

# ON PARTITIONS OF $G$ -SPACES AND $G$ -LATTICES

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ABSTRACT. Given a  $G$ -space  $X$  and a non-trivial  $G$ -invariant ideal  $\mathcal{I}$  of subsets of  $X$ , we prove that for every partition  $X = A_1 \cup \dots \cup A_n$  of  $X$  into  $n \geq 2$  pieces there is a piece  $A_i$  of the partition and a finite set  $F \subset G$  of cardinality  $|F| \leq \phi(n+1) := \max_{1 < x < n+1} \frac{x^{n+1-x}-1}{x-1}$  such that  $G = F \cdot \Delta(A_i)$  where  $\Delta(A_i) = \{g \in G : gA_i \cap A_i \notin \mathcal{I}\}$  is the difference set of the set  $A_i$ . Also we investigate the growth of the sequence  $\phi(n) = \max_{1 < x < n} \frac{x^{n-x}-1}{x-1}$  and show that  $\ln \phi(n+1) = nW(ne) - 2n + \frac{n}{W(ne)} + \frac{W(ne)}{n} + O(\frac{\ln \ln n}{n})$  where  $W(x)$  is the Lambert  $W$ -function, defined implicitly as  $W(x)e^{W(x)} = x$ . This shows that  $\phi(n)$  grows faster than any exponent  $a^n$  but slower than the sequence  $n!$  of factorials.

## 1. MOTIVATION, PRINCIPAL PROBLEMS AND RESULTS

This paper was motivated by the following open problem posed by I.V. Protasov in the Kourovka Notebook [5, 13.44].

**Problem 1.1.** *Is it true that for any partition  $G = A_1 \cup \dots \cup A_n$  of a group  $G$  into  $n$  pieces there is a piece  $A_i$  of the partition such that  $G = FA_iA_i^{-1}$  for some finite set  $F \subset G$  of cardinality  $|F| \leq n$ ?*

A simple measure-theoretic argument shows that the answer to this problem is affirmative for any amenable group  $G$ . So, the problem actually concerns non-amenable groups. Let us recall that a group  $G$  is amenable if it admits a left-invariant finitely additive probability measure  $\mu : \mathcal{P}(X) \rightarrow [0, 1]$  defined on the Boolean algebra  $\mathcal{P}(X)$  of all subsets of  $X$ . In Theorem 12.7 of [7] Protasov and Banakh gave a partial answer to Problem 1.1 proving that for any partition  $G = A_1 \cup \dots \cup A_n$  of a group  $G$  into  $n$  pieces there is a piece  $A_i$  of the partition such that  $G = FA_iA_i^{-1}$  for some finite set  $F \subset G$  of cardinality  $|F| \leq 2^{2^{n-1}-1}$ . They also observed that the answer to Problem 1.1 is affirmative for  $n \leq 2$ .

In [6] Protasov considered an “idealized” version of Problem 1.1. A family  $\mathcal{I}$  of subsets of a set  $X$  is called an *ideal* on  $X$  if for any sets  $A, B \in \mathcal{I}$  and  $C \in \mathcal{P}(X)$  we get  $A \cup B \in \mathcal{I}$  and  $A \cap C \in \mathcal{I}$ . An ideal  $\mathcal{I}$  on  $X$  is trivial if  $X \in \mathcal{I}$ .

Now assume that  $X$  is a  $G$ -space (i.e., a set endowed with a left action of a group  $G$ ) and  $\mathcal{I}$  is a  $G$ -invariant ideal on  $X$ . The  $G$ -invariantness of the ideal  $\mathcal{I}$  means that for every  $g \in G$  and  $A \in \mathcal{I}$  the shift  $gA$  of the set  $A$  belongs to the ideal  $\mathcal{I}$ . For a subset  $A \subset X$  let  $\Delta(A) = \{g \in G : gA \cap A \notin \mathcal{I}\}$  be the  $\mathcal{I}$ -difference set of  $A$ . In [6] Protasov asked the following modification of Problem 1.1.

**Problem 1.2.** *Let  $X$  be an infinite  $G$ -space and  $\mathcal{I}$  be the ideal of finite subsets of  $X$ . Is it true that for any partition  $X = A_1 \cup \dots \cup A_n$  of  $X$  there is a piece  $A_i$  of the partition such that  $G = F \cdot \Delta(A_i)$  for some finite set  $F \subset G$  of cardinality  $|F| \leq n$ ?*

The answer to this problem is affirmative if  $X$  admits a  $G$ -invariant probability measure. Also the upper bound  $2^{2^{n-1}-1}$  on  $|F|$  from Theorem 12.7 [7] generalizes to the “idealized” setting, see [4]. Let us observe that Problem 1.2 actually concerns partitions of the Boolean algebra  $\mathcal{P}(X)/\mathcal{I}$ , so it is natural to consider this problem in context of Boolean algebras or more generally, bounded lattices.

By a *lattice* we understand a set  $X$  endowed with two commutative idempotent associative operations  $\vee, \wedge : X \times X \rightarrow X$  connected by the absorption law:  $x \vee (x \wedge y) = x$  and  $x \wedge (x \vee y) = x$  for all  $x, y \in X$ . Each lattice  $(X, \vee, \wedge)$  carries a natural partial order  $\leq$  in which  $x \leq y$  iff  $x \wedge y = x$  iff  $x \vee y = y$ . A lattice  $X$  is *bounded* if it has the smallest element  $\mathbf{0}$  and the largest element  $\mathbf{1}$ . In the sequel we shall assume that  $\mathbf{0} \neq \mathbf{1}$ . This happens if and only if  $|X| > 1$ . A (bounded) lattice is called *distributive* (resp.  *$\mathbf{0}$ -distributive*) if for any points  $x, y, z \in X$  (with  $x \wedge y = \mathbf{0}$ ) we get  $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ . For a finite subset  $A = \{a_1, \dots, a_n\}$

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of a lattice  $X$  we put  $\bigvee A = a_1 \vee \cdots \vee a_n$  and  $\bigwedge A = a_1 \wedge \cdots \wedge a_n$ . For an element  $a \in X$  of a lattice  $X$  and a natural number  $n \in \mathbb{N}$  the set

$$a/n = \{A \subset X : |A| \leq n \text{ and } \bigvee A = a\}$$

can be thought as the family of  $n$ -element covers of  $a$ .

By a  $G$ -lattice we shall understand a lattice  $X$  endowed with an action  $\alpha : G \times X \rightarrow X$ ,  $\alpha : (g, x) \mapsto gx$ , of a group  $G$  such that for every  $g \in G$  the shift  $\alpha_g : x \rightarrow gx$  of  $X$  is an automorphism of the lattice  $X$ . For a finite subset  $F \subset G$  and an element  $a \in X$  we put

$$Fa = \{fa : f \in F\} \subset X \text{ and } F \cdot a = \bigvee Fa \in X.$$

A basic example of a distributive bounded  $G$ -lattice is the Boolean algebra  $\mathcal{P}(X)$  of a  $G$ -space  $X$  or its quotient  $\mathcal{P}(X)/\mathcal{I}$  by some non-trivial  $G$ -invariant ideal  $\mathcal{I}$ .

For a bounded  $G$ -lattice  $X$  and an element  $a \in X$  let

$$\Delta(a) = \{g \in G : ga \wedge a \neq \mathbf{0}\}$$

be the difference set of  $a$ . This set is not empty if and only if  $a \neq \mathbf{0}$ .

For a non-empty subset  $D$  of a group  $G$  let

$$\text{cov}(D) = \min\{|F| : F \subset G \text{ and } G = F \cdot D\}$$

be the *covering number* of  $D$  in  $G$ . If  $D = \emptyset$ , then we put  $\text{cov}(D)$  be equal to the smallest infinite cardinal greater than  $|G|$ , the cardinality of the group  $G$ .

On the language of lattices, Problem 1.2 can be generalized as follows.

**Problem 1.3.** *Let  $X$  be a bounded  $G$ -lattice and  $A \subset X$  be a finite subset such that  $\bigvee A = \mathbf{1}$ . Is it true that  $\min_{a \in A} \text{cov}(\Delta(a)) \leq |A|$ ?*

Again the answer to this problem is affirmative for amenable bounded  $G$ -lattices. A bounded  $G$ -lattice  $X$  is called *amenable* if it possesses a  $G$ -invariant measure  $\mu : X \rightarrow [0, 1]$ .

Let  $X$  be a bounded  $G$ -lattice. A function  $\mu : X \rightarrow [0, 1]$  is called

- *$G$ -invariant* if  $\mu(ga) = \mu(a)$  for any  $g \in G$  and  $a \in X$ ;
- *monotone* if  $\mu(a) \leq \mu(b)$  for any elements  $a \leq b$  of the lattice  $X$ ;
- *subadditive* if  $\mu(a \vee b) \leq \mu(a) + \mu(b)$  for any elements  $a, b \in X$ ;
- *additive* if  $\mu(a_1 \vee \cdots \vee a_n) = \mu(a_1) + \cdots + \mu(a_n)$  for any elements  $a_1, \dots, a_n \in X$  such that  $a_i \wedge a_j = \mathbf{0}$  for any indices  $1 \leq i < j \leq n$ ;
- a *density* on  $X$  if  $\mu$  is a monotone function such that  $\mu(\mathbf{0}) = 0$  and  $\mu(\mathbf{1}) = 1$ ;
- a *submeasure* on  $X$  if  $\mu$  is a subadditive density on  $X$ ;
- a *measure* on  $X$  if  $\mu$  is an additive submeasure on  $X$ .

For any density  $\mu : X \rightarrow [0, 1]$  on a bounded lattice  $X$  and any natural number  $n \in \mathbb{N}$  the function

$$\partial^n \mu : X \rightarrow [0, 1], \quad \partial^n \mu : x \mapsto \sup_{A \in x/n} \left( \mu(x) - \sum_{a \in A} \mu(a) \right),$$

will be called the  *$n$ -th subadditivity defect* of  $\mu$ . In this definition

$$x/n = \{A \subset X : |A| \leq n \text{ and } \bigvee A = x\}.$$

For any natural numbers  $n \leq m$  the inclusion  $\{x\} = x/1 \subset x/n \subset x/m$  implies that

$$0 \leq \partial^n \mu(x) \leq \partial^m \mu(x) \leq 1 \text{ for every } x \in X.$$

It follows that for any elements  $a_1, \dots, a_n \in X$  and their supremum  $a = \bigvee_{i=1}^n a_i$  we get

$$\mu(a) \leq \partial^n \mu(a) + \sum_{i=1}^n \mu(a_i).$$

The definition of the subadditivity defects implies the following characterization of subadditive densities.

**Proposition 1.4.** *A density  $\mu : X \rightarrow [0, 1]$  on a bounded lattice  $X$*

- (1) *is subadditive if and only if  $\partial^2 \mu \equiv 0$  if and only if  $\partial^n \mu \equiv 0$  for every  $n \geq 2$ ;*
- (2) *has  $\partial^n \mu(\mathbf{1}) = 0$  for all  $n \in \mathbb{N}$  if  $\mu \geq \nu$  for some submeasure  $\nu : X \rightarrow [0, 1]$ .*

It turns out that Problems 1.1–1.3 are related to the problem of evaluating the subadditivity defects of the Protasov density  $p_X : X \rightarrow [0, 1]$  defined on each bounded  $G$ -lattice  $X$  by the formula

$$p_X(a) = \begin{cases} \frac{1}{\text{cov}(\Delta(a))}, & \text{if } 0 < \text{cov}(\Delta(a)) < \omega; \\ 0, & \text{otherwise.} \end{cases}$$

The definitions of the Protasov density and the subadditivity defect imply the following simple:

**Proposition 1.5.** *Let  $X$  be a bounded  $G$ -lattice and  $n \in \mathbb{N}$  be a natural number. If  $\partial^n p_X(\mathbf{1}) = 0$ , then for each subset  $A \subset X$  with  $|A| \leq n$  and  $\bigvee A = \mathbf{1}$ , we get*

$$\sum_{a \in A} p_X(a) \geq 1 \quad \text{and} \quad \min_{a \in A} \text{cov}(\Delta(a)) = \frac{1}{\max p_X|A} \leq n.$$

This proposition suggests another open problem.

**Problem 1.6.** *Let  $X$  be a bounded  $G$ -lattice. Is  $\partial^n p_X(\mathbf{1}) = 0$  for every natural number  $n \in \mathbb{N}$ ?*

The answer to this problem is affirmative for amenable bounded  $G$ -lattices and will be given with help of the upper Banach density  $\bar{u} : X \rightarrow [0, 1]$  defined on each bounded  $G$ -lattice  $X$  by the formula

$$\bar{u}_X(a) = \sup_{\mu} \inf_{g \in G} \mu(ga),$$

where  $\mu$  runs over all measures on  $X$ . If  $X$  has no measure, then we define the Banach density  $\bar{u} : X \rightarrow [0, 1]$  letting  $\bar{u}_X(\mathbf{1}) = 1$  and  $\bar{u}_X(a) = 0$  for all  $a \in X \setminus \{\mathbf{1}\}$ . It is known [2] that each distributive lattice possesses a measure.

It turns out that the upper Banach density  $\bar{u}_X$  bounds from below the Protasov density  $p_X$ .

**Theorem 1.7.** *For any bounded  $G$ -lattice  $X$  we get  $p_X \geq \bar{u}_X$ .*

*Proof.* Given any element  $a \in X$ , we should prove that  $\bar{u}_X(a) \leq p_X(a)$ . Assuming that  $\bar{u}_X(a) > p_X(a)$ , we conclude that  $a \notin \{\mathbf{0}, \mathbf{1}\}$  and  $\bar{u}_X(a) > 0$ , which implies that the set  $M(X)$  of measures on  $X$  is not empty and hence  $p_X(a) < \bar{u}_X(a) = \sup_{\mu \in M(X)} \inf_{g \in G} \mu(ga)$ . Then we can choose  $\varepsilon > 0$  and a measure  $\mu : X \rightarrow [0, 1]$  such that  $\inf_{g \in G} \mu(ga) \geq p_X(a) + \varepsilon$ . By Zorn's Lemma, there is a maximal subset  $F \subset G$  such that  $xa \wedge ya = \mathbf{0}$  for any distinct elements  $x, y \in F$ . The maximality of the set  $F$  implies that for every  $x \in G$  there is an element  $y \in F$  such that  $ya \wedge xa \neq \mathbf{0}$ , which implies that  $a \wedge y^{-1}x \cdot a \neq \mathbf{0}$ . By the definition of the difference set  $\Delta(a)$ , we get  $y^{-1}x \in \Delta(a)$  and hence  $x \in y \cdot \Delta(a) \subset F \cdot \Delta(a)$ . So,  $G = F \cdot \Delta(a)$  and  $\text{cov}(\Delta(a)) \leq |F|$ . By the additivity of the measure  $\mu$ , for any finite subset  $E \subset F$  we get

$$1 = \mu(\mathbf{1}) \geq \mu\left(\bigvee_{x \in E} xa\right) = \sum_{x \in E} \mu(xa) \geq |E| \cdot \inf_{x \in E} \mu(xa) \geq |E| \cdot (p_X(a) + \varepsilon),$$

which implies that  $F$  is a finite set of cardinality  $|F| \leq 1/(p_X(a) + \varepsilon)$ . Then

$$p_X(a) = \frac{1}{\text{cov}(\Delta(a))} \geq \frac{1}{|F|} \geq p_X(a) + \varepsilon > p_X(a),$$

which is a desired contradiction. □

**Corollary 1.8.** *If a bounded  $G$ -lattice  $X$  is amenable, then  $\partial^n p_X(\mathbf{1}) = \partial^n \bar{u}_X(\mathbf{1}) = 0$  for every  $n \in \mathbb{N}$ .*

*Proof.* Fix a  $G$ -invariant measure  $\mu : X \rightarrow [0, 1]$  on  $X$  and observe that for every  $x \in X$  we get

$$\mu(x) = \inf_{g \in G} \mu(gx) \leq \bar{u}_X(x) \leq p_X(x)$$

according to Theorem 1.7. Then for every  $n \in \mathbb{N}$  and a set  $A \in \mathbf{1}/n$  the subadditivity of the measure  $\mu$  implies:

$$1 = \mu(\mathbf{1}) = \mu\left(\bigvee_{a \in A} a\right) \leq \sum_{a \in A} \bar{u}_X(a) \leq \sum_{a \in A} p_X(a).$$

Then  $0 \leq \partial^n p_X(\mathbf{1}) = \sup_{A \in \mathbf{1}/n} (1 - \sum_{a \in A} p_X(a)) \leq 0$  and hence  $\partial^n p_X(\mathbf{1}) = 0$ . By the same reason  $\partial^n \bar{u}_X(\mathbf{1}) = 0$ . □

**Problem 1.9.** *Is a distributive bounded  $G$ -lattice  $X$  amenable if  $\partial^n p_X(\mathbf{1}) = 0$  for all  $n \in \mathbb{N}$ ?*

By [1, §5], for any amenable group  $G$  the upper Banach density  $\bar{u}_X : \mathcal{P}(G) \rightarrow [0, 1]$  on the Boolean algebra  $X = \mathcal{P}(G)$  is subadditive (and coincides with the right Solecki density considered in [1]) and hence has subadditivity defects  $\partial^n \bar{u}_X = 0$  for all  $n \in \mathbb{N}$ . However, for non-amenable groups, the Banach density can be highly non-subadditive: by [1, 3.2] the free group  $G = F_2$  with two generators can be written as the union  $G = A \cup B$  of two sets with  $\bar{u}_X(A) = \bar{u}_X(B) = 0$ . This implies  $\partial^n \bar{u}_X(\mathbf{1}) = 1$  for all  $n \geq 2$ , where  $\mathbf{1} = G$  is the unit of the Boolean algebra  $X = \mathcal{P}(G)$ .

The Protasov density  $p_X : \mathcal{P}(G) \rightarrow [0, 1]$  fails to be subadditive even for nice (abelian) groups. If  $G = A \oplus B$  for infinite subgroups  $A, B \subset G$ , then the sets  $A, B \in \mathcal{P}(G) = X$  have Protasov density  $p_X(A) = p_X(B) = 0$  while their union has  $p_X(A \cup B) = 1$ . This yields  $\partial^2 p_X(A \cup B) = 1$ .

Nonetheless the Protasov density has certain weak subadditivity property at  $\mathbf{1}$ . To describe this property in quantitative terms, consider the function

$$\phi : \mathbb{N} \rightarrow \mathbb{R}, \quad \phi : n \mapsto \sup_{1 < x < n} \frac{x^{n-x} - 1}{x - 1}.$$

For  $n = 1$  we put  $\phi(1) = 0$ .

The main result of this paper is the following theorem, which generalizes and improves Theorem 12.7 [7] and Theorem 1 of [4]. This theorem follows from Theorems 1.15 and 1.16 discussed below.

**Theorem 1.10.** *For any  $\mathbf{0}$ -distributive bounded  $G$ -lattice  $X$  and any subset  $A \subset X$  of finite cardinality  $|A| = n \in \mathbb{N}$  with  $\bigvee A = \mathbf{1}$  there is an element  $a \in A$  with  $\text{cov}(\Delta(a)) \leq \phi(n + 1)$  and  $p_X(a) \geq \frac{1}{\phi(n+1)}$ .*

This theorem yields the following upper bound on the subadditivity defects of the Protasov density  $p_X$  at the unit  $\mathbf{1}$  on any  $\mathbf{0}$ -distributive bounded  $G$ -lattice  $X$ .

**Corollary 1.11.** *For any  $\mathbf{0}$ -distributive bounded  $G$ -lattice  $X$  the Protasov density  $p_X : X \rightarrow [0, 1]$  has the subadditivity defect*

$$\partial^n p_X(\mathbf{1}) \leq 1 - \frac{1}{\phi(n+1)} \quad \text{for every } n \in \mathbb{N}.$$

In light of these results it is important to evaluate the growth of the function  $\phi(n)$  as  $n \rightarrow \infty$ . This will be done in Section 6 with the help of the Lambert  $W$ -function, which is inverse to the function  $y = xe^x$ . So,  $W(y)e^{W(y)} = y$  for each positive real numbers  $y$ . It is known [3] that at infinity the Lambert  $W$ -function  $W(x)$  has asymptotical growth

$$W(x) = L - l + \frac{l}{L} + \frac{l(-2+l)}{2L^2} + \frac{l(6-9l+2l^2)}{6L^3} + \frac{l(-12+36l-22l^2+3l^3)}{12L^4} + O\left[\left(\frac{l}{L}\right)^5\right]$$

where  $L = \ln x$  and  $l = \ln \ln x$ .

The following theorem gives the lower and upper bounds on the (logarithm) of the sequence  $\phi(n + 1)$  and will be proved in Section 6.

