the intuition behind the proof, which is an effectivization of a proof of Mycielski's Theorem along the lines discussed in [56].

We want to build a perfect tree T. Let us ignore for now the complexity of T, as showing that $\emptyset'$ is sufficient to build T will not be difficult. For ease of exposition, let us first discuss only how to make joins of two paths 1-random, the general case below being similar. The idea is to define for each σ a set X_σ and for each n a number k_n so that for each $\sigma \in 2^n$, we have $X_{\sigma^\frown i} \restriction k_n = X_\sigma \restriction k_n$ for $i = 0, 1$ and $X_{\sigma^\frown 0} \restriction k_{n+1} \neq X_{\sigma^\frown 1} \restriction k_{n+1}$. We will then define $T(\sigma) = X_\sigma \restriction k_n$, so that each path through T will be a limit of X_σ's (with respect to the usual metric on 2^ω).

Suppose we were just trying to make each individual path 1-random. The first idea might be to pick each X_σ to be 1-random, but that is not enough because the limit of 1-randoms might not be 1-random. So instead we can fix a Π^0_1 class of 1-randoms $\mathcal{P}_1$ and choose each X_σ to be in $\mathcal{P}_1$, which ensures that so is their limit. Notice that if $X_\sigma \in \mathcal{P}_1$ then $\mathcal{P}_1 \cap [X_\sigma \upharpoonright k]$ has positive measure for any k, so $X_{\sigma^\frown 0}$ and $X_{\sigma^\frown 1}$ can be defined.

To make joins of pairs of distinct paths 1-random, we want to keep all $X_\sigma \oplus X_\tau$, where σ and τ are distinct strings of the same length, inside some Π^0_1 class $\mathcal{P}_2$. This class cannot consist entirely of 1-randoms, because X_σ and X_τ can be arbitrarily close, so to be closed, $\mathcal{P}_2$ must include elements of the form $X \oplus X$. But we can define $\mathcal{P}_2$ so that every element $X \oplus Y$ is 1-random unless $X = Y$, by letting it consist of all sets of the form $X \oplus X$, together with all sets of the form $X \oplus Y$ such that, for the least m such that $X(m) \neq Y(m)$, we have $X = (X \upharpoonright m+1)Z_0$ and $Y = (X \upharpoonright m+1)Z_1$ for some $Z_0 \oplus Z_1 \in \mathcal{R}_m$, where the $\mathcal{R}_m$ are uniformly Π^0_1 classes of 1-randoms. For reasons addressed below, it will be important to choose these classes so that $\mu(\mathcal{R}_m)$ approaches 1 as m increases.

Say that a sequence of sets is *acceptable* if the sets are pairwise distinct and for every pair of distinct sets X and Y in the sequence, $X \oplus Y \in \mathcal{P}_2$. We aim to make $\{X_\sigma : \sigma \in 2^n\}$ acceptable for all n. Then for any two distinct paths X and Y through T, we will have that $X \oplus Y$ is the limit of elements of $\mathcal{P}_2$, and hence is in $\mathcal{P}_2$. Since $X \neq Y$, this will ensure that $X \oplus Y$ is 1-random. We proceed recursively. Suppose that we have defined an acceptable family $\{X_\sigma : \sigma \in 2^n\}$ and want to do the same for $n+1$.

We can do so in a step-by-step fashion as long as we can establish a lemma stating that if $Z_0, \ldots, Z_{n-1}$ is acceptable and $k \in \omega$, then there is an acceptable $Y_0, \ldots, Y_n$ such that $Y_i \upharpoonright k = Z_i \upharpoonright k$ for all $i < n$ and $Y_n \upharpoonright k = Z_{n-1} \upharpoonright k$. To give an example, suppose we can do this, and we have X_0 and X_1 and want to build $X_{00}, X_{01}, X_{10}, X_{11}$ as above. Recall that we also have a parameter k_1. Then we can find an acceptable sequence Y_0, Y_1, Y_2 such that $Y_0 \upharpoonright k_1 = X_0 \upharpoonright k_1$ and $Y_1 \upharpoonright k_1 = Y_2 \upharpoonright k_1 = X_1 \upharpoonright k_1$. Then we repeat this procedure with Y_1, Y_2, Y_0 to get an acceptable sequence Z_0, Z_1, Z_2, Z_3 such that $Z_0 \upharpoonright k_1 = Z_3 \upharpoonright k_1 = Y_0 \upharpoonright k_1 = X_0 \upharpoonright k_1$, while $Z_1 \upharpoonright k_1 = Y_1 \upharpoonright k_1 = X_1 \upharpoonright k_1$ and $Z_2 \upharpoonright k_1 = Y_2 \upharpoonright k_1 = X_1 \upharpoonright k_1$. Finally, we let $X_{00} = Z_0$, $X_{01} = Z_3$, $X_{10} = Z_1$, and $X_{11} = Z_2$.

The necessary lemma can be proved using a measure argument. First, we can argue using Lemma 7.4 that we can assume that $\bigoplus_{i<n} Z_i$ is a density-one point. Let $Z_n = Z_{n-1}$. If we consider a pair of distinct numbers $i, j \leqslant n$ other than $n-1, n$, the fact that $Z_i \oplus Z_j$ is a density-one point implies that the relative measure of $\mathcal{P}_2$ above $(Z_i \oplus Z_j) \upharpoonright m$ goes to 1 as m increases. Thus if m is large enough, all such relative measures will be close to 1.

For the pair $n-1, n$, the relative measure of $\mathcal{P}_2$ above $(Z_{n-1} \oplus Z_n) \upharpoonright m$ also has a positive lim inf by the definition of $\mathcal{P}_2$. To see that this is the case, let $\mathcal{E}_m = \mathcal{P}_2 \cap [(Z_{n-1} \oplus Z_n) \upharpoonright m]$. Note that if $U \oplus V$ extends $(Z_{n-1} \oplus Z_n) \upharpoonright 2m$ and $U(m) \neq V(m)$, and $U \oplus V$ has the form $((U \oplus V) \upharpoonright (2m+2))^\frown R$ for some $R \in \mathcal{R}_m$, then $U \oplus V \in \mathcal{E}_{2m}$. As the three events just mentioned are mutually independent, we have that

$$\mu(\mathcal{E}_{2m}) \geqslant 2^{-2m} \cdot 1/2 \cdot \mu(\mathcal{R}_m),$$

since the three factors on the right-hand side are the respective probabilities of the three events just mentioned. Since $\mathcal{E}_{2m} \subseteq \mathcal{E}_{2m-1}$, we also have

$$\mu(\mathcal{E}_{2m-1}) \geqslant 2^{-2m} \cdot 1/2 \cdot \mu(\mathcal{R}_m) = 2^{-(2m-1)} \cdot 1/4 \cdot \mu(\mathcal{R}_m)$$

for $m > 0$. It follows that

$$d(Z_{n-1} \oplus Z_n \mid \mathcal{P}_2) = \liminf_k 2^k \mu(\mathcal{E}_k) \geqslant 1/4,$$

since $\lim_m \mu(\mathcal{R}_m) = 1$.

Thus we see that if m is sufficiently large, then the classes $\mathcal{C}_{i,j} = \mathcal{P}_2 \cap [(Z_i \oplus Z_j) \upharpoonright m]$ have large enough measure to ensure that the class of all $\bigoplus_{i \leqslant n} Y_i$ such that each $Y_i \oplus Y_j$ for $i \neq j$ is in $\mathcal{C}_{i,j}$ has positive measure, and in particular contains an element such that $Y_i \neq Y_j$ for all $i < j \leqslant n$, since the class of all $\bigoplus_{i \leqslant n} Y_i$ such that $Y_i \neq Y_j$ for all $i < j \leqslant n$ has measure 1.

We now proceed with the full construction. Let $\mathcal{R}_0, \mathcal{R}_1, \ldots$ be uniformly Π^0_1 classes of 1-randoms such that $\mu(\mathcal{R}_m)$ goes to 1 as m increases (for instance, the complements of the levels of a universal Martin-Löf test). Let $\mathcal{P}_1 = \mathcal{R}_0$. For $n > 1$, let $\mathcal{P}_n$ consist of all $\bigoplus_{k<n} X_k$ such that either $X_{n-2} = X_{n-1}$ or for the least m such that $X_{n-2}(m) \neq X_{n-1}(m)$, there is a $Y \in \mathcal{R}_m$ such that $\bigoplus_{k<n} X_k = (\bigoplus_{k<n}(X_k \upharpoonright m+1))^\frown Y$. Note that the $\mathcal{P}_n$ are uniformly Π^0_1 classes. Note also that if $\bigoplus_{k<n} X_k \in \mathcal{P}_n$ and $X_{n-1} \neq X_{n-2}$, then $\bigoplus_{k<n} X_k$ is 1-random.

Let $n \geqslant 1$. For $X = \bigoplus_{i<n} X_i$ and $0 < m \leqslant n$, let $\langle X \rangle^m$ be the set of all $\bigoplus_{j<m} X_{i_j}$ such that the i_j are distinct numbers less than n. We say that X is *n-acceptable* if $X_i \neq X_j$ for every $i < j < n$, and for every $m \leqslant n$, every element of $\langle X \rangle^m$ is in $\mathcal{P}_m$. Note that if X is n-acceptable, then $X \in \langle X \rangle^n \subset \mathcal{P}_n$, so X is 1-random. Note also that if $\bigoplus_{i<n} X_i$ is n-acceptable and π is a permutation of $0, 1, \ldots, n-1$, then $\bigoplus_{i<n} X_{\pi(i)}$ is also n-acceptable.

Lemma 7.6. *Let $\bigoplus_{i<n} X_i$ be n-acceptable, and let $k \in \omega$. Then there is an n-acceptable density-one point $\bigoplus_{i<n} Z_i$ such that $Z_i \upharpoonright k = X_i \upharpoonright k$ for all $i < n$.*

Proof. Let $\mathcal{P}$ be the class of all $Z = \bigoplus_{i<n} Z_i$ such that for every $0 < m \leqslant n$, every element of $\langle Z \rangle^m$ is in $\mathcal{P}_m$, and $Z_i \upharpoonright k = X_i \upharpoonright k$ for all $i < n$. Then $\mathcal{P}$ is a Π^0_1 class containing the 1-random set $\bigoplus_{i<n} X_i$, and hence $\mathcal{P}$ has positive measure, so it contains a density-one 1-random point $Z = \bigoplus_{i<n} Z_i$, by Lemma 7.4. Since Z is 1-random, $Z_i \neq Z_j$ for every $i < j < n$, so Z is n-acceptable. □

Lemma 7.7. *Let $\bigoplus_{i<n} X_i$ be n-acceptable, and let $k \in \omega$. Let $X_n = X_{n-1}$. Then there is an $n+1$-acceptable $\bigoplus_{i \leqslant n} Y_i$ such that $Y_i \upharpoonright k = X_i \upharpoonright k$ for all $i \leqslant n$.*

Proof. By Lemma 7.6, we can assume that $\bigoplus_{i<n} X_i$ is a density-one point.

