International Journal of Bifurcation and Chaos © World Scientific Publishing Company

# A Stochastic Pitchfork Bifurcation in Most Probable Phase Portraits

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Received (to be inserted by publisher)

We study stochastic bifurcation for a system under multiplicative stable Lévy noise (an important class of non-Gaussian noise), by examining the qualitative changes of equilibrium states in its most probable phase portraits. We have found some peculiar bifurcation phenomena in contrast to the deterministic counterpart: (i) When the non-Gaussianity parameter in Lévy noise varies, there is either one, two or none backward pitchfork type bifurcations; (ii) When a parameter in the vector field varies, there are two or three forward pitchfork bifurcations; (iii) The non-Gaussian Lévy noise clearly leads to fundamentally more complex bifurcation scenarios, since in the special case of Gaussian noise, there is only one pitchfork bifurcation which is reminiscent of the deterministic situation.

*Keywords*: Stochastic pitchfork bifurcation; Lévy motion; most probable equilibrium states; nonlocal Fokker-Planck equation; bifurcation diagrams.

#### 1. Introduction

Despite the rapid development in many aspects of stochastic dynamical systems, the investigation of stochastic bifurcation is still in its infancy. A stochastic bifurcation may be defined as a qualitative change in the evolution of a stochastic dynamical system, as a parameter varies. Stochastic bifurcations have been observed in a wide range of nonlinear systems in physical science and engineering. The existing works on stochastic bifurcation mostly are for stochastic dynamical systems with Gaussian noise and focus on the qualitative changes in stationary probability densities [Namachchivaya , 1990] as solutions of steady Fokker-Planck equations, invariant measures (together with their supports and Lyapunov spectra) and random point attractors [Arnold , 2003], or Conley index [Chen *et al.*, 2009].

Random fluctuations are often assumed to have Gaussian distributions [Gui *et al.*, 2016; Süel *et al.*, 2006; Hasty *et al.*, 2000; Liu & Jia , 2004; Li *et al.*, 2014] and are represented by Brownian motion. But the fluctuations in some complex systems, such as temperature evolution in paleoclimate ice-core records

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[Ditlevsen , 1999] and bursty transition in gene expression [Kumar *et al.* , 2015; Dar *et al.* , 2012], are not Gaussian. Then it is more appropriate to model these random fluctuations by a non-Gaussian Lévy motion (i.e.,  $\alpha$ -stable Lévy motion) with heavy tails and bursting sample paths [Zheng *et al.* , 2016; Klafter *et al.* , 2011; Woyczynski , 2001; Chechkin *et al.* , 2007].

A bifurcation in deterministic low dimensional dynamical systems often appears as a qualitative change in phase portraits in state space, and is usually illustrated via a bifurcation diagram in a 'parameter-steady state plane' [Guckenheimer & Holmes , 1983; Wiggins , 2003; Strogatz , 1994].

In this present work, we study stochastic bifurcation in a kind of stochastic phase portraits. However, phase portraits for stochastic differential equations are delicate objects. It turns out that the phase portraits in terms of most probable orbits [Duan, 2015; Cheng *et al.*, 2016] offer a promising option. Thus we propose here to study stochastic bifurcation by examining the qualitative changes (especially the changes in the number and stability type for equilibrium states) in most probable phase portraits. To this end, we consider bifurcation for the prototypical scalar stochastic differential equation with multiplicative  $\alpha$ -stable Lévy motion

$$dX_t = (rX_t - X_t^3)dt + X_t dL_t^{\alpha},$$

where r is a real parameter and the parameter  $\alpha$  is in the interval (0,2). The  $\alpha$ -stable Lévy motion  $L_t^{\alpha}$  will be reviewed in the next section.

The deterministic counterpart  $\dot{x} = rx - x^3$  has the well-known (forward) 'pitchfork' bifurcation [Guckenheimer & Holmes , 1983], as the parameter r increases.



Fig. 1. (Color online) Bifurcation diagram for deterministic dynamical system  $\dot{x} = rx - x^3$ : Equilibrium states vs. parameter r. This is a pitchfork bifurcation at r = 0.