**Theorem 1.12.** *For every  $n \geq 51$*

$$nW(ne) - 2n + \frac{n}{W(ne)} + \frac{W(ne)}{n} < \ln \phi(n + 1) < nW(ne) - 2n + \frac{n}{W(ne)} + \frac{W(ne)}{n} + \frac{\ln \ln(ne)}{n}.$$

In light of Theorem 1.12, it is interesting to compare the growth of the sequence  $\phi(n)$  with the growth of the sequence  $n!$  of factorials. Asymptotical bounds on  $n!$  proved in [8] yield the following lower and upper bounds on the logarithm  $\ln n!$  of  $n!$ :

$$n \ln n - n + \frac{1}{2} \ln 2 + \frac{\ln 2}{2} + \frac{1}{12n+1} < \ln n! < n \ln n - n + \frac{\ln n}{n} + \frac{1}{2} \ln n + \frac{\ln 2}{2} + \frac{1}{12n}.$$

Comparing these two formulas, we see that the sequence  $\phi(n)$  grows faster than any exponent  $a^n$ ,  $a > 1$ , but slower than the sequence of factorials.

The the upper bound  $\sup_{A \in \mathbf{1}/n} \min_{a \in A} \text{cov}(\Delta(a)) \leq \phi(n + 1)$  from Theorem 1.10 will be derived from the inequalities

$$\sup_{A \in \mathbf{1}/n} \min_{a \in A} \text{cov}(\Delta(a)) \leq s_{-\infty}(n) \leq \phi(n + 1)$$

where the number  $s_{-\infty}(n)$  has algorithmic nature and is defined as follows.

Let  $\omega^n$  be the semigroup of all functions  $f : n \rightarrow \omega$ , endowed with the operation of the addition of functions. The semigroup  $\omega^n$  is partially ordered by the relation  $f \leq g$  iff  $f(i) \leq g(i)$  for all  $i \in n$ . Given two functions

$f, g \in \omega^n$  we shall write  $g < f$  if  $g(i) < f(i)$  for all  $i \in n$ , and put  $\downarrow f = \{g \in \omega^n : g < f\}$  be the *strict lower cone* of  $f$  in  $\omega^n$ . In the same way, the set  $\downarrow h$  can be defined by any function  $h : n \rightarrow \Omega$  with values in some set  $\Omega$  of cardinals. Such functions  $h$  will be called *cardinal-valued*. For a cardinal-valued function  $h : n \rightarrow \Omega$  we put  $\downarrow h = \{g \in \omega^n : \forall i \in n \ g(i) < h(i)\}$ .

For subsets  $A_0, \dots, A_{n-1}$  of  $\omega^n$  let

$$\sum_{i \in n} A_i = \left\{ \sum_{i \in n} a_i : \forall i \in n \ a_i \in A_i \right\}$$

be the pointwise sum of the sets  $A_0, \dots, A_n$ . By  $\mathcal{P}(\omega^n)$  we denote the family of all subsets of  $\omega^n$ .

Given a cardinal-valued function  $h : n \rightarrow \Omega$ , for every  $m \in \omega$  consider the functions  $\bar{h}^{\{m\}}, \bar{h}^{[m]} : n \rightarrow \mathcal{P}(\omega^n)$  defined by the recursive formulas

$$\begin{aligned} \bar{h}^{[0]}(i) &= \bar{h}^{\{0\}}(i) = \{1_i\}, \\ \bar{h}^{\{m+1\}}(i) &= \{x - x(i)1_i : x \in (\downarrow h) \cap \sum_{j \in n} \bar{h}^{[m]}(j)\}, \\ \bar{h}^{[m+1]}(i) &= \bar{h}^{\{m+1\}}(i) \cup \bar{h}^{[m]}(i) \end{aligned}$$

for  $i \in n$  and  $m \in \omega$ . Let also  $\bar{h}^{[\omega]}(i) = \bigcup_{m \in \omega} \bar{h}^{\{m\}}(i)$  for all  $i \in n$ . The definition of the functions  $\bar{h}^{[k]}$ ,  $k \in \omega$ , implies that  $\bar{h}^{[\omega]}(i) \subset (\downarrow h) \cup \{1_i\}$  for all  $i \in n$ , which means that the set  $\bar{h}^{[\omega]}(i)$  is finite and is equal to  $\bar{h}^{[k]}(i)$  for some  $k \in \omega$ .

**Definition 1.13.** A cardinal-valued function  $h : n \rightarrow \Omega$  is called *0-generating* if the constant zero function  $0 : n \rightarrow \{0\} \subset \omega$  belongs to the set  $\bigcup_{i \in n} \bar{h}^{[\omega]}(i)$ .

Let us observe that the problem of recognizing 0-generating functions is algorithmically resolvable.

The following theorem (which will be proved in Section 2) is one of two ingredients of the proof of Theorem 1.10.

**Theorem 1.14.** Let  $A = \{a_0, \dots, a_{n-1}\} \subset X \setminus \{0\}$  be a finite subset of a  $\mathbf{0}$ -distributive bounded  $G$ -lattice  $X$  and  $h$  be the cardinal-valued function defined by  $\bar{h}(i) = \text{cov}(\Delta(a_i))$  for  $i \in n$ . If  $\text{sup } A = \mathbf{1}$ , then the function  $h$  is not 0-generating.

For a non-zero function  $f \in \omega^n$  and a real number  $q$  let

$$M_q(f) = \left( \frac{1}{n} \sum_{i \in n} f(i)^q \right)^{\frac{1}{q}}$$

be the mean value of  $f$  of degree  $q$ . Observe that  $M_1(f)$  is the arithmetic mean and  $M_{-1}(f)$  is the harmonic mean of the function  $f$ . For  $q = \pm\infty$  we put

$$M_{-\infty}(f) = \min_{i \in n} f(i) \quad \text{and} \quad M_{+\infty}(f) = \max_{i \in n} f(i).$$

It is known that  $M_p(f) \leq M_q(f)$  for any numbers  $-\infty \leq p \leq q \leq +\infty$ .

For every  $q \in [-\infty, +\infty]$  consider the number

$$s_q(n) = \sup \{M_q(\bar{h}) : \bar{h} \in \omega^n \text{ is not } \mathbf{0}\text{-generating}\} \in [0, +\infty].$$

We shall be especially interested in the numbers  $s_{-\infty}(n)$  and  $s_{-1}(n)$ . These numbers relate as follows:

$$s_{-\infty}(n) \leq s_{-1}(n) \leq n \cdot s_{-\infty}(n).$$

Theorem 1.14 implies:

**Theorem 1.15.** For every  $\mathbf{0}$ -distributive bounded  $G$ -lattice  $X$  and every  $n \in \mathbb{N}$  we get

$$\inf_{A \in \mathbf{1}/n} \sum_{a \in A} p_X(a) \geq \frac{n}{s_{-1}(n)} \geq \frac{1}{s_{-\infty}(n)}, \quad \partial^n p_X(\mathbf{1}) \leq 1 - \frac{n}{s_{-1}(n)} \leq 1 - \frac{1}{s_{-\infty}(n)}$$

and

$$\inf_{A \in \mathbf{1}/n} \max_{a \in A} p_X(a) \geq \frac{1}{s_{-\infty}(n)}, \quad \sup_{A \in \mathbf{1}/n} \min_{a \in A} \text{cov}(\Delta(a)) \leq s_{-\infty}(n).$$

The other ingredient of the proof of Theorem 1.10 is Theorem 1.16 comparing the growth of the sequence  $s_{-\infty}(n)$  with growth of the sequences

$$\varphi(n) = \max_{0 < k < n} \sum_{i=0}^{n-k-1} k^i = \max_{1 < k < n} \frac{k^{n-k} - 1}{k - 1} \in \mathbb{N} \quad \text{and} \quad \phi(n) = \sup_{1 < x < n} \frac{x^{n-x} - 1}{x - 1} \in \mathbb{R}.$$

It is clear that  $\varphi(n) \leq \phi(n)$ . For  $n = 1$  we put  $\varphi(1) = \phi(1) = 0$ .

**Theorem 1.16.** *For every  $n \geq 2$  we have the lower and upper bounds*

$$\varphi(n) \leq \phi(n) < s_{-\infty}(n) \leq \varphi(n+1) \leq \phi(n+1).$$

The upper and lower bound from Theorem 1.16 will be proved in Sections 4 and 5, respectively.

Finally, we present the results of computer calculations of the values of the sequences  $s_{-\infty}(n)$ ,  $s_{-1}(n)$ ,  $\varphi(n)$  and  $1 + \lfloor \phi(n) \rfloor$  for  $n \leq 9$ :

TABLE 1. Values of the numbers  $\varphi(n)$ ,  $1 + \lfloor \phi(n) \rfloor$ ,  $s_{-\infty}(n)$ ,  $s_{-1}(n)$ ,  $\varphi(n+1)$ ,  $n!$  for  $n \leq 9$

$n$	1	2	3	4	5	6	7	8	9
$\varphi(n)$	0	1	2	3	7	15	40	121	364
$1 + \lfloor \phi(n) \rfloor$	1	2	3	4	8	17	42	122	395
$s_{-\infty}(n)$	1	2	3	5	9	19	$\leq 48$	$\leq 141$	?
$s_{-1}(n)$	1	2	3	5	$\geq 9\frac{9}{49}$	$\geq 19$	?	?	?
$\varphi(n+1)$	1	2	3	7	15	40	121	364	1365
$n!$	1	2	6	24	120	720	4320	30240	241920

Here  $\lfloor x \rfloor$  denotes the integer part of the real number  $x$ . For  $n \leq 4$  the values  $s_{-\infty}(n)$  and  $s_{-1}(n)$  will be calculated in Sections 7 and 8.

Combining the results of computer calculations of the numbers  $s_{-\infty}(n)$  for  $n \leq 5$  with Theorem 1.15, we get the following values of the subadditivity defects  $\partial^n p_X(\mathbf{1})$  of the Protasov density  $p_X$  at  $\mathbf{1}$  on each  $\mathbf{0}$ -distributive bounded  $G$ -lattice  $X$ :

TABLE 2. Values of the numbers  $s_{-1}(n)$  and  $\partial^n p_X(\mathbf{1})$  for  $n \leq 8$

$n$	1	2	3	4	5	6	7	8
$s_{-1}(n)$	1	2	3	5	$\geq 9\frac{9}{49}$	$\geq 19$	$\geq 42$	$\geq 122$
$\partial^n p_X(\mathbf{1})$	0	0	0	$\leq \frac{1}{5}$	$\leq \frac{41}{90}$	$\leq \frac{13}{19}$	$\leq \frac{5}{6}$	$\leq \frac{57}{61}$

Theorem 1.16 gives the lower and upper bounds on  $s_{-\infty}(n)$ :

$$\varphi(n) \leq 1 + \lfloor \phi(n) \rfloor \leq s_{-\infty}(n) \leq \varphi(n+1)$$

for every  $n \in \omega$ .

**Problem 1.17.** *Is  $s_{-1}(n) \leq \varphi(n+1)$  for all (sufficiently large) numbers  $n$ ?*

Looking at Table 1 (containing the results of computer calculations), we can observe that  $s_{-\infty}(n) = s_{-1}(n)$  for  $n \leq 4$  but  $s_{-1}(n) > s_{-\infty}(n)$  for  $n = 5$ . The inequality  $s_{-1}(5) \geq 9\frac{9}{49}$  follows from the empirical fact that the vector  $(9, 9, 9, 9, 10)$  is not  $\mathbf{0}$ -generating. On the other hand, the vectors  $(9, 9, 9, 10, 10)$ ,  $(9, 9, 9, 9, 11)$ , and  $(8, 9, 9, 9, 12)$ ,  $(8, 8, 8, 8, 23)$  are  $\mathbf{0}$ -generating.

**Problem 1.18.** *Is  $s_{-1}(5) = 9\frac{9}{49}$ ?*

**Problem 1.19.** *Is  $s_{-\infty}(n) > s_{-1}(n)$  for all sufficiently large  $n$ ? (for all  $n \geq 5$ )?*

Looking at the results of calculations in Table 1, we can see that  $s_{-\infty}(n)$  is more near to the lower bound  $\varphi(n)$  than to the upper bound  $\varphi(n+1)$ .

**Problem 1.20.** *Is  $s_{-\infty}(n) = O(\phi(n))$ ? Is  $s_{-\infty}(n) = (1 + o(1))\phi(n)$ ?*

Now we switch to the proofs of the results announced in the introduction.

## 2. PROOF OF THEOREM 1.14

Let  $X$  be a  $\mathbf{0}$ -distributive  $G$ -lattice and  $A = \{a_0, \dots, a_{n-1}\} \subset X \setminus \{\mathbf{0}\}$  be a subset such that  $\bigvee_{i \in n} a_i = \mathbf{1}$ . We need to check that the cardinal-valued function  $\hbar$  defined by  $\hbar(i) = \text{cov}(\Delta(a_i))$  for  $i \in n$  is not  $\mathbf{0}$ -generating.

For a number  $k \in \mathbb{N}$  by  $[G]^{<k} = \{F \subset G : |F| < k\}$  we shall denote the family of all at most  $(k-1)$ -element subsets of  $G$ . For every  $i \in n$  and a finite set  $F \in [G]^{<\hbar(i)}$  by the definition of  $\text{cov}(\Delta(a_i)) = \hbar(i)$  there is a point  $v_i(F) \in G \setminus (F \cdot \Delta(a_i))$ . It follows that for every  $u \in F$  we get  $v_i(F) \notin u \cdot \Delta(a_i)$  and hence  $u^{-1}v_i(F) a_i \wedge a_i = \mathbf{0}$  and  $a_i \wedge v_i(F)^{-1}u a_i = \mathbf{0}$ . The assignment  $v_i : F \mapsto v_i(F)$  determines a function  $v_i : [G]^{<\hbar(i)} \rightarrow G$  such that

$$a_i \wedge v_i(F)^{-1}u a_i = \mathbf{0} \text{ for every } u \in F \in [G]^{<\hbar(i)}.$$

Now  $\mathbf{0}$ -distributivity of the lattice  $X$  guarantees that

$$(1) \quad a_i \wedge v_i(F)^{-1}F \cdot a_i = \mathbf{0} \text{ for every set } F \in [G]^{<\hbar(i)}.$$

We recall that  $F \cdot a = \bigvee_{f \in F} f a$ .

For every  $i \in n$  consider the function  $\delta_i : n \rightarrow \mathcal{P}(G)$  defined by

$$\delta_i(j) = \begin{cases} \{e_G\} & \text{if } i = j, \\ \emptyset & \text{if } i \neq j, \end{cases}$$

where  $e_G$  denotes the neutral element of the group  $G$ . Let us recall that  $\hbar^{\{0\}}(i) = \{1_i\}$  and define the function  $\Phi_i^{\{0\}} : \hbar^{\{0\}}(i) \rightarrow \mathcal{P}(G)^n$  letting  $\Phi_i^{\{0\}}(1_i) = \delta_i \in \mathcal{P}(G)^n$ . Observe that for the unique point  $x = 1_i$  of the set  $\hbar^{\{0\}}(i)$  and the function  $\Psi = \Phi_i^{\{0\}}(x) = \delta_i$  the following two conditions hold:

- (1<sub>0</sub>)  $|\Psi(j)| \leq x(j)$  for all  $j \in n$ ;
- (2<sub>0</sub>)  $a_i \leq \bigvee_{j \in n} \Psi(j) \cdot a_j$ .

By induction for every  $i \in \omega$  and  $m \geq 1$  we shall construct a function

$$\Phi_i^{\{m\}} : \hbar^{\{m\}}(i) \rightarrow \mathcal{P}(G)^n$$

such that for every  $x \in \hbar^{\{m\}}(i)$  and the function  $\Psi = \Phi_i^{\{m\}}(x) \in \mathcal{P}(G)^n$  the following conditions hold:

- (1<sub>m</sub>)  $|\Psi(k)| \leq x(k)$  for all  $k \in n$ ;
- (2<sub>m</sub>)  $a_i \leq \bigvee_{k \in n} \Psi(k) \cdot a_k$ .

Assume that for some  $m \geq 1$  and all  $i \in n$  and  $k < m$  the functions  $\Phi_i^{\{k\}} : \hbar^{\{k\}}(i) \rightarrow \mathcal{P}(G)^n$  have been constructed. Now for every  $i \in n$  we shall define the function  $\Phi_i^{\{m\}}$ . Given any vector  $x \in \hbar^{\{m\}}(i)$ , find a function  $y \in (\downarrow \hbar) \cap \sum_{j \in n} \hbar^{\{m-1\}}(j)$  such that  $x = y - y(i)1_i$ . It follows that  $y = \sum_{j \in n} y_j$  for some functions  $y_j \in \hbar^{\{m-1\}}(j)$ ,  $j \in n$ . For every  $j \in n$  find a number  $m_j < m$  such that  $y_j \in \hbar^{\{m_j\}}(j)$ . By the inductive hypothesis, for every  $j \in n$  the function  $\Psi_j = \Phi_j^{\{m_j\}}(y_j) \in \mathcal{P}(G)^n$  has two properties:

- (1<sub>m<sub>j</sub></sub>)  $|\Psi_j(k)| \leq y_j(k)$  for all  $k \in n$ ;
- (2<sub>m<sub>j</sub></sub>)  $a_j \leq \bigvee_{k \in n} \Psi_j(k) \cdot a_k$ .

Now consider the function

$$\Upsilon = \bigcup_{j \in n} \Psi_j : n \rightarrow \mathcal{P}(G), \quad \Upsilon : k \mapsto \bigcup_{j \in n} \Psi_j(k).$$

It follows that for every  $k \in n$  the set  $\Upsilon(k) \in \mathcal{P}(G)$  has cardinality

$$|\Upsilon(k)| \leq \sum_{j \in n} |\Psi_j(k)| \leq \sum_{j \in n} y_j(k) = y(k) < \hbar(k).$$

In particular,  $|\Upsilon(i)| < \hbar(i)$ . So,  $\Upsilon(i) \in [G]^{<\hbar(i)}$  and the element  $g_i = v_i(\Upsilon(i)) \in G$  is well-defined and by (1) has the property

$$(2) \quad a_i \wedge g_i^{-1}\Upsilon(i) \cdot a_i = \mathbf{0}.$$

Finally consider the function  $\Psi : n \rightarrow \mathcal{P}(G)^n$  defined by

$$\Psi(k) = \begin{cases} g_i^{-1}\Upsilon(k) & \text{if } k \neq i \\ \emptyset & \text{if } k = i \end{cases}$$

and put  $\Phi_i^{\{m\}}(x) = \Psi$ . It follows that so defined function  $\Psi$  has the property  $(1_m)$  of the inductive construction because for every  $k \in n$  with  $k \neq i$  we get

$$|\Psi(k)| = |g_i^{-1}\Upsilon(k)| = |\Upsilon(k)| \leq y(k) = x(k)$$

and  $0 = |\emptyset| = |\Psi(i)| \leq x(i)$ .

Next, we check that  $\Psi$  also satisfies the condition  $(2_m)$  of the inductive construction. The conditions  $(2_{m_j})$  applied to functions  $\Psi_j$ ,  $j \in n$ , guarantee that

$$\mathbf{1} = \bigvee_{j \in n} a_j \leq \bigvee_{j \in n} \bigvee_{k \in n} \Psi_j(k) \cdot a_k = \bigvee_{k \in n} \bigvee_{j \in n} \Psi_j(k) \cdot a_k = \bigvee_{k \in n} \bigcup_{j \in n} \Psi_j(k) \cdot a_k = \bigvee_{k \in n} \Upsilon(k) \cdot a_k$$

and hence

$$\mathbf{1} = \bigvee_{k \in n} g_i^{-1}\Upsilon(k) \cdot a_k.$$

The  $\mathbf{0}$ -distributivity of the lattice  $X$  and the condition (2) imply that

$$\begin{aligned} a_i \wedge \mathbf{1} &= a_i \wedge \left( \bigvee_{k \in n} g_i^{-1}\Upsilon(k) \cdot a_k \right) = \left( a_i \wedge g_i^{-1}\Upsilon(i) \cdot a_i \right) \vee \left( a_i \wedge \bigvee_{i \neq k \in n} g_i^{-1}\Upsilon(k) \cdot a_k \right) = \\ &= \mathbf{0} \vee \left( a_i \wedge \bigvee_{i \neq k \in n} \Psi(k) \cdot a_k \right) \leq a_i \wedge \left( \bigvee_{k \in n} \Psi(k) \cdot a_k \right), \end{aligned}$$

which implies that  $a_i \leq \bigvee_{k \in n} \Psi(k) \cdot a_k$  and completes the inductive construction.

Now we can complete the proof of Theorem 1.14. Assuming that the function  $\hbar$  is 0-generating, we would conclude that the zero function  $z : n \rightarrow \{0\}$  belong to the set  $\hbar^{\{m\}}(i)$  for some  $m \in \omega$  and  $i \in n$ . For the function  $z$ , consider the function  $\Psi = \Phi_i^{\{m\}}(z)$ . For this function, the conditions  $(1_m)$ ,  $(2_m)$ ,  $m \in \omega$ , of the inductive construction yield:

- (1<sub>z</sub>)  $|\Psi(k)| \leq z(k) = 0$  for all  $k \in n$ ;
- (2<sub>z</sub>)  $a_i \leq \bigvee_{k \in n} \Psi(k) \cdot a_k = \bigvee \emptyset = \mathbf{0}$ ,

which contradicts the choice of the element  $a_i \in X \setminus \{\mathbf{0}\}$ .

