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REALIZATION AND BIFURCATION OF BOOLEAN FUNCTIONS VIA CELLULAR NEURAL NETWORKS*

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In this work, we study the realization and bifurcation of Boolean functions of four variables via a Cellular Neural Network (CNN). We characterize the basic relations between the genes and the offsets of an uncoupled CNN as well as the basis of the binary input vectors set. Based on the analysis, we have rigorously proved that there are exactly 1882 linearly separable Boolean functions of four variables, and found an effective method for realizing all linearly separable Boolean functions via an uncoupled CNN. Consequently, any kind of linearly separable Boolean function can be implemented by an uncoupled CNN, and all CNN genes that are associated with these Boolean functions, called the CNN gene bank of four variables, can be easily determined. Through this work, we will show that the standard CNN invented by Chua and Yang in 1988 indeed is very essential not only in terms of engineering applications but also in the sense of fundamental mathematics.

Keywords: CNN; CNN gene; CNN gene bank; linearly separable Boolean function; bifurcation.

1. Introduction

Cellular Neural Networks (CNN) were originally introduced by Chua and Yang [1988a, 1988b] as an array of dynamical systems, called cells. In a two-dimensional (2-D) configuration, it can be described by the following dynamical equations [Chua, 1997]:

$$\frac{dx_{i,j}}{dt} = -x_{i,j} + z + \sum_{C_{k,l}\in S_{i,j}} a_{k,l}y_{i+k,j+l} + \sum_{C_{k,l}\in S_{i,j}} b_{k,l}u_{i+k,j+l}, \quad i,j \in Z^2 \tag{1}$$

with the output equations

$$y_{i,j} = f(x_{i,j}) = \frac{1}{2}(|x_{i,j}+1| - |x_{i,j}-1|) \tag{2}$$

where $S_{i,j}$ is the sphere of influence of radius $r = 1$; $x_{i,j}$, $y_{i,j}$, $u_{i,j}$ and z are scalars, called respectively state, output, input and threshold of cell $C_{i,j}$; $a_{k,l}$ and $b_{k,l}$ are scalars synaptic weights.

A standard CNN is uniformly defined by a string of "19" real numbers, called a CNN gene, i.e. a uniform threshold z, nine feedback synaptic weights $a_{k,l}$, and nine control synaptic weights $b_{k,l}$, because the string completely determines the

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properties of the CNN. The universe of all CNN genes is called the CNN genome. Many real-world applications, from image processing to brain science to pattern recognition, can be easily implemented by a single CNN gene or a CNN "program" defined by a string of CNN genes called a CNN chromosome.

A Boolean function of n variables is defined as the following binary map:

$$F : \{-1, 1\}^n \to \{-1, 1\}, \quad F(u_1, u_2, \ldots, u_n) = v \tag{3}$$

where $(u_1, u_2, \ldots, u_n) \in \{-1, 1\}^n$ and $v \in \{-1, 1\}$. Obviously, there exist 2^{2^n} Boolean functions for any given $n \in N$.

A CNN gene G is said to be Boolean if, and only if, given any binary input image $U = \{u_{i,j} \in \{-1, 1\}\}$, the steady-state output $y_{i,j}(\infty)$ of each cell $C_{i,j}$ is also binary, and can be uniquely determined by the input pattern of only those $C_{k,l}$ that are located inside the sphere of influence $S_{i,j}$ of $C_{i,j}$ [Chua, 1999].

It is known that only linearly separable CNN genes or linearly separable Boolean functions can be realized by an uncoupled standard CNN. In other words, the class of all uncoupled CNNs with binary inputs and outputs is identical to the linearly separable class of Boolean functions with respect to Boolean input–output maps [Chua, 1997; Chua *et al.*, 2002; Julian *et al.*, 2003]. It is also known that the linearly separable genes are very important for constructing the CNN chromosome; for example, the well-known game-of-life chromosome contains two linearly separable genes and a logical AND gene [Berlekamp *et al.*, 1982; Chua, 1999].

Observe, on the other hand, that the number of Boolean functions quickly increases as the number of variables increases; for instance, there are $2^{2^9} = 2^{512} \approx 1.34078 \times 10^{154}$ distinct Boolean functions or Boolean genes when $n = 9$, which is a number tremendously greater than the size or age of the universe [Chua *et al.*, 2002]. Therefore, a realization of the Boolean functions is a very important but also extremely difficult task.

How many distinct linearly separable CNN genes are there for n input variables? That is, how many Boolean functions of n variables can be realized by an uncoupled CNN? The known results are that there are 14 linearly separable CNN genes of two variables and there are 104 linearly separable ones of three variables [Chua, 1997, 1999]. The corresponding results on linearly separable CNN genes of four or more variables remain a question to be answered today.

In this paper, we study the realization problem for linearly separable Boolean functions of *four* variables via an uncoupled CNN, and analyze the bifurcation of their genes. Because the CNN of four input variables agrees with the simplest 2-D network model, a realization of linearly separable Boolean genes is very essential. In this work, we not only rigorously prove that there are exactly 1882 linearly separable CNN genes in the family of $2^{2^4} = 65536$ Boolean functions of four variables and their bifurcations, but also build up a complete CNN gene bank, which contains all the linearly separable genes of this kind.

The rest of this paper is organized as follows. Section 2 characterizes some essential properties of the input vector set and the structures of the uncoupled CNN of four input variables. Section 3 gives a main method of realization and bifurcation of these linearly separable Boolean functions. Section 4 lists one part of linearly separable Boolean genes of four variables and the binary decoding tapes as well as the decimal codes of the corresponding CNN output patterns. All these genes constitute a complete CNN gene bank of four variables, a part of the CNN genome. Finally, Sec. 5 presents some conclusions.

2. Some Essential Properties of an Uncoupled CNN

The standard uncoupled CNN is described by

$$\frac{dx_{i,j}}{dt} = -x_{i,j} + z + af(x_{i,j}) + \sum_{|k| \le 1, |l| \le 1} b_{k,l} u_{i+k,j+l}, \quad i, j \in Z^2 \tag{4}$$

namely, the feedback template and the control template in its gene $\boxed{\text{z}}\,\boxed{\text{B}}\,\boxed{\text{A}}$ are

$$\boxed{\text{A}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{5}$$

and

$$\boxed{\text{B}} = \begin{bmatrix} b_{1,-1} & b_{1,0} & b_{1,1} \\ b_{0,-1} & b_{0,0} & b_{0,1} \\ b_{-1,-1} & b_{-1,0} & b_{-1,1} \end{bmatrix} \tag{6}$$

respectively.

The simplest model of locally-connected networks on a plane is the one whose cell each links its three nearest neighbors, i.e. the sphere of influence

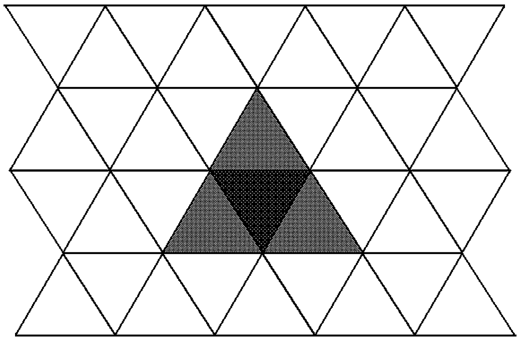

Fig. 1. Cell C_i and its three nearest locally-connected neighbors on a plane.

of cell C_i is a large triangle consisting of four small triangles as shown in Fig. 1.

In this situation, we can extend (4) to the following form:

$$\frac{dx_i}{dt} = -x_i + z + af(x_i) + \sum_{i=1}^{4} b_i u_i, \quad i \in Z \tag{7}$$

This is equivalent as in (6), with

$$\boxed{\mathrm{B}} = \begin{bmatrix} 0 & b_1 & 0 \\ 0 & b_2 & 0 \\ b_3 & 0 & b_4 \end{bmatrix} \tag{8}$$

We only consider the binary input–output operations of (7) in this paper.

Let

$$U = \{u = (u_1, u_2, u_3, u_4)^T | u_i \in \{-1, 1\}, \quad i = 1, 2, 3, 4\} \tag{9}$$

where U is a set of binary input vectors of (7), with

$$U = \{-1, 1\}^4 = \{u^k | k = 0, 1, 2, \ldots, 15\} \tag{10}$$

in which

$$k = \overline{u}_1 2^3 + \overline{u}_2 2^2 + \overline{u}_3 2 + \overline{u}_4, \quad \overline{u}_i = \begin{cases} 1 & \text{if } u_i = 1 \\ 0 & \text{if } u_i = -1 \end{cases}$$

Namely,

$$\begin{aligned} u^0 &= (-1,-1,-1,-1)^T, & u^1 &= (-1,-1,-1,1)^T, \\ u^2 &= (-1,-1,1,-1)^T, & u^3 &= (-1,-1,1,1)^T, \\ u^4 &= (-1,1,-1,-1)^T, & u^5 &= (-1,1,-1,1)^T, \\ u^6 &= (-1,1,1,-1)^T, & u^7 &= (-1,1,1,1)^T, \\ u^8 &= (1,-1,-1,-1)^T, & u^9 &= (1,-1,-1,1)^T, \\ u^{10} &= (1,-1,1,-1)^T, & u^{11} &= (1,-1,1,1)^T, \\ u^{12} &= (1,1,-1,-1)^T, & u^{13} &= (1,1,-1,1)^T, \\ u^{14} &= (1,1,1,-1)^T, & u^{15} &= (1,1,1,1)^T. \end{aligned} \tag{11}$$

Lemma 1. (a) *A subset of* $U, V = \{u^0, u^1, u^2, u^4\}$, *is a linearly independent subset of the binary input vector set* U. (b) *Each vector of* U *can be linearly expressed by the vectors of* V, *i.e.* V *is a basis of the input vector set* U.

Proof

(a) It can be easily seen that the matrix A, which consists of the vectors u^0, u^1, u^2 and u^4, is invertible, where

$$A = \begin{pmatrix} -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & 1 \\ -1 & -1 & 1 & -1 \\ -1 & 1 & -1 & -1 \end{pmatrix} \tag{12}$$

and

$$A^{-1} = \begin{pmatrix} 0.5 & -0.5 & -0.5 & -0.5 \\ -0.5 & 0 & 0 & 0.5 \\ -0.5 & 0 & 0.5 & 0 \\ -0.5 & 0.5 & 0 & 0 \end{pmatrix} \tag{13}$$

Thus, $V = \{u^0, u^1, u^2, u^4\}$ is a linearly independent group of U.

(b) From (11), it is easy to see that

$$\begin{gathered} u^3 = -u^0 + u^1 + u^2, \quad u^5 = -u^0 + u^1 + u^4, \\ u^6 = -u^0 + u^2 + u^4, \\ u^7 = -2u^0 + u^1 + u^2 + u^4, \\ u^8 = 2u^0 - u^1 - u^2 - u^4, \quad u^9 = u^0 - u^2 - u^4, \\ u^{10} = u^0 - u^1 - u^4, \quad u^{11} = -u^4, \\ u^{12} = u^0 - u^1 - u^2, \quad u^{13} = -u^2, \\ u^{14} = -u^1, \quad u^{15} = -u^0. \end{gathered} \tag{14}$$

The proof of the Lemma is thus completed. ■

Next, let

$$\begin{aligned} w_k &= z + \sum_{i=1}^{4} b_i u_i^k \\ &= z + \left(u_1^k,\, u_2^k,\, u_3^k,\, u_4^k\right) \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} \end{aligned} \tag{15}$$

Here, w_k is called the offset level of the CNN (7) with respect to the input vector u^k ($k = 0, 1, 2, \ldots, 15$) [Chua, 1999].