Figure 1 is the bifurcation diagram for this deterministic pitchfork system. For  $r \leq 0$ , x = 0 is the only equilibrium state which is stable. While for r > 0, there exist two stable equilibrium states  $\sqrt{r}$  and  $-\sqrt{r}$  and one unstable equilibrium state x = 0. The bifurcation parameter value is at r = 0.

This paper is organized as follows. In Section 2, we review the definition of a scalar stable Lévy motion  $L_t^{\alpha}$ , the most probable phase portraits, and the numerical methods for bifurcation diagrams. In Section 3, we show bifurcation diagrams for a stochastic pitchfork bifurcation under multiplicative stable Lévy motion. Finally, we summarize our results in Section 4.

#### 2. Methods

#### 2.1. Stable Lévy motion

A scalar stable Lévy motion  $L_t^{\alpha}$ , for  $0 < \alpha < 2$ , is a non-Gaussian stochastic process with the following properties [Duan , 2015; Applebaum , 2009; Sato , 1999; Samorodnitsky & Taqqu , 1994]: (i)  $L_0^{\alpha} = 0$ , almost surely (a.s.); (ii)  $L_t^{\alpha}$  has independent increments;

(iii) $L_t^{\alpha}$  has stationary increments:  $L_t^{\alpha} - L_s^{\alpha}$  has probability distribution  $S_{\alpha}((t-s)^{\frac{1}{\alpha}}, 0, 0)$  for  $s \leq t$ ; in particular,  $L_t^{\alpha}$  has distribution  $S_{\alpha}(t^{\frac{1}{\alpha}}, 0, 0)$ ;

(iv)  $L_t^{\alpha}$  has stochastically continuous sample paths, i.e.,  $L_t^{\alpha} \to L_s^{\alpha}$  in probability, as  $t \to s$ .

Here  $S_{\alpha}(\sigma, \beta, \mu)$  is the so-called stable distribution [Samorodnitsky & Taqqu, 1994; Duan, 2015] and is determined by four parameters, non-Gaussianity index  $\alpha(0 < \alpha < 2)$ , skewness index  $\beta(-1 \le \beta \le 1)$ , shift index  $\mu(-\infty < \mu < +\infty)$  and scale index  $\sigma(\sigma \ge 0)$ .

The stable Lévy motion  $L_t^{\alpha}$  has the jump measure

$$\nu_{\alpha}(dy) = C_{\alpha}|y|^{-(1+\alpha)} \, dy,$$

where the coefficient

$$C_{\alpha} = \frac{\alpha}{2^{1-\alpha}\sqrt{\pi}} \frac{\Gamma(\frac{1+\alpha}{2})}{\Gamma(1-\frac{\alpha}{2})}.$$

Note that the well-known Brownian motion  $B_t$  is a special case corresponding to  $\alpha = 2$ . Brownian motion  $B_t$  has independent and stationary increments, and has continuous sample paths (a.s.). Moreover,  $B_t - B_s$  has normal distribution  $\mathcal{N}(t - s, 0)$  for  $s \leq t$ . In particular,  $B_t$  has normal distribution  $\mathcal{N}(t, 0)$ . That is, Brownian motion is a Gaussian process.

#### 2.2. Nonlocal Fokker-Planck equation and numerical methods

Consider a scalar stochastic differential equation with multiplicative Lévy noise

$$dX_t = f(X_t)dt + \sigma(X_t)dL_t^{\alpha}, \quad X_0 = x_0, \tag{1}$$

where f is a given vector field (or drift) and  $\sigma$  is the noise intensity.