### 3. CHARACTERIZING CONSTANT 0-GENERATING FUNCTIONS

In this section we prove Theorem 3.1 characterizing constant 0-generating functions. This theorem will be used in Section 4 for the proof of the upper bound  $s_{-\infty}(n) \leq \varphi(n+1)$  from Theorem 1.16.

Fix an integer number  $n \geq 2$ . We consider the set  $\omega^n$  as a  $G$ -space endowed with the natural right action  $\omega^n \times G \rightarrow \omega^n$ ,  $(f, \sigma) \mapsto f \circ \sigma$ , of the group  $G = \Sigma_n$  of all permutations of the set  $n = \{0, \dots, n-1\}$ . For a function  $f \in \omega^n$  by

$$\|f\| = \max_{i \in n} f(i)$$

we denote its norm.

For a subset  $J \subset n$  by  $\bar{1}_J : n \rightarrow \{0, 1\}$  we denote the characteristic function of the set  $J$ . This is a unique function such that  $\bar{1}_J^{-1}(1) = J$ .

For a subset  $A \subset \omega^n$  and a number  $k \in \omega$  by  $\sum^k A$  we denote the set-sum of  $k$  copies of  $A$ . If  $k = 0$ , then  $\sum^0 A = \{\mathbf{0}\}$  is the singleton consisting of the constant zero function  $\mathbf{0} \in \omega^n$ . Let also  $A \circ \Sigma_n = \{f \circ \sigma : f \in A, \sigma \in \Sigma_n\}$  and  $\uparrow A = \{f \in \omega^n : \exists g \in A \text{ with } g \leq f\}$ . On the other hand,  $\downarrow f = \{g \in \omega^n : g < f\}$  for a function  $f \in \omega^n$ . We shall identify integer numbers  $c \in \mathbb{N}$  with the constant functions  $\bar{h}_c : n \rightarrow \{c\} \subset \omega$ .

Given a constant function  $\bar{h} \in \omega^n$  consider the sequence of finite subsets  $\bar{h}^{(m)} \subset \omega^n$ ,  $m \in \omega$ , defined inductively as  $\bar{h}^{(0)} = \emptyset$  and

$$\bar{h}^{(m+1)} = \bar{h}^{(m)} \cup \left\{ (x - x(n-1) \cdot 1_{n-1}) \circ \sigma : \sigma \in \Sigma_n, x \in (\downarrow \bar{h}) \cap \bigcup_{0 \leq k < n} \bar{1}_{n \setminus k} + \sum^k \bar{h}^{(m-1)} \right\} \text{ for } m \in \omega.$$

**Theorem 3.1.** *A constant function  $\bar{h} \in \omega^n$  is 0-generating if and only if the constant zero function  $\mathbf{0} : n \rightarrow \{0\}$  belongs to the set  $\bar{h}^{(\omega)} = \bigcup_{m \in \omega} \bar{h}^{(m)}$ .*

*Proof.* Let  $\bar{h} : n \rightarrow \omega$  be a constant function. To prove the theorem it suffices to check that

$$\bigcup_{i \in n} \bar{h}^{\{m\}}(i) \subset \uparrow \bar{h}^{(m)} \subset \bigcup_{i \in n} \uparrow \bar{h}^{[m]}(i)$$



for every  $m \in \mathbb{N}$ . This will be done in Lemmas 3.4 and 3.5, which will be proved with the help of Lemmas 3.2 and 3.3.

**Lemma 3.2.** *For every permutation  $\sigma \in S_n$  and  $m \in \omega$  we get*

$$\hbar^{\{m\}}(i) \circ \sigma \subset \hbar^{\{m\}}(\sigma^{-1}(i)) \text{ for all } i \in n.$$

*Proof.* This lemma will be proved by induction on  $m$ . For  $m = 0$  and every  $i \in n$  the set  $\hbar^{\{0\}}(i)$  contains a unique element  $1_i$ , for which  $1_i \circ \sigma = 1_{\sigma^{-1}(i)}$ . So,  $\hbar^{\{0\}}(i) \circ \sigma = \{1_{\sigma^{-1}(i)}\} = \hbar^{\{0\}}(\sigma^{-1}(i))$ .

Assume that the lemma has been proved for all numbers smaller or equal than some  $m \in \omega$ . To show that  $\hbar^{\{m+1\}}(i) \circ \sigma \subset \hbar^{\{m+1\}}(\sigma^{-1}(i))$  for all  $i \in n$ , take any function  $f \in \hbar^{\{m+1\}}(i)$  and find functions  $g_j \in \hbar^{\{m\}}(j)$ ,  $j \in n$ , such that the function  $g = \sum_{j \in n} g_j$  is strictly smaller than  $\hbar$  and  $f = g - g(i)1_i$ . By the inductive assumption, for every  $j \in n$  the function  $g_j \circ \sigma$  belongs to the set  $\hbar^{\{m\}}(\sigma^{-1}(j))$ . This implies that for every  $k \in n$  the function  $h_k = g_{\sigma(k)} \circ \sigma$  belongs to  $\hbar^{\{m\}}(k)$ . It follows that the function  $h = \sum_{k \in n} h_k = \sum_{k \in n} g_{\sigma(k)} \circ \sigma = g \circ \sigma < \hbar \circ \sigma = \hbar$ . Consequently, for every  $i \in n$  the function  $h - h(\sigma^{-1}(i))1_{\sigma^{-1}(i)}$  belongs to  $\hbar^{\{m+1\}}(\sigma^{-1}(i))$ . Now observe that

$$h \circ \sigma^{-1} = \left( \sum_{k \in n} h_k \right) \circ \sigma^{-1} = \left( \sum_{k \in n} g_{\sigma(k)} \circ \sigma \right) \circ \sigma^{-1}(i) = \sum_{k \in n} g_{\sigma(k)} = g$$

and  $h \circ \sigma^{-1}(i) = g(i)$ . So,

$$f \circ \sigma = (g - g(i)1_i) \circ \sigma = g \circ \sigma - g(i)1_{\sigma^{-1}(i)} = h - h(\sigma^{-1}(i))1_{\sigma^{-1}(i)} \in \hbar^{\{m+1\}}(\sigma^{-1}(i))$$

and we are done.  $\square$

**Lemma 3.3.** *For every  $m \in \mathbb{N}$ , permutation  $\sigma \in \Sigma_n$ , index  $i \in n$  and a non-zero function  $f \in \hbar^{\{m\}}(i)$  the function  $f \circ \sigma$  belongs to the set  $\uparrow \hbar^{\{m\}}(j)$  for every index  $j \in n$ .*

*Proof.* If  $f \circ \sigma(j) > 0$ , then  $f \circ \sigma \geq 1_j$  and hence  $f \circ \sigma \in \uparrow \hbar^{\{0\}}(j)$ . So, we assume that  $f \circ \sigma(j) = 0$ . If  $\sigma^{-1}(i) = j$ , then  $f \circ \sigma \in \hbar^{\{m\}}(\sigma^{-1}(i)) \subset \hbar^{\{m\}}(j)$  by Lemma 3.2. So, we assume that  $\sigma^{-1}(i) \neq j$ . It follows from  $f \in \hbar^{\{m\}}(i)$  that  $f(i) = 0$ . Let  $\tau \in \Sigma_n$  be the permutation such that  $\tau^{-1}(j) = \tau(j) = \sigma^{-1}(i)$  and  $\tau(k) = k$  for any  $k \in n \setminus \{j, \sigma^{-1}(i)\}$ . Lemma 3.2 implies that  $f \circ \sigma \circ \tau \in \hbar^{\{m\}}((\sigma \circ \tau)^{-1}(i)) = \hbar^{\{m\}}(j)$ . It remains to check that  $f \circ \sigma = f \circ \sigma \circ \tau$ .

Fix any index  $k \in n$ . If  $k \notin \{j, \sigma^{-1}(i)\}$ , then  $f \circ \sigma \circ \tau(k) = f \circ \sigma(k)$ . If  $k = j$ , then  $f \circ \sigma \circ \tau(j) = f \circ \sigma(\sigma^{-1}(i)) = f(i) = 0 = f \circ \sigma(j)$ . If  $k = \sigma^{-1}(i)$ , then  $f \circ \sigma \circ \tau(k) = f \circ \sigma(j) = 0 = f(i) = f \circ \sigma(k)$ .  $\square$

**Lemma 3.4.**  $\bigcup_{i \in n} \hbar^{\{m\}}(i) \subset \uparrow \hbar^{\{m\}}$  for every  $m \geq 1$ .

*Proof.* First we check the lemma for  $m = 1$ . In this case for every  $i \in n$  the set  $\hbar^{\{1\}}(i)$  consists of a single function  $x$ , which coincides with the characteristic function  $\bar{1}_{n \setminus \{i\}}$  of the set  $n \setminus \{i\}$ . Let  $\sigma \in \Sigma_n$  be the transposition exchanging  $i$  and  $n - 1$ . Then

$$x = \bar{1}_{n-1} \circ \sigma = (\bar{1}_n - \bar{1}_n(n-1) \cdot 1_{n-1}) \circ \sigma \in \hbar^{\{1\}}.$$

Now assume that the lemma has been proved for all numbers smaller or equal than some  $m \in \mathbb{N}$ . To prove the lemma for  $m + 1$ , take any  $i \in n$  and a function  $x \in \hbar^{\{m+1\}}(i)$ . By the definition of the set  $\hbar^{\{m+1\}}(i)$  there is a function  $y \in (\downarrow \hbar) \cap \sum_{j \in n} \hbar^{\{m\}}(j)$  such that  $x = y - y(i) \cdot 1_i$ . Find functions  $y_j \in \hbar^{\{m\}}(j)$ ,  $j \in n$ , such that  $y = \sum_{j \in n} y_j$  and consider the set  $J = \{j \in n : y_j = 1_j\}$ . Then  $y = \bar{1}_J + \sum_{j \in n \setminus J} y_j$ . For every  $j \in n \setminus J$  the function  $y_j \neq 1_j$  belongs to  $\hbar^{\{m_j\}}(j)$  for some positive  $m_j \leq m$ . By the inductive assumption,  $y_j \in \hbar^{\{m_j\}}(j) \subset \hbar^{\{m_j\}} \subset \hbar^{\{m\}}$ .

Choose a permutation  $\sigma \in \Sigma_n$  such that  $\sigma^{-1}(i) = n - 1$  and  $\sigma^{-1}(\{i\} \cup J) = n \setminus k$  for some  $k \leq n$ . Separately we shall consider two cases.

1) If  $i \in J$ , then  $n - 1 = \sigma^{-1}(i) \in \sigma^{-1}(J) = n \setminus k$  and

$$y \circ \sigma = \bar{1}_J \circ \sigma + \sum_{j \in n \setminus J} y_j \circ \sigma \in \bar{1}_{n \setminus k} + \sum_{j \in n \setminus J} \hbar^{\{m_j\}}(j) \circ \sigma \subset \bar{1}_{n \setminus k} + \sum_{j \in n \setminus J} \hbar^{\{m\}} \circ \sigma = \bar{1}_{n \setminus k} + \sum^k \hbar^{\{m\}}.$$

Since  $y \circ \sigma \leq \|y \circ \sigma\| = \|y\| < \hbar$ , we conclude that the function  $x \circ \sigma = (y - y(i) \cdot 1_i) \circ \sigma = y \circ \sigma - y \circ \sigma(n-1)1_{n-1} \in \hbar^{\{m+1\}}$  and hence  $x \in \hbar^{\{m+1\}} \circ \Sigma_n = \hbar^{\{m+1\}}$ .

2) Next, we assume that  $i \notin J$ . If  $y_i \circ \sigma(n-1) = 0$ , then  $y \geq y_i$  implies

$$x \circ \sigma = y \circ \sigma - y \circ \sigma(n-1) \cdot 1_{n-1} = y \circ \sigma \geq y_i \circ \sigma \in \hbar^{\{m\}} \circ \sigma$$

and hence  $x \in \uparrow \bar{h}^{(m)}$ .

If  $y_i \circ \sigma(n-1) > 0$ , then  $y_i \circ \sigma \geq 1_{n-1}$  and

$$\begin{aligned} y \circ \sigma &= \bar{1}_J \circ \sigma + \sum_{j \in n \setminus J} y_j \circ \sigma = \bar{1}_{\sigma^{-1}(J)} + y_i \circ \sigma + \sum_{i \neq j \in n \setminus J} y_j \circ \sigma \geq \\ &\geq \bar{1}_{(n-1) \setminus k} + 1_{n-1} + \sum_{i \neq j \in n \setminus J} y_j \circ \sigma \geq \bar{1}_{n \setminus k} + \sum_{i \neq j \in n \setminus J} \bar{h}^{\{m_j\}}(j) \circ \sigma \subset \\ &\subset \bar{1}_{n \setminus k} + \sum_{i \neq j \in n \setminus J} \bar{h}^{(m)} \circ \sigma = \bar{1}_{n \setminus k} + \sum^k \bar{h}^{(m)}. \end{aligned}$$

Since  $y \circ \sigma \leq \|y \circ \sigma\| = \|y\| < \bar{h}$ , we conclude that  $x \circ \sigma = y \circ \sigma - y \circ \sigma(n-1) \cdot 1_{n-1} \in \uparrow \bar{h}^{(m)}$  and then  $x \in \uparrow \bar{h}^{(m)} \circ \sigma = \uparrow \bar{h}^{(m)}$ .  $\square$

**Lemma 3.5.** *For every  $m \in \omega$  and every  $i \in n$  we get  $\bar{h}^{(m)} \subset \uparrow \bar{h}^{[m]}(i)$ .*

*Proof.* For  $m = 0$  this inclusion is trivial. Assume that the inclusion from the lemma has been proved for some  $m \geq 0$ . To prove it for  $m+1$ , take any function  $x \in \bar{h}^{(m+1)}$ . If  $x \in \bar{h}^{(m)}$ , then  $x \in \uparrow \bar{h}^{(m)}(i) \subset \uparrow \bar{h}^{[m]}(i)$  by the inductive assumption. If  $x \in \bar{h}^{(m+1)} \setminus \bar{h}^{(m)}$ , then there is a number  $k < n$  and a function  $y \in \bar{1}_{n \setminus k} + \sum^k \bar{h}^{(m)}$  such that  $y < \bar{h}$  and  $x = (y - y(n-1)) \cdot 1_{n-1} \circ \sigma$  for some permutation  $\sigma \in \Sigma_n$ . Write  $y$  as the sum  $y = \bar{1}_{n \setminus k} + \sum_{j \in k} y_j$  for some functions  $y_j \in \bar{h}^{(m)}$ ,  $j \in k$ . By the inductive assumption, for every  $j \in k$  the function  $y_j \in \bar{h}^{(m)}$  belongs to the set  $\uparrow \bar{h}^{[m]}(j)$ . Letting  $y_j = 1_j$  for  $j \in k$ , we see that  $y = \sum_{j \in n} y_j \in \sum_{j \in n} \uparrow \bar{h}^{[m]}(j)$  and hence  $y - y(n-1) \cdot 1_{n-1} \in \uparrow \bar{h}^{\{m+1\}}(n-1)$ . By Lemma 3.3, the function  $x = (y - y(n-1)) \cdot 1_{n-1} \circ \sigma$  belongs to  $\uparrow \bar{h}^{[m+1]}(i)$ .  $\square$

#### 4. THE PROOF OF THE UPPER BOUND $s_{-\infty}(n) \leq \varphi(n+1)$ FROM THEOREM 1.16

To prove the upper bound  $s_{-\infty}(n) \leq \varphi(n+1)$  from Theorem 1.16, it suffices to check that for  $n \in \mathbb{N}$  the constant function  $\bar{h} : n \rightarrow \{1 + \varphi(n+1)\}$  is 0-generating. In order to do that, we shall construct a special double sequence of functions  $f_{k,m} \in \downarrow \bar{h}$  defined as follows.

We recall that

$$\varphi(n+1) = \max_{0 < k \leq n} \sum_{i=0}^{n-k} k^i = \max_{0 < k < n+1} \frac{k^{n+1-k} - 1}{k-1}.$$

For  $n = 1$  the 0-generacy of the constant function  $\bar{h} \equiv 1 + \varphi(2) = 2$  is trivial, so we shall assume that  $n \geq 2$ . Denote by  $\sigma \in \Sigma_n$  the cyclic permutation of  $n$  defined by

$$\sigma(i) = \begin{cases} n-1 & \text{if } i = 0 \\ i-1 & \text{otherwise} \end{cases}$$

and consider the map  $\vec{S} : \omega^n \rightarrow \omega^n$  assigning to each function  $f \in \omega^n$  the function  $\vec{S}f = (f - f(n-1)) \cdot 1_{n-1} \circ \sigma$ . It is easy to check that for every  $i \in n$  we get

$$\vec{S}f(i) = \begin{cases} 0 & \text{for } i = 0, \\ f(i-1) & \text{for } i > 0. \end{cases}$$

This observation and the definition of the set  $\bar{h}^{(\omega)} = \bigcup_{m \in \omega} \bar{h}^{(m)}$  imply:

**Lemma 4.1.** *For any non-negative  $k < n$  and a function  $f \in \omega^n$  with  $\vec{S}f \in \bar{h}^{(\omega)}$  and  $\bar{1}_{n \setminus k} + k \cdot \vec{S}f < \bar{h}$  we get*

$$\vec{S}(\bar{1}_{n \setminus k} + k \cdot \vec{S}f) \in \bar{h}^{(\omega)}.$$

Let  $f_0 = \bar{1}_n$  and for every  $0 < k \leq n$  consider the function  $f_k \in \omega^n$  defined by

$$f_k(i) = \begin{cases} 0, & \text{if } 0 \leq i < k, \\ \sum_{j=0}^{i-k} k^j, & \text{if } k \leq i < n. \end{cases}$$

It follows that  $f_n \equiv 0$  and

$$f_k(i) = \frac{k^{i-k+1} - 1}{k-1} \leq \varphi(i+1) \leq \varphi(n) < \bar{h}$$

for  $2 \leq k \leq i < n$ . We shall put  $\frac{k^m-1}{k-1} = m$  for  $k = 1$  and  $m \in \omega$ .

**Lemma 4.2.**  $f_k = \bar{1}_{n \setminus k} + k \cdot \vec{S}f_k$  for any  $0 < k \leq n$ .

*Proof.* If  $i < k$ , then  $f_k(i) = 0 = \bar{1}_{n \setminus k}(i) + k \cdot \vec{S}f_k(i)$ .

If  $i = k$ , then  $\bar{1}_{n \setminus k}(k) + k \cdot \vec{S}f_k(k) = 1 + k \cdot f_k(k-1) = 1 + k \cdot 0 = 1 = k^0 = f_k(k)$ .

If  $k < i < n$ , then

$$\bar{1}_{n \setminus k}(i) + k \cdot \vec{S}f_k(i) = 1 + k \cdot f_k(i-1) = 1 + k \cdot \sum_{j=0}^{i-1-k} k^j = \sum_{j=0}^{i-k} k^j = f_k(i).$$

□

For every  $0 < k \leq n$  let  $f_{k,0} = f_{k-1}$  and  $f_{k,m+1} = \bar{1}_{n \setminus k} + k \cdot \vec{S}(f_{k,m})$  for  $m \in \omega$ .

**Lemma 4.3.** For every  $0 < k \leq n$  and  $0 < m \leq n - k + 1$  we get

$$f_{k,m}(i) = \begin{cases} 0 & \text{if } i < k \\ f_k(i) & \text{if } k \leq i < k + m - 1 \\ k^m \cdot \sum_{j=0}^{i-k-m+1} (k-1)^j + \sum_{j=0}^{m-1} k^j & \text{if } k + m - 1 \leq i < n. \end{cases}$$

*Proof.* For  $m = 1$ , we get  $f_{k,1} = \bar{1}_{n \setminus k} + k \cdot \vec{S}f_{k-1}$ , which implies  $f_{k,1}(i) = 0$  for  $i < k$  and

$$f_{k,1}(i) = 1 + k \cdot f_{k-1}(i-1) = k \cdot \sum_{j=0}^{i-k} (k-1)^j + 1 = k^m \cdot \sum_{j=0}^{i-k-m+1} (k-1)^j + \sum_{j=0}^{m-1} k^j$$

for  $k = k + m - 1 \leq i < n$ .

Assume that the claim has been proved for some  $0 < m < n - k - 1$ . To prove it for  $m + 1$ , take any number  $i \in n$  and consider the value  $f_{k,m+1}(i) = \bar{1}_{n \setminus k}(i) + k \cdot \vec{S}f_{k,m}(i)$ .

If  $i = 0$ , then  $f_{k,m+1}(i) = f_{k,m+1}(0) = \bar{1}_{n \setminus k}(0) + k \cdot \vec{S}f_{k,m}(0) = 0 + k \cdot 0 = 0$ .

If  $0 < i < k$ , then  $f_{k,m+1}(i) = 0$  as  $\bar{1}_{n \setminus k}(i) = 0$  and  $\vec{S}f_{k,m}(i) = f_{k,m}(i-1) = 0$  by the inductive assumption.