Lemma 2. *Assume $a > 1$. If, for all $u^k \in U$ $(k = 0, 1, 2, 3, \ldots, 15)$,*

$$|w_k| > a - 1 \tag{16}$$

then the CNN (7) has a constant steady-state output $y_i(+\infty) = \lim_{t\to+\infty} y_i(t)$ for each cell C_i, which is independent of the initial state $x_i(0)$ and can be expressed in terms of the constant binary input $u^k \in U$ via the following formula:

$$y_i(+\infty) = \operatorname{sgn}(w_k), \quad (k = 0, 1, 2, 3, \ldots, 15). \tag{17}$$

Proof. Because the DP plot (an acronym for the driving-point plot [Chua *et al.*, 1985]) of $\dot{x}_i = -x_i + af(x_i) + w_k$ depends only on two parameters, namely, the self-feedback coefficient a and the offset level w_k, the proof is similar to that of Theorem 2.8.1 in [Chua, 1999]. Also, $y_i(+\infty)$, the output of x_i, depends only on the sign of w_k. For simplicity, we omit the details. ■

Theorem 1. *For the uncoupled CNN (7), we have the following*:

(1) *the following relations among A in (12), consisting of $V = \{u^0, u^1, u^2, u^4\}$, the basis of U, the offset levels w_k $(k = 0, 1, 2, 4)$, and the threshold z, where*

$$A\begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} = \begin{pmatrix} w_0 - z \\ w_1 - z \\ w_2 - z \\ w_4 - z \end{pmatrix} \tag{18}$$

or

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} = A^{-1}\begin{pmatrix} w_0 - z \\ w_1 - z \\ w_2 - z \\ w_4 - z \end{pmatrix} \tag{19}$$

(2) *the offset levels w_k of (7), except w_0, w_1, w_2 and w_4, given by*

$$\begin{aligned} w_3 &= -w_0 + w_1 + w_2, & w_5 &= -w_0 + w_1 + w_4, \\ w_6 &= -w_0 + w_2 + w_4, & w_7 &= -w_0 + w_3 + w_4, \\ w_8 &= 2z - w_7, & w_9 &= 2z - w_6, \\ w_{10} &= 2z - w_5, & w_{11} &= 2z - w_4, \\ w_{12} &= 2z - w_3, & w_{13} &= 2z - w_2, \\ w_{14} &= 2z - w_1, & w_{15} &= 2z - w_0. \end{aligned} \tag{20}$$

Proof. (1) The formula (18) or (19) can be directly obtained from (15).

(2) From (14) and (15), we have

$$\begin{aligned} w_3 &= z + u^3\begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} \\ &= z + (-u^0 + u^1 + u^2)\begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} \\ &= z - (w_0 - z) + (w_1 - z) + (w_2 - z) \\ &= -w_0 + w_1 + w_2 \end{aligned} \tag{21}$$

and the calculations of w_5, w_6 and w_7 are similar to w_3. As to w_8, from (14) and (15), we have $u^8 = -u^7$ and

$$\begin{aligned} w_8 = z + u^8\begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} &= z - u^7\begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} \\ &= z - (w_7 - z) = 2z - w_7. \end{aligned} \tag{22}$$

Similarly, we obtain other w_k $(k = 9, 10, \ldots, 15)$ as shown in (20). The proof is thus completed. ■

Each binary input vector u^k of U is called a Boolean window.

If its corresponding binary output is $v_k, v_k \in \{-1, 1\}$, then every truth table shown in Table 1 is equivalent to a Boolean function or a Boolean CNN gene. Obviously, there are $2^{2^4} = 65536$ different Boolean-function truth tables of four variables.

From Lemma 2 and Theorem 1, we know that if $|w_k| > a - 1 > 0$ then the truth table of the input–output operation of CNN (7) can be obtained as shown in Table 2.

Thus, applying Theorem 1, we can immediately get the following result.

Theorem 2. *A Boolean function $F(u^k) = v_k$ $(k = 0, 1, 2, \ldots, 15)$ is linearly separable if, and only if, there exist constants w_0, w_1, w_2, w_4 and z such that $v_k = \operatorname{sgn}(w_k)$ $(w_k \neq 0,\ k = 0, 1, 2, \ldots, 15)$, where w_k satisfies formulas (20) in Theorem 1.*

Theorems 1 and 2 will be the most important results for realizing Boolean genes via the uncoupled CNN (7), as further discussed in the next section.

Table 1. Boolean-function truth table of four variables.

k	Boolean Window	Output Pattern
0	u^0	v_0
1	u^1	v_2
3	u^3	v_3
4	u^4	v_4
5	u^5	v_5
6	u^6	v_6
7	u^7	v_7
8	u^8	v_8
9	u^9	v_9
10	u^{10}	v_{10}
11	u^{11}	v_{11}
12	u^{12}	v_{12}
13	u^{13}	v_{13}
14	u^{14}	v_{14}
15	u^{15}	v_{15}

Table 2. Truth table of the input–output operations of CNN (7).

k	Boolean Window	Output Pattern	w_k
0	u^0	$\text{sgn}(w_0)$	w_0 *any given*
1	u^1	$\text{sgn}(w_1)$	w_1 *any given*
2	u^2	$\text{sgn}(w_2)$	w_2 *any given*
3	u^3	$\text{sgn}(w_3)$	$w_3 = w_1 + w_2 - w_0$
4	u^4	$\text{sgn}(w_4)$	w_4 *any given*
5	u^5	$\text{sgn}(w_5)$	$w_5 = w_1 + w_4 - w_0$
6	u^6	$\text{sgn}(w_6)$	$w_6 = w_2 + w_4 - w_0$
7	u^7	$\text{sgn}(w_7)$	$w_7 = w_3 + w_4 - w_0$
8	u^8	$\text{sgn}(w_8)$	$w_8 = 2z - w_7$
9	u^9	$\text{sgn}(w_9)$	$w_9 = 2z - w_6$
10	u^{10}	$\text{sgn}(w_{10})$	$w_{10} = 2z - w_5$
11	u^{11}	$\text{sgn}(w_{11})$	$w_{11} = 2z - w_4$
12	u^{12}	$\text{sgn}(w_{12})$	$w_{12} = 2z - w_3$
13	u^{13}	$\text{sgn}(w_{13})$	$w_{13} = 2z - w_2$
14	u^{14}	$\text{sgn}(w_{14})$	$w_{14} = 2z - w_1$
15	u^{15}	$\text{sgn}(w_{15})$	$w_{15} = 2z - w_0$

3. Realization and Bifurcation of Boolean Functions

Firstly, we present some elementary Lemmas.

Lemma 3. *For any three different real numbers w_0, w_1 and w_2, let $w_3 = w_1 + w_2 - w_0$. Then, all the allowable orders of the four numbers w_0, w_1, w_2 and w_3 are listed as follows*:

$$\begin{aligned} &(a)\ w_0 < w_1 < w_2 < w_3, \quad &(b)\ w_0 < w_2 < w_1 < w_3,\\ &(c)\ w_2 < w_0 < w_3 < w_1, \quad &(d)\ w_2 < w_3 < w_0 < w_1,\\ &(e)\ w_1 < w_0 < w_3 < w_2, \quad &(f)\ w_1 < w_3 < w_0 < w_2,\\ &(g)\ w_3 < w_1 < w_2 < w_0, \quad &(h)\ w_3 < w_2 < w_1 < w_0, \end{aligned} \tag{23}$$

or, in a simple form,

$$w_{i_0} < w_{i_1} < w_{i_2} < w_{i_3},$$

where (i_0, i_1, i_2, i_3) are

$$\begin{aligned} &(a)\ (0,1,2,3), \quad &(b)\ (0,2,1,3),\\ &(c)\ (2,0,3,1), \quad &(d)\ (2,3,0,1),\\ &(e)\ (1,0,3,2), \quad &(f)\ (1,3,0,2),\\ &(g)\ (3,1,2,0), \quad &(h)\ (3,2,1,0), \end{aligned} \tag{24}$$

respectively.

Lemma 4. *For any three different real numbers w_0, w_1, w_2, let $w_3 = w_1 + w_2 - w_0$ and for any w_4, $w_4 \neq w_j$ $(j = 0, 1, 2, 3)$, let $w_5 = w_4 + w_1 - w_0$, $w_6 = w_4 + w_2 - w_0$, $w_7 = w_4 + w_3 - w_0$. Then the eight numbers w_k $(k = 0, 1, 2, \ldots, 7)$ satisfy the following two properties*:

Property A

$$w_0 + w_7 = w_1 + w_6 = w_2 + w_5 = w_3 + w_4$$

and for any allowable ordering of w_k $(k = 0, 1, 2, \ldots, 7)$, namely,

$$w_{i_0} < w_{i_1} < w_{i_2} < w_{i_3} < w_{i_4} < w_{i_5} < w_{i_6} < w_{i_7},$$

we have

$$w_{i_0} + w_{i_7} = w_{i_1} + w_{i_6} = w_{i_2} + w_{i_5} = w_{i_3} + w_{i_4}$$

and

$$i_0 + i_7 = i_1 + i_6 = i_2 + i_5 = i_3 + i_4 = 7,$$

where $(i_0, i_1, i_2, i_3, i_4, i_5, i_6, i_7)$ is an arrangement of $(0, 1, 2, 3, 4, 5, 6, 7)$.