The generator for this stochastic differential equation is

$$A\varphi(x) = f(x)\varphi'(x) + \int_{\mathbb{R}^1 \setminus \{0\}} [\varphi(x + y\sigma(x)) - \varphi(x)]\nu_\alpha(dy).$$
(2)

Let  $z = y\sigma(x)$ . The generator becomes

$$A\varphi(x) = f(x)\varphi'(x) + |\sigma(x)|^{\alpha} \int_{\mathbb{R}^1 \setminus \{0\}} [\varphi(x+z) - \varphi(x)]\nu_{\alpha}(dz).$$

The Fokker-Planck equation for this stochastic differential equation, i.e., the probability density p(x,t) for the solution process  $X_t$  with initial condition  $X_0 = x_0$  is [Duan, 2015]

$$p_t = A^* p, \ p(x,0) = \delta(x - x_0),$$
(3)

where  $A^*$  is the adjoint operator of the generator A in Hilbert space  $L^2(R^1)$ , as defined by

$$\int_{\mathbb{R}^1 \setminus \{0\}} A\varphi(x)u(x)dx = \int_{\mathbb{R}^1 \setminus \{0\}} \varphi(x)A^*u(x)dx.$$

Then via integration by parts, we get the adjoint operator for A

$$A^* u(x) = \int_{\mathbb{R}^1 \setminus \{0\}} [|\sigma(x-y)|^{\alpha} u(x-y) - |\sigma(x)|^{\alpha} u(x)] \nu_{\alpha}(dy).$$
(4)

Thus we have the nonlocal Fokker-Planck equation

$$p_t = -(f(x)p(x,t))_x + \int_{\mathbb{R}^1 \setminus \{0\}} [|\sigma(x-y)|^{\alpha} p(x-y,t) - |\sigma(x)|^{\alpha} p(x,t)] \nu_{\alpha}(dy).$$
(5)

When the stable Lévy motion is replaced by Brownian motion, we have the following stochastic differential equation

$$dX_t = f(X_t)dt + \sigma(X_t)dB_t, \quad X_0 = x_0.$$
(6)

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The corresponding Fokker-Planck equation is a local partial differential equation

$$p_t = -(f(x)p(x,t))_x + \frac{1}{2}(\sigma^2(x)p(x,t))_{xx}, \quad p(x,0) = \delta(x-x_0).$$
(7)

We use a numerical finite difference method developed in Gao et al. [Gao *et al.*, 2016] to simulate the nonlocal Fokker-Planck equation (5) and use the standard finite difference method to simulate the local Fokker-Planck equation (7).

### 2.3. Most probable phase portraits

As the solution of the Fokker-Planck equation, the probability density function p(x,t) is a surface in the (x,t,p)-space. At a given time instant t, the maximizer  $x_m(t)$  for p(x,t) indicates the most probable (i.e., maximal likely) location of this orbit at time t. The orbit traced out by  $x_m(t)$  is called a most probable orbit starting at  $x_0$ . Thus, the deterministic orbit  $x_m(t)$  follows the top ridge of the surface in the (x,t,p)-space as time goes on. For more information, see [Duan , 2015; Cheng *et al.* , 2016].

*Definition*: A most probable equilibrium state is a state which either attracts or repels all nearby orbits. When it attracts all nearby orbits, it is called a most probable *stable* equilibrium state, while if it repels all nearby orbits, it is called a most probable *unstable* equilibrium state.

A phase portrait for a stochastic dynamical system, in the sense of most probable orbits, consists of representative orbits (including invariant objects such as most probable equilibrium states) in the state space. Both most probable phase portraits and most probable equilibrium states are deterministic geometric objects. As in the study of bifurcation for deterministic dynamical systems [Guckenheimer & Holmes , 1983; Wiggins , 2003; Strogatz , 1994], we examine the qualitative changes in the most probable phase portraits as a parameter varies. A simple qualitative change is the change in the 'number' and 'stability type' of 'most probable equilibrium states'.