Let S be the set of all nonempty sequences of distinct numbers less than or equal to n. For each $s = (i_0, \ldots, i_{m-1}) \in S$, let $\mathcal{C}_s$ be the class of all $Y = \bigoplus_{i \leqslant n} Y_i$ such that $\bigoplus_{j<m} Y_{i_j} \in \mathcal{P}_m$. Our goal is to show that there is an element $\bigoplus_{i \leqslant n} Y_i$ of the intersection of all of these classes such that $Y_i \neq Y_j$ for all $i < j \leqslant n$ and $Y_i \upharpoonright k = X_i \upharpoonright k$ for all $i \leqslant n$. Our strategy is first to define a sequence of strings $\{\sigma_l\}_{l \in \omega}$ in such a way that we can show that, for all $s \in S$, the relative measure of $\mathcal{C}_s$ above σ_l approaches 1 as l grows. Let

$$\sigma_l = (X_0 \upharpoonright l+1) \oplus \cdots \oplus (X_{n-2} \upharpoonright l+1) \oplus ((X_{n-1} \upharpoonright l)^\frown 0) \oplus ((X_n \upharpoonright l)^\frown 1).$$

Let $s = (i_0, \ldots, i_{m-1}) \in S$ and suppose that s does not contain both $n-1$ and n. Then $\bigoplus_{j<m} X_{i_j}$ is a density-one point, by Lemma 7.3. Furthermore, since $\bigoplus_{i<n} X_i$ is n-acceptable, $\bigoplus_{j<m} X_{i_j} \in \mathcal{P}_m$. Thus $\bigoplus_{i \leqslant n} X_i \in \mathcal{C}_s$, and hence, by

Lemma 7.3,

$$d\left(\bigoplus_{i\leqslant n} X_i \mid \mathcal{C}_s\right) = 1.$$

We claim that it now follows from Lemma 7.2 that the relative measure of $\mathcal{C}_s$ above σ_l approaches 1 as l increases. To justify this claim, let $X = \bigoplus_{i\leqslant n} X_i$, so that $d(X \mid \mathcal{C}_s) = 1$. Then $\bigoplus_{i\leqslant n}(X_i \upharpoonright l) = X \upharpoonright l(n+1)$. For all l, the string σ_l is obtained from the latter string by adding $n+1$ new bits at the end. So applying Lemma 7.2 with $c = n+1$ and values of t of the form $l(n+1)$ shows that the relative measure of $\mathcal{C}_s$ above σ_l approaches 1 as l increases.

If $s = (i_0, \ldots, i_{m-1}) \in S$ contains both $n-1$ and n, then let $a < b$ be such that $\{i_a, i_b\} = \{n-1, n\}$. For any l, any $\bigoplus_{j<m} Z_j \in \mathcal{R}_l$, and any $c_0, \ldots, c_{m-1} \in \{0,1\}$ such that $c_a \neq c_b$, we have $\bigoplus_{j<m}((X_{i_j} \upharpoonright l)^\frown c_j^\frown Z_j) \in \mathcal{P}_m$. Thus

$$\mu([\sigma_l] \cap \mathcal{C}_s) \geqslant 2^{-|\sigma_l|}\mu(\mathcal{R}_l).$$

Since $\mu(\mathcal{R}_l)$ goes to 1 as l increases, it follows that the relative measure of $\mathcal{C}_s$ above σ_l approaches 1 as l increases.

It follows from the previous two paragraphs that for *all* $s \in \mathcal{S}$ the relative measure of $\mathcal{C}_s$ above σ_l approaches 1 as l increases. Hence the relative measure of $\bigcap_{s\in\mathcal{S}} \mathcal{C}_s$ above σ_l approaches 1 as l increases.

Let k be as in the hypothesis of the lemma, and let $\beta = \bigoplus_{i\leqslant n}(X_i \upharpoonright k)$. If $l \geqslant k$, then σ_l extends β. Fix l sufficiently large so that $l \geqslant k$ and $\mu([\sigma_l] \cap \bigcap_{s\in\mathcal{S}} \mathcal{C}_s) > 0$. Then

$$\mu\left([\beta] \cap \bigcap_{s\in S} \mathcal{C}_s\right) > 0.$$

Let $\mathcal{C}$ be the class of all $n+1$-acceptable $\bigoplus_{i\leqslant n} Y_i$ such that $Y_i \upharpoonright k = X_i \upharpoonright k$ for all $i \leqslant n$, and let $\mathcal{D} = \{\bigoplus_{i\leqslant n} Y_i : (\forall i < j \leqslant n)[Y_i \neq Y_j]\}$. Then $\mathcal{D}$ has measure 1, and

$$\mathcal{C} = [\beta] \cap \bigcap_{s\in S} \mathcal{C}_s \cap \mathcal{D},$$

so $\mathcal{C}$ has positive measure, and in particular is nonempty. □

We now build a $\emptyset'$-computable perfect tree as follows. For each $\sigma \in 2^n$, we will define a set X_σ, and for each n we will define a number k_n so that

(1) $\bigoplus_{\sigma\in 2^n} X_\sigma$ is 2^n-acceptable,
(2) $X_{\sigma^\frown i} \upharpoonright k_n = X_\sigma \upharpoonright k_n$ for $i = 0, 1$, and
(3) $X_\sigma \upharpoonright k_n \neq X_\tau \upharpoonright k_n$ for all distinct $\sigma, \tau \in 2^n$.

We then take $T(\sigma) = X_\sigma \upharpoonright k_n$. Note that, by the closure of n-acceptability under permutations mentioned above Lemma 7.6, it does not matter in item (1) how we order 2^n.

For the empty string λ, let $X_\lambda \in \mathcal{P}_1$ and let $k_n = 0$. Given k_n and X_σ for each $\sigma \in 2^n$, apply Lemma 7.7 repeatedly to obtain sets X_τ for each $\tau \in 2^{n+1}$ so that $\bigoplus_{\tau\in 2^{n+1}} X_\tau$ is 2^{n+1}-acceptable, and each $X_{\sigma^\frown i}$ extends $X_\sigma \upharpoonright k_n$. (Here we again use the fact that n-acceptability is closed under permutations.) We can do this $\emptyset'$-computably because the class $\mathcal{C}$ in the proof of Lemma 7.7 is nonempty and the class $\mathcal{D}$ in that proof is Σ^0_1, so using $\emptyset'$ we can find a σ such that $[\sigma] \subset \mathcal{D}$ and $[\sigma] \cap \mathcal{C}$ is nonempty, and then $[\sigma] \cap \mathcal{C}$ is a Π^0_1 class, so $\emptyset'$ can find a path on it. Let k_{n+1} be sufficiently large so that $X_\sigma \upharpoonright k_{n+1} \neq X_\tau \upharpoonright k_{n+1}$ for all distinct $\sigma, \tau \in 2^{n+1}$.

Then T is a perfect tree. Let $Y_0, \dots, Y_n$ be distinct paths through T. Then each Y_i is the limit of sets $X_{\tau_{i,k}}$ with $|\tau_{i,k}| = k$. If k is sufficiently large, then $\tau_{0,k}, \dots, \tau_{n,k}$ are pairwise distinct, so $\bigoplus_{i \leqslant n} X_{\tau_{i,k}} \in \mathcal{P}_{n+1}$. Since $\mathcal{P}_{n+1}$ is closed, it follows that the limit $\bigoplus_{i \leqslant n} Y_i$ is also in $\mathcal{P}_{n+1}$, and hence is 1-random. □

Corollary 7.8. *For each $n \geqslant 1$, there is a $\emptyset^{(n)}$-computable perfect tree T such that for any nonempty finite collection $\mathcal{F}$ of pairwise distinct paths through T, the set $\bigoplus_{Y \in \mathcal{F}} Y$ is n-random, and hence any two finite, disjoint, nonempty collections of pairwise distinct paths through T are mutually n-random.*

As another consequence, we get a proof of Mycielski's Theorem 2.13, as noticed earlier by Miller and Yu [46], who gave their own direct proof of Corollary 2.16, though without a bound on the complexity of the perfect tree: Let $\mathcal{M}_0, \mathcal{M}_1, \dots$ be such that each $\mathcal{M}_i$ is a measure-0 subset of $(2^\omega)^{n_i}$ for some $n_i \geqslant 1$. Then each $\mathcal{N}_i = \{\bigoplus_{j<n_i} X_i : (X_0, \dots, X_{n_i-1}) \in \mathcal{M}_i\}$ is a measure-0 subset of 2^ω, and so is contained in a measure-0 G_δ subset $\mathcal{C}_i$ of 2^ω. Each $\mathcal{C}_i$ is the intersection of an A_i-Martin-Löf test for some A_i. Let $A = \bigoplus_i A_i$, and let T be as in the theorem. If $X_0, \dots, X_{n_i-1}$ are distinct paths through T, then $\bigoplus_{j<n_i} X_i$ is 1-random relative to A, and hence relative to A_i, and so is not in $\mathcal{C}_i$, and hence $(X_0, \dots, X_{n_i-1})$ is not in $\mathcal{M}_i$.

Indeed, this proof shows that Theorem 7.5 is basically a "quantitative version" of Mycielski's Theorem.

Notice that in the proof of Theorem 7.5, $\mathcal{P}_1 = \mathcal{R}_0$ can be any nonempty Π^0_1 class of 1-randoms. So if we are given a Π^0_1 class $\mathcal{C}$ of positive measure, then we can intersect $\mathcal{C}$ with a Π^0_1 class of 1-randoms of sufficiently large measure, then take that intersection as $\mathcal{P}_1$. Thus the theorem still holds if we require that the paths through T be in some given Π^0_1 class of positive measure. This fact implies for instance that T can be chosen to be *pathwise-random*, as defined by Barmpalias and Wang [6], which means that there is a c such that every path X through T has randomness deficiency at most c (i.e., $K(X \upharpoonright n) \geqslant n - c$ for all n, where K is prefix-free Kolmogorov complexity). (They call trees all of whose paths are 1-random *weakly pathwise-random*.)

Kučera [40] showed that if $\mathcal{C}$ is a Π^0_1 class of positive measure, then every 1-random has a tail in $\mathcal{C}$. The fact that Theorem 7.5 still holds if we require that the paths through T be in some given Π^0_1 class of positive measure also follows from the following extension of Kučera's result, which has also been noted by Barmpalias and Wang [6]. For a tree T and $\sigma \in T$, we write T_σ for the tree consisting of all τ such that $\sigma^\frown\tau \in T$. We say that $\sigma \in T$ is *extendible* if T_σ is infinite.

Proposition 7.9. *Let T be an infinite binary tree such that each path through T is 1-random and let $\mathcal{C}$ be a Π^0_1 class of positive measure. Then there is an extendible $\sigma \in T$ such that every path through T_σ is in $\mathcal{C}$.*

Proof. This proof is a minor variation on that of the aforementioned result of Kučera [40]. Suppose that no such σ exists. We can assume that $\mathcal{C} \neq 2^\omega$. Let W be a prefix-free set of strings generating the complement of $\mathcal{C}$ (i.e., this complement is $\bigcup_{\sigma \in W}[\sigma]$). Let $S_0 = W$ and $S_{n+1} = \{\sigma^\frown\tau : \sigma \in S_n \ \&\ \tau \in W\}$. Let $\mathcal{U}_n$ be the Σ^0_1 class generated by S_n. Then the $\mathcal{U}_n$ are uniformly Σ^0_1 classes, and $\mu(\mathcal{U}_{n+1}) = \mu(\mathcal{U}_n)\mu(\mathcal{U}_0) = \mu(\mathcal{U}_0)^{n+2}$, so we can find a Martin-Löf test $\mathcal{U}_{n_0}, \mathcal{U}_{n_1}, \dots,$ since $\mu(U_0) = 1 - \mu(\mathcal{C}) < 1$.

Since not every path through T is in $\mathcal{C}$, there is an extendible $\sigma_0 \in T$ that is in S_0. Since not every path through T_{σ_0} is in $\mathcal{C}$, there is a σ_1 such that $\sigma_0^\frown\sigma_1$ is an extendible element of T and is in S_1. Proceeding in this way, we build $\sigma_0, \sigma_1, \dots$ such that $\sigma_0^\frown \cdots {}^\frown \sigma_n$ is an extendible element of T and $T_{\sigma_0^\frown \cdots {}^\frown \sigma_n} \subseteq \mathcal{U}_n$. Then $\sigma_0^\frown\sigma_1^\frown \cdots$ is a path through T, but is also in every $\mathcal{U}_n$, and hence is not 1-random. □

It is interesting to consider whether A' is the best we can do in Theorem 7.5. While we do not know the answer to this question, we can give a lower bound using the following fact. A degree $\mathbf{x}$ is a *strong minimal cover* of a degree $\mathbf{a}$ if $\mathbf{a} < \mathbf{x}$ and every degree strictly below $\mathbf{x}$ is below $\mathbf{a}$.