Property B. *There are 96 different kinds of allowable orderings of the eight numbers w_k $(k = 0, 1, 2, 3, 4, 5, 6, 7)$, namely,*

(a) *if $w_0 < w_1 < w_2 < w_3$, then $w_4 < w_5 < w_6 < w_7$, and all the allowable orderings of eight numbers w_0, w_1, w_2, w_3, w_4, w_5, w_6 and w_7 are as follows*:

$$w_{i_0} < w_{i_1} < w_{i_2} < w_{i_3} < w_{i_4} < w_{i_5} < w_{i_6} < w_{i_7},$$

where $(i_0, i_1, i_2, i_3, i_4, i_5, i_6, i_7)$ *are*

$$\begin{array}{ll} (1)\ (0,1,2,3,4,5,6,7), & (2)\ (0,1,2,4,3,5,6,7),\\ (3)\ (0,1,4,2,5,3,6,7), & (4)\ (0,4,1,5,2,6,3,7),\\ (5)\ (4,0,5,1,6,2,7,3), & (6)\ (4,5,0,6,1,7,2,3),\\ (7)\ (4,5,6,0,7,1,2,3), & (8)\ (4,5,6,7,0,1,2,3),\\ (9)\ (0,1,4,5,2,3,6,7), & (10)\ (0,4,1,2,5,6,3,7),\\ (11)\ (4,0,5,6,1,2,7,3), & (12)\ (4,5,0,1,6,7,2,3), \end{array} \tag{25}$$

respectively;

(b) *if* $w_0 < w_2 < w_1 < w_3$, *then* $w_4 < w_6 < w_5 < w_7$, *and all the allowable orderings of eight numbers* w_0, w_1, w_2, w_3, w_4, w_5, w_6 *and* w_7 *are as follows*:

$$w_{i_0} < w_{i_1} < w_{i_2} < w_{i_3} < w_{i_4} < w_{i_5} < w_{i_6} < w_{i_7},$$

where $(i_0, i_1, i_2, i_3, i_4, i_5, i_6, i_7)$ *are*

$$\begin{array}{ll} (13)\ (0,2,1,3,4,6,5,7), & (14)\ (0,2,1,4,3,6,5,7),\\ (15)\ (0,2,4,1,6,3,5,7), & (16)\ (0,4,2,6,1,5,3,7),\\ (17)\ (4,0,6,2,5,1,7,3), & (18)\ (4,6,0,5,2,7,1,3),\\ (19)\ (4,6,5,0,7,2,1,3), & (20)\ (4,6,5,7,0,2,1,3),\\ (21)\ (0,2,4,6,1,3,5,7), & (22)\ (0,4,2,1,6,5,3,7),\\ (23)\ (4,0,6,5,2,1,7,3), & (24)\ (4,6,0,2,5,7,1,3), \end{array} \tag{26}$$

respectively;

(c) *if* $w_2 < w_0 < w_3 < w_1$, *then* $w_6 < w_4 < w_7 < w_5$, *and all the allowable orderings of eight numbers* w_0, w_1, w_2, w_3, w_4, w_5, w_6 *and* w_7 *are as follows*:

$$w_{i_0} < w_{i_1} < w_{i_2} < w_{i_3} < w_{i_4} < w_{i_5} < w_{i_6} < w_{i_7},$$

where $(i_0, i_1, i_2, i_3, i_4, i_5, i_6, i_7)$ *are*

$$\begin{array}{ll} (25)\ (2,0,3,1,6,4,7,5), & (26)\ (2,0,3,6,1,4,7,5),\\ (27)\ (2,0,6,3,4,1,7,5), & (28)\ (2,6,0,4,3,7,1,5),\\ (29)\ (6,2,4,0,7,3,5,1), & (30)\ (6,4,2,7,0,5,3,1),\\ (31)\ (6,4,7,2,5,0,3,1), & (32)\ (6,4,7,5,2,0,3,1),\\ (33)\ (2,0,6,4,3,1,7,5), & (34)\ (2,6,0,3,4,7,1,5),\\ (35)\ (6,2,4,7,0,3,5,1), & (36)\ (6,4,2,0,7,5,3,1), \end{array} \tag{27}$$

respectively;

(d) *if* $w_2 < w_3 < w_0 < w_1$, *then* $w_6 < w_7 < w_4 < w_5$, *and all the allowable orderings of eight numbers* w_0, w_1, w_2, w_3, w_4, w_5, w_6 *and* w_7 *are as follows*:

$$w_{i_0} < w_{i_1} < w_{i_2} < w_{i_3} < w_{i_4} < w_{i_5} < w_{i_6} < w_{i_7},$$

where $(i_0, i_1, i_2, i_3, i_4, i_5, i_6, i_7)$ *are*

$$\begin{array}{ll} (37)\ (2,3,0,1,6,7,4,5), & (38)\ (2,3,0,6,1,7,4,5),\\ (39)\ (2,3,6,0,7,1,4,5), & (40)\ (2,6,3,7,0,4,1,5),\\ (41)\ (6,2,7,3,4,0,5,1), & (42)\ (6,7,2,4,3,5,0,1),\\ (43)\ (6,7,4,2,5,3,0,1), & (44)\ (6,7,4,5,2,3,0,1),\\ (45)\ (2,3,6,7,0,1,4,5), & (46)\ (2,6,3,0,7,4,1,5),\\ (47)\ (6,2,7,4,3,0,5,1), & (48)\ (6,7,2,3,4,5,0,1), \end{array} \tag{28}$$

respectively;

(e) *if* $w_1 < w_0 < w_3 < w_2$, *then* $w_5 < w_4 < w_7 < w_6$, *and all the allowable orderings of eight numbers* w_0, w_1, w_2, w_3, w_4, w_5, w_6 *and* w_7 *are as follows*:

$$w_{i_0} < w_{i_1} < w_{i_2} < w_{i_3} < w_{i_4} < w_{i_5} < w_{i_6} < w_{i_7},$$

where $(i_0, i_1, i_2, i_3, i_4, i_5, i_6, i_7)$ *are*

$$\begin{array}{ll} (49)\ (1,0,3,2,5,4,7,6), & (50)\ (1,0,3,5,2,4,7,6),\\ (51)\ (1,0,5,3,4,2,7,6), & (52)\ (1,5,0,4,3,7,2,6),\\ (53)\ (5,1,4,0,7,3,6,2), & (54)\ (5,4,1,7,0,6,3,2),\\ (55)\ (5,4,7,1,6,0,3,2), & (56)\ (5,4,7,6,1,0,3,2),\\ (57)\ (1,0,5,4,3,2,7,6), & (58)\ (1,5,0,3,4,7,2,6),\\ (59)\ (5,1,4,7,0,3,6,2), & (60)\ (5,4,1,0,7,6,3,2), \end{array} \tag{29}$$

respectively;

(f) *if* $w_1 < w_3 < w_0 < w_2$, *then* $w_5 < w_7 < w_4 < w_6$, *and all the allowable orderings of eight numbers* w_0, w_1, w_2, w_3, w_4, w_5, w_6 *and* w_7 *are as follows*:

$$w_{i_0} < w_{i_1} < w_{i_2} < w_{i_3} < w_{i_4} < w_{i_5} < w_{i_6} < w_{i_7},$$

where $(i_0, i_1, i_2, i_3, i_4, i_5, i_6, i_7)$ *are*

$$\begin{array}{ll} (61)\ (1,3,0,2,5,7,4,6), & (62)\ (1,3,0,5,2,7,4,6),\\ (63)\ (1,3,5,0,7,2,4,6), & (64)\ (1,5,3,7,0,4,2,6),\\ (65)\ (5,1,7,3,4,0,6,2), & (66)\ (5,7,1,4,3,6,0,2),\\ (67)\ (5,7,4,1,6,3,0,2), & (68)\ (5,7,4,6,1,3,0,2),\\ (69)\ (1,3,5,7,0,2,4,6), & (70)\ (1,5,3,0,7,4,2,6),\\ (71)\ (5,1,7,4,3,0,6,2), & (72)\ (5,7,1,3,4,6,0,2), \end{array} \tag{30}$$

respectively;

(g) *if* $w_3 < w_1 < w_2 < w_0$, *then* $w_7 < w_5 < w_6 < w_4$, *and all the allowable orderings of eight* *numbers* $w_0,\ w_1,\ w_2,\ w_3,\ w_4,\ w_5,\ w_6$ *and* w_7 *are as follows*:

$$w_{i_0} < w_{i_1} < w_{i_2} < w_{i_3} < w_{i_4} < w_{i_5} < w_{i_6} < w_{i_7},$$

where $(i_0, i_1, i_2, i_3, i_4, i_5, i_6, i_7)$ *are*

$$\begin{array}{ll} (73)\ (3,1,2,0,7,5,6,4), & (74)\ (3,1,2,7,0,5,6,4), \\ (75)\ (3,1,7,2,5,0,6,4), & (76)\ (3,7,1,5,2,6,0,4), \\ (77)\ (7,3,5,1,6,2,4,0), & (78)\ (7,5,3,6,1,4,2,0), \\ (79)\ (7,5,6,3,4,1,2,0), & (80)\ (7,5,6,4,3,1,2,0), \\ (81)\ (3,1,7,5,2,0,6,4), & (82)\ (3,7,1,2,5,6,0,4), \\ (83)\ (7,3,5,6,1,2,4,0), & (84)\ (7,5,3,1,6,4,2,0), \end{array} \tag{31}$$

respectively;

(h) *if* $w_3 < w_2 < w_1 < w_0$, *then* $w_7 < w_6 < w_5 < w_4$, *and all the allowable orderings of eight numbers* $w_0,\ w_1,\ w_2,\ w_3,\ w_4,\ w_5,\ w_6$ *and* w_7 *are as follows*:

$$w_{i_0} < w_{i_1} < w_{i_2} < w_{i_3} < w_{i_4} < w_{i_5} < w_{i_6} < w_{i_7},$$

where $(i_0, i_1, i_2, i_3, i_4, i_5, i_6, i_7)$ *are*

$$\begin{array}{ll} (85)\ (3,2,1,0,7,6,5,4), & (86)\ (3,2,1,7,0,6,5,4), \\ (87)\ (3,2,7,1,6,0,5,4), & (88)\ (3,7,2,6,1,5,0,4), \\ (89)\ (7,3,6,2,5,1,4,0), & (90)\ (7,6,3,5,2,4,1,0), \\ (91)\ (7,6,5,3,4,2,1,0), & (92)\ (7,6,5,4,3,2,1,0), \\ (93)\ (3,2,7,6,1,0,5,4), & (94)\ (3,7,2,1,6,5,0,4), \\ (95)\ (7,3,6,5,2,1,4,0), & (96)\ (7,6,3,2,5,4,1,0), \end{array} \tag{32}$$

respectively.

Lemmas 3 and 4 can be easily proved, we omit them for simplicity.

Theorem 3. *Nine Boolean functions can be realized via the uncoupled CNN (7) by changing the threshold* z *for any allowable ordering of the eight numbers* w_k $(k = 0,1,2,3,4,5,6,7)$ *in Lemma 4, namely,*

$$w_{i_0} < w_{i_1} < w_{i_2} < w_{i_3} < w_{i_4} < w_{i_5} < w_{i_6} < w_{i_7}.$$

Proof. Firstly, nine open intervals can be given on $(-\infty, +\infty)$ according to the numbers w_{i_k} $(k = 0,1,2,3,4,5,6,7)$; they are:

$$I_0 = \left(-\infty, \frac{w_{i_0}}{2}\right), \quad I_1 = \left(\frac{w_{i_0}}{2}, \frac{w_{i_1}}{2}\right),$$

$$I_2 = \left(\frac{w_{i_1}}{2}, \frac{w_{i_2}}{2}\right), \quad I_3 = \left(\frac{w_{i_2}}{2}, \frac{w_{i_3}}{2}\right),$$

$$I_4 = \left(\frac{w_{i_3}}{2}, \frac{w_{i_4}}{2}\right), \quad I_5 = \left(\frac{w_{i_4}}{2}, \frac{w_{i_5}}{2}\right),$$

$$I_6 = \left(\frac{w_{i_5}}{2}, \frac{w_{i_6}}{2}\right), \quad I_7 = \left(\frac{w_{i_6}}{2}, \frac{w_{i_7}}{2}\right),$$

$$I_8 = \left(\frac{w_{i_7}}{2}, +\infty\right).$$

For every interval I_j $(j = 0,1,2,\ldots,8)$, and for any given $z_j,\ z_j \in I_j$, we can calculate $(b_1, b_2, b_3, b_4)^T$ by using the $w_0,\ w_1,\ w_2,\ w_4$ and z_j in formula (19), as follows:

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} = A^{-1} \begin{pmatrix} w_0 - z_j \\ w_1 - z_j \\ w_2 - z_j \\ w_4 - z_j \end{pmatrix}, \tag{33}$$

where A^{-1} is the inverse matrix of A in (12).