#### 3. Results

We now investigate the bifurcation for the scalar stochastic differential equation with multiplicative Lévy noise

$$dX_t = f(r, X_t)dt + X_t dL_t^{\alpha}, \tag{8}$$

where  $f(r, X_t) = rX_t - X_t^3$ , r is a real parameter, and the non-Gaussianity parameter  $\alpha \in (0, 2)$ . We also compare this bifurcation diagram with that of the same system under multiplicative Brownian noise

$$dX_t = f(r, X_t)dt + X_t \ dB_t. \tag{9}$$

The existing relevant works. The stochastic bifurcation for  $dX_t = f(r, X_t)dt + B_t$ , with additive Brownian noise, was studied in [Crauel & Flandoli, 1998; Callaway *et al.*, 2017] by examining the qualitative changes in invariant measure and their spectral stability. The stochastic bifurcation for  $dX_t = f(r, X_t)dt + X_t B_t$ , with *multiplicative* Brownian noise, was considered in [Xu, 1995] by examining the qualitative changes in invariant measures with supports, and in [Wang, 2015] by examining the qualitative changes in random complete quasi-solutions. Moreover, the stochastic bifurcation for  $dX_t = f(r, X_t)dt + L_t^{\alpha}$ , with additive Lévy noise, was studied in [Chen *et al.*, 2012] by considering steady probability distributions for the solutions.

# 3.1. Bifurcation diagram: System under stable Lévy motion $L_t^{\alpha}$

In the present work, we consider the case for a stochastic bifurcation in system (8), with *multiplicative*  $\alpha$ -stable Lévy motion, using most probable phase portraits (especially most probable equilibrium states) as a parameter r in vector field or the non-Gaussianity parameter  $\alpha$  varies. As the analytical results for most probable equilibrium states are lacking at this time [Cheng *et al.*, 2016], we conduct numerical simulations to generate bifurcation diagrams.

For this system (8), 0 is always a most probable equilibrium state. Figure 2 shows the most probable orbits, starting from several initial points, with one or two most probable equilibrium states. To generate



Fig. 2. (Color online) Most probable orbits and 'most probable equilibrium states' for system (8): (a)  $\alpha = 0.3, r = -0.9$ , together with equilibrium state  $x_m \approx 0$ . (b)  $\alpha = 0.3, r = 0.8$ , together with equilibrium states  $x_m \approx 1$  and  $x_m \approx -1$ .

a bifurcation diagram, we plot all possible equilibrium states versus a parameter r or  $\alpha$  in the 'parameterequilibrium states plane'.

Figure 3 shows the most probable equilibrium states with respect to  $\alpha$ . We divide the real line r into five intervals, in each interval the system (8) has the same bifurcation phenomenon.

(a) For  $r \leq -0.5$ , the system (8) has only the stable equilibrium state 0 with all  $\alpha$  and there is no bifurcation.

(b) For  $-0.5 \leq r \leq -0.2$ , there is a backward pitchfork bifurcation at  $\alpha_1 \approx 0.93$ : with two stable equilibrium states and one unstable equilibrium state 0 when  $\alpha < \alpha_1$  but only one unstable equilibrium state 0. (c) For  $-0.2 < r \leq 0.2$ , there is a backward pitchfork bifurcation at  $\alpha_{21} \approx 1.07$  (two stable equilibrium states and one unstable equilibrium state 0). Then there is a 'collapsing' bifurcation (as if three equilibrium states collapse into one) at  $\alpha_{22} \approx 0.23$  when two stable equilibrium states disappear but the equilibrium state 0 remains and becomes stable.

(d) For  $0.2 \leq r < 2.5$ , there is a backward pitchfork bifurcation at  $\alpha_{31} \approx 0.69$ , a forward pitchfork bifurcation at  $\alpha_{32} \approx 0.95$ , and finally a 'collapsing' bifurcation at  $\alpha_{33} \approx 1.5$  when two stable equilibrium states disappear but the equilibrium state 0 remains and becomes stable.

(e) For  $r \gtrsim 2.5$ , the system (8) has two stable equilibrium states and one unstable equilibrium state 0 and there is no bifurcation.

Figure 4 shows the most probable equilibrium states with respect to r. The parameter  $\alpha$  can be divided into two parts, with  $\alpha = 1$  as the critical or borderline value.