Theorem 7.10. *If a degree has a strong minimal cover then it does not compute any perfect tree all of whose paths are* 1*-random.*

Proof. Suppose the degree $\mathbf{a}$ has a strong minimal cover $\mathbf{x}$. If $X \in \mathbf{x}$ then X cannot be 1-random, as otherwise if we write $X = X_0 \oplus X_1$ then both X_0 and X_1 have degree strictly below $\mathbf{x}$, and hence are $\mathbf{a}$-computable, whence so is X.

Suppose that there is an $\mathbf{a}$-computable perfect tree T all of whose paths are 1-random. Let $X \in \mathbf{x}$ and let $B = T(X)$. Then $B \leqslant_{\mathrm{T}} T \oplus X \equiv_{\mathrm{T}} X$, but B is a path through T, and hence is 1-random, so in fact $B <_{\mathrm{T}} X$, and hence B is $\mathbf{a}$-computable. From B and T we can compute X, however, so X is $\mathbf{a}$-computable, which is a contradiction. □

Lewis [44] showed that there is a 1-random degree with a strong minimal cover. Indeed, Barmpalias and Lewis [4] showed that every 2-random degree has a strong minimal cover. Thus we have the following corollary, which can be seen as an analog to the fact that in the set-theoretic context, adding a random real does not necessarily add a perfect set of random reals. (See for instance Bartoszynski and Judah [7].)

Corollary 7.11. *There is a* 1*-random that does not compute any perfect tree all of whose paths are* 1*-random. Indeed, every* 2*-random has this property.*

Barmpalias and Wang [6] have independently proved a stronger version of this result, showing that 2-randomness can be replaced by the weaker notion of difference randomness, shown by Franklin and Ng [24] to be equivalent to being 1-random and not computing $\emptyset'$. Notice that if T is a perfect tree such that every path through T is 1-random, then since $\mathcal{C} = \{X : K(X \upharpoonright n) \geqslant n - c\}$ is a Π^0_1 class of positive measure for all sufficiently large c, it follows from Proposition 7.9 that there is an extendible $\sigma \in T$ such that every path through T_σ is in $\mathcal{C}$. Then T_σ is a T-computable pathwise-random tree that is perfect, and hence has infinitely many paths.

Theorem 7.12 (Barmpalias and Wang [6])**.** *Let $X \not\geqslant_{\mathrm{T}} \emptyset'$ be* 1*-random. Then X does not compute a pathwise-random tree with infinitely many paths. Thus X does not compute a perfect tree all of whose paths are* 1*-random.*

Chong, Li, Wang, and Yang [15, Question 4.2] asked whether there is a (computable or not) tree T such that the set of paths through T has positive measure but there are only measure-0 many oracles that compute a perfect subtree of T. Corollary 7.11 and Theorem 7.12 give strong positive answers to this question, as they show that no 2-random, and even no difference random, can compute a perfect subtree of a tree all of whose paths are 1-random.

If we weaken Theorem 7.5 to say only that each individual path on T is 1-random, then it has an obvious proof, since we can take a computable binary tree R all of whose paths are 1-random, and use $\emptyset'$ to find a perfect subtree of R, using the fact that R has no isolated paths. But in this case we can improve the theorem from A' to any set that has PA degree relative to A, by work of Greenberg, Miller, and Nies [25] and of Chong, Li, Wang, and Yang [15]. Say that a binary tree has positive measure if the set of paths through T does. The former group of authors showed that if T is a tree of positive measure and B has PA degree relative to T, then there is a B-computable nonempty subtree S of T such that for every $\sigma \in S$, the tree S_σ has positive measure, and such an S is perfect (i.e., it has no dead ends and no isolated paths). The latter group of authors showed directly and independently that if T is a tree of positive measure and B has PA degree relative to T, then T has a B-computable perfect subtree (which can be chosen to have positive measure). Greenberg, Miller, and Nies showed that the PA degrees are not a lower bound for their result, and Chong, Li, Wang, and Yang noted that Patey did the same in their context. We will return to these results in the setting of reverse mathematics below.

Another way we could weaken Theorem 7.5 is to replace 1-randomness by a weaker notion of randomness. Of course, notions of computability-theoretic randomness (as opposed to, say, complexity-theoretic randomness) generally do not admit computable instances, so for such notions we can never have a fully effective version of the theorem. However, for the weak notion of independence in Theorem 2.15, which was the original motivation for the work in this section, we have the following result.

Theorem 7.13. *There is a computable perfect tree T such that every path through T has density $1/2$, and the symmetric difference of any two distinct paths through T also has density $1/2$.*

Proof. Call $\sigma \in 2^{<\omega}$ *balanced* if $|\sigma^{-1}(0)| = |\sigma^{-1}(1)|$, i.e. σ has the same number of 0's as 1's. Call a pair (σ, τ) of strings *balanced* if σ and τ have the same length and agree on exactly half of their arguments. If σ is a string and $n \in \omega$, let σ^n denote the concatenation of n copies of σ, and call σ^n a *power* of σ. Note that any power of a balanced string is balanced.

Let $\mu_i = 0^{2^i}1^{2^i}$. Then μ_i is balanced and has length 2^{i+1}. Our tree T will have the following properties:

(1) $T(\sigma)$ is balanced for every σ.
(2) The length of $T(\sigma)$ depends only on the length of σ. Let l_n denote the length of $T(\sigma)$ for all σ of length n.
(3) For all strings σ and $i \leqslant 1$, the string $T(\sigma)^\frown i$ will be of the form $T(\sigma)^\frown \mu_j^k$ for some j and k.

We now give the definition of T. Let $T(\lambda) = \lambda$, where λ is the empty string. Assume inductively that $T(\sigma)$ is defined for every string σ of length n, and that all of these strings are balanced and have the same length l_n. We will choose $l_{n+1} = l_n + 2^{2^{n+4}}$. (The reason for this outlandish choice will become clear later.)

Let the strings of length n be $\sigma_1, \sigma_2, \dots, \sigma_{2^n}$. For each $k \leqslant 2^n$ and $i \leqslant 1$, define $T(\sigma_k^\frown i)$ to be $T(\sigma_k)^\frown \gamma$, where γ is the unique power of μ_{2k+i-1} of length $l_{n+1} - l_n$. (There is such a power because $l_{n+1} - l_n = 2^{2^{n+4}}$ is a multiple of $|\mu_{2k+i-1}| = 2^{2k+i}$.)

It follows immediately from the construction by induction that if $|\sigma| = n$ then $|T(\sigma)| = l_n$. It is also easy to see that $T(\sigma)$ is balanced for all σ.

Let $b_n = |\mu_{2^n}| = 2^{2^n+1}$. Call a number l *good* if every string on T of length l is balanced. We have already remarked that l_n is good for every n. The next lemma gives further examples of good numbers.

Lemma 7.14. *If $l_n \leqslant j \leqslant l_{n+1}$ and $j - l_n$ is divisible by b_n, then j is good.*

Proof. To prove the lemma, assume that j is as in its hypothesis, and let γ be a string on T of length j. We must show that γ is balanced. Write γ as $T(\sigma)^\frown\nu$, where σ has length n. Since $T(\sigma)$ is balanced, it suffices to show that ν is balanced. Let $i \leqslant 1$ be such that $\gamma \preccurlyeq T(\sigma^\frown i)$. Then by construction, ν is extended by a power of μ_{2k+i-1}, where $\sigma = \sigma_k$. Since $|\mu_{2k+i-1}|$ divides b_n, and b_n divides $k - l_n = |\gamma|$, it follows that $|\mu_{2k+i-1}|$ divides $|\nu|$. Since ν is extended by a power of μ_{2k+i-1} and $|\mu_{2k+i-1}|$ divides $|\nu|$, it follows that ν is a power of μ_{2k+i-1}. Since μ_{2k+i-1} is balanced, and powers of balanced strings are balanced, ν is balanced. □

Lemma 7.15. *Every path through T has density* $1/2$.

Proof. To prove the lemma, let C be a path through T, so $C \upharpoonright k$ is on T for every k. Let $l_n \leqslant k < l_{n+1}$. Let j be maximal such that $j \leqslant k$ and $j - l_n$ is divisible by b_n. Then $k - j \leqslant b_n$. Also, j is good by Lemma 7.14, so $|C \upharpoonright j| = j/2$. It follows that

$$|C \upharpoonright k| \leqslant |C \upharpoonright j| + b_n.$$

Dividing through by k, we obtain that

$$\rho_k(C) \leqslant \frac{|C \upharpoonright j|}{k} + \frac{b_n}{k} \leqslant \frac{|C \upharpoonright j|}{j} + \frac{b_n}{l_n} = \frac{1}{2} + \frac{b_n}{l_n}.$$

It follows from the definitions of b_n and l_n and a straightforward computation that $\lim_n \frac{b_n}{l_n} = 0$, and so $\overline{\rho}(C) \leqslant 1/2$. Replacing C by its complement $\neg C$ in the above argument we obtain that $\overline{\rho}(\neg C) \leqslant 1/2$, and so $\underline{\rho}(C) \geqslant 1/2$. It follows that $\rho(C) = 1/2$. □

It remains to be shown that if A and B are distinct branches of T, then $\rho(A \triangle B) = 1/2$. The following fact will be useful.

Lemma 7.16. *Let $i \neq j$, and let σ and τ be such that $|\sigma| = |\tau|$. If σ is a power of μ_i and τ is a power of μ_j then (σ, τ) is balanced.*

Proof. Suppose that $i < j$. Break up the numbers less than $|\tau|$ into consecutive intervals of length 2^j. This is possible because τ is a power of μ_j, and hence 2^j divides $|\tau|$. On each such interval, τ has the form 0^{2^j} or 1^{2^j}, and σ is a power of $0^{2^i}1^{2^i}$. Hence σ and τ agree on exactly half of each interval, so (σ, τ) is balanced. □

Lemma 7.17. *Let A and B be distinct paths through T. Then $\rho(A \triangle B) = 1/2$.*

Proof. For every sufficiently large n, there are distinct strings σ and τ of length n such that A extends $T(\sigma)$ and B extends $T(\tau)$. We assume that this fact holds for every $n > 0$, since the general case is essentially the same. Then, by induction and the previous lemma, $(A \upharpoonright l_n, B \upharpoonright l_n)$ is balanced for every $n > 0$. If $l_n \leqslant j \leqslant l_{n+1}$ and $j - l_n$ is divisible by b_n, then as in the proof of Lemma 7.14, $A \upharpoonright j$ is of the form $(A \upharpoonright l_n)^\frown\nu_0$ for a power ν_0 of some μ_{i_0}, and $B \upharpoonright j$ is of the form $(B \upharpoonright l_n)^\frown\nu_1$ for a power ν_1 of some μ_{i_1}, and it follows from the definition of T that $i_0 \neq i_1$. So by the previous lemma, $(A \upharpoonright j, B \upharpoonright j)$ is balanced. Now essentially the same argument as that in the proof of Lemma 7.15 shows that $\rho(A \triangle B) = 1/2$. □

The theorem follows from Lemmas 7.15 and 7.17. □

We can also consider other variants on Theorem 7.5 where we relax the conditions on T, possibly leading to better upper bounds on its complexity. One possibility is to require only that pairs of distinct paths be mutually 1-random. Another is to replace the condition that T be perfect by the condition that T be infinite and have no isolated paths (which still ensures that the set of paths through T is perfect), and of course we can do both at the same time. We do not know whether any of these variants give rise to better upper bounds. For infinite trees with no isolated paths, we also no longer have the same lower bound, but notice that if T is a computable infinite tree all of whose paths are 1-random, then there are paths through T that are not mutually 1-random: Take incompatible strings σ and τ in T such that T is infinite above both σ and τ. Then the leftmost paths of T above σ and above τ are both left-c.e. 1-randoms, so they have the same degree, namely $\mathbf{0}'$, and hence cannot be mutually 1-random.