If  $i = k$ , then  $f_{k,m+1}(k) = \bar{1}_{n \setminus k}(k) + k \cdot \vec{S}f_{k,m}(k-1) = 1 + 0 = \sum_{j=0}^{i-k} k^j = f_k(i)$ .

If  $k < i < k + (m + 1) - 1$ , then  $k \leq i - 1 < k + m - 1$  and by the inductive assumption

$$f_{k,m+1}(i) = \bar{1}_{n \setminus k}(i) + k \cdot \vec{S}f_{k,m}(i) = 1 + k \cdot f_{k,m}(i-1) = 1 + k \cdot \sum_{j=0}^{i-1-k} k^j = \sum_{j=0}^{i-k} k^j = f_k(i).$$

If  $k + (m + 1) - 1 \leq i < n$ , then  $k + m - 1 \leq i - 1 < n - 1$  and then

$$f_{k,m+1}(i) = 1 + k \cdot f_{k,m}(i-1) = k \cdot \left( k^m \cdot \sum_{j=0}^{i-k-m} (k-1)^j + \sum_{j=0}^{m-1} k^j \right) + 1 = k^{m+1} \cdot \sum_{j=0}^{i-(m+1)-k+1} (k-1)^j + \sum_{j=0}^m k^j.$$

□

The following lemma combined with Theorem 3.1 and the fact that  $\vec{S}f_n = f_n = \mathbf{0}$  implies that the constant function  $\hbar \equiv \varphi(n+1) + 1$  is 0-generating and hence  $s_{-\infty}(n) \leq \varphi(n+1)$ .

**Lemma 4.4.** For every  $0 \leq k \leq n$  the function  $\vec{S}f_k$  belongs to the set  $\hbar^{(\omega)}$ .

*Proof.* The proof is by induction on  $k$ . For  $k = 0$  the function  $\vec{S}f_0 = \bar{1}_{n \setminus 1}$  belongs to  $\hbar^{(1)} \subset \hbar^{(\omega)}$  by the definition of  $\hbar^{(1)}$ . Assume that for some positive number  $k < n$  we have proved that the function  $\vec{S}f_{k-1}$  belongs to  $\hbar^{(\omega)}$ .

By induction on  $m \leq n - k + 1$  we shall prove that the function  $\vec{S}f_{k,m}$  belongs to  $\hbar^{(\omega)}$ . For  $m = 0$  this follows from the inductive assumption as  $f_{k,0} = f_{k-1}$ . Assume that for some  $m \leq n - k + 1$  we have proved that  $\vec{S}f_{k,m} \in \hbar^{(\omega)}$ . By Lemma 4.3,

$$\|f_{k,m+1}\| = f_{k,m+1}(n-1) = k^m \cdot \sum_{j=0}^{n-k-m} (k-1)^j + \sum_{j=0}^{m-1} k^j \leq k^m \sum_{j=0}^{n-k-m} k^j + \sum_{j=0}^{m-1} k^j = \sum_{j=0}^{n-k} k^j \leq \varphi(n-m+1) < \hbar.$$

By Lemma 4.1,  $\vec{S}f_{k,m+1} = \bar{1}_{n \setminus k} + k \cdot \vec{S}f_{k,m} \in \hbar^{(\omega)}$ . Thus  $\vec{S}f_{k,m} \in \hbar^{(\omega)}$  for all  $m \leq n - k + 1$ . In particular,  $\vec{S}f_{k+1} = \vec{S}f_{k,n-k+1} \in \hbar^{(\omega)}$ . □

5. THE PROOF OF THE LOWER BOUND  $\phi(n) < s_{-\infty}(n)$  FROM THEOREM 1.16

In this section for every  $n \geq 2$  we prove the lower bound  $\phi(n) < s_{-\infty}(n)$  from Theorem 1.16.

If  $n \leq 3$ , then  $1 + \lfloor \phi(n) \rfloor = n$ . So, it suffices to check that  $n \leq s_{-\infty}(n)$ . For this consider any group  $G$  of order  $n$ . The Boolean algebra  $\mathcal{P}(G)$  consisting of all subsets of  $G$  is a distributive  $G$ -lattice. Taking into account that  $p_X(A) \geq \frac{1}{|G|} = \frac{1}{n}$  for any non-empty subset  $A \subset G$  and  $p_X(\{a\}) = \frac{1}{n}$  for any singleton  $\{a\} \subset G$ , we see that

$$\frac{1}{n} = \inf_{A \in \mathbf{1}/n} \max_{a \in A} p_X(a) \leq \frac{1}{s_{-\infty}(n)}$$

according to Theorem 1.15, which implies the desired lower bound  $s_{-\infty}(n) \geq n > \phi(n)$  for  $n \leq 3$ .

Next, we consider the case  $n \geq 4$ . We recall that  $\phi(n)$  is the maximum of the function

$$\phi_n(x) = \frac{x^{n-x} - 1}{x - 1}$$

on the interval  $]1, n]$ . By standard methods of Calculus, it can be shown that the function  $\phi_n(x)$  attains its maximal value at a unique point  $\lambda \in ]1, n]$ .

Given any positive number  $c \leq \frac{\lambda^{n-1} - 1}{\lambda - 1}$ , consider the function  $\xi_c : [1, n] \rightarrow \mathbb{R}$  defined by

$$\xi_c(x) = (x - \lambda)c + \frac{\lambda^{n-x} - 1}{\lambda - 1}$$

and find its minimum. For this observe that

$$\xi'_c(x) = c - \frac{\lambda^{n-x} \ln(\lambda)}{\lambda - 1}$$

is an increasing function, equal to zero at a point  $x = x_c$  such that

$$\lambda^{-x} = \frac{c(\lambda - 1)}{\lambda^n \ln(\lambda)}.$$

This implies that at the point

$$x_c = n + \frac{\ln \ln(\lambda) - \ln(\lambda - 1) - \ln(c)}{\ln(\lambda)}$$

the function  $\xi_c$  attains its minimal value:

$$\begin{aligned} \xi_c(x_c) &= (x_c - \lambda)c + \frac{\lambda^{n-x_c} - 1}{\lambda - 1} = \left( n - \lambda + \frac{\ln \ln(\lambda) - \ln(\lambda - 1) - \ln(c)}{\ln(\lambda)} \right) c + \frac{c}{\ln(\lambda)} - \frac{1}{\lambda - 1} = \\ &= \left( n - \lambda + \frac{\ln \ln(\lambda) - \ln(\lambda - 1) + 1}{\ln(\lambda)} \right) c - \frac{\ln(c)}{\ln(\lambda)} c - \frac{1}{\lambda - 1}. \end{aligned}$$

Now consider the function

$$\zeta(c) = \min_{1 < x < n} \xi_c(x) = \xi_c(x_c)$$

and find its maximum. This function has derivative:

$$\zeta'(c) = n - \lambda + \frac{\ln \ln(\lambda) - \ln(\lambda - 1) + 1}{\ln(\lambda)} - \frac{\ln(c)}{\ln(\lambda)} - \frac{1}{\ln(\lambda)}$$

which is a decreasing function, equal to zero at a unique point  $c_\lambda$  such that

$$\ln(c_\lambda) = (n - \lambda) \ln(\lambda) + \ln \ln(\lambda) - \ln(\lambda - 1) \quad \text{and} \quad c_\lambda = \frac{\lambda^{n-\lambda} \ln(\lambda)}{\lambda - 1}.$$

Consequently, at this point the function  $\zeta(c)$  attains its maximal value:

$$\begin{aligned} \zeta(c_\lambda) &= \left( n - \lambda + \frac{\ln \ln(\lambda) - \ln(\lambda - 1) + 1 - \ln(c_\lambda)}{\ln(\lambda)} \right) c_\lambda - \frac{1}{\lambda - 1} = \\ &= \left( n - \lambda + \frac{\ln \ln(\lambda) - \ln(\lambda - 1) + 1 - ((n - \lambda) \ln(\lambda) + \ln \ln(\lambda) - \ln(\lambda - 1))}{\ln(\lambda)} \right) \frac{\lambda^{n-\lambda} \ln(\lambda)}{\lambda - 1} - \frac{1}{\lambda - 1} = \\ &= \frac{1}{\ln(\lambda)} \frac{\lambda^{n-\lambda} \ln(\lambda)}{\lambda - 1} - \frac{1}{\lambda - 1} = \frac{\lambda^{n-\lambda} - 1}{\lambda - 1} = \phi_n(\lambda). \end{aligned}$$

Then for the number

$$c_\lambda = \frac{\lambda^{n-\lambda} \ln(\lambda)}{\lambda - 1}$$

we get

$$(k - \lambda)c_\lambda + \frac{\lambda^{n-k} - 1}{\lambda - 1} \geq \min_{1 < x < n} \xi_{c_\lambda}(x) = \zeta(c_\lambda) = \phi_n(\lambda) = \phi(n)$$

for every  $1 < k < n$ . This inequality can be rewritten in the form

$$(3) \quad \frac{1}{\lambda} \left( -\phi(n) + \frac{\lambda^{n-k} - 1}{\lambda - 1} + kc_\lambda \right) \geq c_\lambda$$

which will be used in the proof of the lower bound  $\phi(n) \leq s(n)$  from Theorem 1.16.

**Lemma 5.1.** *If  $n \geq 4$ , then*

$$c_\lambda \leq \frac{\lambda^{n-1} - 1}{\lambda - 1}.$$

*Proof.* For  $n \leq 7$  the inequality from lemma can be verified by computer calculations, which give the following results:

$n =$	3	4	5	6	7	8
$\lambda \approx$	0.49	1.48	1.93	2.34	2.72	3.07
$\phi_n(\lambda) \approx$	1.29	3.51	7.01	16.01	41.53	121.31
$c_\lambda \approx$	0.23	2.19	5.32	14.24	42.14	136.61
$\frac{\lambda^{n-1}-1}{\lambda-1} \approx$	-0.17	2.48	5.48	19.26	86.61	456.78

If  $n \geq 8$ , then the function  $\phi_n(x)$  is increasing at  $x = 3$ , which implies that  $\lambda^{n-1} \geq \lambda > 3$  and then

$$\frac{\lambda^{n-1} - 1}{c_\lambda(\lambda - 1)} = \frac{\lambda^{n-1} - 1}{\lambda^{n-\lambda} \ln(\lambda)} \geq \frac{\lambda^{n-1} - \frac{1}{2}\lambda^{n-1}}{\lambda^{n-\lambda} \ln(\lambda)} = \frac{\lambda^{\lambda-1}}{2 \ln(\lambda)} \geq \frac{\lambda^2}{2 \ln(\lambda)} > 1.$$

□

With the help of the real numbers  $\lambda$  and  $c_\lambda$ , we can introduce the notion of *weight*  $w(f)$  of a function  $f \in \omega^n$  letting

$$w(f) = \min_{\sigma \in \Sigma_n} \sum_{i=0}^{n-1} \lambda^i \cdot f \circ \sigma(i).$$

Here  $\Sigma_n$  denote the group of all permutations of the set  $n = \{0, \dots, n-1\}$ . The definition of the weight  $w$  implies:

**Lemma 5.2.** *The weight  $w : \omega^n \rightarrow \mathbb{R}$  is a monotone and  $\Sigma_n$ -invariant function on  $\omega^n$ .*

The lower bound  $\phi(n) < s_{-\infty}(n)$  will be proved as soon as we check that the constant function

$$\hbar : n \rightarrow \{1 + \lfloor \phi(n) \rfloor\} \subset \omega$$

is not 0-generating. This is done in the following lemma.

**Lemma 5.3.** *For any  $m \in \mathbb{N}$  and any  $x \in \bigcup_{i \in n} \hbar^{\{m\}}(i)$  we get  $w(x) \geq c_\lambda > 0$ , which implies that  $x \neq 0$  and  $\hbar$  is not 0-generating.*

*Proof.* The proof is by induction on  $m \in \omega$ . For  $m = 1$  and every  $i \in n$  the set  $\hbar^{\{1\}}(i)$  consists of a unique function  $x$ , which coincides with the characteristic function  $\bar{1}_{n \setminus \{i\}}$  of the set  $n \setminus \{i\}$  and has weight

$$w(x) = \sum_{j=0}^{n-2} \lambda^j = \frac{\lambda^{n-1} - 1}{\lambda - 1} \geq c_\lambda$$

according to Lemma 5.1.

Assume that the lemma was proved for some  $m \geq 0$ . To prove it for  $m+1$ , take any function  $x \in \bigcup_{i \in n} \hbar^{\{m+1\}}(i)$ . We need to check that  $w(x) \geq c_\lambda$ . Find an index  $i \in n$  such that  $x \in \hbar^{\{m+1\}}(i)$ .

By the definition of  $\hbar^{\{m+1\}}(i)$ , there are functions  $y_j \in \hbar^{\{m\}}(j)$ ,  $j \in n$ , such that the sum  $y = y_0 + \dots + y_{n-1}$  is strictly smaller than  $\hbar$  and  $x = y - y(i) \cdot 1_i$ . Taking into account that  $y$  is an integer-valued function with  $y < 1 + \lfloor \phi(n) \rfloor$ , we conclude that  $y \leq \phi(n)$ . Replacing  $y$  by  $y \circ \sigma$  for a suitable permutation  $\sigma \in \Sigma_n$  we can

assume that  $w(y) = \sum_{i \in n} \lambda^i \cdot y(i)$ . In this case the function  $y$  is non-increasing. Let  $K = \{j \in n : y_j = 1_j\}$  and put  $k = |K|$ . Observe that the characteristic function  $\bar{1}_K : n \rightarrow \{0, 1\}$  of the set  $K \subset n$  has weight

$$w(\bar{1}_K) = w(\bar{1}_k) = \sum_{i=0}^{k-1} \lambda^i = \frac{\lambda^k - 1}{\lambda - 1}.$$

Since  $y$  is non-increasing,  $y(0)$  is the maximal value of the function  $y \leq \phi(n)$  and then

$$\begin{aligned} w(x) &= w(y - y(i) \cdot 1_i) \geq w(y - y(0) \cdot 1_0) = \sum_{i=1}^{n-1} \lambda^{i-1} y(i) = \frac{1}{\lambda} \left( -y(0) + \sum_{i=0}^{n-1} \lambda^i y(i) \right) > \\ &> \frac{1}{\lambda} \left( -\phi(n) + \sum_{i=0}^{n-1} \lambda^i \sum_{j=0}^{n-1} y_j(i) \right) = \frac{1}{\lambda} \left( -\phi(n) + \sum_{j \in K} \sum_{i=0}^{n-1} \lambda^i y_j(i) + \sum_{j \in n \setminus K} \sum_{i=0}^{n-1} \lambda^i y_j(i) \right) \geq \\ &\geq \frac{1}{\lambda} \left( -\phi(n) + \sum_{i=0}^{n-1} \lambda^i \sum_{j \in K} 1_j(i) + \sum_{j \in n \setminus K} w(y_j) \right) \geq \frac{1}{\lambda} \left( -\phi(n) + \sum_{i=0}^{n-1} \lambda^i \bar{1}_K(i) + \sum_{j=n \setminus K} c_\lambda \right) \geq \\ &\geq \frac{1}{\lambda} \left( -\phi(n) + w(\bar{1}_K) + (n-k)c_\lambda \right) \geq \frac{1}{\lambda} \left( -\phi(n) + \frac{\lambda^k - 1}{\lambda - 1} + (n-k)c_\lambda \right) \geq c_\lambda \end{aligned}$$

according to the inequality (3). □

## 6. PROOF OF THEOREM 1.12

In this section we shall prove Theorem 1.12 evaluating the growth of the sequence  $\phi(n)$ .

This will be done with the help of the Lambert W-function  $W(x)$ , which is the solution of the equation

$$W(x)e^{W(x)} = x.$$

This equation is equivalent to

$$(4) \quad e^{W(x)} = \frac{x}{W(x)}.$$

It is easy to check that

$$(5) \quad \ln x - \ln \ln x < W(x) < \ln x \quad \text{for all } x > e.$$

With the help of the Lambert W-function we shall calculate the maximal value of the function  $\psi_n(x) = x^{n-x}$  which has the same growth order as the function  $\phi_{n+1}(x) = \frac{x^{n+1-x}-1}{x-1}$ , whose maximum on the interval  $]1, n+1[$  is equal to  $\phi(n+1)$ .

**Lemma 6.1.** *The function  $\ln \psi_n(x) = (n-x) \ln x$  attains its maximum*

$$nW(ne) - 2n + \frac{n}{W(ne)} \quad \text{at the point } x_\psi = \frac{n}{W(ne)}.$$

*Proof.* Observe that

$$\frac{d}{dx} \ln \psi_n(x) = \frac{n-x}{x} - \ln x.$$

Consequently the point of maximum of the function  $\psi_n(x)$  can be found from the equation

$$0 = n - x - x \ln x = n - x \ln(xe).$$

Multiplying this equation by  $e$  and substituting  $\ln(xe) = y$ , we get

$$0 = en - xe \ln(xe) = ne - ye^y,$$

which implies that  $y = W(ne)$  and

$$xe = e^y = e^{W(ne)} = \frac{ne}{W(ne)}$$

according to the equation (4).

The value of the function  $\ln \psi_n(x) = (n-x) \ln(x)$  at the point  $x_\psi = \frac{n}{W(ne)} = e^{W(ne)-1}$  equals

$$\left( n - \frac{n}{W(ne)} \right) \cdot (W(ne) - 1) = nW(ne) - 2n + \frac{n}{W(ne)}.$$

□

**Lemma 6.2.** *If  $n \geq 51$ , then the function  $\phi_{n+1}(x) = \frac{x^{n+1-x}-1}{x-1}$  attains its maximum at a point  $x_\phi$  such that*

$$\frac{n}{\ln n} + 1 < x_\phi < \frac{n}{W(ne)}.$$

*Proof.* It can be shown that the derivative of the function  $\phi_{n+1}(x)$ :

$$\begin{aligned} \phi'_{n+1}(x) &= \frac{1}{(x-1)^2} \left( e^{(n+1-x)\ln(x)} \left( \frac{n+1-x}{x} - \ln(x) \right) (x-1) - e^{(n+1-x)\ln(x)} + 1 \right) = \\ &= \frac{1}{(x-1)^2} \left( e^{(n+1-x)\ln(x)} \left( n+1-x - \frac{n+1}{x} - (x-1)\ln(x) \right) + 1 \right) \end{aligned}$$

has a unique zero  $x_\phi$  (at which the function  $\phi_{n+1}(x)$  attains its maximum).

By computer calculations one can show that for  $x = \frac{n}{\ln n} + 1$  we get

$$\begin{aligned} n+1-x - \frac{n+1}{x} - (x-1)\ln(x) &= n - \frac{n}{\ln n} - \frac{(n+1)\ln n}{n+\ln n} - \frac{n}{\ln n} \ln \left( 1 + \frac{n}{\ln n} \right) = \\ &= \frac{n}{\ln n} \left( \ln n - 1 - \left( 1 + \frac{1}{n} \right) \frac{\ln^2 n}{n+\ln n} - \ln \left( \frac{n}{\ln n} + 1 \right) \right) > 0 \end{aligned}$$

if  $n \geq 51$ . This means that the function  $\phi_{n+1}(x)$  is increasing at the point  $x = \frac{n}{\ln n} + 1$ , which implies that  $x < x_\phi$ .

On the other hand, for the point  $x = \frac{n}{W(ne)} = e^{W(ne)-1}$  we get

$$n+1-x - \frac{n+1}{x} - (x-1)\ln(x) = n+1 - \frac{n}{W(ne)} - \frac{n+1}{n}W(ne) - \left( \frac{n}{W(ne)} - 1 \right) (W(ne) - 1) = -\frac{W(ne)}{n} < 0,$$

which implies that  $\phi'_{n+1}(x) = \frac{1}{(x-1)^2} (-x^{n+1-x} \frac{1}{x} + 1) < 0$ , the function  $\phi_{n+1}(x)$  is decreasing at  $x = \frac{n}{W(ne)}$  and hence  $x_\phi < \frac{n}{W(ne)}$ .  $\square$

Our strategy is to evaluate the maximum of the function  $\phi_{n+1}(x) = (x^{n+1-x} - 1)/(x - 1)$  using known information on the maximal value of the function  $\psi_n(x) = x^{n-x}$ . For this we establish some lower and upper bounds on the logarithm of the fraction  $\frac{\phi_{n+1}(x)}{\psi_n(x)}$ . We recall that  $x_\phi$  (resp.  $x_\psi$ ) stands for the point at which the function  $\phi_{n+1}(x)$  (resp.  $\psi_n(x)$ ) attains its maximal value. By Lemmas 6.1 and 6.2,

$$x_\psi = \frac{n}{W(ne)} \quad \text{and} \quad \frac{n}{\ln n} + 1 < x_\phi < \frac{n}{W(ne)}.$$

**Lemma 6.3.** *If  $n \geq 51$ , then*

- (1)  $\ln \frac{\phi_{n+1}(x_\phi)}{\psi_n(x_\phi)} < \frac{\ln n}{n}$ .
- (2)  $\ln \frac{\phi_{n+1}(x_\psi)}{\psi_n(x_\psi)} > \frac{W(ne)}{n}$ .