(a) For  $0 < \alpha \leq 1$ , the stochastic dynamical system (8) has a forward pitchfork bifurcation at  $r_{11} \approx -0.25$ , a 'collapsing' bifurcation at  $r_{12} \approx 0.73$  when two stable equilibrium states disappear and the equilibrium state 0 becomes stable, and finally a forward pitchfork bifurcation at  $r_{13} \approx 1.05$ .

(b) For  $1 < \alpha < 2$ , there is a forward pitchfork bifurcation at  $r_{21} \approx 0.20$  and then a 'collapsing' bifurcation at  $r_{22} \approx 1.17$ , suddenly a small forward pitchfork bifurcation at  $r_{23} \approx 1.29$ , again a 'collapsing' bifurcation at  $r_{24} \approx 1.31$ , and finally a forward pitchfork bifurcation at  $r_{25} \approx 1.48$ .

# 3.2. Bifurcation diagram: System under Brownian motion $B_t$

Figure 5 shows the bifurcation diagram, i.e., the most probable equilibrium states versus parameter r, for the system (9) with multiplicative Brownian motion. There is a pitchfork bifurcation at  $r \approx -1.6$ ,



Fig. 3. (Color online) Bifurcation diagram for system (8) with respect to non-Gaussianity parameter  $\alpha$ : (a)  $r \leq -0.5$  (showing here r = -0.8). (b)  $-0.5 \leq r \leq -0.2$  (showing here r = -0.2). (c)  $-0.2 < r \leq 0.2$  (showing here r = 0). (d)  $0.2 \leq r < 2.5$  (showing here r = 0.8). (e)  $r \geq 2.5$  (showing here r = 5).

and this bifurcation was also detected in [Xu, 1995, Fig. 2(b)] by examining the support of the invariant measures. This bifurcation diagram is qualitatively the same as the bifurcation diagram in Figure 1 for the corresponding deterministic system  $\dot{x} = rx - x^3$ , although the bifurcation value is different due to the



Fig. 4. (Color online) Bifurcation diagram for system (8) with respect to parameter r in vector field: (a)  $0 < \alpha \lesssim 1$  ( showing here  $\alpha = 0.9$ ). (b)  $1 < \alpha < 2$  (showing here  $\alpha = 1.2$ ).



Fig. 5. (Color online) Bifurcation diagram for system (9) with multiplicative Brown noise: Stochastic pitchfork bifurcation at  $r \approx -1.6$ .

effect of noise. More significantly, this bifurcation is fundamentally different from the bifurcations under  $\alpha$ -stable Lévy noise, as shown in Figure 4.

# 4. Conclusion

Although bifurcation studies for deterministic dynamical systems have a long history, the stochastic bifurcation investigation is still in its early stage. One reason for this slow development in stochastic bifurcation is due to the lack of appropriate phase portraits, in contrast to deterministic dynamical systems.

One promising option for phase portraits of stochastic dynamical systems is the so-called most probable phase portraits [Duan, 2015; Cheng *et al.*, 2016]. We thus conduct stochastic bifurcation study with the help of these phase portraits.

To demonstrate this stochastic bifurcation approach, we study the bifurcation for a system under multiplicative stable Lévy noise (non-Gaussian). The deterministic counterpart of this system has the wellknown pitchfork bifurcation. The existing works in this topic is for the case of Brownian noise (Gaussian) and in terms of the qualitative changes of invariant measures or point attractors. But analytical studies of invariant measures, together with their spectra and supports, are not easily available for stochastic dynamical systems with Lévy noise. This also motivates us to investigate stochastic bifurcation by most

#### 8 REFERENCES

probable phase portraits, especially their invariant structures such as most probable equilibrium states. By numerically examining the qualitative changes of equilibrium states in its most probable phase portraits, we have detected some bifurcation phenomena such as the double or triple pitchfork bifurcation and a collapsing bifurcation, when a parameter in the vector field or in Lévy noise varies.

# Acknowledgments

We would like to thank Xiujun Cheng and Jian Ren for helpful discussions. This work was partly supported by the National Science Foundation grant 1620449, and the National Natural Science Foundation of China grants 11531006 and 11771449.

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