If we go even further and require only that T be an infinite tree, or an infinite tree with no dead ends, then we do get a better upper bound for fairly trivial reasons: If X is 1-random relative to A then the set of all strings $X \upharpoonright n$ is the desired tree. (It does not change things if we add the condition that T have infinitely many paths, because if we write $X = \bigoplus_i X_i$ then we can form a tree by taking X_0 and appending a copy of X_{i+1} with root $(X \upharpoonright i)^\frown(1 - X(i))$ for each i.) If we do require that T have no dead ends, then 1-randomness is also a lower bound.

7.2. **Mycielski's Theorem for measure and reverse mathematics.** We now turn to reverse mathematics. From now on, all implications and equivalences we mention are over the usual weak base system RCA_0. A binary tree T is *positive* if there is a $q > 0$ such that $\frac{|T \cap 2^n|}{2^n} > q$ for all n. The system WWKL_0 consists of RCA_0 together with WWKL, the principle that every positive tree has an infinite path. This principle is equivalent to the one stating that for each A, there is a set that is 1-random relative to A, where the latter notion is formalized using Martin-Löf tests. (See Avigad, Dean, and Rute [3] for details.) We will implicitly use van Lambalgen's Theorem (for instance in not having to distinguish between saying that A and B are relatively 1-random and saying that $A \oplus B$ is 1-random), so it is worth noting that a standard proof of this theorem, for instance the one given in [20, Section 6.9], can be carried out in RCA_0.

Corresponding to their computability-theoretic work mentioned above, Greenberg, Miller, and Nies [25] defined the following principles, whose strength they showed to be strictly intermediate between WWKL_0 and WKL_0.

WSWWKL: Every positive tree has a positive subtree with no dead ends.

SWWKL: Every positive tree has a positive subtree T such that for every $\sigma \in T$, the restriction of T to strings compatible with σ is positive.

At about the same time, Chong, Li, Wang, and Yang [15] studied the following principle, which has also been studied by Barmpalias and Wang [6], who denoted it by P.

PSUB: Every positive tree has a perfect subtree.

Chong, Li, Wang, and Yang [15] showed that PSUB is provable in WKL_0 and noted that Patey gave an argument showing that it does not imply WKL. (The fact that PSUB is provable in WKL_0 is also implicit in an earlier proof by Barmpalias, Lewis, and Ng [5]; see also [20, proof of Theorem 8.8.8].) These facts also follow from the results of Greenberg, Miller, and Nies [25], since WSWWKL clearly implies PSUB. Indeed, it implies the following principle, which was shown by Barmpalias and Wang [6] (who denote it by P^+) to be strictly stronger than PSUB.

SPSUB: Every positive tree has a positive perfect subtree.

Clearly, PSUB implies WWKL, since from a perfect subtree of a positive tree all of whose paths are 1-random relative to A, we can compute a set that is 1-random relative to A. Chong, Li, Wang, and Yang [15, Question 4.1] asked whether PSUB is provable in WWKL_0. We can answer this question with the following result, which has also been obtained by Barmpalias and Wang [6]. (Here an *ω-model* is a structure in the language of second-order arithmetic whose first-order part is standard.)

Theorem 7.18. *There is an ω-model of* WWKL_0 *that is not a model of* PSUB*.*

Proof. As noted above, WWKL is equivalent over RCA_0 to the principle that for each X, there is a set that is 1-random relative to X. If Z is 1-random then let $Z_i = \{n : \langle i, n\rangle \in Z\}$, and let $\mathcal{S} = \{Y : (\exists k)[Y \leqslant_{\mathrm{T}} \bigoplus_{i \leqslant k} Z_i]\}$. Then $\mathcal{S}$ is a Turing ideal, and hence is an ω-model of RCA_0. Furthermore, by van Lambalgen's Theorem, each Z_i is 1-random relative to $\bigoplus_{j \neq i} Z_j$, so $\mathcal{S}$ is in fact a model of WWKL_0.

Let A be a 1-random as in Corollary 7.11. By the previous paragraph, there is an ω-model of WWKL_0 all of whose members are A-computable. If PSUB holds in this model then, applying it to a computable positive tree all of whose paths are 1-random, we obtain an A-computable perfect tree all of whose paths are 1-random, contradicting the choice of A. □

Corollary 7.19. RCA_0 + PSUB *is strictly intermediate between* WWKL_0 *and* WKL_0*.*

Notice that, by Corollary 7.11, the set A in the proof of Theorem 7.18 can be chosen to be any 2-random, and hence to have any level of randomness we would like. So, for example, PSUB is not implied by the statement that for each A, there is a set that is arithmetically random relative to A.

We can begin to connect these principles with Mycielski's Theorem by showing that PSUB is equivalent to the following statement.

PTR: For each A, there is a perfect tree T such that every path through T is 1-random relative to A.

To see that PTR and PSUB are equivalent, we use the following version of Proposition 7.9. The restriction to trees with no dead ends is necessary because in the absence of WKL, an infinite tree can satisfy the condition that all of its paths be 1-random simply by not having any paths, an issue we will return to below.

Proposition 7.20. RCA_0 *proves that if S is a positive tree, and T is an infinite binary tree with no dead ends such that each path through T is 1-random relative to S, then there is an extendible $\sigma \in T$ such that T_σ is a subtree of S.*

Proof. The proof of Proposition 7.9 can be carried out in RCA_0. The only thing to note is that, since we do not have to worry about extendibility when T has no dead ends, the existence of the strings σ_i in that proof follows by Σ^0_1-induction, and then the sequence $\sigma_0, \sigma_1, \dots$ can be obtained computably in T. □

Thus PTR is equivalent to the statement that for each A, each positive tree S has a perfect subtree T such that every path through T is 1-random relative to A. Since (provably in RCA_0) for each A there is a positive tree whose paths are 1-random relative to A, PTR and PSUB are equivalent. Thus, however we formalize Mycielski's Theorem for measure using perfect trees, it should imply PSUB.

Theorem 7.5 and the proof of Mycielski's Theorem for measure using this theorem suggest that the following is a reasonable way to formulate Mycielski's Theorem for measure as a statement of second-order arithmetic.

MYC-M: For each A, there is a perfect tree T such that for any nonempty finite collection $\mathcal{F}$ of pairwise distinct paths through T, the set $\bigoplus_{Y \in \mathcal{F}} Y$ is 1-random relative to A.

By Proposition 7.20, MYC-M is equivalent to the statement that for each A, each positive tree S has a perfect subtree T such that for any nonempty finite collection $\mathcal{F}$ of pairwise distinct paths through T, the set $\bigoplus_{Y \in \mathcal{F}} Y$ is 1-random relative to A.

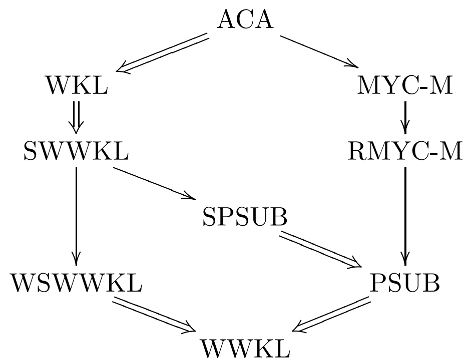
We can also consider the following version of MYC-M restricted to joins of pairs of paths, which corresponds to Corollary 2.14.

RMYC-M: For each A, there is a perfect tree T such that any two distinct paths through T are mutually 1-random relative to A.

RMYC-M implies PSUB, since it clearly implies PTR, and hence it properly implies WWKL. The proof of Theorem 7.5 shows that MYC-M is provable in ACA_0: Showing that the class $\mathcal{C}$ in the proof of Lemma 7.7 has positive measure, under the assumption that $\bigoplus_{i<n} X_i$ is a density-one point, requires only (the relativized version of) the fact that if $d(Z \mid \mathcal{C}_i) = 1$ for each of finitely many Π^0_1 classes $\mathcal{C}_0, \dots, \mathcal{C}_n$, then $d(Z \mid \bigcap_{i \leqslant n} \mathcal{C}_i) = 1$, for which arithmetical induction is more than enough. Then building T arithmetically is straightforward. So the only possible issue is Lemma 7.6, but all the proof of that lemma requires is (the relativized version of) the fact that every Π^0_1 class $\mathcal{P}$ of positive measure contains a density-one 1-random point X. This fact is provable in ACA_0, for instance by proving the Lebesgue Density Theorem as in [15, Proposition 3.3], applied to the intersection $\mathcal{C}$ of $\mathcal{P}$ with a sufficiently large Π^0_1 class of 1-randoms, then taking $\sigma_0 \prec \sigma_1 \prec \cdots$ such that the relative measure of $\mathcal{C}$ above σ_n goes to 1, and defining $X = \bigcup_n \sigma_n$.

We thus have the following diagram, where double arrows represent implications that are known not to reverse, single arrows represent implications for which we do not know whether the reversal holds, and all missing arrows not implied by transitivity represent open questions.

(7.1)



Although PTR and PSUB are equivalent, there is a significant formal difference between them: the former statement is Π^1_3, while the latter is Π^1_2. We often think of a Π^1_2 statement of the form $(\forall X)[\Phi(X) \rightarrow (\exists Y)\Psi(X,Y)]$ with Φ and Ψ arithmetic as a *problem*. An *instance* is an X such that $\Phi(X)$ holds, and a *solution* to this instance is a Y such that $\Psi(X,Y)$ holds. (For the sake of this discussion, let us assume we are working over the standard natural numbers.) We can do the same with a statement of this form where Ψ is now Π^1_1, but we then need to be careful what we mean by saying that $\Psi(X,Y)$ holds, since that might be model-dependent. That is, for an instance X in an ω-model $\mathcal{M}$, there might be a $Y \in \mathcal{M}$ such that $\mathcal{M} \vDash \Psi(X,Y)$ but $\mathcal{N} \nvDash \Psi(X,Y)$ for some $\mathcal{N} \supset \mathcal{M}$. This fact leads to some peculiar situations.

For example, let us consider reverse-mathematical analogs of some of the variants of MYC-M discussed following the proof of Theorem 7.13. The one where T is required to be a tree with no dead ends is equivalent to WWKL_0, by the same argument as in the computability-theoretic case, but the one where T is required only to be an infinite tree is provable in RCA_0: It is provable in WWKL_0, but also in RCA_0 plus the negation of WWKL, or even just of WKL, because if T is an infinite tree with no infinite paths, then it satisfies any universal condition on paths, an issue already mentioned above Proposition 7.20. This proof, which is made possible by the Π^1_3 form of this statement, feels like something of a cheat, particularly as it allows us to prove the statement in RCA_0 without giving us any more idea of its computability-theoretic complexity than we previously had.

A similar kind of cheating allows us to show that even the version where T is required to be an infinite tree with no isolated paths is provable in RCA_0 (despite the fact mentioned above that every computable infinite tree all of whose paths are 1-random has paths that are not mutually 1-random). One can define the notion of having no isolated paths by quantifying over paths, in which case trees with no paths automatically have no isolated paths. A first-order way to define this notion is to say that a tree T has an isolated path if there is a $\sigma \in T$ such that T is infinite above σ, and for each $\tau \succcurlyeq \sigma$, at most one of T above $\tau^\frown 0$ and T above $\tau^\frown 1$ is infinite. However, even with this definition we can still prove in RCA_0 that if T is infinite but has no infinite paths, then it cannot have an isolated path: Assume T has an isolated path in this sense, and let σ be as above. Then we can produce a path on T computably in T by beginning with σ, and given τ, waiting for T to terminate above some $\tau^\frown i$, then continuing on to $\tau^\frown(1-i)$. It follows by Π^0_1 induction that this process produces an infinite path. Thus for the following proposition it does not matter which definition of having no isolated paths we take.