*Proof.* It follows that for  $x = x_\phi$  we get

$$\ln \frac{\phi_{n+1}(x)}{\psi_n(x)} = \ln \frac{x^{n+1-x} - 1}{x^{n-x}(x-1)} < \ln \frac{x^{n+1-x}}{x^{n-x}(x-1)} = \ln \left( 1 - \frac{1}{x-1} \right) < \frac{1}{x-1} < \frac{\ln n}{n}$$

according to Lemma 6.2.

On the other hand, the inequality  $n \geq 51 > 2e$  implies that for the point  $x = x_\psi = n/W(ne) = e^{W(ne)-1}$  of maximum of the function  $\psi_n(x)$  we get  $W(ne)e^{W(ne)} = ne \geq 2e^2$ . In this case  $W(ne) \geq 2$  and

$$n+1-x = n+1 - \frac{n}{W(ne)} \geq n+1 - \frac{n}{2} > 3$$

and hence  $x^{n+1-x} > x^3$ . Also  $x = e^{W(ne)-1} \geq e$  implies that

$$\frac{1}{2} - \frac{1}{x} - \frac{1}{2x^2} \geq \frac{1}{2} - \frac{1}{e} - \frac{1}{2e^2} > 0.$$

Using the known lower bound  $\ln(1+z) > z - \frac{1}{2}z^2$  holding for all  $z > 0$ , we conclude that

$$\begin{aligned} \ln \frac{\phi_{n+1}(x)}{\psi_n(x)} &= \ln \frac{x^{n+1-x} - 1}{x^{n-x}(x-1)} = \ln \left( \frac{1 - x^{x-n-1}}{1 - x^{-1}} \right) > \ln \left( \frac{1 - x^{-3}}{1 - x^{-1}} \right) = \ln \left( 1 + \frac{1}{x} + \frac{1}{x^2} \right) > \\ &> \frac{1}{x} + \frac{1}{x^2} - \frac{1}{2} \left( \frac{1}{x} + \frac{1}{x^2} \right)^2 = \frac{1}{x} + \frac{1}{x^2} \left( \frac{1}{2} - \frac{1}{x} - \frac{1}{2x^2} \right) \geq \frac{1}{x} + \frac{1}{x^2} \left( \frac{1}{2} - \frac{1}{e} - \frac{1}{2e^2} \right) > \frac{1}{x} = \frac{W(ne)}{n}. \end{aligned}$$

□

Now Theorem 1.12 follows from:

**Lemma 6.4.** *For every  $n \geq 51$  we get*

- (1)  $\ln \phi(n+1) > nW(ne) - 2n + \frac{n}{W(ne)} + \frac{W(ne)}{n}$ ;
- (2)  $\ln \phi(n+1) < nW(ne) - 2n + \frac{n}{W(ne)} + \frac{W(ne)}{n} + \frac{\ln \ln(ne)}{n}$ .

*Proof.* 1. By Lemmas 6.1 and 6.3(2),

$$\ln \phi(n+1) = \ln \phi_{n+1}(x_\phi) \geq \ln \phi_{n+1}(x_\psi) = \ln \psi_n(x_\psi) + \ln \frac{\phi_{n+1}(x_\psi)}{\psi_n(x_\psi)} > nW(ne) - 2n + \frac{n}{W(ne)} + \frac{W(ne)}{n}.$$

2. By Lemmas 6.1 and 6.3(1),

$$\begin{aligned} \ln \phi(n+1) &= \ln \phi_{n+1}(x_\phi) = \ln \psi(x_\phi) + \ln \frac{\phi_{n+1}(x_\phi)}{\psi(x_\phi)} < \ln \psi(x_\psi) + \frac{\ln n}{n} = \\ &= nW(ne) - 2n + \frac{n}{W(ne)} + \frac{W(ne)}{n} - \frac{W(ne)}{n} + \frac{\ln n}{n}. \end{aligned}$$

It remains to find an upper bound on the difference  $\frac{\ln n}{n} - \frac{W(ne)}{n}$ . Taking into account that  $W(ne) > \ln(ne) - \ln \ln(ne)$  we see that

$$\frac{\ln n}{n} - \frac{W(ne)}{n} < \frac{\ln n}{n} - \frac{1 + \ln(n) - \ln \ln(ne)}{n} < \frac{\ln \ln(ne)}{n}.$$

□

## 7. EVALUATING THE NUMBERS $s_{-\infty}(n)$ FOR $n \leq 5$

In this section we shall calculate the values of the numbers  $s_{-\infty}(n)$ ,  $n \leq 5$ , from Table 1. Each function  $x \in \omega^n$  will be identified with the sequence  $(x(0), \dots, x(n-1))$ .

**7.1. Lower bounds.** Theorem 1.16 yields the lower bound  $1 + \lfloor \phi(n) \rfloor \leq s_{-\infty}(n)$  which is equal to  $s_{-\infty}(n)$  for  $n \leq 3$ . For  $n = 4$  this does not work as  $1 + \lfloor \phi(n) \rfloor = 4$  while  $s_{-\infty}(4) = 5$ . To see that  $s_{-\infty}(4) \geq 5$ , consider the set

$$M_4 = \{(0, 0, 1, 2), (0, 0, 0, 4)\} \circ \Sigma_4 \subset \omega^4.$$

By routine calculations it can be shown that for the constant function  $\hbar : 4 \rightarrow \{5\} \subset \omega$  we get

$$\{(x - x(3)1_3) \circ \sigma : \sigma \in \Sigma_4, x \in (\downarrow \hbar) \cap \bigcup_{0 \leq k < 4} (\bar{1}_{4 \setminus k} + \sum^k M_4)\} \subset \uparrow M_4.$$

This implies  $\hbar^{(\omega)} \subset \uparrow M_4$  and  $(0, 0, 0, 0) \notin \hbar^{(\omega)}$ . Then Theorem 3.1 guarantees that the constant function  $\hbar : 4 \rightarrow \{5\} \subset \omega$  is not 0-generating and hence  $s_{-\infty}(4) \geq 5$ .

For  $n = 5$  the inequality  $s_{-\infty}(n) \geq 9$  follows from the observation that for the set

$$M_5 = \{(0, 0, 1, 1, 2), (0, 0, 0, 1, 6), (0, 0, 0, 2, 4), (0, 0, 0, 3, 3)\} \circ \Sigma_5$$

and the constant function  $\hbar : 5 \rightarrow \{9\} \subset \omega$  we get

$$\{(x - x(4) \cdot 1_4) \circ \sigma : \sigma \in \Sigma_5, x \in (\downarrow \hbar) \cap \bigcup_{0 \leq k < 5} (\bar{1}_{5 \setminus k} + \sum^k M_5)\} \subset \uparrow M_5.$$



**7.2. Upper bounds.** According to Theorem 3.1, to show that  $s_{-\infty}(n) < \hbar$  for some constant  $\hbar \in \mathbb{N}$ , it suffices to find a sequence of functions  $(f_i)_{i=1}^m$  such that  $f_m$  is the zero function and each function  $f_i$ ,  $1 \leq i \leq m$ , is equal to  $(\hat{f}_i - \hat{f}_i(n-1) \cdot 1_{n-1}) \circ \sigma$  for some permutation  $\sigma \in \Sigma_n$  and some function  $\hat{f}_i \in \bigcup_{0 \leq k < n} (\bar{1}_{n \setminus k} + \sum_{1 \leq j < i}^k \{f_j\})$  with  $\hat{f}_i < \hbar$ .

1) For  $n = 1$  the inequality  $s_{-\infty}(1) \leq 1$  is witnessed by the sequence  $(f_i)_{i=1}^1$  of length 1:

TABLE 3. A witness for  $s_{-\infty}(1) \leq 1$ 

$f_i$	$\hat{f}_i$	$\bar{1}_{n \setminus k} + \sum_{j \in k} f_j$	$k$
(0)	(1)	(1)	0

2) For  $n = 2$  the inequality  $s_{-\infty}(2) \leq 2$  is witnessed by the sequence  $(f_i)_{i=1}^2$  of length 2:

TABLE 4. A witness for  $s_{-\infty}(2) \leq 2$ 

$f_i$	$\hat{f}_i$	$\bar{1}_{n \setminus k} + \sum_{j \in k} f_j$	$k$
(1,0)	(1,1)	(1,1)	0
(0,0)	(0,2)	(0,1)+(0,1)	1

3) For  $n = 3$  the sequence witnessing that  $s_{-\infty}(3) \leq 3$  has length 4:

TABLE 5. A witness for  $s_{-\infty}(3) \leq 3$ 

$f_i$	$\hat{f}_i$	$\bar{1}_{n \setminus k} + \sum_{j \in k} f_j$	$k$
(1,1,0)	(1,1,1)	(1,1,1)	0
(0,2,0)	(0,2,2)	(0,1,1)+(0,1,1)	1
(0,1,0)	(1,1,3)	(0,1,1)+(0,0,2)	1
(0,0,0)	(0,0,3)	(0,0,1)+(0,0,1)+(0,0,1)	2

4) For  $n = 4$  the sequence witnessing that  $s_{-\infty}(4) \leq 5$  has length 8:

TABLE 6. A witness for  $s_{-\infty}(4) \leq 5$ 

$f_i$	$\hat{f}_i$	$\bar{1}_{n \setminus k} + \sum_{j \in k} f_j$	$k$
(1,1,1,0)	(1,1,1,1)	(1,1,1,1)	0
(0,2,2,0)	(0,2,2,2)	(0,1,1,1)+(0,1,1,1)	1
(0,1,3,0)	(0,1,3,3)	(0,1,1,1)+(0,0,2,2)	1
(0,1,2,0)	(0,1,2,4)	(0,1,1,1)+(0,0,1,3)	1
(0,0,3,0)	(0,0,3,5)	(0,0,1,1)+(0,0,1,2)+(0,0,1,2)	2
(0,1,1,0)	(0,1,1,4)	(0,1,1,1)+(0,0,0,3)	1
(0,0,2,0)	(0,0,2,5)	(0,0,1,1)+(0,0,1,1)+(0,0,0,3)	2
(0,0,0,0)	(0,0,0,5)	(0,0,1,1)+(0,0,0,2)+(0,0,0,2)	2

5) For  $n = 5$  the sequence witnessing that  $s_{-\infty}(5) \leq 9$  has length 23 and is presented in Table 7.

For  $n = 6$  the length of the annihilating sequence found by computer is equal to 143. So, it is too long to be presented here.

TABLE 7. A witness for  $s_{-\infty}(5) \leq 9$ 

$f_i$	$\hat{f}_i$	$\bar{1}_{n \setminus k} + \sum_{j \in k} f_j$	$k$
(1,1,1,1,0)	(1,1,1,1,1)	(1,1,1,1,1)	0
(0,2,2,2,0)	(0,2,2,2,2)	(0,1,1,1,1)+(0,1,1,1,1)	1
(0,1,3,3,0)	(0,1,3,3,3)	(0,1,1,1,1)+(0,0,2,2,2)	1
(0,1,2,4,0)	(0,1,2,4,4)	(0,1,1,1,1)+(0,0,1,3,3)	1
(0,1,2,3,0)	(0,1,2,3,5)	(0,1,1,1,1)+(0,0,1,2,4)	1
(0,0,3,5,0)	(0,0,3,5,7)	(0,0,1,1,1)+(0,0,1,2,3)+(0,0,1,2,3)	2
(0,1,1,4,0)	(0,1,1,4,6)	(0,1,1,1,1)+(0,0,0,3,5)	1
(0,0,3,3,0)	(0,0,3,3,9)	(0,0,1,1,1)+(0,0,1,1,4)+(0,0,1,1,4)	2
(0,0,1,7,0)	(0,0,1,7,7)	(0,0,1,1,1)+(0,0,0,3,3)+(0,0,0,3,3)	2
(0,1,1,2,0)	(0,1,1,2,8)	(0,1,1,1,1)+(0,0,0,1,7)	1
(0,0,2,4,0)	(0,0,2,4,9)	(0,0,1,1,1)+(0,0,1,2,1)+(0,0,0,1,7)	2
(0,0,1,5,0)	(0,0,1,5,9)	(0,0,1,1,1)+(0,0,0,2,4)+(0,0,0,2,4)+	2
(0,1,1,2,0)	(0,1,1,2,8)	(0,1,1,1,1)+(0,0,0,1,5)	1
(0,0,2,3,0)	(0,0,2,3,8)	(0,0,1,1,1)+(0,0,1,1,2)+(0,0,0,1,5)	2
(0,0,1,4,0)	(0,0,1,4,9)	(0,0,1,1,1)+(0,0,0,1,5)+(0,0,0,2,3)	2
(0,0,1,3,0)	(0,0,1,3,9)	(0,0,1,1,1)+(0,0,0,1,4)+(0,0,0,1,4)	2
(0,0,2,2,0)	(0,0,2,2,9)	(0,0,1,1,1)+(0,0,0,1,3)+(0,0,1,0,3)	2
(0,0,0,5,0)	(0,0,0,5,9)	(0,0,0,1,1)+(0,0,0,1,3)+(0,0,0,1,3)+(0,0,0,2,2)	3
(0,0,1,2,0)	(0,0,1,2,9)	(0,0,1,1,1)+(0,0,0,1,3)+(0,0,0,0,5)	2
(0,0,0,4,0)	(0,0,0,4,9)	(0,0,0,1,1)+(0,0,0,1,2)+(0,0,0,2,1)+(0,0,0,0,5)	3
(0,0,0,3,0)	(0,0,0,3,9)	(0,0,0,1,1)+(0,0,0,1,2)+(0,0,0,1,2)+(0,0,0,0,4)	3
(0,0,0,2,0)	(0,0,0,2,9)	(0,0,0,1,1)+(0,0,0,1,2)+(0,0,0,0,3)+(0,0,0,0,3)	3
(0,0,0,0,0)	(0,0,0,0,9)	(0,0,0,0,1)+(0,0,0,0,2)+(0,0,0,0,2)+(0,0,0,0,2)+(0,0,0,0,2)	4

8. EVALUATING THE NUMBERS  $s_{-1}(n)$  FOR  $n \leq 4$ 

In this section we calculate the values of the numbers  $s_{-1}(n)$  for  $n \leq 4$ , presented in Table 1. We recall that

$$s_{-1}(n) = \sup \{M_{-1}(x) : x \in \omega^n \text{ is not 0-generating}\}$$

is the maximal value of the harmonic means

$$M_{-1}(x) = \frac{n}{\frac{1}{x(0)} + \cdots + \frac{1}{x(n-1)}}$$

of the values of functions  $x \in \omega^n$  which are not 0-generating. The inequality  $M_{-\infty}(x) \leq M_{-1}(x)$ ,  $x \in \omega^n$ , implies that  $s_{-\infty}(n) \leq s_{-1}(n)$  for all  $n \in \mathbb{N}$ . So, it suffices to check that  $s_{-1}(n) \leq s_{-\infty}(n)$  for  $n \leq 4$ . A vector  $x \in \omega^n$  will be called *monotone* if  $x(i) \leq x(j)$  for any  $0 \leq i < j < n$ . It can be shown that a vector  $x \in \omega^n$  is 0-generating if and only if some monotone vector  $y \in x \circ \Sigma_n$  is 0-generating.

8.1. **Case  $n = 2$ .** It can be shown that each monotone vector  $x \in \omega^2$  with  $M_{-1}(x) > 2$  is greater or equal to the vector  $(2, 3)$ . So, the inequality  $s_{-1}(n) \leq 2$  will follow as soon as we check that the vectors  $(2, 3)$  is 0-generating. This is witnessed by the following annullating sequence:

TABLE 8. A witness that the vector  $(2, 3)$  is 0-generating

$m$	$\hat{h}^{[m]}(0)$	$\hat{h}^{[m]}(1)$	$\sum_{i \in 2} \hat{h}^{[m]}(i)$	$\hat{h}^{\{m+1\}}(0)$	$\hat{h}^{\{m+1\}}(1)$
0	(1,0)	(0,1)	(1,1)	(0,1)	
1	(0,1)	(0,1)	(0,2)		<b>(0,0)</b>

8.2. **Case  $n = 3$ .** In this case consider the 3-element subset

$$A_3 = \{(2, 3, 7), (2, 4, 5), (3, 3, 4)\}.$$

**Lemma 8.1.** *For each monotone vector  $x \in \omega^3$  with the harmonic mean  $M_{-1}(x) > 3$  there is a vector  $y \in A_3$  such that  $x \geq y$ .*

*Proof.* It follows from  $M_{-1}(x) > 3$  that

$$\frac{1}{x(0)} + \frac{1}{x(1)} + \frac{1}{x(2)} < 1.$$

This implies that  $x(0) \geq 2$ .

If  $x(0) = 2$ , then the above inequality implies that  $\frac{1}{x(1)} + \frac{1}{x(2)} < 1 - \frac{1}{2} = \frac{1}{2}$  and hence  $x(1) \geq 3$ . If  $x(1) = 3$ , then  $\frac{1}{x(2)} < \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$  and hence  $x(2) \geq 7$ . In this case we get  $x \geq (2, 3, 7)$ . If  $x(1) = 4$ , then  $\frac{1}{x(2)} < \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$  and  $x(2) \geq 5$ . In this case  $x \geq (2, 4, 5)$ . If  $x(1) \geq 5$ , then  $x \geq (2, 5, 5) \geq (2, 4, 5)$ .

If  $x(0) = 3$  and  $x(1) = 3$ , then  $\frac{1}{x(2)} < 1 - \frac{2}{3} = \frac{1}{3}$  and hence  $x(1) \geq 4$ . In this case  $x \geq (3, 3, 4)$ . If  $x(0) = 3$  and  $x(1) \geq 4$ , the  $x \geq (3, 4, 4) \geq (3, 3, 4)$ .  $\square$

By Lemma 8.1 the upper bound  $s_{-1}(3) \leq 3$  will be proved as soon as we check that each vector  $x \in A_3$  is 0-generating. This is witnessed by the annihilating sequences given in Tables 9–11.

TABLE 9. A sequence witnessing that the vector  $\bar{h} = (2, 3, 7)$  is 0-generating

$m$	$\bar{h}^{[m]}(0)$	$\bar{h}^{[m]}(1)$	$\bar{h}^{[m]}(2)$	$\sum_{i \in \mathbb{Z}} \bar{h}^{[m]}(i)$	$\bar{h}^{\{m+1\}}(0)$	$\bar{h}^{\{m+1\}}(1)$	$\bar{h}^{\{m+1\}}(2)$
0	(1,0,0)	(0,1,0)	(0,0,1)	(1,1,1)	(0,1,1)		
1	(0,1,1)	(0,1,0)	(0,0,1)	(0,2,2)		(0,0,2)	
2	(1,0,0)	(0,0,2)	(0,0,1)	(1,0,3)	(0,0,3)		
3	(0,0,3)	(0,0,2)	(0,0,1)	(0,0,6)			<b>(0,0,0)</b>

TABLE 10. A sequence witnessing that the vector  $\bar{h} = (2, 4, 5)$  is 0-generating

$m$	$\bar{h}^{[m]}(0)$	$\bar{h}^{[m]}(1)$	$\bar{h}^{[m]}(2)$	$\sum_{i \in \mathbb{Z}} \bar{h}^{[m]}(i)$	$\bar{h}^{\{m+1\}}(0)$	$\bar{h}^{\{m+1\}}(1)$	$\bar{h}^{\{m+1\}}(2)$
0	(1,0,0)	(0,1,0)	(0,0,1)	(1,1,1)	(0,1,1)		
1	(0,1,1)	(0,1,0)	(0,0,1)	(0,2,2)		(0,0,2)	
2	(0,1,1)	(0,0,2)	(0,0,1)	(0,1,4)			(0,1,0)
3	(1,0,0)	(0,1,0)	(0,1,0)	(1,2,0)	(0,2,0)		
4	(0,2,0)	(0,1,0)	(0,0,1)	(0,3,1)		(0,0,1)	
5	(1,0,0)	(0,0,1)	(0,0,1)	(1,0,2)	(0,0,2)		
6	(0,0,2)	(0,0,1)	(0,0,1)	(0,0,4)			<b>(0,0,0)</b>

TABLE 11. A sequence witnessing that the vector  $\bar{h} = (3, 3, 4)$  is 0-generating

$m$	$\bar{h}^{[m]}(0)$	$\bar{h}^{[m]}(1)$	$\bar{h}^{[m]}(2)$	$\sum_{i \in \mathbb{Z}} \bar{h}^{[m]}(i)$	$\bar{h}^{\{m+1\}}(0)$	$\bar{h}^{\{m+1\}}(1)$	$\bar{h}^{\{m+1\}}(2)$
0	(1,0,0)	(0,1,0)	(0,0,1)	(1,1,1)		(1,0,1)	
1	(1,0,0)	(1,0,1)	(0,0,1)	(2,0,2)	(0,0,2)		
2	(0,0,2)	(0,1,0)	(0,0,1)	(0,1,3)			(0,1,0)
3	(1,0,0)	(0,1,0)	(0,1,0)	(1,2,0)		(1,0,0)	
4	(1,0,0)	(1,0,0)	(0,0,1)	(2,0,1)	(0,0,1)		
5	(1,0,0)	(1,0,0)	(0,1,0)	(2,1,0)	(0,1,0)		
6	(0,1,0)	(0,1,0)	(0,0,1)	(0,2,1)		(0,0,1)	
7	(0,0,1)	(0,0,1)	(0,0,1)	(0,0,3)			<b>(0,0,0)</b>

8.3. **Case  $n = 4$ .** Finally, we consider the case  $n = 4$ . We should prove that  $s_{-1}(4) \leq 5$ . For this consider the following 11-element subset of  $\omega^4$

$$A_4 = \{(2, 4, 12, 15), (2, 5, 9, 13), (2, 6, 8, 13), (2, 7, 7, 11), (3, 3, 8, 11), (3, 4, 5, 12), (3, 4, 6, 10), (4, 4, 4, 12), (4, 4, 5, 9), (4, 5, 5, 7), (4, 5, 6, 6), (5, 5, 5, 6)\}.$$

Each vector  $x \in A_4$  is 0-generating as witnessed by the annihilating sequences presented in Tables 12–23 in Appendix. This fact combined with the following elementary lemma implies that  $s_{-1}(4) \leq 5$ .