Clearly, from Lemma 4, we have

$$\begin{aligned} w_3 &= -w_0 + w_1 + w_2, & w_5 &= -w_0 + w_1 + w_4, \\ w_6 &= -w_0 + w_2 + w_4, & w_7 &= -w_0 + w_3 + w_4, \\ w_8 &= 2z_j - w_7, & w_9 &= 2z_j - w_6, \\ w_{10} &= 2z_j - w_5, & w_{11} &= 2z_j - w_4, \\ w_{12} &= 2z_j - w_3, & w_{13} &= 2z_j - w_2, \\ w_{14} &= 2z_j - w_1, & w_{15} &= 2z_j - w_0. \end{aligned}$$

Next, let

$$a = 1 + \frac{1}{2}\min\{|w_k| \mid k = 0,1,2,\ldots,15\}. \tag{34}$$

Then, we can construct the uncoupled CNN (7) by using the six numbers $b_1,\ b_2,\ b_3,\ b_4,\ z_j$ and a, its gene is given by

z_j	b_1	b_2	b_3	b_4	a

which can realize a Boolean function whose binary output pattern is

$$(\mathrm{sgn}(w_0), \mathrm{sgn}(w_1), \mathrm{sgn}(w_2), \ldots, \mathrm{sgn}(w_{15})).$$

The proof is thus completed. ■

Example 1. If $w_0 = 2,\ w_1 = 6,\ w_2 = 10,\ w_4 = 18$, then $w_3 = 14,\ w_5 = 22,\ w_6 = 26,\ w_7 = 30$. Consequently, $z_1 \in I_1 = (w_0/2, w_1/2) = (1, 3)$. Take

$z_1 = 2$, the center point of the interval I_1. Then,

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} = A^{-1} \begin{pmatrix} w_0 - z_1 \\ w_1 - z_1 \\ w_2 - z_1 \\ w_4 - z_1 \end{pmatrix} = A^{-1} \begin{pmatrix} 0 \\ 4 \\ 8 \\ 16 \end{pmatrix}$$

$$= \begin{pmatrix} -14 \\ 8 \\ 4 \\ 2 \end{pmatrix}, \tag{35}$$

and $w_8 = -26$, $w_9 = -22$, $w_{10} = -18$, $w_{11} = -14$, $w_{12} = -10$, $w_{13} = -6$, $w_{14} = -2$, $w_{15} = 2$, $a = 1 + (1/2)\min\{|w_k| \| k = 0, 1, 2, \ldots, 15\} = 2$. Therefore, we can design the uncoupled CNN (7) with gene

2	−14	8	4	2	2

and output pattern

$$\begin{aligned}(\mathrm{sgn}(w_0), \mathrm{sgn}(w_1), \ldots, \mathrm{sgn}(w_{15}))\\ = (1,1,1,1,1,1,1,1,\\ -1,-1,-1,-1,-1,-1,-1,1).\end{aligned} \tag{36}$$

Similarly, we can list the other eight genes and the corresponding output patterns, binary decoding tapes, and decimal codes of the CNN. Details are shown in Table 3.

Remark 1. If the output pattern of CNN (7) is $(v_0, v_1, v_2, \ldots, v_{15})$, then its binary decoding tape is $\overline{v}_0\overline{v}_1\overline{v}_2\cdots\overline{v}_{15}$, where $\overline{v}_k = 1$ if $v_k = 1$, $\overline{v}_k = 0$ if $v_k = -1$, and its decimal code is

$$p = \overline{v}_0 2^{14} + \overline{v}_1 2^{13} + \cdots + \overline{v}_{14} 2 + \overline{v}_{15}.$$

We should pay more attention to the values $w_0/2$, $w_1/2$, $w_2/2, \ldots, w_7/2$ in the above theorem and example. They are the bifurcation values that yield Boolean functions. This bifurcation phenomenon is generated by changing only one parameter, i.e. the threshold z of the CNN (7).

Next, we will classify all the allowable orderings of w_k $(k = 0, 1, 2, \ldots, 7)$ in Lemma 4 into nine classes where the sign of w_k is determined.

$$\begin{aligned}(\mathrm{I}): \quad & w_{i_0} < w_{i_1} < w_{i_2} < w_{i_3} < w_{i_4}\\ & < w_{i_5} < w_{i_6} < w_{i_7} < 0\end{aligned} \tag{37}$$

where $(i_0, i_1, \ldots, i_7)$ denotes all the 96 arrangements of (1) to (96) in Lemma 4;

$$\begin{aligned}(\mathrm{II}): \quad & w_{i_0} < w_{i_1} < w_{i_2} < w_{i_3} < w_{i_4}\\ & < w_{i_5} < w_{i_6} < 0 < w_{i_7}\end{aligned} \tag{38}$$

where $(i_0, i_1, \ldots, i_7)$ denotes all the 96 arrangements of (1) to (96) in Lemma 4;

$$\begin{aligned}(\mathrm{III}): \quad & w_{i_0} < w_{i_1} < w_{i_2} < w_{i_3} < w_{i_4}\\ & < w_{i_5} < 0 < w_{i_6} < w_{i_7}\end{aligned} \tag{39}$$

where $(i_0, i_1, \ldots, i_7)$ denotes all the 96 arrangements of (1) to (96) in Lemma 4;

$$\begin{aligned}(\mathrm{IV}): \quad & w_{i_0} < w_{i_1} < w_{i_2} < w_{i_3} < w_{i_4}\\ & < 0 < w_{i_5} < w_{i_6} < w_{i_7}\end{aligned} \tag{40}$$

where $(i_0, i_1, \ldots, i_7)$ denotes all the 96 arrangements of (1) to (96) in Lemma 4;

$$\begin{aligned}(\mathrm{V}): \quad & w_{i_0} < w_{i_1} < w_{i_2} < w_{i_3} < 0\\ & < w_{i_4} < w_{i_5} < w_{i_6} < w_{i_7}\end{aligned} \tag{41}$$

Table 3. Genes, output patterns, binary decoding tapes and decimal codes of Example 1.

Gene	Output Pattern	Binary Code Tape	Decimal Code
16 | 0 | 8 | 4 | 2 | 2	(1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1)	1111111111111111	65535
14 | −2 | 8 | 4 | 2 | 2	(1,1,1,1,1,1,1,1,−1,1,1,1,1,1,1,1)	1111111101111111	65407
12 | −4 | 8 | 4 | 2 | 2	(1,1,1,1,1,1,1,1,−1, −1,1,1,1,1,1,1)	1111111100111111	65343
10 | −6 | 8 | 4 | 2 | 2	(1,1,1,1,1,1,1,1,−1, −1, −1,1,1,1,1,1)	1111111100011111	65311
8 | −8 | 8 | 4 | 2 | 2	(1,1,1,1,1,1,1,1,−1, −1, −1, −1,1,1,1,1)	1111111100001111	65295
6 | −10 | 8 | 4 | 2 | 2	(1,1,1,1,1,1,1,1,−1, −1, −1, −1, −1,1,1,1)	1111111100000111	65287
4 | −12 | 8 | 4 | 2 | 2	(1,1,1,1,1,1,1,1,−1, −1, −1, −1, −1, −1,1,1)	1111111100000011	65283
2 | −14 | 8 | 4 | 2 | 2	(1,1,1,1,1,1,1,1,−1, −1, −1, −1, −1, −1, −1,1)	1111111100000001	65281
0 | −16 | 8 | 4 | 2 | 2	(1,1,1,1,1,1,1,1,−1, −1, −1, −1, −1, −1, −1, −1)	1111111100000000	65280

where $(i_0, i_1, \ldots, i_7)$ denotes all the 96 arrangements of (1) to (96) in Lemma 4;

$$\begin{aligned}\text{(VI)}: \quad & w_{i_0} < w_{i_1} < w_{i_2} < 0 < w_{i_3} \\ & < w_{i_4} < w_{i_5} < w_{i_6} < w_{i_7}\end{aligned} \tag{42}$$

where $(i_0, i_1, \ldots, i_7)$ denotes all the 96 arrangements of (1) to (96) in Lemma 4;

$$\begin{aligned}\text{(VII)}: \quad & w_{i_0} < w_{i_1} < 0 < w_{i_2} < w_{i_3} \\ & < w_{i_4} < w_{i_5} < w_{i_6} < w_{i_7}\end{aligned} \tag{43}$$

where $(i_0, i_1, \ldots, i_7)$ denotes all the 96 arrangements of (1) to (96) in Lemma 4;

$$\begin{aligned}\text{(VIII)}: \quad & w_{i_0} < 0 < w_{i_1} < w_{i_2} < w_{i_3} \\ & < w_{i_4} < w_{i_5} < w_{i_6} < w_{i_7}\end{aligned} \tag{44}$$

where $(i_0, i_1, \ldots, i_7)$ denotes all the 96 arrangements of (1) to (96) in Lemma 4;

$$\begin{aligned}\text{(IX)}: \quad & 0 < w_{i_0} < w_{i_1} < w_{i_2} < w_{i_3} \\ & < w_{i_4} < w_{i_5} < w_{i_6} < w_{i_7}\end{aligned} \tag{45}$$

where $(i_0, i_1, \ldots, i_7)$ denotes all the 96 arrangements of (1) to (96) in Lemma 4.