Proposition 7.21. RCA_0 *proves that for each A, there is an infinite tree T with no isolated paths such that for any nonempty finite collection $\mathcal{F}$ of pairwise distinct paths through T, the set $\bigoplus_{Y\in\mathcal{F}} Y$ is 1-random relative to A.*

Proof. If ACA holds then so does MYC-M, so we are done. If WKL fails then there is an infinite tree T with no paths. This tree cannot have an isolated path, as otherwise it would be able to compute such a path, and it vacuously satisfies the condition on randomness of joins of paths. So we can assume that WKL (or even just WWKL) holds but ACA fails. Then there is a B such that B' does not exist.

Now fix A, and let $X = \bigoplus_i X_i$ be 1-random relative to A. We define a tree S as follows. Fix a B-computable enumeration of B'. Also fix an ordering of $2^{<\omega}$. For a tree R, say that σ is an *off-node* of R if σ is not in R but its immediate predecessor is. Let S_0 consist of every initial segment of $X_{\langle 0,0\rangle}$. Given S_{n-1}, let $\sigma_0, \sigma_1, \dots$ be the off-nodes of S_{n-1} in order. Form S_n by appending to S_{n-1} a copy of $X_{\langle n,k\rangle}$ with root σ_k for each k. Let $S = \bigcup_n S_n$. It is not difficult to see that S can be built computably from X.

Now define T as follows. Start with S, and whenever we find at a stage s that n is in B', truncate S so that if $k < s$ then T is finite above the kth off-node of S_n. Again, it is easy to see that such a T can be built X-computably. It is also easy to see that T has no isolated paths.

Let $Y_0, \dots, Y_m$ be distinct paths on T. For each $i \leqslant m$, it cannot be the case that Y_i goes through an off-node of S_n for every n, since then we could compute B' from knowing which off-nodes Y_i goes through, so each Y_i is equal to $\sigma^\frown X_{j_i}$ for some σ and j_i. By construction, the j_i's are pairwise distinct, so $\bigoplus_{i\leqslant m} X_{j_i}$ is 1-random, and hence so is $\bigoplus_{i\leqslant m} Y_i$. □

Natural Π^1_3 statements can exhibit even stranger behavior. Consider for instance the following statement. (See [3] for a formalization of the notion of 2-randomness.)

T2R: For each A, there is an infinite tree T such that every path through T is 2-random relative to A.

For the same reason as above, T2R is implied by the negation of WKL (or equivalently, RCA_0 is the infimum of T2R and WKL). It is also follows from 2-RAND, the principle stating that for every A, there is a set that is 2-random relative to A, since if X is 2-random relative to A, then the initial segments of X form the required T. Indeed, T2R is equivalent to $\neg\mathrm{WKL} \vee$ 2-RAND, since T2R and WKL together imply 2-RAND. Unlike the previous principles for 1-randomness, however, T2R is not provable in RCA_0, or even in WKL_0, because if WKL_0 proves T2R then it proves the existence of 2-randoms, and there are ω-models of WKL_0 not containing any 2-randoms (e.g., any ω-model of WKL_0 below a Δ^0_2 PA degree). Nevertheless, suppose we take any model $\mathcal{N}$ of Σ^0_1-PA (the first-order part of RCA_0) and consider the model $\mathcal{M}$ of RCA_0 with first-order part $\mathcal{N}$ and second-order part consisting of all Δ^0_1-definable subsets of $\mathcal{N}$. Then $\mathcal{M}$ is not a model of WKL, so it is a model of T2R, and hence any principle that does not hold in $\mathcal{M}$ is not implied by T2R. Almost all natural principles not provable in RCA_0 that have been studied have this property for some $\mathcal{M}$ of this form, so the position of T2R in the reverse-mathematical universe is rather strange. We will discuss genericity below, but note here that the principle that for each A, there is an infinite tree T such that every path through T is 1-generic relative to A has the same properties. These are not the first examples of these kinds of "monsters in the reverse mathematics zoo": Belanger [8, 9] found several natural model-theoretic principles equivalent to $\neg$WKL $\vee$ ACA.

With this discussion in mind, it is worth seeking a Π^1_2 version of MYC-M. We can be guided here by the proof of Theorem 7.5, as well as by the formulation of PSUB. Assume that the Π^0_1 classes $\mathcal{R}_0, \mathcal{R}_1, \dots$ in that proof are nested, i.e., $\mathcal{R}_0 \subseteq \mathcal{R}_i \subseteq \cdots$. The construction of T in that proof ensures that if $\sigma_0, \dots, \sigma_n$ are pairwise distinct strings of the same length, then there is an m such that for all paths $T(\sigma_0)^\frown X_0, \dots, T(\sigma_n)^\frown X_n$ through T, we have $\bigoplus_{i \leqslant n} X_i \in \mathcal{R}_m$. By taking the $\mathcal{R}_m$'s to be the complements of the levels of a universal Martin-Löf test, this fact gives us a way to talk about the 1-randomness of joins of paths through T without quantifying over the paths themselves. There are several ways in which we could formalize this idea as a reverse-mathematical statement, and it is not clear how much of a difference the exact choice makes to the strength of the resulting principle. The following seems like a reasonable way to do it, which is also adaptable to the variants of MYC-M discussed above, and is weak enough to be as close to MYC-M as possible. Here we think of a perfect tree as a binary tree with no dead ends and no isolated paths. Let T be any binary tree. Recall that for $\sigma \in T$, we write T_σ for the tree consisting of all τ such that $\sigma^\frown\tau \in T$. For $\sigma_0, \dots, \sigma_n \in T$, we write $T_{(\sigma_0,\dots,\sigma_n)}$ for the tree consisting of the closure under initial segments of the set of all strings of the form $\bigoplus_{i \leqslant n} \tau_n$ where $\tau_0 \in T_{\sigma_0}, \dots, \tau_n \in T_{\sigma_n}$ are strings of the same length. By a *bar* for T we mean a finite collection F of pairwise incompatible elements of T such that every element of T is compatible with some element of F.

MYC-M$^+$: Let A be a set and let $S_0, S_1, \dots$ be positive trees such that for each k, we have $\frac{|S_k \cap 2^m|}{2^m} > 2^{-(k+1)}$ for all m. Then there is a perfect tree T with the following property. For any pairwise distinct $\sigma_0, \dots, \sigma_n \in T$ of the same length, there are bars $F_0, \dots, F_n$ for $T_{\sigma_0}, \dots, T_{\sigma_n}$, respectively, such that for each $\tau_0, \dots, \tau_n$ with $\tau_i \in F_i$ for $i \leqslant n$, there is a k for which $T_{\tau_0,\dots,\tau_n}$ is a subtree of S_k.

We can similarly define RMYC-M$^+$ by restricting the above to $n = 1$. Clearly MYC-M$^+$ and RMYC-M$^+$ imply MYC-M and RMYC-M, respectively, and Diagram (7.1) remains unchanged if we replace MYC-M and RMYC-M by MYC-M$^+$ and RMYC-M$^+$, respectively. We can also define versions of the weakenings of MYC-M and RMYC-M to infinite trees with no isolated paths, or just to infinite trees. For example, we can consider the following principle.

WMYC-M$^+$: Let A be a set and let $S_0, S_1, \dots$ be positive trees such that for each k, we have $\frac{|S_k \cap 2^m|}{2^m} > 2^{-(k+1)}$ for all m. Then there is an infinite tree T with no isolated paths satisfying the following property. For any pairwise distinct $\sigma_0, \dots, \sigma_n \in T$ of the same length, there are bars $F_0, \dots, F_n$ for $T_{\sigma_0}, \dots, T_{\sigma_n}$, respectively, such that for each $\tau_0, \dots, \tau_n$ with $\tau_i \in F_i$ for $i \leqslant n$, there is a k for which $T_{\tau_0,\dots,\tau_n}$ is either finite or is a subtree of S_k.

We do not know what relationships hold between these various principles beyond the obvious ones, and the fact that even the weakening of WMYC-M$^+$ obtained by removing the requirement that T have no isolated paths is not provable in RCA$_0$, by the fact mentioned above that every computable infinite tree all of whose paths are 1-random has paths that are not mutually 1-random. It might also be interesting to consider the computability-theoretic and reverse-mathematical content of versions of Corollary 5.9, given that mutual 1-randomness is sufficient but not necessary to ensure Hausdorff distance 1/2.

7.3. **Mycielski's Theorem for category.** As noted in Theorem 2.17, Mycielski's Theorem has an analog for category in place of measure that is much easier to prove. As in the measure case, we can do so via the effective case, by considering 1-genericity and then relativizing. A useful fact here is the analog of van Lambalgen's Theorem for 1-genericity in place of 1-randomness proved by Yu [57]. It is straightforward to check that the proof given in that paper can be carried out in RCA_0.

Theorem 7.22 (Mycielski [48])**.** *Let $\mathcal{C}_0 \subseteq (2^\omega)^{n_0}, \mathcal{C}_1 \subseteq (2^\omega)^{n_1}, \ldots$ be comeager. Then there is a perfect subset P of 2^ω such that $P^{n_i} \in \mathcal{C}_i$ for all i.*

Proof. By a *finite partial perfect tree* we mean the restriction of a perfect tree to $\sigma \in 2^{<n}$ for some n. It is straightforward to encode finite partial perfect trees as binary strings so that if σ and τ encode S and T respectively, then $\tau \succ \sigma$ if and only if T properly extends S. Let $W_0, W_1, \ldots$ list the c.e. sets of binary strings. Let D_e be the set of all σ such that σ encodes a finite partial perfect tree T and for every tuple $\tau_0, \ldots, \tau_n$ of distinct leaves of T, we have that $\tau = \bigoplus_{i \leqslant n} \tau_i$ meets or avoids W_e (i.e., either τ has an initial segment in W_e, or no extension of τ is in W_e).

We claim that each D_e is dense, so that if X is sufficiently generic and T is the union of the finite partial perfect trees encoded by initial segments of X, then T is a perfect tree such that any join of finitely many pairwise distinct paths through T is 1-generic. To prove the claim, fix a string σ. By extending σ if needed, we can assume that σ encodes a tree S. Let $S(\tau_0), \ldots, S(\tau_n)$ be the leaves of S. Let P be the set of all tuples of pairwise distinct pairs (i,j) with $i \leqslant n$ and $j \in \{0,1\}$. Let $F_0, \ldots, F_{m-1} \in P$ be pairwise distinct, and consider the following procedure. For $i \leqslant n$ and $j \in \{0,1\}$, let $\mu_{i,j,0} = S(\tau_j)^\frown i$. If we have defined all the strings $\mu_{i,j,k}$ for some $k < m$, search for strings $\nu_{i,j}$ for $j \leqslant n$ and $i \in \{0,1\}$ such that each $\nu_{i,j}$ extends $\mu_{i,j,k}$, and $\bigoplus_{(i,j) \in F_k} \nu_{i,j} \in W_e$ (where this join is ordered as in F_k). If such strings are found then let $\mu_{i,j,k+1} = \nu_{i,j}$. Let Q be the set of all pairwise distinct $F_0, \ldots, F_{m-1} \in P$ such that this procedure ends up defining the strings $\mu_{i,j,m}$. Take an element of Q of maximal size, use that element to define the strings $\mu_{i,j,m}$, extend S by defining $S(\tau_i^\frown j) = \mu_{i,j,m+1}$, and let ρ be a string encoding this new tree. It is easy to see that ρ is an extension of σ in D_e.

Now, each $\mathcal{C}_i$ contains the intersection of dense open sets $\mathcal{U}_i^0, \mathcal{U}_i^1, \ldots$. Let V_i^j be a set of strings generating the open set $\{\bigoplus_{k<n_i} Y_k : (Y_0, \ldots, Y_{n_i-1}) \in \mathcal{U}_i^j\}$. Each V_i^j is Σ_1^{0,A_i^j} for some A_i^j. Let $A = \bigoplus_{i,j} A_i^j$. Then relativizing the above argument to A produces a perfect tree T such that for each i and each sequence $Y_0, \ldots, Y_{n_i-1}$ of distinct paths through T, we have $(Y_0, \ldots, Y_{n_i-1}) \in \mathcal{U}_i^j$ for all j, and hence $(Y_0, \ldots, Y_{n_i-1}) \in \mathcal{C}_i$. □

The proof of Theorem 7.22 shows that, in contrast with the second part of Theorem 7.11, if X is sufficiently generic, then it computes a perfect tree such that the join of any nonempty finite collection of distinct paths is 1-generic. (This distinction mirrors the set-theoretic one, as, unlike in the case of random reals, adding a single Cohen generic real does add a perfect set of mutually Cohen generic reals (see e.g. [27]).) A natural question now is how generic such an X needs to be. It is not quite enough to have X be 1-generic, because Kumabe [41] showed that there is a 1-generic degree with a strong minimal cover, so the proof of Theorem 7.10 with "1-random" replaced by "1-generic" throughout yields the following result.