**Lemma 8.2.** *For any monotone vector  $x \in \omega^4$  with  $M_{-1}(x) > 5$  there is a vector  $y \in A_4$  such that  $x \geq y$ .*

In the proof of this lemma we shall use another elementary lemma.

**Lemma 8.3.** *Let  $x \leq y$  be two positive integer numbers such that  $\frac{1}{x} + \frac{1}{y} < a$  for some real number  $a$ . Then  $(x, y) > (\frac{1}{a}, \frac{2}{a})$ .*

*Proof.* The inequality  $x > a$  follows immediately from  $\frac{1}{x} + \frac{1}{y} < a$ . Since  $x \leq y$ , we get  $\frac{2}{y} \leq \frac{1}{x} + \frac{1}{y} < a$  and hence  $y > \frac{2}{a}$ .  $\square$

*Proof of Lemma 8.2.* Given a monotone vector  $x \in \omega^4$  with  $M_{-1}(x) > 5$ , we should find a vector  $y \in A$  with  $x \geq y$ . Observe that the strict inequality  $M_{-1}(x) > 5$  is equivalent to

$$\frac{1}{x(0)} + \frac{1}{x(1)} + \frac{1}{x(2)} + \frac{1}{x(3)} < \frac{4}{5}.$$

This implies  $x(0) \geq 2$ . Now we shall consider four cases:

1)  $x(0) = 2$ . In this case we get

$$\frac{1}{x(1)} + \frac{1}{x(2)} + \frac{1}{x(3)} < \frac{4}{5} - \frac{1}{2} = \frac{3}{10},$$

which implies  $x(1) \geq 4$ . Now consider four subcases:

1a) If  $x(1) = 4$ , then  $\frac{1}{x(2)} + \frac{1}{x(3)} < \frac{3}{10} - \frac{1}{4} = \frac{1}{20}$  and  $(x(2), x(3)) \geq (21, 41)$  according to Lemma 8.3. In this case  $x \geq (2, 4, 21, 41) \geq (2, 4, 12, 15) \in A_4$ .

1b) If  $x(1) = 5$ , then  $\frac{1}{x(2)} + \frac{1}{x(3)} < \frac{3}{10} - \frac{1}{5} = \frac{1}{10}$  and  $(x(2), x(3)) \geq (11, 21)$  according to Lemma 8.3. In this case  $x \geq (2, 5, 11, 21) \geq (2, 5, 9, 13) \in A_4$ .

1c) If  $x(1) = 6$ , then  $\frac{1}{x(2)} + \frac{1}{x(3)} < \frac{3}{10} - \frac{1}{6} = \frac{2}{15}$  and  $(x(2), x(3)) \geq (8, 16)$  according to Lemma 8.3. In this case  $x \geq (2, 6, 8, 16) \geq (2, 6, 8, 13) \in A_4$ .

1d) If  $x(1) \geq 7$ , then  $\frac{1}{x(2)} + \frac{1}{x(3)} < \frac{3}{10} - \frac{1}{7} = \frac{11}{70}$  and then  $(x(2), x(3)) \geq (7, 13)$  according to Lemma 8.3. In this case  $x \geq (2, 7, 7, 13) \geq (2, 7, 7, 11) \in A_4$ .

2)  $x(0) = 3$ . This case has two subcases.

2a) If  $x(1) = 3$ , then  $\frac{1}{x(2)} + \frac{1}{x(3)} < \frac{4}{5} - \frac{2}{3} = \frac{2}{15}$  and  $(x(2), x(3)) \geq (8, 16)$  according to Lemma 8.3. In this case  $x \geq (3, 3, 8, 16) \geq (3, 3, 8, 11) \in A_4$ .

2b) If  $x(1) = 4$  then  $\frac{1}{x(2)} + \frac{1}{x(3)} < \frac{4}{5} - \frac{1}{3} - \frac{1}{4} = \frac{13}{60}$  and hence  $x(2) \geq 5$ . If  $x(2) = 5$ , then  $\frac{1}{x(3)} < \frac{13}{60} - \frac{1}{5} = \frac{1}{60}$  and  $x \geq (3, 4, 5, 61) \geq (3, 4, 5, 12) \in A_4$ . If  $x(2) \geq 6$ , then  $\frac{1}{x(3)} < \frac{13}{60} - \frac{1}{6} = \frac{1}{20}$  and  $x \geq (3, 4, 6, 21) \geq (3, 4, 6, 10) \in A_4$ .

3)  $x(0) = 4$ . This case has three subcases.

3a)  $x(1) = 4$ . If  $x(2) = 4$ , then  $\frac{1}{x(3)} < \frac{4}{5} - \frac{3}{4} = \frac{1}{20}$  and then  $x \geq (4, 4, 4, 21) \geq (4, 4, 4, 12) \in A_4$ . If  $x(2) \geq 5$ , then  $\frac{1}{x(3)} < \frac{4}{5} - \frac{2}{4} - \frac{1}{5} \leq \frac{1}{10}$  and hence  $x \geq (4, 4, 5, 11) \geq (4, 4, 5, 9) \in A_4$ .

3b)  $x(1) = 5$ . If  $x(2) = 5$ , then  $\frac{1}{x(3)} < \frac{4}{5} - \frac{1}{4} - \frac{2}{5} = \frac{3}{20}$  and  $x \geq (4, 5, 5, 7) \in A_4$ . If  $x(2) \geq 6$ , then  $x \geq (4, 5, 6, 6) \in A_4$ .

3c)  $x(1) \geq 6$  In this case  $x \geq (4, 6, 6, 6) \geq (4, 5, 6, 6) \in A_4$ .

4)  $x(0) = 5$ . In this case the inequality  $M_{-1}(x) > 5$  implies  $x \geq (5, 5, 5, 6) \in A_4$ .  $\square$

## 9. ACKNOWLEDGEMENTS

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## APPENDIX A. COMPUTER ASSISTED PROOFS OF 0-GENERACY OF SOME SEQUENCES

TABLE 12. A sequence witnessing that the function  $\hbar = (2, 4, 12, 15)$  is 0-generating

$m$	$\hbar^{[m]}(0)$	$\hbar^{[m]}(1)$	$\hbar^{[m]}(2)$	$\hbar^{[m]}(3)$	$\sum_{i \in \mathbb{3}} \hbar^{[m]}(i)$	$\hbar^{\{m+1\}}(0)$	$\hbar^{\{m+1\}}(1)$	$\hbar^{\{m+1\}}(2)$	$\hbar^{\{m+1\}}(3)$
0	(1,0,0,0)	(0,1,0,0)	(0,0,1,0)	(0,0,0,1)	(1,1,1,1)	(0,1,1,1)			
1	(0,1,1,1)	(0,1,0,0)	(0,0,1,0)	(0,0,0,1)	(0,2,2,2)		(0,0,2,2)		
2	(1,0,0,0)	(0,0,2,2)	(0,0,1,0)	(0,0,0,1)	(1,0,3,3)	(0,0,3,3)			
3	(0,1,1,1)	(0,0,2,2)	(0,0,1,0)	(0,0,0,1)	(0,1,4,4)			(0,1,0,4)	
4	(0,1,1,1)	(0,1,0,0)	(0,1,0,4)	(0,0,0,1)	(0,3,1,6)		(0,0,1,6)		
5	(0,1,1,1)	(0,0,1,6)	(0,0,1,0)	(0,0,0,1)	(0,1,3,8)				(0,1,3,0)
6	(0,1,1,1)	(0,1,0,0)	(0,0,1,0)	(0,1,3,0)	(0,3,5,1)		(0,0,5,1)		
7	(0,1,1,1)	(0,0,5,1)	(0,0,1,0)	(0,0,0,1)	(0,1,7,3)			(0,1,0,3)	
8	(0,1,1,1)	(0,1,0,0)	(0,1,0,3)	(0,0,0,1)	(0,3,1,5)		(0,0,1,5)		
9	(0,0,3,3)	(0,0,2,2)	(0,0,1,0)	(0,0,0,1)	(0,0,6,6)			(0,0,0,6)	
10	(0,1,1,1)	(0,0,1,5)	(0,0,0,6)	(0,0,0,1)	(0,1,2,13)				(0,1,2,0)
11	(0,1,1,1)	(0,1,0,0)	(0,0,1,0)	(0,1,2,0)	(0,3,4,1)		(0,0,4,1)		
12	(1,0,0,0)	(0,0,4,1)	(0,0,1,0)	(0,0,0,1)	(1,0,5,2)	(0,0,5,2)			
13	(0,0,3,3)	(0,0,2,2)	(0,0,0,6)	(0,0,0,1)	(0,0,5,12)				(0,0,5,0)
14	(0,1,1,1)	(0,0,4,1)	(0,0,1,0)	(0,0,5,0)	(0,1,11,2)			(0,1,0,2)	
15	(1,0,0,0)	(0,1,0,0)	(0,1,0,2)	(0,0,0,1)	(1,2,0,3)	(0,2,0,3)			
16	(0,1,1,1)	(0,1,0,0)	(0,1,0,2)	(0,0,0,1)	(0,3,1,4)		(0,0,1,4)		
17	(1,0,0,0)	(0,0,1,4)	(0,0,1,0)	(0,0,0,1)	(1,0,2,5)	(0,0,2,5)			
18	(0,0,5,2)	(0,0,4,1)	(0,0,1,0)	(0,0,0,1)	(0,0,10,4)			(0,0,0,4)	
19	(0,2,0,3)	(0,1,0,0)	(0,0,0,4)	(0,0,0,1)	(0,3,0,8)		(0,0,0,8)		
20	(0,1,1,1)	(0,0,0,8)	(0,0,0,4)	(0,0,0,1)	(0,1,1,14)				(0,1,1,0)
21	(1,0,0,0)	(0,1,0,0)	(0,0,1,0)	(0,1,1,0)	(1,2,2,0)	(0,2,2,0)			
22	(0,1,1,1)	(0,1,0,0)	(0,0,1,0)	(0,1,1,0)	(0,3,3,1)		(0,0,3,1)		
23	(1,0,0,0)	(0,0,3,1)	(0,0,1,0)	(0,0,0,1)	(1,0,4,2)	(0,0,4,2)			
24	(0,0,2,5)	(0,0,1,4)	(0,0,0,4)	(0,0,0,1)	(0,0,3,14)				(0,0,3,0)
25	(0,2,2,0)	(0,1,0,0)	(0,0,1,0)	(0,0,3,0)	(0,3,6,0)		(0,0,6,0)		
26	(0,1,1,1)	(0,0,6,0)	(0,0,1,0)	(0,0,3,0)	(0,1,11,1)			(0,1,0,1)	
27	(1,0,0,0)	(0,1,0,0)	(0,1,0,1)	(0,0,0,1)	(1,2,0,2)	(0,2,0,2)			
28	(0,1,1,1)	(0,1,0,0)	(0,1,0,1)	(0,0,0,1)	(0,3,1,3)		(0,0,1,3)		
29	(1,0,0,0)	(0,0,1,3)	(0,0,1,0)	(0,0,0,1)	(1,0,2,4)	(0,0,2,4)			
30	(0,0,4,2)	(0,0,3,1)	(0,0,1,0)	(0,0,3,0)	(0,0,11,3)			(0,0,0,3)	
31	(1,0,0,0)	(0,1,0,0)	(0,0,0,3)	(0,0,0,1)	(1,1,0,4)	(0,1,0,4)			
32	(0,2,0,2)	(0,1,0,0)	(0,0,0,3)	(0,0,0,1)	(0,3,0,6)		(0,0,0,6)		
33	(0,1,0,4)	(0,0,0,6)	(0,0,0,3)	(0,0,0,1)	(0,1,0,14)				(0,1,0,0)
34	(1,0,0,0)	(0,1,0,0)	(0,0,1,0)	(0,1,0,0)	(1,2,1,0)	(0,2,1,0)			
35	(0,2,1,0)	(0,1,0,0)	(0,0,1,0)	(0,0,0,1)	(0,3,2,1)		(0,0,2,1)		
36	(1,0,0,0)	(0,0,2,1)	(0,0,1,0)	(0,0,0,1)	(1,0,3,2)	(0,0,3,2)			
37	(0,0,2,4)	(0,0,0,6)	(0,0,0,3)	(0,0,0,1)	(0,0,2,14)				(0,0,2,0)
38	(1,0,0,0)	(0,1,0,0)	(0,0,1,0)	(0,0,2,0)	(1,1,3,0)	(0,1,3,0)			
39	(0,2,1,0)	(0,1,0,0)	(0,0,1,0)	(0,0,2,0)	(0,3,4,0)		(0,0,4,0)		
40	(0,1,3,0)	(0,0,4,0)	(0,0,1,0)	(0,0,2,0)	(0,1,10,0)			(0,1,0,0)	
41	(1,0,0,0)	(0,1,0,0)	(0,1,0,0)	(0,0,0,1)	(1,2,0,1)	(0,2,0,1)			
42	(0,2,0,1)	(0,1,0,0)	(0,0,1,0)	(0,0,0,1)	(0,3,1,2)		(0,0,1,2)		
43	(0,0,3,2)	(0,0,4,0)	(0,0,1,0)	(0,0,2,0)	(0,0,10,2)			(0,0,0,2)	
44	(1,0,0,0)	(0,0,1,2)	(0,0,0,2)	(0,0,0,1)	(1,0,1,5)	(0,0,1,5)			
45	(0,2,0,1)	(0,1,0,0)	(0,0,0,2)	(0,0,0,1)	(0,3,0,4)		(0,0,0,4)		
46	(0,0,1,5)	(0,0,0,4)	(0,0,0,2)	(0,0,0,1)	(0,0,1,12)				(0,0,1,0)
47	(1,0,0,0)	(0,1,0,0)	(0,0,1,0)	(0,0,1,0)	(1,1,2,0)	(0,1,2,0)			
48	(0,1,2,0)	(0,1,0,0)	(0,0,1,0)	(0,1,0,0)	(0,3,3,0)		(0,0,3,0)		
49	(1,0,0,0)	(0,0,3,0)	(0,0,1,0)	(0,0,1,0)	(1,0,5,0)	(0,0,5,0)			
50	(0,0,5,0)	(0,0,3,0)	(0,0,1,0)	(0,0,1,0)	(0,0,10,0)				<b>(0,0,0,0)</b>

TABLE 13. A sequence witnessing that the function  $\tilde{h} = (2, 5, 9, 13)$  is 0-generating

$m$	$\tilde{h}^{[m]}(0)$	$\tilde{h}^{[m]}(1)$	$\tilde{h}^{[m]}(2)$	$\tilde{h}^{[m]}(3)$	$\sum_{i \in \mathbb{3}} \tilde{h}^{[m]}(i)$	$\tilde{h}^{\{m+1\}}(0)$	$\tilde{h}^{\{m+1\}}(1)$	$\tilde{h}^{\{m+1\}}(2)$	$\tilde{h}^{\{m+1\}}(3)$
0	(1,0,0,0)	(0,1,0,0)	(0,0,1,0)	(0,0,0,1)	(1,1,1,1)	(0,1,1,1)			
1	(0,1,1,1)	(0,1,0,0)	(0,0,1,0)	(0,0,0,1)	(0,2,2,2)				(0,2,2,0)
2	(0,1,1,1)	(0,1,0,0)	(0,0,1,0)	(0,2,2,0)	(0,4,4,1)		(0,0,4,1)		
3	(0,1,1,1)	(0,0,4,1)	(0,0,1,0)	(0,0,0,1)	(0,1,6,3)			(0,1,0,3)	
4	(1,0,0,0)	(0,1,0,0)	(0,1,0,3)	(0,0,0,1)	(1,2,0,4)	(0,2,0,4)			
5	(0,2,0,4)	(0,1,0,0)	(0,1,0,3)	(0,0,0,1)	(0,4,0,8)		(0,0,0,8)		
6	(0,1,1,1)	(0,0,0,8)	(0,0,1,0)	(0,0,0,1)	(0,1,2,10)				(0,1,2,0)
7	(1,0,0,0)	(0,1,0,0)	(0,0,1,0)	(0,1,2,0)	(1,2,3,0)	(0,2,3,0)			
8	(0,2,3,0)	(0,1,0,0)	(0,0,1,0)	(0,1,2,0)	(0,4,6,0)		(0,0,6,0)		
9	(0,1,1,1)	(0,0,6,0)	(0,0,1,0)	(0,0,0,1)	(0,1,8,2)			(0,1,0,2)	
10	(1,0,0,0)	(0,1,0,0)	(0,1,0,2)	(0,0,0,1)	(1,2,0,3)	(0,2,0,3)			
11	(0,1,1,1)	(0,0,0,8)	(0,1,0,2)	(0,0,0,1)	(0,2,1,12)				(0,2,1,0)
12	(0,1,1,1)	(0,1,0,0)	(0,0,1,0)	(0,2,1,0)	(0,4,3,1)		(0,0,3,1)		
13	(1,0,0,0)	(0,0,3,1)	(0,0,1,0)	(0,0,0,1)	(1,0,4,2)	(0,0,4,2)			
14	(0,2,0,3)	(0,1,0,0)	(0,1,0,2)	(0,0,0,1)	(0,4,0,6)		(0,0,0,6)		
15	(0,0,4,2)	(0,0,3,1)	(0,0,1,0)	(0,0,0,1)	(0,0,8,4)			(0,0,0,4)	
16	(1,0,0,0)	(0,1,0,0)	(0,0,0,4)	(0,0,0,1)	(1,1,0,5)	(0,1,0,5)			
17	(0,1,1,1)	(0,0,0,6)	(0,0,0,4)	(0,0,0,1)	(0,1,1,12)				(0,1,1,0)
18	(1,0,0,0)	(0,1,0,0)	(0,0,1,0)	(0,1,1,0)	(1,2,2,0)	(0,2,2,0)			
19	(0,2,2,0)	(0,1,0,0)	(0,0,1,0)	(0,1,1,0)	(0,4,4,0)		(0,0,4,0)		
20	(0,1,1,1)	(0,0,4,0)	(0,0,1,0)	(0,1,1,0)	(0,2,7,1)			(0,2,0,1)	
21	(0,1,1,1)	(0,1,0,0)	(0,2,0,1)	(0,0,0,1)	(0,4,1,3)		(0,0,1,3)		
22	(1,0,0,0)	(0,0,1,3)	(0,0,1,0)	(0,0,0,1)	(1,0,2,4)	(0,0,2,4)			
23	(0,0,2,4)	(0,0,1,3)	(0,0,0,4)	(0,0,0,1)	(0,0,3,12)				(0,0,3,0)
24	(1,0,0,0)	(0,1,0,0)	(0,0,1,0)	(0,0,3,0)	(1,1,4,0)	(0,1,4,0)			
25	(0,1,4,0)	(0,1,0,0)	(0,0,1,0)	(0,0,3,0)	(0,2,8,0)			(0,2,0,0)	
26	(1,0,0,0)	(0,1,0,0)	(0,2,0,0)	(0,0,0,1)	(1,3,0,1)	(0,3,0,1)			
27	(0,1,1,1)	(0,1,0,0)	(0,2,0,0)	(0,0,0,1)	(0,4,1,2)		(0,0,1,2)		
28	(1,0,0,0)	(0,0,1,2)	(0,0,1,0)	(0,0,0,1)	(1,0,2,3)	(0,0,2,3)			
29	(0,1,0,5)	(0,1,0,0)	(0,0,0,4)	(0,0,0,1)	(0,2,0,10)				(0,2,0,0)
30	(0,1,1,1)	(0,1,0,0)	(0,0,1,0)	(0,2,0,0)	(0,4,2,1)		(0,0,2,1)		
31	(1,0,0,0)	(0,0,2,1)	(0,0,1,0)	(0,0,0,1)	(1,0,3,2)	(0,0,3,2)			
32	(0,0,3,2)	(0,0,4,0)	(0,0,1,0)	(0,0,0,1)	(0,0,8,3)			(0,0,0,3)	
33	(1,0,0,0)	(0,1,0,0)	(0,0,0,3)	(0,0,0,1)	(1,1,0,4)	(0,1,0,4)			
34	(0,3,0,1)	(0,1,0,0)	(0,0,0,3)	(0,0,0,1)	(0,4,0,5)		(0,0,0,5)		
35	(0,0,2,3)	(0,0,0,5)	(0,0,0,3)	(0,0,0,1)	(0,0,2,12)				(0,0,2,0)
36	(0,1,1,1)	(0,0,4,0)	(0,0,1,0)	(0,0,2,0)	(0,1,8,1)			(0,1,0,1)	
37	(1,0,0,0)	(0,1,0,0)	(0,1,0,1)	(0,0,0,1)	(1,2,0,2)	(0,2,0,2)			
38	(0,2,0,2)	(0,1,0,0)	(0,1,0,1)	(0,0,0,1)	(0,4,0,4)		(0,0,0,4)		
39	(0,1,0,4)	(0,0,0,4)	(0,0,0,3)	(0,0,0,1)	(0,1,0,12)				(0,1,0,0)
40	(1,0,0,0)	(0,1,0,0)	(0,0,1,0)	(0,1,0,0)	(1,2,1,0)	(0,2,1,0)			
41	(0,2,1,0)	(0,1,0,0)	(0,0,1,0)	(0,1,0,0)	(0,4,2,0)		(0,0,2,0)		
42	(1,0,0,0)	(0,0,2,0)	(0,0,1,0)	(0,0,0,1)	(1,0,3,1)	(0,0,3,1)			
43	(0,0,3,1)	(0,0,2,0)	(0,0,1,0)	(0,0,2,0)	(0,0,8,1)			(0,0,0,1)	
44	(1,0,0,0)	(0,1,0,0)	(0,0,0,1)	(0,1,0,0)	(1,2,0,1)	(0,2,0,1)			
45	(0,2,0,1)	(0,1,0,0)	(0,0,0,1)	(0,1,0,0)	(0,4,0,2)		(0,0,0,2)		
46	(1,0,0,0)	(0,0,0,2)	(0,0,0,1)	(0,0,0,1)	(1,0,0,4)	(0,0,0,4)			
47	(0,0,0,4)	(0,0,0,2)	(0,0,0,1)	(0,0,0,1)	(0,0,0,8)				<b>(0,0,0,0)</b>