Lemma 5. *For every one of the above ordering classes of w_k $(k = 0, 1, 2, \ldots, 7)$, property A of Lemma 4 holds. Moreover,*

(i) *for a pair of allowable orders* (a) *and* (b) *in the same class, if the order* (b) *is obtained by exchanging one pair of numbers of* (a), *i.e.*

$$\begin{cases}\text{(a)}\ w_{i_0} < w_{i_1} < w_{i_2} < w_{i_3} < w_{i_4} \\ \qquad < w_{i_5} < w_{i_6} < w_{i_7} \\ \text{(b)}\ w_{i_0} < w_{i_1} < w_{i_2} < \underline{w_{i_4} < w_{i_3}} \\ \qquad < w_{i_5} < w_{i_6} < w_{i_7}\end{cases} \tag{46}$$

then, according to Theorem 3, (b) *can only realize a new Boolean function that is different from the Boolean functions yielded by* (a). *This situation is denoted by*

$$\text{(a)} \overset{1}{\text{——}} \text{(b)}; \tag{47}$$

(ii) *for a pair of allowable orders* (a) *and* (b) *in the same class, if the order* (b) *is obtained by exchanging two pairs of numbers of* (a), *i.e.*

$$\begin{cases}\text{(a)}\ w_{i_0} < w_{i_1} < w_{i_2} < w_{i_3} < w_{i_4} \\ \qquad < w_{i_5} < w_{i_6} < w_{i_7} \\ \text{(b)}\ w_{i_0} < w_{i_1} < \underline{w_{i_3} < w_{i_2}} < \underline{w_{i_5} < w_{i_4}} \\ \qquad < w_{i_6} < w_{i_7}\end{cases} \tag{48}$$

or

$$\begin{cases}\text{(a)}\ w_{i_0} < w_{i_1} < w_{i_2} < w_{i_3} < w_{i_4} \\ \qquad < w_{i_5} < w_{i_6} < w_{i_7} \\ \text{(b)}\ w_{i_0} < \underline{w_{i_2} < w_{i_1}} < w_{i_3} < w_{i_4} \\ \qquad < \underline{w_{i_6} < w_{i_5}} < w_{i_7}\end{cases} \tag{49}$$

then, according to Theorem 3, (b) *can only realize two new Boolean functions that are different from the Boolean functions yielded by* (a). *This situation is denoted by*

$$\text{(a)} \overset{2}{\text{——}} \text{(b)}; \tag{50}$$

(iii) *for a pair of allowable orders* (a) *and* (b) *in the same class, if the order* (b) *is obtained by exchanging three pairs of numbers of* (a), *i.e.*

$$\begin{cases}\text{(a)}\ w_{i_0} < w_{i_1} < w_{i_2} < w_{i_3} < w_{i_4} \\ \qquad < w_{i_5} < w_{i_6} < w_{i_7} \\ \text{(b)}\ w_{i_0} < \underline{w_{i_2} < w_{i_1}} < \underline{w_{i_4} < w_{i_3}} \\ \qquad < \underline{w_{i_6} < w_{i_5}} < w_{i_7}\end{cases} \tag{51}$$

then, according to Theorem 3, (b) *can only realize three new Boolean functions that are different from the Boolean functions yielded by* (a). *This situation is denoted by*

$$\text{(a)} \overset{3}{\text{——}} \text{(b)}; \tag{52}$$

(iv) *for a pair of allowable orders* (a) *and* (b) *in the same class, if the order* (b) *is obtained by exchanging four of pairs numbers of* (a), *i.e.*

$$\begin{cases}\text{(a)}\ w_{i_0} < w_{i_1} < w_{i_2} < w_{i_3} < w_{i_4} \\ \qquad < w_{i_5} < w_{i_6} < w_{i_7} \\ \text{(b)}\ \underline{w_{i_1} < w_{i_0}} < \underline{w_{i_3} < w_{i_2}} < \underline{w_{i_5} < w_{i_4}} \\ \qquad < \underline{w_{i_7} < w_{i_6}}\end{cases} \tag{53}$$

then, according to Theorem 3, (b) *can only realize four new Boolean functions that are different from the Boolean functions yielded by* (a). *This situation is denoted by*

$$\text{(a)} \overset{4}{\text{——}} \text{(b)}; \tag{54}$$

(v) *for a pair of allowable orders* (a) *and* (b) *in the same class, assume that they can be indicated in the following form*:

$$\begin{cases}\text{(a)}\ w_{i_0} < \cdots < w_{i_p} < w_{i_{p+1}} < \cdots < w_{i_7} \\ \text{(b)}\ w_{j_1} < \cdots < w_{j_p} < w_{j_{p+1}} < \cdots < w_{j_7}\end{cases} \tag{55}$$

where

$$\{i_1,\ldots,i_p\}\cup\{i_{p+1},\ldots,i_7\}$$
$$=\{j_1,\ldots,j_p\}\cup\{j_{p+1},\ldots,j_7\}$$
$$=\{0,1,2,3,4,5,6,7\},$$

and the subscript sets $\{j_1,\ldots,j_p\}$ *are a permutation of* $\{i_1,\ldots,i_p\}$ *and* $\{j_{p+1},\ldots,j_7\}$ *are a permutation of* $\{i_{p+1},\ldots,i_7\}$*, respectively, then* (a) *and* (b) *will realize the same output pattern when* $z\in(w_{i_p}/2,w_{i_{p+1}}/2)$ *in* (a) *and* $z\in(w_{j_p}/2,w_{j_{p+1}}/2)$ *in* (b)*.*

Remark 2. It is allowable that one subscript set is an empty set.

Proof. (i) Firstly, the signs of w_k $(k=0,1,2,\ldots,7)$ are completely determined in the order pair (a) and (b), because they are in the same order class (I) to (IX). If $z\in(w_{i_3}/2,w_{i_4}/2)$ in (a), then $\mathrm{sgn}(2z-w_{i_3})=1$, $\mathrm{sgn}(2z-w_{i_4})=-1$; if $z\in(w_{i_4}/2,w_{i_3}/2)$ in (b), then $\mathrm{sgn}(2z-w_{i_3})=-1$, $\mathrm{sgn}(2z-w_{i_4})=1$; but all the signs of $(2z-w_{i_k})$ $(k=0,1,\ldots,7)$ are not changed regardless of (a) or (b) when z belongs to the remaining eight intervals appeared in the proof of Theorem 3. Thus, only one output pattern of (b) is different from those of (a).

The proofs of (ii)–(iv) are similar.

For (v), we only note that when $z\in(w_{i_p}/2,\ w_{i_{p+1}}/2)$ in (a) and $z\in(w_{j_p}/2,\ w_{j_{p+1}}/2)$ in (b), all signs of $(2z-w_{i_k})$ $(k=0,1,\ldots,7)$ are fixed. The proof of the Lemma is thus completed. ■

For convenience, we denote the linearly separable Boolean functions as LSBF, and the linearly separable Boolean genes as LSBG, in the following.

Lemma 6

(i) *The ordering class* (I) *can realize* 104 *LSBF via the uncoupled CNN* (*7*);

(ii) *the ordering class* (II) *can realize* 160 *LSBF via the uncoupled CNN* (*7*);

(iii) *the ordering class* (III) *can realize* 240 *LSBF via the uncoupled CNN* (*7*);

(iv) *the ordering class* (IV) *can realize* 288 *LSBF via the uncoupled CNN* (*7*);

(v) *the ordering class* (V) *can realize* 298 *LSBF via the uncoupled CNN* (*7*);

(vi) *the ordering class* (VI) *can realize* 288 *LSBF via the uncoupled CNN* (*7*);

(vii) *the ordering class* (VII) *can realize* 240 *LSBF via the uncoupled CNN* (*7*);

(viii) *the ordering class* (VIII) *can realize* 160 *LSBF via the uncoupled CNN* (*7*);

(ix) *the ordering class* (IX) *can realize* 104 *LSBF via the uncoupled CNN* (*7*)*.*

Proof

(i) The class (I) is

$$w_{i_0}<w_{i_1}<w_{i_2}<w_{i_3}<w_{i_4}$$
$$<w_{i_5}<w_{i_6}<w_{i_7}<0.$$

If we use the notations of Lemma 5, then there are some related chains of the orderings in Lemma 4, namely,

$$\text{(a)}:(1)\overset{1}{\text{——}}(2)\overset{2}{\text{——}}(3)\overset{3}{\text{——}}(4)\overset{4}{\text{——}}(5)\overset{3}{\text{——}}(6)\overset{2}{\text{——}}(7)\overset{1}{\text{——}}(8),$$

Applying Theorem 3 and Lemma 5, we know that the order (1) can realize 9 LSBF. The other orders in the chain, (2)–(8), can yield 16 new LSBF. But other orders out of the chain, (9)–(12), cannot yield any new LSBF, because the LSBFs realized do appear in the chain. Thus, the whole chain (a) can generate 25 LSBF.

Further, we have other chains of w_k as shown below:

$$\text{(b)}:\ \underline{(1)}\overset{2}{\text{——}}(13)\text{——}(14)\overset{2}{\text{——}}(15)\overset{1}{\text{——}}(16)\overset{2}{\text{——}}(17)\text{——}(18)\text{——}(19)\text{——}(20);$$

$$\text{(c)}:\ \underline{(13)}\overset{4}{\text{——}}(25)\overset{1}{\text{——}}(26)\overset{2}{\text{——}}(27)\overset{2}{\text{——}}(28)\overset{4}{\text{——}}(29)\overset{1}{\text{——}}(30)\overset{2}{\text{——}}(31)\text{——}(32);$$

$$\text{(d)}:\ \underline{(25)}\overset{2}{\text{——}}(37)\text{——}(38)\overset{2}{\text{——}}(39)\overset{1}{\text{——}}(40)\overset{2}{\text{——}}(41)\overset{2}{\text{——}}(42)\text{——}(43)\text{——}(44);$$

$$\text{(e)}:\ \underline{(1)}\overset{4}{\text{——}}(49)\overset{1}{\text{——}}(50)\overset{2}{\text{——}}(51)\overset{2}{\text{——}}(52)\overset{4}{\text{——}}(53)\overset{1}{\text{——}}(54)\overset{2}{\text{——}}(55)\text{——}(56);$$

$$\text{(f)}: \ \underline{(49)} \overset{2}{\text{——}} (61) \text{——} (62) \overset{2}{\text{——}} (63) \overset{1}{\text{——}} (64) \overset{2}{\text{——}} (65) \overset{2}{\text{——}} (66) \text{——} (67) \text{——} (68);$$

$$\text{(g)}: \ \underline{(61)} \overset{4}{\text{——}} (73) \overset{1}{\text{——}} (74) \overset{2}{\text{——}} (75) \overset{2}{\text{——}} (76) \overset{4}{\text{——}} (77) \overset{1}{\text{——}} (78) \overset{2}{\text{——}} (79) \text{——} (80);$$

$$\text{(h)}: \ \underline{(73)} \text{——} (85) \text{——} (86) \overset{2}{\text{——}} (87) \text{——} (88) \overset{2}{\text{——}} (89) \text{——} (90) \text{——} (91) \text{——} (92). \quad \blacksquare$$

Remark 3. The underline above indicates that the order has appeared in the preceding orders. If no script on the midline between two orders, it implies that the latter cannot yield a new LSBF.

Applying Theorem 3 and Lemma 5 repeatedly, we can see that the chain (b) can realize 9 LSBF, the chain (c) can realize 16 LSBF, and likewise, (d) : 9, (e) : 16, (f) : 9, (g) : 16 and (h) : 4, respectively. Thus, the class (I) can realize a total of 104 LSBF.