Theorem 7.23. *There is a* 1*-generic that does not compute any perfect tree all of whose paths are* 1*-generic.*

The sets D_e above are uniformly Δ^0_2, and indeed can be replaced by uniformly Π^0_1 dense sets, because if a Δ^0_2 set D is dense then so is the Π^0_1 set $C = \{\tau : (\forall s > |\tau|)(\exists \sigma \preccurlyeq \tau)[\sigma \in D[s]]\}$, and clearly a set meets D if and only if it meets C. Thus the notion of Π^0_1-genericity becomes relevant here. (We will return to this notion in the reverse-mathematical context below.) We can analyze these sets a bit further, though.

A set of strings D is *pb-dense* if there is a function f that is computable from $\emptyset'$ with a primitive recursive bound on the use function, such that for each σ, we have that $f(\sigma) \in D$ and $f(\sigma) \succcurlyeq \sigma$. A set is *pb-generic* if it meets every pb-dense set of strings. Downey, Jockusch, and Stob [22, Theorem 3.2] showed that a degree $\mathbf{a}$ computes a pb-generic if and only if it is array noncomputable, which means that for each $g \leqslant_{\text{wtt}} \emptyset'$, there is an $\mathbf{a}$-computable function h such that $h(n) \geqslant g(n)$ for infinitely many n.

The argument in the proof of Theorem 7.22 that the sets D_e are dense shows that given σ, we can find an extension τ of σ in W_e by asking enough existential questions to determine the elements of Q, since once we have an element of Q of maximal length, we can find such a τ computably. There clearly is a primitive recursive bound on the numbers n for which we need to query $\emptyset'(n)$ to answer all of these questions. Thus each D_e is pb-dense, and hence any array noncomputable degree can compute a set that meets all of them. Relativizing this argument and that of Downey, Jockusch, and Stob [22], we have the following result.

Theorem 7.24. *For any A and any B that is array noncomputable relative to A, there is a B-computable perfect tree T such that for any nonempty finite collection $\mathcal{F}$ of paths through T, the set $\bigoplus_{Y \in \mathcal{F}} Y$ is* 1*-generic relative to A, and hence the joins of any two finite, disjoint, nonempty collections of paths through T are mutually* 1*-generic relative to A.*

On the reverse mathematics side, the following principles have been well-studied. (See e.g. [29, Section 9.3] for more on the reverse mathematics of model theory.)

$\mathbf{\Pi^0_1 G}$: For any family of uniformly Π^0_1 dense predicates $P_0, P_1, \ldots$ on $2^{<\omega}$, there is a set G such that each P_i holds of some initial segment of G.

AMT: Every complete atomic theory has an atomic model.

It follows from computability-theoretic work of Csima, Hirschfeldt, Knight, and Soare [17] and Conidis [16] that Π^0_1G implies AMT, and that the two are equivalent over RCA_0 together with Σ^0_2 induction. Hirschfeldt, Shore, and Slaman [32] showed that AMT does not imply Π^0_1G over RCA_0 alone. They also showed that both principles, while provable in ACA_0, are incomparable with WKL and WWKL. The existence of pb-generics does not seem to have been studied from this point of view. It is not difficult to show that it follows from Π^0_1G, but we do not know whether it is strictly weaker.

Corresponding to the principles for measure in the previous subsection, we have the following principles.

MYC-C: For each A, there is a perfect tree T such that for any nonempty finite collection $\mathcal{F}$ of pairwise distinct paths through T, the set $\bigoplus_{Y \in \mathcal{F}} Y$ is 1-generic relative to A.

RMYC-C: For each A, there is a perfect tree T such that any two distinct paths through T are mutually 1-generic relative to A.

PTG: For each A, there is a perfect tree T such that every path through T is 1-generic relative to A.

These also have natural versions that avoid quantification over paths. (Here a string σ *meets or avoids* a predicate P if either $P(\sigma)$ holds or $P(\tau)$ fails to hold for every $\tau \succcurlyeq \sigma$.)

MYC-C$^+$: For each A, there is a perfect tree T such that for every Σ^0_1 predicate P on binary strings, every n, and every k, there is an $m > k$ such that for every pairwise distinct $\sigma_0, \dots, \sigma_n \in T \cap 2^m$, the string $\bigoplus_{i \leqslant n} \sigma_n$ meets or avoids P.

RMYC-C$^+$: For each A, there is a perfect tree T such that for every Σ^0_1 predicate P on binary strings and every k, there is an $m > k$ such that for every distinct $\sigma_0, \sigma_1 \in T \cap 2^m$, the string $\sigma_0 \oplus \sigma_1$ meets or avoids P.

PTG$^+$: For each A, there is a perfect tree T such that for every Σ^0_1 predicate P on binary strings and every k, there is an $m > k$ such that every $\sigma \in T \cap 2^m$ meets or avoids P.

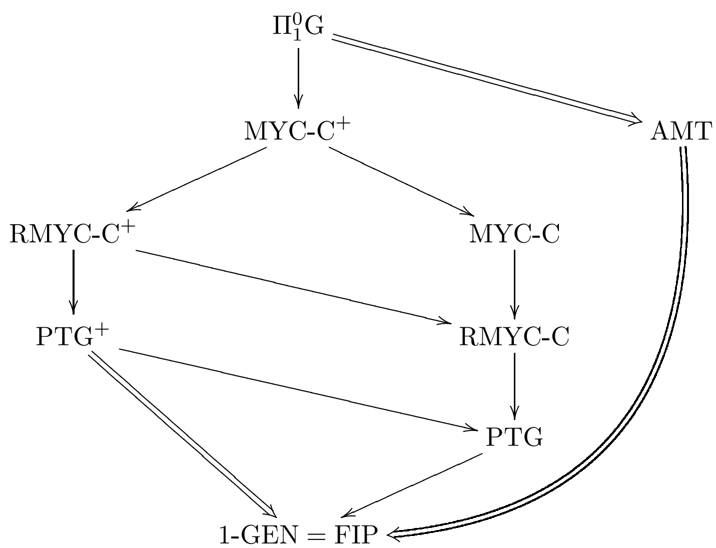
The argument we gave above that Π^0_1-genericity is enough for the proof of Theorem 7.22 can be carried out over RCA_0 to show that Π^0_1G implies MYC-C$^+$. (In the proof of Theorem 7.22, the existence of the set Q is ensured by bounded Σ^0_1-comprehension, which holds in RCA_0.) PTG clearly implies 1-GEN, the principle stating that for every A there is a set that is 1-generic relative to A. The latter principle has been shown by Cholak, Downey, and Igusa [14] to be equivalent to the Finite Intersection Principle (FIP) studied by Dzhafarov and Mummert [23]. Day, Dzhafarov, and Miller [unpublished] and Greenberg and Hirschfeldt [unpublished] showed that AMT implies 1-GEN. We have mentioned that Π^0_1G is incomparable with WKL and WWKL. The same is true of 1-GEN, since it follows from Π^0_1G, and hence cannot imply WWKL, and is not implied by WKL because there are hyperimmune-free PA degrees, which cannot compute 1-generics. Thus we are in a different section of the reverse-mathematical universe as in the measure case.

For each 1-generic A, there is an ω-model of 1-GEN all of whose elements are computable in A, by the same argument as in the first paragraph of the proof of Theorem 7.18, with randomness replaced by genericity. Thus Theorem 7.23 has the following consequence.

Corollary 7.25. *There is an ω-model of* RCA_0 + 1-GEN *that is not a model of* PTG$^+$.

Thus we have the following picture, where the arrows and missing arrows have the same meaning as in Diagram (7.1).

(7.2)



As in the measure case, we can also define versions of the above principles where we replace perfect trees by infinite trees without isolated paths, or just by infinite trees. We will not study these principles further here. It might also be interesting to consider the computability-theoretic and reverse-mathematical content of versions of Corollary 5.3, given that relative 1-genericity is sufficient but not necessary to ensure Hausdorff distance 1.

8. Cauchy sequences in $(\mathcal{S}, \delta)$

In this section we pursue the question raised following Theorem 2.6 regarding the effectiveness of the completeness of $(\mathcal{S}, \delta)$. We need to work with particular representatives of coarse equivalence classes, so, keeping in mind that δ is defined (as a pseudo-metric) on sets as well as on elements of $\mathcal{S}$, we say that $A_0, A_1, \dots$ form a *δ-Cauchy sequence* if for every k there is an m such that $\delta(A_m, A_n) \leqslant 2^{-k}$ for all $n > m$. We say that $A_0, A_1, \dots$ form a *strongly δ-Cauchy sequence* if $\delta(A_m, A_n) \leqslant 2^{-m}$ for all $m < n$. We say that A is a *limit* of the δ-Cauchy sequence $A_0, A_1, \dots$ if for every k there is an m such that $\delta(A_n, A) \leqslant 2^{-k}$ for all $n > m$. By Theorem 2.6, every δ-Cauchy sequence has a limit. While this limit is not unique, all the limits of a given δ-Cauchy sequence are coarsely equivalent (i.e., are coarse descriptions of each other).

For the purposes of reverse mathematics, these definitions need to be rephrased a bit, because the existence of the δ function cannot be proven in RCA_0. (Indeed, because it involves the existence of the upper density, and there is a computable set whose density computes $\emptyset'$, as shown in [36, Theorem 2.21], the existence of the δ function can be shown to be equivalent to ACA_0.) However, the definitions above do not actually require the δ function to exist (i.e., they can be written in classically equivalent forms that do not mention δ). So when working in RCA_0, we say that $A_0, A_1, \dots$ form a *δ-Cauchy sequence* if

$$(\forall k)(\exists m)(\forall n > m)(\exists i)(\forall j > i)[\rho_j(A_m \triangle A_n) \leqslant 2^{-k}]$$

and that it is a *strongly δ-Cauchy sequence* if

$$(\forall m)(\forall n > m)(\exists i)(\forall j > i)[\rho_j(A_m \triangle A_n) \leqslant 2^{-m}].$$

We say that A is a *limit* of the δ-Cauchy sequence $A_0, A_1, \dots$ if

$$(\forall k)(\exists m)(\forall n > m)(\exists i)(\forall j > i)[\rho_j(A_m \triangle A) \leqslant 2^{-k}].$$

The reason we consider strongly δ-Cauchy sequences is the following theorem, which was stated in a different form in [31, Theorem 5.9].

Theorem 8.1 (Miller, see [31, Theorem 5.9])**.** *Every strongly δ-Cauchy sequence has a limit computable from the sequence.*

The proof of this theorem (as given in [31]) consists of taking a strongly δ-Cauchy $C_0, C_1, \dots$, defining the notion of C_m *trusting* C_n on J_k in terms of the density function d_k (where J_k and d_k are as in Definition 2.4), and then defining a limit C of this sequence by letting $C \upharpoonright J_k = C_n \upharpoonright J_k$ for the largest $n \leqslant k$ such that C_n is trusted on J_k by all C_m with $m < n$. Since the definition of trusting is computable, this definition of C can be carried out in RCA_0. The rest of the proof is a verification that C is indeed a limit of the sequence, using Lemma 2.5, which can also be carried out in RCA_0. Thus we have the following fact.