TABLE 14. A sequence witnessing that the function  $\hbar = (2, 6, 8, 13)$  is 0-generating

$m$	$\hbar^{[m]}(0)$	$\hbar^{[m]}(1)$	$\hbar^{[m]}(2)$	$\hbar^{[m]}(3)$	$\sum_{i \in \mathbb{3}} \hbar^{[m]}(i)$	$\hbar^{\{m+1\}}(0)$	$\hbar^{\{m+1\}}(1)$	$\hbar^{\{m+1\}}(2)$	$\hbar^{\{m+1\}}(3)$
0	(1,0,0,0)	(0,1,0,0)	(0,0,1,0)	(0,0,0,1)	(1,1,1,1)	(0,1,1,1)			
1	(0,1,1,1)	(0,1,0,0)	(0,0,1,0)	(0,0,0,1)	(0,2,2,2)		(0,0,2,2)	(0,2,0,2)	
2	(0,1,1,1)	(0,0,2,2)	(0,0,1,0)	(0,0,0,1)	(0,1,4,4)			(0,1,0,4)	(0,1,4,0)
3	(0,1,1,1)	(0,1,0,0)	(0,0,1,0)	(0,1,4,0)	(0,3,6,1)			(0,3,0,1)	
4	(0,1,1,1)	(0,1,0,0)	(0,3,0,1)	(0,0,0,1)	(0,5,1,3)		(0,0,1,3)		
5	(0,1,1,1)	(0,1,0,0)	(0,1,0,4)	(0,0,0,1)	(0,3,1,6)				(0,3,1,0)
6	(0,1,1,1)	(0,1,0,0)	(0,0,1,0)	(0,3,1,0)	(0,5,3,1)		(0,0,3,1)		
7	(0,1,1,1)	(0,0,3,1)	(0,0,1,0)	(0,0,0,1)	(0,1,5,3)			(0,1,0,3)	
8	(1,0,0,0)	(0,1,0,0)	(0,1,0,3)	(0,0,0,1)	(1,2,0,4)	(0,2,0,4)			
9	(0,2,0,4)	(0,1,0,0)	(0,1,0,3)	(0,0,0,1)	(0,4,0,8)		(0,0,0,8)		
10	(0,1,1,1)	(0,0,0,8)	(0,0,1,0)	(0,0,0,1)	(0,1,2,10)				(0,1,2,0)
11	(0,2,0,4)	(0,1,0,0)	(0,2,0,2)	(0,0,0,1)	(0,5,0,7)		(0,0,0,7)		
12	(0,1,1,1)	(0,0,0,7)	(0,1,0,3)	(0,0,0,1)	(0,2,1,12)				(0,2,1,0)
13	(1,0,0,0)	(0,1,0,0)	(0,0,1,0)	(0,2,1,0)	(1,3,2,0)	(0,3,2,0)			
14	(0,3,2,0)	(0,1,0,0)	(0,0,1,0)	(0,1,2,0)	(0,5,5,0)		(0,0,5,0)		
15	(0,1,1,1)	(0,0,5,0)	(0,0,1,0)	(0,0,0,1)	(0,1,7,2)			(0,1,0,2)	
16	(1,0,0,0)	(0,1,0,0)	(0,1,0,2)	(0,0,0,1)	(1,2,0,3)	(0,2,0,3)			
17	(0,2,0,3)	(0,1,0,0)	(0,1,0,2)	(0,0,0,1)	(0,4,0,6)		(0,0,0,6)		
18	(0,2,0,3)	(0,0,0,6)	(0,1,0,2)	(0,0,0,1)	(0,3,0,12)				(0,3,0,0)
19	(0,1,1,1)	(0,1,0,0)	(0,0,1,0)	(0,3,0,0)	(0,5,2,1)		(0,0,2,1)		
20	(1,0,0,0)	(0,0,2,1)	(0,0,1,0)	(0,0,0,1)	(1,0,3,2)	(0,0,3,2)			
21	(0,0,3,2)	(0,0,2,1)	(0,0,1,0)	(0,0,0,1)	(0,0,6,4)			(0,0,0,4)	
22	(1,0,0,0)	(0,1,0,0)	(0,0,0,4)	(0,0,0,1)	(1,1,0,5)	(0,1,0,5)			
23	(0,1,1,1)	(0,0,0,6)	(0,0,0,4)	(0,0,0,1)	(0,1,1,12)				(0,1,1,0)
24	(1,0,0,0)	(0,1,0,0)	(0,0,1,0)	(0,1,1,0)	(1,2,2,0)	(0,2,2,0)			
25	(0,2,2,0)	(0,1,0,0)	(0,0,1,0)	(0,1,1,0)	(0,4,4,0)		(0,0,4,0)		
26	(0,1,1,1)	(0,0,4,0)	(0,0,1,0)	(0,1,1,0)	(0,2,7,1)			(0,2,0,1)	
27	(1,0,0,0)	(0,1,0,0)	(0,2,0,1)	(0,0,0,1)	(1,3,0,2)	(0,3,0,2)			
28	(0,3,0,2)	(0,1,0,0)	(0,1,0,2)	(0,0,0,1)	(0,5,0,5)		(0,0,0,5)		
29	(0,0,3,2)	(0,0,1,3)	(0,0,0,4)	(0,0,0,1)	(0,0,4,10)				(0,0,4,0)
30	(0,2,2,0)	(0,1,0,0)	(0,0,1,0)	(0,0,4,0)	(0,3,7,0)			(0,3,0,0)	
31	(0,1,1,1)	(0,1,0,0)	(0,3,0,0)	(0,0,0,1)	(0,5,1,2)		(0,0,1,2)		
32	(1,0,0,0)	(0,0,1,2)	(0,0,1,0)	(0,0,0,1)	(1,0,2,3)	(0,0,2,3)			
33	(0,1,0,5)	(0,1,0,0)	(0,0,0,4)	(0,0,0,1)	(0,2,0,10)				(0,2,0,0)
34	(1,0,0,0)	(0,1,0,0)	(0,0,1,0)	(0,2,0,0)	(1,3,1,0)	(0,3,1,0)			
35	(0,3,1,0)	(0,1,0,0)	(0,0,1,0)	(0,1,1,0)	(0,5,3,0)		(0,0,3,0)		
36	(1,0,0,0)	(0,0,3,0)	(0,0,1,0)	(0,0,0,1)	(1,0,4,1)	(0,0,4,1)			
37	(0,0,4,1)	(0,0,2,1)	(0,0,1,0)	(0,0,0,1)	(0,0,7,3)			(0,0,0,3)	
38	(0,0,2,3)	(0,0,0,5)	(0,0,0,3)	(0,0,0,1)	(0,0,2,12)				(0,0,2,0)
39	(1,0,0,0)	(0,1,0,0)	(0,0,1,0)	(0,0,2,0)	(1,1,3,0)	(0,1,3,0)			
40	(0,1,1,1)	(0,0,3,0)	(0,0,1,0)	(0,0,2,0)	(0,1,7,1)			(0,1,0,1)	
41	(0,1,3,0)	(0,1,0,0)	(0,0,1,0)	(0,0,2,0)	(0,2,6,0)			(0,2,0,0)	
42	(1,0,0,0)	(0,1,0,0)	(0,2,0,0)	(0,0,0,1)	(1,3,0,1)	(0,3,0,1)			
43	(0,3,0,1)	(0,1,0,0)	(0,1,0,1)	(0,0,0,1)	(0,5,0,3)		(0,0,0,3)		
44	(1,0,0,0)	(0,0,0,3)	(0,0,1,0)	(0,0,0,1)	(1,0,1,4)	(0,0,1,4)			
45	(0,0,1,4)	(0,0,0,3)	(0,0,0,3)	(0,0,0,1)	(0,0,1,11)				(0,0,1,0)
46	(1,0,0,0)	(0,1,0,0)	(0,0,1,0)	(0,0,1,0)	(1,1,2,0)	(0,1,2,0)			
47	(0,1,2,0)	(0,0,3,0)	(0,0,1,0)	(0,0,1,0)	(0,1,7,0)			(0,1,0,0)	
48	(1,0,0,0)	(0,1,0,0)	(0,1,0,0)	(0,0,0,1)	(1,2,0,1)	(0,2,0,1)			
49	(0,2,0,1)	(0,1,0,0)	(0,1,0,0)	(0,0,0,1)	(0,4,0,2)		(0,0,0,2)		
50	(1,0,0,0)	(0,0,0,2)	(0,1,0,0)	(0,0,0,1)	(1,1,0,3)	(0,1,0,3)			
51	(0,1,0,3)	(0,0,0,2)	(0,0,0,3)	(0,0,0,1)	(0,1,0,9)				(0,1,0,0)
52	(1,0,0,0)	(0,1,0,0)	(0,1,0,0)	(0,1,0,0)	(1,3,0,0)	(0,3,0,0)			
53	(0,3,0,0)	(0,1,0,0)	(0,0,1,0)	(0,1,0,0)	(0,5,1,0)		(0,0,1,0)		
54	(1,0,0,0)	(0,0,1,0)	(0,0,1,0)	(0,0,1,0)	(1,0,3,0)	(0,0,3,0)			
55	(0,0,3,0)	(0,0,1,0)	(0,0,1,0)	(0,0,1,0)	(0,0,6,0)		<b>(0,0,0,0)</b>		

TABLE 15. A sequence witnessing that the function  $\bar{h} = (2, 7, 7, 11)$  is 0-generating

$m$	$\bar{h}^{[m]}(0)$	$\bar{h}^{[m]}(1)$	$\bar{h}^{[m]}(2)$	$\bar{h}^{[m]}(3)$	$\sum_{i \in \mathbb{3}} \bar{h}^{[m]}(i)$	$\bar{h}^{\{m+1\}}(0)$	$\bar{h}^{\{m+1\}}(1)$	$\bar{h}^{\{m+1\}}(2)$	$\bar{h}^{\{m+1\}}(3)$
0	(1,0,0,0)	(0,1,0,0)	(0,0,1,0)	(0,0,0,1)	(1,1,1,1)	(0,1,1,1)			
1	(0,1,1,1)	(0,1,0,0)	(0,0,1,0)	(0,0,0,1)	(0,2,2,2)			(0,2,0,2)	
2	(0,1,1,1)	(0,1,0,0)	(0,2,0,2)	(0,0,0,1)	(0,4,1,4)				(0,4,1,0)
3	(0,1,1,1)	(0,1,0,0)	(0,0,1,0)	(0,4,1,0)	(0,6,3,1)		(0,0,3,1)		
4	(0,1,1,1)	(0,0,3,1)	(0,0,1,0)	(0,0,0,1)	(0,1,5,3)			(0,1,0,3)	
5	(1,0,0,0)	(0,1,0,0)	(0,1,0,3)	(0,0,0,1)	(1,2,0,4)	(0,2,0,4)			
6	(0,2,0,4)	(0,1,0,0)	(0,1,0,3)	(0,0,0,1)	(0,4,0,8)		(0,0,0,8)		(0,4,0,0)
7	(0,1,1,1)	(0,1,0,0)	(0,0,1,0)	(0,4,0,0)	(0,6,2,1)		(0,0,2,1)		
8	(1,0,0,0)	(0,0,2,1)	(0,0,1,0)	(0,0,0,1)	(1,0,3,2)	(0,0,3,2)			
9	(0,1,1,1)	(0,0,0,8)	(0,0,1,0)	(0,0,0,1)	(0,1,2,10)				(0,1,2,0)
10	(1,0,0,0)	(0,1,0,0)	(0,0,1,0)	(0,1,2,0)	(1,2,3,0)	(0,2,3,0)			
11	(0,2,3,0)	(0,1,0,0)	(0,0,1,0)	(0,1,2,0)	(0,4,6,0)			(0,4,0,0)	
12	(0,1,1,1)	(0,1,0,0)	(0,4,0,0)	(0,0,0,1)	(0,6,1,2)		(0,0,1,2)		
13	(1,0,0,0)	(0,0,1,2)	(0,0,1,0)	(0,0,0,1)	(1,0,2,3)	(0,0,2,3)			
14	(0,0,2,3)	(0,0,1,2)	(0,0,1,0)	(0,0,0,1)	(0,0,4,6)				(0,0,4,0)
15	(0,1,1,1)	(0,1,0,0)	(0,0,1,0)	(0,0,4,0)	(0,2,6,1)			(0,2,0,1)	
16	(1,0,0,0)	(0,1,0,0)	(0,2,0,1)	(0,0,0,1)	(1,3,0,2)	(0,3,0,2)			
17	(0,3,0,2)	(0,1,0,0)	(0,2,0,1)	(0,0,0,1)	(0,6,0,4)		(0,0,0,4)		
18	(0,0,2,3)	(0,0,0,4)	(0,0,1,0)	(0,0,0,1)	(0,0,3,8)				(0,0,3,0)
19	(0,0,3,2)	(0,0,2,1)	(0,0,1,0)	(0,0,0,1)	(0,0,6,4)			(0,0,0,4)	
20	(1,0,0,0)	(0,1,0,0)	(0,0,0,4)	(0,0,0,1)	(1,1,0,5)	(0,1,0,5)			
21	(0,1,1,1)	(0,0,0,4)	(0,0,0,4)	(0,0,0,1)	(0,1,1,10)				(0,1,1,0)
22	(1,0,0,0)	(0,1,0,0)	(0,0,1,0)	(0,1,1,0)	(1,2,2,0)	(0,2,2,0)			
23	(0,2,2,0)	(0,1,0,0)	(0,0,1,0)	(0,0,3,0)	(0,3,6,0)			(0,3,0,0)	
24	(0,1,0,5)	(0,1,0,0)	(0,0,0,4)	(0,0,0,1)	(0,2,0,10)				(0,2,0,0)
25	(1,0,0,0)	(0,1,0,0)	(0,0,1,0)	(0,2,0,0)	(1,3,1,0)	(0,3,1,0)			
26	(0,3,1,0)	(0,1,0,0)	(0,0,1,0)	(0,2,0,0)	(0,6,2,0)		(0,0,2,0)		
27	(1,0,0,0)	(0,0,2,0)	(0,0,1,0)	(0,0,0,1)	(1,0,3,1)	(0,0,3,1)			
28	(0,0,3,1)	(0,0,2,0)	(0,0,1,0)	(0,0,0,1)	(0,0,6,2)			(0,0,0,2)	
29	(1,0,0,0)	(0,1,0,0)	(0,0,0,2)	(0,0,0,1)	(1,1,0,3)	(0,1,0,3)			
30	(0,1,0,3)	(0,0,0,4)	(0,0,0,2)	(0,0,0,1)	(0,1,0,10)				(0,1,0,0)
31	(1,0,0,0)	(0,1,0,0)	(0,0,1,0)	(0,1,0,0)	(1,2,1,0)	(0,2,1,0)			
32	(0,2,1,0)	(0,1,0,0)	(0,3,0,0)	(0,0,0,1)	(0,6,1,1)		(0,0,1,1)		
33	(1,0,0,0)	(0,0,1,1)	(0,0,1,0)	(0,0,0,1)	(1,0,2,2)	(0,0,2,2)			
34	(0,0,2,2)	(0,0,0,4)	(0,0,0,2)	(0,0,0,1)	(0,0,2,9)				(0,0,2,0)
35	(0,2,1,0)	(0,0,2,0)	(0,0,1,0)	(0,0,2,0)	(0,2,6,0)			(0,2,0,0)	
36	(0,2,1,0)	(0,1,0,0)	(0,2,0,0)	(0,1,0,0)	(0,6,1,0)		(0,0,1,0)		
37	(1,0,0,0)	(0,0,1,0)	(0,0,1,0)	(0,1,0,0)	(1,1,2,0)	(0,1,2,0)			
38	(0,1,2,0)	(0,0,1,0)	(0,0,1,0)	(0,0,2,0)	(0,1,6,0)			(0,1,0,0)	
39	(1,0,0,0)	(0,1,0,0)	(0,1,0,0)	(0,1,0,0)	(1,3,0,0)	(0,3,0,0)			
40	(0,3,0,0)	(0,1,0,0)	(0,1,0,0)	(0,1,0,0)	(0,6,0,0)		<b>(0,0,0,0)</b>		

TABLE 16. A sequence witnessing that the function  $\hbar = (3, 3, 8, 11)$  is 0-generating