(ii) Similar to (i), for class (II):

$$w_{i_0} < w_{i_1} < w_{i_2} < w_{i_3} < w_{i_4} < w_{i_5} < w_{i_6} < 0 < w_{i_7}.$$

For the pair (a) and (b) in Lemma 4, we have two related chains:

$$(4) \overset{3}{\text{——}} (3) \overset{2}{\text{——}} (2) \overset{1}{\text{——}} (1) \overset{2}{\text{——}} (13) \text{——} (14) \overset{2}{\text{——}} (15) \overset{1}{\text{——}} (16)$$

and

$$(5) \overset{3}{\text{——}} (6) \overset{2}{\text{——}} (7) \overset{1}{\text{——}} (8) \overset{2}{\text{——}} (20) \text{——} (19) \overset{2}{\text{——}} (18) \overset{1}{\text{——}} (17);$$

for the pair (c) and (d) in Lemma 4, we have

$$(28) \overset{3}{\text{——}} (27) \overset{2}{\text{——}} (26) \overset{1}{\text{——}} (25) \overset{2}{\text{——}} (37) \text{——} (38) \overset{2}{\text{——}} (39) \overset{1}{\text{——}} (40)$$

and

$$(29) \overset{3}{\text{——}} (30) \overset{2}{\text{——}} (31) \overset{1}{\text{——}} (32) \overset{2}{\text{——}} (44) \text{——} (43) \overset{2}{\text{——}} (42) \overset{1}{\text{——}} (41);$$

for the pair (e) and (f) in Lemma 4, we have

$$(52) \overset{3}{\text{——}} (51) \overset{2}{\text{——}} (50) \overset{1}{\text{——}} (49) \overset{2}{\text{——}} (61) \text{——} (62) \overset{2}{\text{——}} (63) \overset{1}{\text{——}} (64)$$

and

$$(53) \overset{3}{\text{——}} (54) \overset{2}{\text{——}} (55) \overset{1}{\text{——}} (56) \overset{2}{\text{——}} (68) \text{——} (67) \overset{2}{\text{——}} (66) \overset{1}{\text{——}} (65);$$

for the pair (g) and (h) in Lemma 4, we have

$$(76) \overset{3}{\text{——}} (75) \overset{2}{\text{——}} (74) \overset{1}{\text{——}} (73) \overset{2}{\text{——}} (85) \text{——} (86) \overset{2}{\text{——}} (87) \overset{1}{\text{——}} (88)$$

and

$$(77) \overset{3}{\text{——}} (78) \overset{2}{\text{——}} (79) \overset{1}{\text{——}} (80) \overset{2}{\text{——}} (92) \text{——} (91) \overset{2}{\text{——}} (90) \overset{1}{\text{——}} (89).$$

Based on Theorem 3 and Lemma 5, each pair of orders can yield 40 LSBF. Thus, the class (II) can realize exactly 160 LSBF.

Remark 4. Other orders out of the above chains cannot yield any new LSBF.

(iii) Class (III) is:

$$w_{i_0} < w_{i_1} < w_{i_2} < w_{i_3} < w_{i_4} < w_{i_5} < 0 < w_{i_6} < w_{i_7}.$$

We have 12 chains of the orders shown in Lemma 4:

$$(9)\overset{1}{\text{---}}(3)\overset{2}{\text{---}}(2)\overset{1}{\text{---}}(1)\overset{4}{\text{---}}(49)\overset{1}{\text{---}}(50)\overset{2}{\text{---}}(51)\text{---}(57),$$

$$(12)\overset{1}{\text{---}}(6)\overset{2}{\text{---}}(7)\overset{1}{\text{---}}(8)\overset{4}{\text{---}}(56)\overset{1}{\text{---}}(55)\overset{2}{\text{---}}(54)\text{---}(60),$$

$$(16)\overset{1}{\text{---}}(22)\overset{2}{\text{---}}(10)\overset{1}{\text{---}}(4)\overset{4}{\text{---}}(5)\overset{1}{\text{---}}(11)\overset{2}{\text{---}}(23)\text{---}(17),$$

$$(21)\overset{1}{\text{---}}(15)\overset{2}{\text{---}}(14)\overset{1}{\text{---}}(13)\overset{4}{\text{---}}(25)\overset{1}{\text{---}}(26)\overset{2}{\text{---}}(27)\text{---}(33),$$

$$(24)\overset{1}{\text{---}}(18)\overset{2}{\text{---}}(19)\overset{1}{\text{---}}(20)\overset{4}{\text{---}}(32)\overset{1}{\text{---}}(31)\overset{2}{\text{---}}(30)\text{---}(36),$$

$$(40)\overset{1}{\text{---}}(46)\overset{2}{\text{---}}(34)\overset{1}{\text{---}}(28)\overset{4}{\text{---}}(29)\overset{1}{\text{---}}(35)\overset{2}{\text{---}}(47)\text{---}(41),$$

$$(44)\overset{1}{\text{---}}(43)\overset{2}{\text{---}}(42)\overset{1}{\text{---}}(48)\overset{4}{\text{---}}(96)\overset{1}{\text{---}}(90)\overset{2}{\text{---}}(91)\text{---}(92),$$

$$(45)\overset{1}{\text{---}}(39)\overset{2}{\text{---}}(38)\overset{1}{\text{---}}(37)\overset{4}{\text{---}}(85)\overset{1}{\text{---}}(86)\overset{2}{\text{---}}(87)\text{---}(93),$$

$$(52)\overset{1}{\text{---}}(58)\overset{2}{\text{---}}(70)\overset{1}{\text{---}}(64)\overset{4}{\text{---}}(65)\overset{1}{\text{---}}(71)\overset{2}{\text{---}}(59)\text{---}(53),$$

$$(68)\overset{1}{\text{---}}(67)\overset{2}{\text{---}}(66)\overset{1}{\text{---}}(72)\overset{4}{\text{---}}(84)\overset{1}{\text{---}}(78)\overset{2}{\text{---}}(79)\text{---}(80),$$

$$(69)\overset{1}{\text{---}}(63)\overset{2}{\text{---}}(62)\overset{1}{\text{---}}(61)\overset{4}{\text{---}}(73)\overset{1}{\text{---}}(74)\overset{2}{\text{---}}(75)\text{---}(81),$$

$$(88)\overset{1}{\text{---}}(94)\overset{2}{\text{---}}(82)\overset{1}{\text{---}}(76)\overset{4}{\text{---}}(77)\overset{1}{\text{---}}(83)\overset{2}{\text{---}}(95)\text{---}(89).$$

Every one of the above chains is independent. By Theorem 3 and Lemma 5, each chain can yield 20 LSBF, so that situation (III) can realize a total of 240 LSBF.

(iv) Class (IV) is:

$$w_{i_0} < w_{i_1} < w_{i_2} < w_{i_3} < w_{i_4} < 0 < w_{i_5} < w_{i_6} < w_{i_7}.$$

At this time, there are 24 independent chains of the orders shown in Lemma 4. They are:

$$(1)\overset{1}{\text{---}}(2)\overset{2}{\text{---}}(14)\text{---}(13), \quad (3)\overset{3}{\text{---}}(4)\text{---}(10)\text{---}(9),$$

$$(5)\overset{3}{\text{---}}(6)\text{---}(12)\text{---}(11), \quad (7)\overset{1}{\text{---}}(8)\overset{2}{\text{---}}(20)\text{---}(19),$$

$$(15)\overset{3}{\text{---}}(16)\text{---}(22)\text{---}(21), \quad (17)\overset{3}{\text{---}}(18)\text{---}(24)\text{---}(23),$$

$$(25)\overset{1}{\text{---}}(26)\overset{2}{\text{---}}(38)\text{---}(37), \quad (27)\overset{3}{\text{---}}(28)\text{---}(34)\text{---}(33),$$

$$(29)\overset{3}{\text{---}}(30)\text{---}(36)\text{---}(35), \quad (31)\overset{1}{\text{---}}(32)\overset{2}{\text{---}}(44)\text{---}(43),$$

$$(39)\overset{3}{\text{---}}(40)\text{---}(46)\text{---}(45), \quad (41)\overset{3}{\text{---}}(42)\text{---}(48)\text{---}(47),$$

$$(49)\overset{1}{\text{---}}(50)\overset{2}{\text{---}}(62)\text{---}(61), \quad (51)\overset{3}{\text{---}}(52)\text{---}(58)\text{---}(57),$$

$$(53)\overset{3}{\text{---}}(54)\text{---}(60)\text{---}(59), \quad (55)\overset{1}{\text{---}}(56)\overset{2}{\text{---}}(68)\text{---}(67),$$

$$(63)\overset{3}{\text{---}}(64)\text{---}(70)\text{---}(69), \quad (65)\overset{3}{\text{---}}(66)\text{---}(72)\text{---}(71),$$

$$(73)\overset{1}{\text{---}}(74)\overset{2}{\text{---}}(86)\text{---}(85), \quad (75)\overset{3}{\text{---}}(76)\text{---}(82)\text{---}(81),$$

$$(77) \overset{3}{\text{——}} (78) \text{——} (84) \text{——} (83), \quad (79) \overset{1}{\text{——}} (80) \overset{2}{\text{——}} (92) \text{——} (91),$$

$$(87) \overset{3}{\text{——}} (88) \text{——} (94) \text{——} (93), \quad (89) \overset{3}{\text{——}} (90) \text{——} (96) \text{——} (95).$$

From Theorem 3 and Lemma 5, this situation can realize exactly 288 LSBF.

(v) Class (V) is:

$$w_{i_0} < w_{i_1} < w_{i_2} < w_{i_3} < 0 < w_{i_4} < w_{i_5} < w_{i_6} < w_{i_7}.$$

Similar to the preceding classes, there are 14 independent order chains in Lemma 4. They are:

$$(37) \overset{2}{\text{——}} (25) \overset{4}{\text{——}} (13) \overset{2}{\text{——}} (1) \overset{4}{\text{——}} (49) \overset{2}{\text{——}} (61) \overset{4}{\text{——}} (73) \text{——} (85),$$

$$(12) \overset{2}{\text{——}} (5) \overset{4}{\text{——}} (4) \overset{2}{\text{——}} (9) \overset{4}{\text{——}} (57) \overset{2}{\text{——}} (52) \overset{4}{\text{——}} (53) \text{——} (60),$$

$$(44) \overset{2}{\text{——}} (32) \overset{4}{\text{——}} (20) \overset{2}{\text{——}} (8) \overset{4}{\text{——}} (56) \overset{2}{\text{——}} (68) \overset{4}{\text{——}} (80) \text{——} (92),$$

$$(29) \overset{2}{\text{——}} (36) \overset{4}{\text{——}} (24) \overset{2}{\text{——}} (17) \overset{4}{\text{——}} (16) \overset{2}{\text{——}} (21) \overset{4}{\text{——}} (33) \text{——} (28),$$

$$(48) \overset{2}{\text{——}} (41) \overset{4}{\text{——}} (40) \overset{2}{\text{——}} (45) \overset{4}{\text{——}} (93) \overset{2}{\text{——}} (88) \overset{4}{\text{——}} (89) \text{——} (96),$$

$$(72) \overset{2}{\text{——}} (65) \overset{4}{\text{——}} (64) \overset{2}{\text{——}} (69) \overset{4}{\text{——}} (81) \overset{2}{\text{——}} (76) \overset{4}{\text{——}} (77) \text{——} (84),$$

$$(10) \overset{2}{\text{——}} (3) \overset{2}{\text{——}} (2) \overset{2}{\text{——}} (14) \overset{2}{\text{——}} (15) \text{——} (22),$$

$$(11) \overset{2}{\text{——}} (6) \overset{2}{\text{——}} (7) \overset{2}{\text{——}} (19) \overset{2}{\text{——}} (18) \text{——} (23),$$

$$(34) \overset{2}{\text{——}} (27) \overset{2}{\text{——}} (26) \overset{2}{\text{——}} (38) \overset{2}{\text{——}} (39) \text{——} (46),$$

$$(58) \overset{2}{\text{——}} (51) \overset{2}{\text{——}} (50) \overset{2}{\text{——}} (62) \overset{2}{\text{——}} (63) \text{——} (70),$$

$$(35) \overset{2}{\text{——}} (30) \overset{2}{\text{——}} (31) \overset{2}{\text{——}} (43) \overset{2}{\text{——}} (42) \text{——} (47),$$

$$(59) \overset{2}{\text{——}} (54) \overset{2}{\text{——}} (55) \overset{2}{\text{——}} (67) \overset{2}{\text{——}} (66) \text{——} (71),$$

$$(82) \overset{2}{\text{——}} (75) \overset{2}{\text{——}} (74) \overset{2}{\text{——}} (86) \overset{2}{\text{——}} (87) \text{——} (94),$$

$$(83) \overset{2}{\text{——}} (78) \overset{2}{\text{——}} (79) \overset{2}{\text{——}} (91) \overset{2}{\text{——}} (90) \text{——} (95).$$

Thus, these chains can yield exactly 298 LSBF according to Theorem 3 and Lemma 5.