Theorem 8.2. RCA_0 *proves that every strongly δ-Cauchy sequence has a limit.*

Theorem 8.1 cannot be strengthened by dropping the requirement that the δ-Cauchy sequence be strongly δ-Cauchy, as we now show.

Theorem 8.3. *There is a computable δ-Cauchy sequence such that every limit is high.*

Proof. Recall the sets $\mathcal{R}(A)$ from Definition 3.7. By Theorem 2.19 of [36], if $A \leqslant_{\mathrm{T}} \emptyset'$ then $\mathcal{R}(A)$ is coarsely computable. In fact, the proof of that theorem is uniform, and hence shows that if $A_0, A_1, \dots$ are uniformly $\emptyset'$-computable, then there are uniformly computable sets $C_0, C_1, \dots$ such that C_i is a coarse description of $\mathcal{R}(A_i)$.

Let $A_0, A_1, \dots$ be a $\emptyset'$-computable approximation to $\emptyset''$. Then $\mathcal{R}(A_0), \mathcal{R}(A_1), \dots$ is a δ-Cauchy sequence with limit $\mathcal{R}(\emptyset'')$. Let $C_0, C_1, \dots$ be as above. Then $C_0, C_1, \dots$ is a computable δ-Cauchy sequence that also has $\mathcal{R}(\emptyset'')$ as a limit. By Lemma 3.9, any coarse description of $\mathcal{R}(\emptyset'')$ is high. □

One way to get an upper bound on the minimal complexity of limits of computable δ-Cauchy sequences is to consider the complexity of passing from a δ-Cauchy sequence to a strongly δ-Cauchy subsequence. If $A_0, A_1, \dots$ is a computable δ-Cauchy sequence then, using $\emptyset'''$ as an oracle, for each k we can find an m_k such that $(\forall n > m_k)(\exists i)(\forall j > i)[\rho_j(A_{m_k} \triangle A_n) \leqslant 2^{-k}]$. Thus $A_0, A_1, \dots$ has a $\emptyset'''$-computable strongly δ-Cauchy subsequence. On the other hand, it is straightforward to define a computable δ-Cauchy sequence such that any strongly δ-Cauchy subsequence computes $\emptyset'$, and to use this construction to show that the statement that every δ-Cauchy sequence has a strongly δ-Cauchy subsequence is equivalent to ACA_0. We can do better, however, by also allowing ourselves to replace the elements of our δ-Cauchy sequence by coarsely equivalent ones.

Theorem 8.4. *Let $A_0, A_1, \dots$ be a δ-Cauchy sequence, and let $(\bigoplus_i A_i)''' \leqslant_{\mathrm{T}} B'$. Then there is a B-computable strongly δ-Cauchy sequence $C_0, C_1, \dots$ such that for some sequence $i_0 < i_1 < \dots$, each C_j is a coarse description of A_{i_j}.*

Proof. We can B-computably approximate the function $k \mapsto m_k$ defined above. We can assume that this function is strictly increasing. Let $m_k[s]$ be the stage s approximation to m_k. Define C_k by letting $C_k(n) = A_{m_k[s]}(n)$. Then $C_k =^* A_{m_k}$ for all k, so it is easy to check that $C_0, C_1, \dots$ has the desired properties. (Recall that $=^*$ is equality up to finitely many elements.) □

Corollary 8.5. *If $A_0, A_1, \dots$ is a δ-Cauchy sequence and $(\bigoplus_i A_i)''' \leqslant_{\mathrm{T}} B'$, then $A_0, A_1, \dots$ has a B-computable limit.*

Proof. Let $C_0, C_1, \dots$ be as in the theorem. Then $C_0, C_1, \dots$ has the same limits as $A_0, A_1, \dots$, so we can apply Theorem 8.1. □

There is a gap between Theorem 8.3 and Corollary 8.5, which we can close as follows.

Theorem 8.6. *There is a computable δ-Cauchy sequence such that for any limit C, we have $\emptyset''' \leqslant_{\mathrm{T}} C'$.*

Proof. We first outline the proof and then fill in the details. To begin, we construct a certain function f such that every function that majorizes f computes $0'''$. We then partition the natural numbers into uniformly computable sets $U_0, U_1, \dots$ and define a computable δ-Cauchy sequence $C_0, C_1, \dots$ with the following properties for all n and k:

(i) If $k < f(n)$ or $f(n) = 0$, then $C_k \cap U_n$ is finite.
(ii) If $f(n) > 0$, then $\overline{\rho}(C_j \cap U_n) \geqslant 2^{-n-1}$ for all sufficiently large j.

These properties suffice to prove the theorem. To see that this is the case, let C be a limit of $C_0, C_1, \dots$. We must show that $\emptyset''' \leqslant_{\mathrm{T}} C'$. Let n be given. Using a C'-oracle, find a k such that $\delta(C, C_k) < 2^{-n-2}$. Such k exist by the choice of C, and one can be computed from C' since the predicate $\delta(C, C_k)2^{-n-2}$ is c.e. in C'. We claim that $k \geqslant f(n)$. This fact is obvious if $f(n) = 0$, so assume that $f(n) > 0$, and also assume for the sake of a contradiction that $k < f(n)$. Hence by (i), $C_k \cap U_n$ is finite. Now choose j so large that $\overline{\rho}(C_j \cap U_n) \geqslant 2^{-n-1}$ and $\delta(C, C_j) < 2^{-n-2}$. We have that

$$\delta(C_k, C_j) \geqslant \delta(C_k \cap U_n, C_j \cap U_n) = \delta(\emptyset, C_j \cap U_n) = \overline{\rho}(C_j \cap U_n) \geqslant 2^{-n-1}.$$

However, by the triangle inequality,

$$\delta(C_k, C_j) \leqslant \delta(C_k, C) + \delta(C, C_j) < 2^{-n-2} + 2^{-n-2} = 2^{-n-1}.$$

This contradiction proves the claim. Thus, if we let $g(n)$ be the first such k that is found, then g majorizes f, and we have $0''' \leqslant_T g \leqslant_{\mathrm{T}} C'$, which completes the proof of the theorem from the above properties.

It remains to construct f, and sets U_n and C_j as above.

Fix e_1, e_2, e_3 such that $W_{e_1} = \emptyset'$, $W_{e_2}^{\emptyset'} = \emptyset''$, and $W_{e_3}^{\emptyset''} = \emptyset'''$. Define a function f as follows. Let $f(3n) = 0$ if $n \notin \emptyset'$ and otherwise let $f(3n)$ be the least $s+1$ such that $n \in W_{e_1,s}$. Let $f(3n+1) = 0$ if $n \notin \emptyset''$ and otherwise let $f(3n+1)$ be the least $s+1$ such that $n \in W_{e_2,s}^{\emptyset'}$. Let $f(3n+2) = 0$ if $n \notin \emptyset'''$ and otherwise let $f(3n+2)$ be the least $s+1$ such that $n \in W_{e_3,s}^{\emptyset''}$. It is not difficult to see that if $f(k) \leqslant g(k)$ for all k, then $\emptyset''' \leqslant_{\mathrm{T}} g$.

Note that the predicate $f(n) = s > 0$ is a $\emptyset''$-computable predicate of n and s, and hence is Δ_3^0. Thus there is a Π_2^0 predicate $P(n, s, x)$ such that for all n and s,

$$f(n) = s > 0 \iff (\exists x) P(n, s, x).$$

We will need the above x to be unique when it exists to prove that $C_0, C_1, \dots$ is a δ-Cauchy sequence, which we can achieve by modifying P. Since $f(n) = s > 0$ is a $\emptyset''$-computable predicate of n and s, we can apply the limit lemma relative to $\emptyset'$ to obtain a $\emptyset'$-computable function $g_0(n, s, t)$ such that, for all n and s, if $f(n) = s > 0$ then $\lim_t g_0(n, s, t) = 1$, and otherwise $\lim_t g_0(n, s, t) = 0$. Now define $P_0(n, s, x)$ to hold if $(\forall t \geqslant x)[g_0(n, s, t) = 1]$ and either $x = 0$ or $g_0(n, s, x-1) = 0$. Then P_0 is a Π^0_2 predicate, and $P_0(n, s, x)$ holds if and only if x is minimal with the property that $(\forall t \geqslant x)[g(n, s, x) = 1]$. Hence, for all n and s,

$$f(n) = s > 0 \iff (\exists x)P_0(n, s, x) \iff (\exists! x)P_0(n, s, x).$$

Since $\{e : W_e \text{ is infinite}\}$ is Π^0_2 complete (with respect to 1-reducibility), there is a computable function $h(n, s, x)$ such that for all n and s,

$$f(n) = s > 0 \iff (\exists x)[W_{h(n,s,x)} \text{ is infinite}] \iff (\exists! x)[W_{h(n,s,x)} \text{ is infinite}].$$

Recall that J_s is the interval $[2^z - 1, 2^{z+1} - 1)$. Let $U_n = \bigcup_t J_{\langle n,t\rangle}$. We assume that our pairing function is bijective, so to define C_j it suffices to define $C_j \upharpoonright U_n$ for each n. The various values of n will be treated independently of each other. To define $C_j \upharpoonright U_n$ one might attempt to make the conclusion of condition (ii) hold whenever $j > f(n) > 0$. This is difficult to achieve since the condition $j > f(n) > 0$ is only Δ^0_3. To overcome this problem, we require that j not only bound $f(n)$ but also bound an x witnessing that $j > f(n) = s > 0$ as in the above displayed formula. Doing so replaces the Δ^0_3 condition above by a Π^0_2 condition and is crucial for the proof. This Π^0_2 condition is true if and only if there are infinitely many stages t at which we think it is true, and at each such stage we can add a finite set to $C_j \cap U_n$, chosen to make progress towards satisfying the conclusion of condition (ii).

For a nonempty finite set F and a real number r with $0 \leqslant r \leqslant 1$, let $F[r]$ be the shortest initial segment G of F such that $|G| \geqslant r|F|$. Note that $2^{-r} \leqslant d_w(J_w[r]) \leqslant 2^{-r+1}$ for all w and all $r \in [0, 1]$, where d is as in Definition 2.4.

We can now define $C_j \upharpoonright U_n$, for which it suffices to define $C_j \upharpoonright J_{\langle n,t\rangle}$ for each t. To do this, check whether there exist s and x less than j such that $W_{h(n,s,x),t+1} \neq W_{h(n,s,x),t}$. If so, let $C_j \upharpoonright J_{\langle n,t\rangle} = J_{\langle n,t\rangle}[2^{-n}]$. Otherwise, let $C_j \upharpoonright J_{\langle n,t\rangle} = \emptyset$. Clearly the sets $C_0, C_1, \dots$ are uniformly computable.

We now check that conditions (i) and (ii) from the beginning of the proof hold. To prove (i), assume that $k < f(n)$. Then we must prove that $C_k \cap U_n$ is finite, i.e., $C_k \cap J_{\langle n,t\rangle}$ is empty for all sufficiently large t. This fact is clear since there are only finitely many pairs (s, x) with s and x each less than j, and for each such pair the set $W_{h(n,s,x)}$ is finite. The case in (i) where $f(n) = 0$ is similar.

To prove (ii), assume that $f(n) = s > 0$. Fix x such that $W_{h(n,s,x)}$ is infinite. Then if j is greater than both x and s, there are infinitely many t such that $C_j \upharpoonright C_{\langle n,t\rangle} = J_{\langle n,t\rangle}[2^{-n}]$. For such t we have $d_{\langle n,t\rangle}(C_j \cap U_n) \geqslant 2^{-n}$. It follows that $\overline{d}(C_j \cap U_n) \geqslant 2^{-n}$. By Lemma 2.5 it follows that $\overline{\rho}(C_j \cap U_n) \geqslant 2^{-n-1}$, which completes the proof of (ii).