As to the classes (VI)–(IX), since the signs of the orders of these classes are symmetric with respect to (IV), (III), (II) and (I), respectively, they can realize 288, 240, 160 and 104 LSBF, respectively.

The proof of Lemma 6 is thus completed.

From Lemma 6, we immediately obtain the following realization theorem for LSBF:

Theorem 4. *There are only* 1882 *linearly separable Boolean functions or Boolean genes of four variables*, *i.e.* 1882 *Boolean functions can be realized by a standard uncoupled CNN of four input variables.*

We will list the binary decoding tapes, decimal codes and genes of output patterns of classes (I) and (IX) among these 1882 LSBF in the next section, leaving the rest to a separate supplement [Chen & Chen, 2004].

4. Genes, Binary Decoding Tapes and Decimal Codes of LSBF

Based on the analysis and results given in the preceding sections, we are now able to realize exactly 1882 LSBF via the CNN (7), and calculate the corresponding genes, binary decoding tapes and decimal codes. The procedure is as follows.

Step 1. Take three different real numbers, w_0, w_1 and w_2, and let $w_3 = -w_0 + w_1 + w_2$. After that, take another real number w_4 such that $w_4 \neq w_i$ $(i = 0, 1, 2, 3)$, and let $w_5 = -w_0 + w_1 + w_4$, $w_6 = -w_0 + w_2 + w_4$, $w_7 = -w_0 + w_3 + w_4$. Then, the eight numbers w_k $(k = 0, 1, 2, \ldots, 7)$ form one of the 96 orders in the nine classes shown in Lemma 4:

$$w_{i_0} < w_{i_1} < w_{i_2} < w_{i_3} < w_{i_4} < w_{i_5} < w_{i_6} < w_{i_7}.$$

Step 2. Divide $(-\infty, +\infty)$ into nine open intervals by $w_{i_j}/2$ $(j = 0, 1, 2, \ldots, 7)$, as follows:

$$I_0 = \left(-\infty, \frac{w_{i_0}}{2}\right), \quad I_1 = \left(\frac{w_{i_0}}{2}, \frac{w_{i_1}}{2}\right),$$
$$I_2 = \left(\frac{w_{i_1}}{2}, \frac{w_{i_2}}{2}\right), \quad I_3 = \left(\frac{w_{i_2}}{2}, \frac{w_{i_3}}{2}\right),$$
$$I_4 = \left(\frac{w_{i_3}}{2}, \frac{w_{i_4}}{2}\right), \quad I_5 = \left(\frac{w_{i_4}}{2}, \frac{w_{i_5}}{2}\right),$$
$$I_6 = \left(\frac{w_{i_5}}{2}, \frac{w_{i_6}}{2}\right), \quad I_7 = \left(\frac{w_{i_6}}{2}, \frac{w_{i_7}}{2}\right),$$
$$I_8 = \left(\frac{w_{i_7}}{2}, +\infty\right).$$

Step 3. For every interval I_j $(j = 0, 1, 2, \ldots, 7)$, take a number $z_j \in I_j$ and calculate b_1, b_2, b_3 and b_4 by using w_0, w_1, w_2, w_4 and z_j in the formula (19):

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} = A^{-1} \begin{pmatrix} w_0 - z_j \\ w_1 - z_j \\ w_2 - z_j \\ w_4 - z_j \end{pmatrix}.$$

Then, let

$$w_8 = 2z_j - w_7, \quad w_9 = 2z_j - w_6,$$
$$w_{10} = 2z_j - w_5, \quad w_{11} = 2z_j - w_4,$$
$$w_{12} = 2z_j - w_3, \quad w_{13} = 2z_j - w_2,$$
$$w_{14} = 2z_j - w_1, \quad w_{15} = 2z_j - w_0.$$

At last, calculate the self-feedback coefficient a:

$$a = 1 + \frac{1}{2}\min\{|w_k| | k = 0, 1, 2, \ldots, 15\}.$$

Thus, the six numbers, z_j, b_1, b_2, b_3, b_4 and a, constitute a gene of CNN (7):

z_j	b_1	b_2	b_3	b_4	a

Step 4. Calculate the output pattern of the corresponding gene in Step 3:

$$(v_0, v_1, v_2, \ldots, v_{15}) = (\mathrm{sgn}(w_0), \mathrm{sgn}(w_1), \mathrm{sgn}(w_2), \ldots, \mathrm{sgn}(w_{15}))$$

and its binary decoding tapes $\overline{v}_0\overline{v}_1\overline{v}_2 \cdots \overline{v}_{15}$ and decimal code

$$p = \overline{v}_0 2^{14} + \overline{v}_1 2^{13} + \cdots + \overline{v}_{14} 2 + \overline{v}_{15},$$

where $\overline{v}_k = 1$ if $v_k = 1$, $\overline{v}_k = 0$ if $v_k = -1$.

Remark 5. (1) It is easy to see that we can obtain $96 \times 9 \times 9 = 7776$ CNN genes from Steps 1 to 4, based on Theorem 3, but there are only 1882 output pattern of the CNN (7) according to Lemma 5 and Theorem 4. In other words, different genes can connect to the same output pattern. In such a situation, we only take a gene as the representative of the output pattern in the gene bank of the CNN (7).

(2) To be more precise, in general, eight numbers w_i $(i = 0, 1, 2, 3, 4, 5, 6, 7)$ are taken as even numbers, such that $a = 1 + (1/2)\min\{|w_k| | k = 0, 1, 2, \ldots, 15\} = 2$, and let z_j be the center value of the interval I_j $(j \neq 0, 8)$.

(3) The gene (or template) design method described above in this paper may be referred to as a threshold bifurcation method, which is different from the nine design tools provided in [Chua *et al.*, 2002]. Obviously, the method presented here is more mathematically rigorous.

It would be desirable to list all the genes, binary decoding tapes, and decimal codes of the 1882 LSBF that can be realized via CNN. However, since the list is too long, only two classes (I) and (IX) will be shown in Tables 4 and 5, respectively, for demonstration. The rest will be supplied elsewhere [Chen & Chen, 2004].

Table 4. Genes, binary decoding tapes, and decimal codes of LSBF of class (I).

No.	Decimal Code	Gene: z	b_1	b_2	b_3	b_4	a	Binary Decoding Tape
1	0	−16	0	8	4	2	2	0000000000000000
2	1	−14	2	8	4	2	2	0000000000000001
3	2	−14	2	8	4	−2	2	0000000000000010
4	3	−12	2	8	4	2	2	0000000000000011
5	4	−14	2	8	−2	4	2	0000000000000100
6	5	−12	4	8	2	4	2	0000000000000101
7	7	−10	6	8	4	2	2	0000000000000111
8	8	−14	2	8	−2	−4	2	0000000000001000
9	10	−12	4	8	2	−4	2	0000000000001010
10	11	−10	6	8	4	−2	2	0000000000001011
11	12	−12	4	8	−4	2	2	0000000000001100
12	13	−10	6	8	−2	4	2	0000000000001101
13	14	−10	6	8	−2	−4	2	0000000000001110
14	15	−8	8	8	4	2	2	0000000000001111
15	16	−14	2	−2	8	2	2	0000000000010000
16	17	−12	4	2	8	2	2	0000000000010001
17	19	−12	8	6	8	4	2	0000000000010011
18	21	−12	8	6	4	8	2	0000000000010101
19	23	−12	12	10	8	4	2	0000000000010111
20	31	−6	10	8	4	2	2	0000000000011111
21	32	−14	2	−2	8	4	2	0000000000100000
22	34	−12	4	2	8	−4	2	0000000000100010
23	35	−12	8	6	8	−4	2	0000000000100011
24	42	−12	4	6	4	−4	2	0000000000101010
25	43	−12	12	10	8	−4	2	0000000000101011
26	47	−6	10	8	4	−2	2	0000000000101111
27	48	−14	6	−6	8	2	2	0000000000110000
28	49	−10	6	−2	8	2	2	0000000000110001
29	50	−10	6	−2	8	4	2	0000000000110010
30	51	−8	8	2	8	2	2	0000000000110011
31	55	−8	12	6	8	4	2	0000000000110111
32	59	−8	12	6	8	−4	2	0000000000111011
33	63	−4	12	8	4	2	2	0000000000111111
34	64	−14	2	−2	−4	8	2	0000000001000000
35	68	−12	4	2	−4	8	2	0000000001000100

Table 4. (*Continued*)

No.	Decimal Code	Gene: z	b_1	b_2	b_3	b_4	a	Binary Decoding Tape
36	69	−12	8	6	−4	8	2	0000000001000101
37	76	−12	8	6	−8	4	2	0000000001001100
38	77	−12	12	10	−4	8	2	0000000001001101
39	79	−6	10	8	−2	4	2	0000000001001111
40	80	−14	6	−6	4	8	2	0000000001010000
41	81	−10	6	−2	4	8	2	0000000001010001
42	84	−10	6	−2	−4	8	2	0000000001010100
43	85	−8	8	2	4	8	2	0000000001010101
44	87	−8	12	6	4	8	2	0000000001010111
45	93	−8	12	6	−4	8	2	0000000001011101
46	95	−4	12	8	2	4	2	0000000001011111
47	112	−14	10	−10	8	2	2	0000000001110000
48	113	−10	10	−6	8	2	2	0000000001110001
49	115	−6	10	−2	8	2	2	0000000001110011
50	117	−6	10	−2	4	8	2	0000000001110101
51	119	−4	12	2	8	2	2	0000000001110111
52	127	−2	14	8	4	2	2	0000000001111111
53	128	−14	2	−2	−4	−8	2	0000000010000000
54	136	−10	6	2	−4	−8	2	0000000010001000
55	138	−12	8	6	−4	−8	2	0000000010001010
56	140	−12	8	6	−8	−4	2	0000000010001100
57	142	−12	12	10	−4	−8	2	0000000010001110
58	143	−6	10	8	−2	−4	2	0000000010001111
59	160	−14	6	−6	4	8	2	0000000010100000
60	162	−10	6	−2	4	−8	2	0000000010100010
61	168	−10	6	−2	−4	−8	2	0000000010101000
62	170	−8	8	2	4	−8	2	0000000010101010
63	171	−8	8	6	4	−4	2	0000000010101011
64	174	−8	12	6	−4	−8	2	0000000010101110
65	175	−4	12	8	2	−4	2	0000000010101111
66	176	−14	10	−10	8	−4	2	0000000010110000
67	178	−10	10	−6	8	−4	2	0000000010110010
68	179	−6	10	−2	8	4	2	0000000010110011
69	186	−6	10	−2	4	−8	2	0000000010111010
70	187	−4	12	2	8	−4	2	0000000010111011
71	191	−2	14	8	4	−2	2	0000000010111111

Table 4. (*Continued*)

No.	Decimal Code	Gene: z	b_1	b_2	b_3	b_4	a	Binary Decoding Tape
72	192	−14	6	−6	−8	4	2	0000000011000000
73	196	−10	6	−2	−8	4	2	0000000011000100
74	200	−10	6	−2	−8	−4	2	0000000011001000
75	204	−8	8	2	−8	4	2	0000000011001100
76	205	−8	12	6	−8	4	2	0000000011001101
77	206	−8	12	6	−8	−4	2	0000000011001110
78	207	−4	12	8	−4	2	2	0000000011001111
79	208	−14	10	−10	−4	8	2	0000000011010000
80	212	−10	10	−6	−4	8	2	0000000011010100
81	213	−6	10	−2	−4	8	2	0000000011010101
82	220	−6	10	−2	−8	4	2	0000000011011100
83	221	−4	12	2	−4	8	2	0000000011011101
84	223	−2	14	8	−2	4	2	0000000011011111
85	224	−14	10	−10	−4	−8	2	0000000011100000
86	232	−10	10	−6	−4	−8	2	0000000011101000
87	234	−6	10	−2	−4	−8	2	0000000011101010
88	236	−6	10	−2	−8	−4	2	0000000011101100
89	238	−6	10	2	−4	−8	2	0000000011101110
90	239	−2	14	8	−2	−4	2	0000000011101111
91	240	−8	8	−8	4	2	2	0000000011110000
92	241	−10	14	−10	8	2	2	0000000011110001
93	242	−10	14	−10	8	−4	2	0000000011110010
94	243	−6	14	−6	−8	2	2	0000000011110011
95	244	−10	14	−10	−4	8	2	0000000011110100
96	245	−6	14	−6	4	8	2	0000000011110101
97	247	−2	14	−2	8	2	2	0000000011110111
98	248	−10	14	−10	−4	−8	2	0000000011111000
99	250	−6	14	−6	4	8	2	0000000011111010
100	251	−2	14	−2	8	4	2	0000000011111011
101	252	−6	14	−6	−8	4	2	0000000011111100
102	253	−2	14	−2	−4	8	2	0000000011111101
103	254	−2	14	−2	−4	−8	2	0000000011111110
104	255	0	16	8	4	2	2	0000000011111111

Table 5. Genes, binary decoding tapes, and decimal codes of LSBF of class (IX).

No.	Decimal Code	z	b_1	b_2	b_3	b_4	a	Binary Decoding Tape
1779	65280	0	−16	8	4	2	2	1111111100000000
1780	65281	2	−14	8	4	2	2	1111111100000001
1781	65282	2	−14	8	4	−2	2	1111111100000010
1782	65283	2	−12	8	4	2	2	1111111100000011
1783	65284	2	−14	8	−2	4	2	1111111100000100
1784	65285	4	−12	8	2	4	2	1111111100000101
1785	65287	6	−10	8	4	2	2	1111111100000111
1786	65288	2	−14	8	−2	−4	2	1111111100001000
1787	65290	4	−12	8	2	−4	2	1111111100001010
1788	65291	6	−10	8	4	−2	2	1111111100001011
1789	65292	4	−12	8	−4	2	2	1111111100001100
1790	65293	6	−10	8	−2	4	2	1111111100001101
1791	65294	6	−10	8	−2	−4	2	1111111100001110
1792	65295	8	−8	8	4	2	2	1111111100001111
1793	65296	2	−14	−2	8	2	2	1111111100010000
1794	65297	4	−12	2	8	2	2	1111111100010001
1795	65299	8	−12	6	8	4	2	1111111100010011
1796	65301	8	−12	6	4	8	2	1111111100010101
1797	65303	12	−12	10	8	4	2	1111111100010111
1798	65311	10	−6	8	4	2	2	1111111100011111
1799	65312	2	−14	−2	8	4	2	1111111100100000
1800	65314	4	−12	2	8	−4	2	1111111100100010
1801	65315	8	−12	6	8	−4	2	1111111100100011
1802	65322	4	−12	6	4	−4	2	1111111100101010
1803	65323	12	−12	10	8	−4	2	1111111100101011
1804	65327	10	−6	8	4	−2	2	1111111100101111
1805	65328	6	−14	−6	8	2	2	1111111100110000
1806	65329	6	−10	−2	8	2	2	1111111100110001
1807	65330	6	−10	−2	8	4	2	1111111100110010
1808	65331	8	−8	2	8	2	2	1111111100110011
1809	65335	12	−8	6	8	4	2	1111111100110111
1810	65339	12	−8	6	8	−4	2	1111111100111011
1811	65343	12	−4	8	4	2	2	1111111100111111
1812	65344	2	−14	−2	−4	8	2	1111111101000000
1813	65348	4	−12	2	−4	8	2	1111111101000100
1814	65349	8	−12	6	−4	8	2	1111111101000101

Table 5. (*Continued*)

No.	Decimal Code	z	b_1	b_2	b_3	b_4	a	Binary Decoding Tape
1815	65356	8	−12	6	−8	4	2	1111111101001100
1816	65357	12	−12	10	−4	8	2	1111111101001101
1817	65359	10	−6	8	−2	4	2	1111111101001111
1818	65360	6	−14	−6	4	8	2	1111111101010000
1819	65361	6	−10	−2	4	8	2	1111111101010001
1820	65364	6	−10	−2	−4	8	2	1111111101010100
1821	65365	8	−8	2	4	8	2	1111111101010101
1822	65367	12	−8	6	4	8	2	1111111101010111
1823	65373	12	−8	6	−4	8	2	1111111101011101
1824	65375	12	−4	8	2	4	2	1111111101011111
1825	65392	10	−14	−10	8	2	2	1111111101110000
1826	65393	10	−10	−6	8	2	2	1111111101110001
1827	65395	10	−6	−2	8	2	2	1111111101110011
1828	65397	10	−6	−2	4	8	2	1111111101110101
1829	65399	12	−4	2	8	2	2	1111111101110111
1830	65407	14	−2	8	4	2	2	1111111101111111
1831	65408	2	−14	−2	−4	−8	2	1111111110000000
1832	65416	6	−10	2	−4	−8	2	1111111110001000
1833	65418	8	−12	6	−4	−8	2	1111111110001010
1834	65420	8	−12	6	−8	−4	2	1111111110001100
1835	65422	12	−12	10	−4	−8	2	1111111110001110
1836	65423	10	−6	8	−2	−4	2	1111111110001111
1837	65440	6	−14	−6	4	8	2	1111111110100000
1838	65442	6	−10	−2	4	−8	2	1111111110100010
1839	65448	6	−10	−2	−4	−8	2	1111111110101000
1840	65450	8	−8	2	4	−8	2	1111111110101010
1841	65451	8	−8	6	4	−4	2	1111111110101011
1842	65454	12	−8	6	−4	−8	2	1111111110101110
1843	65455	12	−4	8	2	−4	2	1111111110101111
1844	65456	10	−14	−10	8	−4	2	1111111110110000
1845	65458	10	−10	−6	8	−4	2	1111111110110010
1846	65459	10	−6	−2	8	4	2	1111111110110011
1847	65466	10	−6	−2	4	−8	2	1111111110111010
1848	65467	12	−4	2	8	−4	2	1111111110111011
1849	65471	14	−2	8	4	−2	2	1111111110111111
1850	65472	6	−14	−6	−8	4	2	1111111111000000

Table 5. (*Continued*)

No.	Decimal Code	Gene: z	b_1	b_2	b_3	b_4	a	Binary Decoding Tape
1851	65476	6	−10	−2	−8	4	2	1111111111000100
1852	65480	6	−10	−2	−8	−4	2	1111111111001000
1853	65484	8	−8	2	−8	4	2	1111111111001100
1854	65485	12	−8	6	−8	4	2	1111111111001101
1855	65486	12	−8	6	−8	−4	2	1111111111001110
1856	65487	12	−4	8	−4	2	2	1111111111001111
1857	65488	10	−14	−10	−4	8	2	1111111111010000
1858	65492	10	−10	−6	−4	8	2	1111111111010100
1859	65493	10	−6	−2	−4	8	2	1111111111010101
1860	65500	10	−6	−2	−8	4	2	1111111111011100
1861	65501	12	−4	2	−4	8	2	1111111111011101
1862	65503	14	−2	8	−2	4	2	1111111111011111
1863	65504	10	−14	−10	−4	−8	2	1111111111100000
1864	65512	10	−10	−6	−4	−8	2	1111111111101000
1865	65514	10	−6	−2	−4	−8	2	1111111111101010
1866	65516	10	−6	−2	−8	−4	2	1111111111101100
1867	65518	10	−6	2	−4	−8	2	1111111111101110
1868	65519	14	−2	8	−2	−4	2	1111111111101111
1869	65520	8	−8	−8	4	2	2	1111111111110000
1870	65521	14	−10	−10	8	2	2	1111111111110001
1871	65522	14	−10	−10	8	−4	2	1111111111110010
1872	65523	14	−6	−6	−8	2	2	1111111111110011
1873	65524	14	−10	−10	−4	8	2	1111111111110100
1874	65525	14	−6	−6	4	8	2	1111111111110101
1875	65527	14	−2	−2	8	2	2	1111111111110111
1876	65528	14	−10	−10	−4	−8	2	1111111111111000
1877	65530	14	−6	−6	4	8	2	1111111111111010
1878	65531	14	−2	−2	8	4	2	1111111111111011
1879	65532	14	−6	−6	−8	4	2	1111111111111100
1880	65533	14	−2	−2	−4	8	2	1111111111111101
1881	65534	14	−2	−2	−4	−8	2	1111111111111110
1882	65535	16	0	8	4	2	2	1111111111111111

5. Conclusions

In this paper, we have characterized some essential properties of an uncoupled CNN of four input variables. We have not only rigorously proved that the uncoupled CNN can realize exactly 1882 linearly separable Boolean functions (LSBF) or linearly separable Boolean genes (LSBG), but have also developed an effective method for generating all these CNN genes. In particular, we have established the CNN gene bank of four input variables, which contains all LSBG.

It is well known that a single CNN gene is the most important element for constructing the CNN chromosome. The more the CNN genes, the shorter the length of the corresponding CNN chromosome, and the more convenient the corresponding implementation task. The game-of-life chromosome, for example, consists of only two LSBG and one logical AND gene, which is already very powerful.

The CNN considered in this paper is the simplest possible two-dimensional locally-connected network. It can be easily implemented in applications such as image processing, brain science and pattern recognition. It is also possible that some functions of input–output operations of the CNN with nine input variables can be replaced by that of the CNN with four input variables studied in this paper.

Future research along the same line includes the establishment of more general results about the standard CNN of n input variables and the discovery of more CNN genes that can realize all LSBF of n variables, especially for the case of $n = 9$. Such a huge bank of CNN genes is needed to be built, with which the CNN technology will find more engineering and technological applications.

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