It remains to be shown that $C_0, C_1, \dots$ is a δ-Cauchy sequence. Fix n. We claim first that, if j and k are sufficiently large, then $(C_j \triangle C_k) \cap U_n$ is finite. This is clear from condition (i) if $f(n) = 0$. Suppose now that $f(n) = s > 0$. Let x be the unique number such that $W_{h(n,s,x)}$ is infinite. Then if j is greater than both s and x, the set $C_j \cap U_n$ differs only finitely from $\bigcup\{J_{\langle n,i\rangle} : W_{h(n,s,x),i+1} \neq W_{h(n,s,x),i}\}$. Since the latter set does not depend on j, the claim follows. By applying it to all $m < n$, we see that if j and k are sufficiently large, then $\bigcup_{m<n}((C_j \triangle C_k) \cap U_m)$ is finite. Hence, if j

and k are sufficiently large, then $\delta(C_j, C_k) = \overline{\rho}(C_j \triangle C_k) = \overline{\rho}((C_j \triangle C_k) \cap \bigcup_{m \geqslant n} U_m)$. Note that $C_j \cap \bigcup_{m \geqslant n} U_m$ is a union of finite sets F with $d_{\langle m,t \rangle}(F) \leqslant 2^{-m+1}$ for some $m \geqslant n$. It follows that $\overline{d}(C_j \cap \bigcup_{m \geqslant n} U_m) \leqslant 2^{-n+1}$. By Lemma 2.5, we have that $\overline{\rho}(C_j \cap \bigcup_{m \geqslant n} U_m) \leqslant 2^{-n+2}$. The same fact holds with j replaced by k. The symmetric difference of two sets is a subset of their union, and the upper density of their union is at most the sum of their upper densities. Hence, for all n, if j and k are sufficiently large, then $\delta(C_j, C_k) \leqslant 2^{-n+3}$. It follows that $C_0, C_1, \dots$ is a δ-Cauchy sequence. □

For the purposes of reverse mathematics, let us take the completeness of $(\mathcal{S}, \delta)$ to be the statement that every δ-Cauchy sequence has a limit. Let HIGH be the statement that for every A there is a B such that $A'' \leqslant_{\mathrm{T}} B'$. More precisely, we can take this statement as saying that for every A there is a 0, 1-valued binary function f such that for each e, we have that $\lim_s f(e, s)$ exists, and is equal to 1 if and only if Φ_e^A is total. (Basic computability theory, including the notion of a (partial) X-computable function, an enumeration $\Phi_0^X, \Phi_1^X, \dots$ of all partial X-computable functions, and so on, can be developed in RCA_0 using universal Σ_1^0 formulas.)

Hölzl, Raghavan, Stephan, and Zhang [33] and Hölzl, Jain, and Stephan [34] studied the principle DOM, which they stated in terms of the notion of weakly represented families of functions, but can be equivalently stated as saying that for every X, there is a function that dominates all X-computable functions. Martin [45] showed that a set is high if and only if it computes a set that dominates all computable functions. His proof, as given for instance in [20, Theorem 2.23.7], can be relativized and carried out in RCA_0 to show that DOM and HIGH are equivalent over RCA_0. We state our results in terms of DOM since that is the name used in the aforementioned papers, in which the authors obtain several results on the reverse-mathematical strength of this principle. For instance, they show that DOM implies the cohesive set principle COH over $\mathrm{RCA}_0 + \mathrm{B}\Sigma_2^0$, but does not imply SRT_2^2 (stable Ramsey's Theorem for pairs), even over ω-models. They also show that DOM is restricted Π_2^1-conservative over RCA_0 (see [33] for the definition), but implies full arithmetic induction over $\mathrm{B}\Sigma_2^0$, and that it is equivalent to several results in the theory of inductive inference.

Recall that two statements of second-order arithmetic are *ω-equivalent* if they hold in the same ω-models of RCA_0.

Theorem 8.7. *The completeness of* $(\mathcal{S}, \delta)$ *is ω-equivalent to* DOM*.*

Proof. Let $\mathcal{M}$ be a Turing ideal. If every δ-Cauchy sequence in $\mathcal{M}$ has a limit in $\mathcal{M}$, then relativizing the proof of Theorem 8.3 shows that $\mathcal{M}$ is a model of HIGH, and hence of DOM.

Conversely, suppose that $\mathcal{M}$ is a model of DOM, and hence of HIGH, and let $A_0, A_1, \dots$ be a δ-Cauchy sequence in $\mathcal{M}$. Then there is a $D \in \mathcal{M}$ such that $(\bigoplus A_i)'' \leqslant_{\mathrm{T}} D'$. Applying HIGH again, there is a $B \in \mathcal{M}$ such that $(\bigoplus A_i)''' \leqslant_{\mathrm{T}} D'' \leqslant_{\mathrm{T}} B'$. By Corollary 8.5, $A_0, A_1, \dots$ has a limit in $\mathcal{M}$. □

By Theorem 8.2, the second half of the above proof carries through in RCA_0.

Theorem 8.8. DOM *implies the completeness of* $(\mathcal{S}, \delta)$ *over* RCA_0*.*

In the proof of Theorem 8.3, the sequence $\mathcal{R}(A_0), \mathcal{R}(A_1), \dots$ is δ-Cauchy because the approximation to $\emptyset''$ settles on initial segments, i.e., for each n, there is an s

such that $A_t(m) = \emptyset''(m)$ for all $t > s$ and $m < n$. This will not generally be the case over nonstandard models, however, so that proof does not immediately give us an implication over RCA_0. We do have the following, however.

Theorem 8.9. *The completeness of* $(\mathcal{S}, \delta)$ *implies* DOM *over* $\mathrm{RCA}_0 + \mathrm{I}\Sigma^0_2$.

Proof. Fix a set A. Working in $\mathrm{RCA}_0 + \mathrm{I}\Sigma^0_2$ together with the completeness of $(\mathcal{S}, \delta)$, we show that there is a function dominating all A-computable functions. We use the notation in Observation 2.9.

Let $B_{e,n} = \{s : (\forall m < n)[\Phi^A_e(n)[s]\downarrow]\}$ and let $C_n = \bigoplus^{\mathcal{R}}_e B_{e,n}$. Given k, we can use bounded Σ^0_2 comprehension (which is provable in $\mathrm{RCA}_0 + \mathrm{I}\Sigma^0_2$), to obtain the set F of all $e \leqslant k$ such that $\Phi^A_e(m)[s]\uparrow$ for some m. We can then use $\mathrm{B}\Sigma^0_2$ to obtain a b such that for each $e \in F$, we have $\Phi^A_e(m)[s]\uparrow$ for some $m < b$. Let $n > b$. Then $B_{e,n} = B_{e,b} = \emptyset$ for all $e \in F$. Furthermore, there is a t such that $s \in B_{e,n}$ and $s \in B_{e,b}$ for all $s \geqslant t$ and $e \leqslant k$ with $e \notin F$, so $\delta(C_b, C_n) \leqslant 2^{-k}$.

Thus $C_0, C_1, \dots$ is a δ-Cauchy sequence. Let C be a limit of this sequence. If Φ^A_e is total then $B_{e,n} =^* \mathbb{N}$ for all n, so $C \restriction R_n$ has density 1. Otherwise, $B_{e,n} = \emptyset$ for all sufficiently large n, so $C \restriction R_n$ has density 0. Thus we can define the function f as follows. Given n, search for an $s \geqslant n$ such that for each $e \leqslant n$, either $\Phi^A_e(n)[s]\downarrow$ or $\rho_s(C \restriction R_n) \leqslant 1/2$. Such an s must exist. Let $f(n) = \max\{\Phi^A_e(n) : \Phi^A_e(n)[s]\downarrow\} + 1$.

If Φ^A_e is total, then $f(n) > \Phi^A_e(n)$ for all sufficiently large n, since $\rho_s(C \restriction R_n) > 1/2$ for all sufficiently large s. □

Notice that by Theorem 8.8 and the conservativity of DOM, the completeness of $(\mathcal{S}, \delta)$ does not imply $\mathrm{I}\Sigma^0_2$. We do not know whether it implies DOM over RCA_0, or over $\mathrm{RCA}_0 + \mathrm{B}\Sigma^0_2$.

It follows from the results on DOM mentioned above and Theorem 8.9 that the completeness of $(\mathcal{S}, \delta)$ implies both arithmetic induction and COH over $\mathrm{RCA}_0 + \mathrm{I}\Sigma^0_2$. We do not know whether these implications hold over $\mathrm{RCA}_0 + \mathrm{B}\Sigma^0_2$ or, in the case of COH, over RCA_0.

9. Open Questions

In this section, we gather some open questions discussed above.

Question 9.1. Without assuming CH, is there a nonempty $\mathcal{U} \subsetneq \mathcal{S}$ that is generated by a Turing invariant set U and is closed in $(\mathcal{S}, \delta)$? What if U is required to be an ideal? (See the discussion at the end of Section 3).

Recall the notions of attractive and dispersive sets from Definition 5.10.

Question 9.2. Is there a natural characterization of the attractive degrees that does not mention Hausdorff distance? Do the notions of being attractive and being almost everywhere dominating coincide for c.e. degrees? [Raised by T. A. Slaman:] Is every attractive c.e. degree high? Can A be dispersive without it being the case that almost every set computes a set that is weakly 1-generic relative to A? Is every 1-generic set dispersive? Does every attractive set of hyperimmune-free degree compute a 1-random set?

Question 9.3. Which finite (or countable) metric spaces can be isometrically embedded in $(\mathcal{D}, H)$?

It seems conceivable that every finite metric space with every distance equal to 0, 1/2, or 1 is isometrically embeddable in $(\mathcal{D}, H)$. As mentioned above, an interesting test case is the $0, 1/2, 1$-valued metric space $\mathcal{M}$ such that $G_{\mathcal{M}}$ is a cycle of length 5.

One way to answer this question would be to show that if $\mathcal{A}$ and $\mathcal{B}$ are disjoint finite sets of degrees, then there is a degree $\mathbf{c}$ such that $H(\mathbf{a}, \mathbf{c}) = 1/2$ for all $\mathbf{a} \in \mathcal{A}$ and $H(\mathbf{b}, \mathbf{c}) = 1$ for all $\mathbf{b} \in \mathcal{B}$. Corollary 6.9 shows that this is not the case, but it might still be true within some class of degrees. The 1-random degrees seem potentially promising in this regard.

Question 9.4. Let $\mathcal{A}$ and $\mathcal{B}$ be disjoint finite sets of 1-random degrees. Must there be a 1-random degree $\mathbf{c}$ such that $H(\mathbf{a}, \mathbf{c}) = 1/2$ for all $\mathbf{a} \in \mathcal{A}$ and $H(\mathbf{b}, \mathbf{c}) = 1$ for all $\mathbf{b} \in \mathcal{B}$?

However, even the following basic question remains open.

Question 9.5. If $\mathbf{a}$ is 1-random, must there be a 1-random $\mathbf{b}$ that is incomparable with $\mathbf{a}$ such that $H(\mathbf{a}, \mathbf{b}) = 1$?

Question 9.6. If $\mathbf{a}$ is hyperimmune-free and $\mathbf{b}$ is a hyperimmune PA degree, must $H(\mathbf{a}, \mathbf{b}) = 1$?

Question 9.7. What is the diameter of $G^{\mathrm{c}}_{\mathcal{D}}$? We know by Corollary 6.10 and Theorem 6.6 that it is 3 or 4.

Question 9.8. Can we improve on the A' bound in Theorem 7.5? In particular, can we replace A' by any set that has PA degree relative to A? Relatedly, is MYC-M provable in WKL_0? Does it imply WKL_0? Does it imply ACA_0?

Question 9.9. Can we improve the bound in Theorem 7.24?

Question 9.10. What else can we say about the computability-theoretic and reverse-mathematical strength of the principles discussed in Section 7?

Question 9.11. Does the completeness of $(\mathcal{S}, \delta)$ imply DOM over RCA_0, or over $\mathrm{RCA}_0 + \mathrm{B}\Sigma^0_2$? Does it imply COH over either of these systems? Does it imply arithmetic induction over over $\mathrm{RCA}_0 + \mathrm{B}\Sigma^0_2$?

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