$m$	$\hbar^{[m]}(0)$	$\hbar^{[m]}(1)$	$\hbar^{[m]}(2)$	$\hbar^{[m]}(3)$	$\sum_{i \in \mathbb{3}} \hbar^{[m]}(i)$	$\hbar^{\{m+1\}}(0)$	$\hbar^{\{m+1\}}(1)$	$\hbar^{\{m+1\}}(2)$	$\hbar^{\{m+1\}}(3)$
0	(1,0,0,0)	(0,1,0,0)	(0,0,1,0)	(0,0,0,1)	(1,1,1,1)	(0,1,1,1)	(1,0,1,1)		
1	(1,0,0,0)	(1,0,1,1)	(0,0,1,0)	(0,0,0,1)	(2,0,2,2)	(0,0,2,2)			
2	(0,1,1,1)	(0,1,0,0)	(0,0,1,0)	(0,0,0,1)	(0,2,2,2)		(0,0,2,2)		
3	(0,0,2,2)	(0,0,2,2)	(0,0,1,0)	(0,0,0,1)	(0,0,5,5)			(0,0,0,5)	
4	(1,0,0,0)	(0,0,2,2)	(0,0,0,5)	(0,0,0,1)	(1,0,2,8)				(1,0,2,0)
5	(1,0,0,0)	(0,1,0,0)	(0,0,1,0)	(1,0,2,0)	(2,1,3,0)	(0,1,3,0)			
6	(0,1,3,0)	(0,1,0,0)	(0,0,1,0)	(0,0,0,1)	(0,2,4,1)		(0,0,4,1)		
7	(1,0,0,0)	(0,0,4,1)	(0,0,1,0)	(0,0,0,1)	(1,0,5,2)			(1,0,0,2)	
8	(1,0,0,0)	(0,1,0,0)	(1,0,0,2)	(0,0,0,1)	(2,1,0,3)	(0,1,0,3)			
9	(0,1,0,3)	(0,1,0,0)	(0,0,1,0)	(0,0,0,1)	(0,2,1,4)		(0,0,1,4)		
10	(1,0,0,0)	(0,0,1,4)	(0,0,0,5)	(0,0,0,1)	(1,0,1,10)				(1,0,1,0)
11	(1,0,0,0)	(0,1,0,0)	(0,0,1,0)	(1,0,1,0)	(2,1,2,0)	(0,1,2,0)			
12	(0,1,2,0)	(0,1,0,0)	(0,0,1,0)	(0,0,0,1)	(0,2,3,1)		(0,0,3,1)		
13	(0,1,2,0)	(0,0,3,1)	(0,0,1,0)	(0,0,0,1)	(0,1,6,2)			(0,1,0,2)	
14	(1,0,0,0)	(0,1,0,0)	(0,1,0,2)	(0,0,0,1)	(1,2,0,3)		(1,0,0,3)		
15	(1,0,0,0)	(1,0,0,3)	(0,0,1,0)	(0,0,0,1)	(2,0,1,4)	(0,0,1,4)			
16	(0,0,2,2)	(0,0,3,1)	(0,0,1,0)	(0,0,0,1)	(0,0,6,4)			(0,0,0,4)	
17	(0,0,1,4)	(0,1,0,0)	(0,0,0,4)	(0,0,0,1)	(0,1,1,9)				(0,1,1,0)
18	(1,0,0,0)	(0,1,0,0)	(0,0,1,0)	(0,1,1,0)	(1,2,2,0)		(1,0,2,0)		
19	(1,0,0,0)	(1,0,2,0)	(0,0,1,0)	(0,0,0,1)	(2,0,3,1)	(0,0,3,1)			
20	(0,0,3,1)	(0,0,3,1)	(0,0,1,0)	(0,0,0,1)	(0,0,7,3)			(0,0,0,3)	
21	(0,0,2,2)	(0,0,1,4)	(0,0,0,3)	(0,0,0,1)	(0,0,3,10)				(0,0,3,0)
22	(1,0,0,0)	(0,0,3,1)	(0,0,1,0)	(0,0,3,0)	(1,0,7,1)			(1,0,0,1)	
23	(1,0,0,0)	(0,1,0,0)	(1,0,0,1)	(0,0,0,1)	(2,1,0,2)	(0,1,0,2)			
24	(0,1,0,2)	(0,1,0,0)	(0,0,1,0)	(0,0,0,1)	(0,2,1,3)		(0,0,1,3)		
25	(0,1,0,2)	(0,1,0,0)	(0,0,0,3)	(0,0,0,1)	(0,2,0,6)		(0,0,0,6)		
26	(1,0,0,0)	(0,0,0,6)	(0,0,0,3)	(0,0,0,1)	(1,0,0,10)				(1,0,0,0)
27	(1,0,0,0)	(0,1,0,0)	(0,0,1,0)	(1,0,0,0)	(2,1,1,0)	(0,1,1,0)			
28	(0,1,1,0)	(0,1,0,0)	(0,0,1,0)	(0,0,0,1)	(0,2,2,1)		(0,0,2,1)		
29	(0,1,1,0)	(0,0,2,1)	(0,0,1,0)	(0,0,3,0)	(0,1,7,1)			(0,1,0,1)	
30	(1,0,0,0)	(0,1,0,0)	(0,1,0,1)	(0,0,0,1)	(1,2,0,2)		(1,0,0,2)		
31	(1,0,0,0)	(1,0,0,2)	(0,0,1,0)	(0,0,0,1)	(2,0,1,3)	(0,0,1,3)			
32	(0,0,1,3)	(0,0,1,3)	(0,0,0,3)	(0,0,0,1)	(0,0,2,10)				(0,0,2,0)
33	(0,1,1,0)	(0,1,0,0)	(0,0,1,0)	(0,0,2,0)	(0,2,4,0)		(0,0,4,0)		
34	(1,0,0,0)	(0,0,4,0)	(0,0,1,0)	(0,0,2,0)	(1,0,7,0)			(1,0,0,0)	
35	(1,0,0,0)	(0,1,0,0)	(1,0,0,0)	(0,0,0,1)	(2,1,0,1)	(0,1,0,1)			
36	(0,1,0,1)	(0,1,0,0)	(0,0,0,3)	(0,0,0,1)	(0,2,0,5)		(0,0,0,5)		
37	(0,1,0,1)	(0,0,0,5)	(0,0,0,3)	(0,0,0,1)	(0,1,0,10)				(0,1,0,0)
38	(1,0,0,0)	(0,1,0,0)	(0,0,1,0)	(0,1,0,0)	(1,2,1,0)		(1,0,1,0)		
39	(1,0,0,0)	(1,0,1,0)	(0,0,1,0)	(0,0,0,1)	(2,0,2,1)	(0,0,2,1)			
40	(1,0,0,0)	(1,0,1,0)	(0,0,1,0)	(0,0,2,0)	(2,0,4,0)	(0,0,4,0)			
41	(0,0,2,1)	(0,0,2,1)	(0,0,1,0)	(0,0,2,0)	(0,0,7,2)			(0,0,0,2)	
42	(0,1,0,1)	(0,1,0,0)	(0,0,0,2)	(0,0,0,1)	(0,2,0,4)		(0,0,0,4)		
43	(0,0,4,0)	(0,1,0,0)	(0,0,1,0)	(0,0,2,0)	(0,1,7,0)			(0,1,0,0)	
44	(1,0,0,0)	(0,1,0,0)	(0,1,0,0)	(0,0,0,1)	(1,2,0,1)		(1,0,0,1)		
45	(1,0,0,0)	(1,0,0,1)	(0,0,1,0)	(0,0,0,1)	(2,0,1,2)	(0,0,1,2)			
46	(0,0,1,2)	(0,0,0,4)	(0,0,0,2)	(0,0,0,1)	(0,0,1,9)				(0,0,1,0)
47	(1,0,0,0)	(1,0,1,0)	(0,0,1,0)	(0,0,1,0)	(2,0,3,0)	(0,0,3,0)			
48	(0,1,1,0)	(0,1,0,0)	(0,0,1,0)	(0,0,1,0)	(0,2,3,0)		(0,0,3,0)		
49	(0,0,3,0)	(0,0,3,0)	(0,0,1,0)	(0,0,0,1)	(0,0,7,1)			(0,0,0,1)	
50	(1,0,0,0)	(1,0,0,1)	(0,0,0,1)	(0,0,0,1)	(2,0,0,3)	(0,0,0,3)			
51	(0,0,0,3)	(0,0,0,3)	(0,0,0,1)	(0,0,0,1)	(0,0,0,8)				<b>(0,0,0,0)</b>

TABLE 17. A sequence witnessing that the function  $\tilde{h} = (3, 4, 5, 12)$  is 0-generating

$m$	$\tilde{h}^{[m]}(0)$	$\tilde{h}^{[m]}(1)$	$\tilde{h}^{[m]}(2)$	$\tilde{h}^{[m]}(3)$	$\sum_{i \in \mathbb{3}} \tilde{h}^{[m]}(i)$	$\tilde{h}^{\{m+1\}}(0)$	$\tilde{h}^{\{m+1\}}(1)$	$\tilde{h}^{\{m+1\}}(2)$	$\tilde{h}^{\{m+1\}}(3)$
0	(1,0,0,0)	(0,1,0,0)	(0,0,1,0)	(0,0,0,1)	(1,1,1,1)	(0,1,1,1)			
1	(0,1,1,1)	(0,1,0,0)	(0,0,1,0)	(0,0,0,1)	(0,2,2,2)		(0,0,2,2)		
2	(0,1,1,1)	(0,0,2,2)	(0,0,1,0)	(0,0,0,1)	(0,1,4,4)			(0,1,0,4)	
3	(0,1,1,1)	(0,1,0,0)	(0,1,0,4)	(0,0,0,1)	(0,3,1,6)		(0,0,1,6)		
4	(1,0,0,0)	(0,0,1,6)	(0,0,1,0)	(0,0,0,1)	(1,0,2,7)				(1,0,2,0)
5	(1,0,0,0)	(0,1,0,0)	(0,0,1,0)	(1,0,2,0)	(2,1,3,0)	(0,1,3,0)			
6	(0,1,3,0)	(0,1,0,0)	(0,0,1,0)	(0,0,0,1)	(0,2,4,1)			(0,2,0,1)	
7	(1,0,0,0)	(0,1,0,0)	(0,2,0,1)	(0,0,0,1)	(1,3,0,2)		(1,0,0,2)		
8	(1,0,0,0)	(1,0,0,2)	(0,0,1,0)	(0,0,0,1)	(2,0,1,3)	(0,0,1,3)			
9	(0,0,1,3)	(0,1,0,0)	(0,0,1,0)	(0,0,0,1)	(0,1,2,4)			(0,1,0,4)	
10	(0,0,1,3)	(0,1,0,0)	(0,1,0,4)	(0,0,0,1)	(0,2,1,8)				(0,2,1,0)
11	(1,0,0,0)	(0,1,0,0)	(0,0,1,0)	(0,2,1,0)	(1,3,2,0)		(1,0,2,0)		
12	(1,0,0,0)	(1,0,2,0)	(0,0,1,0)	(0,0,0,1)	(2,0,3,1)	(0,0,3,1)			
13	(0,0,3,1)	(0,1,0,0)	(0,0,1,0)	(0,0,0,1)	(0,1,4,2)			(0,1,0,2)	
14	(0,1,1,1)	(0,1,0,0)	(0,1,0,2)	(0,0,0,1)	(0,3,1,4)		(0,0,1,4)		
15	(0,0,1,3)	(0,0,2,2)	(0,0,1,0)	(0,0,0,1)	(0,0,4,6)			(0,0,0,6)	
16	(1,0,0,0)	(0,0,1,4)	(0,0,0,6)	(0,0,0,1)	(1,0,1,11)				(1,0,1,0)
17	(1,0,0,0)	(0,1,0,0)	(0,0,1,0)	(1,0,1,0)	(2,1,2,0)	(0,1,2,0)			
18	(0,1,2,0)	(0,1,0,0)	(0,0,1,0)	(0,0,0,1)	(0,2,3,1)		(0,0,3,1)		
19	(1,0,0,0)	(0,0,3,1)	(0,0,1,0)	(0,0,0,1)	(1,0,4,2)			(1,0,0,2)	
20	(1,0,0,0)	(0,1,0,0)	(1,0,0,2)	(0,0,0,1)	(2,1,0,3)	(0,1,0,3)			
21	(0,1,0,3)	(0,1,0,0)	(0,1,0,2)	(0,0,0,1)	(0,3,0,6)		(0,0,0,6)		
22	(0,1,0,3)	(0,1,0,0)	(0,0,0,6)	(0,0,0,1)	(0,2,0,10)				(0,2,0,0)
23	(1,0,0,0)	(0,1,0,0)	(0,0,1,0)	(0,2,0,0)	(1,3,1,0)		(1,0,1,0)		
24	(1,0,0,0)	(1,0,1,0)	(0,0,1,0)	(0,0,0,1)	(2,0,2,1)	(0,0,2,1)			
25	(0,0,2,1)	(0,0,0,6)	(0,0,1,0)	(0,0,0,1)	(0,0,3,8)				(0,0,3,0)
26	(1,0,0,0)	(0,1,0,0)	(0,0,1,0)	(0,0,3,0)	(1,1,4,0)			(1,1,0,0)	
27	(1,0,0,0)	(0,1,0,0)	(1,1,0,0)	(0,0,0,1)	(2,2,0,1)	(0,2,0,1)			
28	(0,2,0,1)	(0,1,0,0)	(0,0,1,0)	(0,0,0,1)	(0,3,1,2)		(0,0,1,2)		
29	(0,0,2,1)	(0,0,1,2)	(0,0,1,0)	(0,0,0,1)	(0,0,4,4)			(0,0,0,4)	
30	(1,0,0,0)	(0,0,0,6)	(0,0,0,4)	(0,0,0,1)	(1,0,0,11)				(1,0,0,0)
31	(1,0,0,0)	(0,1,0,0)	(0,0,1,0)	(1,0,0,0)	(2,1,1,0)	(0,1,1,0)			
32	(0,1,1,0)	(0,0,0,6)	(0,0,0,4)	(0,0,0,1)	(0,1,1,11)				(0,1,1,0)
33	(0,1,1,0)	(0,1,0,0)	(0,0,1,0)	(0,1,1,0)	(0,3,3,0)		(0,0,3,0)		
34	(1,0,0,0)	(0,0,3,0)	(0,0,1,0)	(0,0,0,1)	(1,0,4,1)			(1,0,0,1)	
35	(1,0,0,0)	(0,1,0,0)	(1,0,0,1)	(0,0,0,1)	(2,1,0,2)	(0,1,0,2)			
36	(0,1,0,2)	(0,1,0,0)	(0,1,0,2)	(0,0,0,1)	(0,3,0,5)		(0,0,0,5)		
37	(0,0,2,1)	(0,0,0,5)	(0,0,0,4)	(0,0,0,1)	(0,0,2,11)				(0,0,2,0)
38	(0,1,1,0)	(0,1,0,0)	(0,0,1,0)	(0,0,2,0)	(0,2,4,0)			(0,2,0,0)	
39	(1,0,0,0)	(0,1,0,0)	(0,2,0,0)	(0,0,0,1)	(1,3,0,1)		(1,0,0,1)		
40	(1,0,0,0)	(1,0,0,1)	(0,0,1,0)	(0,0,0,1)	(2,0,1,2)	(0,0,1,2)			
41	(1,0,0,0)	(1,0,0,1)	(0,0,0,4)	(0,0,0,1)	(2,0,0,6)	(0,0,0,6)			
42	(0,0,0,6)	(0,1,0,0)	(0,0,0,4)	(0,0,0,1)	(0,1,0,11)				(0,1,0,0)
43	(0,1,1,0)	(0,1,0,0)	(0,0,1,0)	(0,1,0,0)	(0,3,2,0)		(0,0,2,0)		
44	(0,1,1,0)	(0,0,2,0)	(0,0,1,0)	(0,0,0,1)	(0,1,4,1)			(0,1,0,1)	
45	(0,0,1,2)	(0,0,2,0)	(0,0,1,0)	(0,0,0,1)	(0,0,4,3)			(0,0,0,3)	
46	(0,0,1,2)	(0,0,0,5)	(0,0,0,3)	(0,0,0,1)	(0,0,1,11)				(0,0,1,0)
47	(1,0,0,0)	(0,0,2,0)	(0,0,1,0)	(0,0,1,0)	(1,0,4,0)			(1,0,0,0)	
48	(1,0,0,0)	(0,1,0,0)	(1,0,0,0)	(0,0,0,1)	(2,1,0,1)	(0,1,0,1)			
49	(0,1,0,1)	(0,1,0,0)	(0,1,0,1)	(0,0,0,1)	(0,3,0,3)		(0,0,0,3)		
50	(0,0,0,4)	(0,0,0,3)	(0,0,0,3)	(0,0,0,1)	(0,0,0,11)				<b>(0,0,0,0)</b>

TABLE 18. A sequence witnessing that the function  $\tilde{h} = (3, 4, 6, 10)$  is 0-generating

$m$	$\tilde{h}^{[m]}(0)$	$\tilde{h}^{[m]}(1)$	$\tilde{h}^{[m]}(2)$	$\tilde{h}^{[m]}(3)$	$\sum_{i \in \mathbb{3}} \tilde{h}^{[m]}(i)$	$\tilde{h}^{\{m+1\}}(0)$	$\tilde{h}^{\{m+1\}}(1)$	$\tilde{h}^{\{m+1\}}(2)$	$\tilde{h}^{\{m+1\}}(3)$
0	(1,0,0,0)	(0,1,0,0)	(0,0,1,0)	(0,0,0,1)	(1,1,1,1)	(0,1,1,1)			
1	(0,1,1,1)	(0,1,0,0)	(0,0,1,0)	(0,0,0,1)	(0,2,2,2)		(0,0,2,2)		
2	(0,1,1,1)	(0,0,2,2)	(0,0,1,0)	(0,0,0,1)	(0,1,4,4)			(0,1,0,4)	
3	(0,1,1,1)	(0,1,0,0)	(0,1,0,4)	(0,0,0,1)	(0,3,1,6)		(0,0,1,6)		
4	(1,0,0,0)	(0,0,1,6)	(0,0,1,0)	(0,0,0,1)	(1,0,2,7)				(1,0,2,0)
5	(1,0,0,0)	(0,1,0,0)	(0,0,1,0)	(1,0,2,0)	(2,1,3,0)	(0,1,3,0)			
6	(0,1,3,0)	(0,1,0,0)	(0,0,1,0)	(0,0,0,1)	(0,2,4,1)		(0,0,4,1)		
7	(1,0,0,0)	(0,0,4,1)	(0,0,1,0)	(0,0,0,1)	(1,0,5,2)			(1,0,0,2)	
8	(1,0,0,0)	(0,1,0,0)	(1,0,0,2)	(0,0,0,1)	(2,1,0,3)	(0,1,0,3)			
9	(0,1,0,3)	(0,1,0,0)	(0,0,1,0)	(0,0,0,1)	(0,2,1,4)		(0,0,1,4)		(0,2,1,0)
10	(1,0,0,0)	(0,1,0,0)	(0,0,1,0)	(0,2,1,0)	(1,3,2,0)		(1,0,2,0)		
11	(1,0,0,0)	(1,0,2,0)	(0,0,1,0)	(0,0,0,1)	(2,0,3,1)	(0,0,3,1)			
12	(0,0,3,1)	(0,1,0,0)	(0,0,1,0)	(0,0,0,1)	(0,1,4,2)			(0,1,0,2)	
13	(0,0,3,1)	(0,0,1,4)	(0,0,1,0)	(0,0,0,1)	(0,0,5,6)			(0,0,0,6)	
14	(1,0,0,0)	(0,1,0,0)	(0,0,0,6)	(0,0,0,1)	(1,1,0,7)				(1,1,0,0)
15	(1,0,0,0)	(0,1,0,0)	(0,0,1,0)	(1,1,0,0)	(2,2,1,0)	(0,2,1,0)			
16	(0,2,1,0)	(0,1,0,0)	(0,0,1,0)	(0,0,0,1)	(0,3,2,1)		(0,0,2,1)		
17	(0,1,0,3)	(0,1,0,0)	(0,1,0,2)	(0,0,0,1)	(0,3,0,6)		(0,0,0,6)		
18	(0,0,3,1)	(0,0,0,6)	(0,0,1,0)	(0,0,0,1)	(0,0,4,8)				(0,0,4,0)
19	(1,0,0,0)	(0,1,0,0)	(0,0,1,0)	(0,0,4,0)	(1,1,5,0)			(1,1,0,0)	
20	(1,0,0,0)	(0,1,0,0)	(1,1,0,0)	(0,0,0,1)	(2,2,0,1)	(0,2,0,1)			
21	(0,2,0,1)	(0,1,0,0)	(0,0,1,0)	(0,0,0,1)	(0,3,1,2)		(0,0,1,2)		
22	(0,0,3,1)	(0,0,1,2)	(0,0,1,0)	(0,0,0,1)	(0,0,5,4)			(0,0,0,4)	
23	(0,1,0,3)	(0,1,0,0)	(0,0,0,4)	(0,0,0,1)	(0,2,0,8)				(0,2,0,0)
24	(1,0,0,0)	(0,1,0,0)	(0,0,1,0)	(0,2,0,0)	(1,3,1,0)		(1,0,1,0)		
25	(1,0,0,0)	(1,0,1,0)	(0,0,1,0)	(0,0,0,1)	(2,0,2,1)	(0,0,2,1)			
26	(0,0,2,1)	(0,0,1,2)	(0,0,0,4)	(0,0,0,1)	(0,0,3,8)				(0,0,3,0)
27	(0,0,2,1)	(0,0,2,1)	(0,0,1,0)	(0,0,0,1)	(0,0,5,3)			(0,0,0,3)	
28	(0,2,0,1)	(0,1,0,0)	(0,0,0,3)	(0,0,0,1)	(0,3,0,5)		(0,0,0,5)		
29	(1,0,0,0)	(0,0,0,5)	(0,0,0,3)	(0,0,0,1)	(1,0,0,9)				(1,0,0,0)
30	(1,0,0,0)	(0,1,0,0)	(0,0,1,0)	(1,0,0,0)	(2,1,1,0)	(0,1,1,0)			
31	(0,1,1,0)	(0,1,0,0)	(0,0,1,0)	(0,0,3,0)	(0,2,5,0)			(0,2,0,0)	
32	(1,0,0,0)	(0,1,0,0)	(0,2,0,0)	(0,0,0,1)	(1,3,0,1)		(1,0,0,1)		
33	(1,0,0,0)	(1,0,0,1)	(0,0,1,0)	(0,0,0,1)	(2,0,1,2)	(0,0,1,2)			
34	(1,0,0,0)	(1,0,0,1)	(0,0,0,3)	(0,0,0,1)	(2,0,0,5)	(0,0,0,5)			
35	(0,1,1,0)	(0,0,0,5)	(0,0,0,3)	(0,0,0,1)	(0,1,1,9)				(0,1,1,0)
36	(0,1,1,0)	(0,1,0,0)	(0,0,1,0)	(0,1,1,0)	(0,3,3,0)		(0,0,3,0)		
37	(0,1,1,0)	(0,0,3,0)	(0,0,1,0)	(0,0,0,1)	(0,1,5,1)			(0,1,0,1)	
38	(0,0,1,2)	(0,0,1,2)	(0,0,0,3)	(0,0,0,1)	(0,0,2,8)				(0,0,2,0)
39	(0,0,0,5)	(0,1,0,0)	(0,0,0,3)	(0,0,0,1)	(0,1,0,9)				(0,1,0,0)
40	(0,1,1,0)	(0,1,0,0)	(0,0,1,0)	(0,1,0,0)	(0,3,2,0)		(0,0,2,0)		
41	(1,0,0,0)	(0,0,2,0)	(0,0,1,0)	(1,0,0,0)	(2,0,3,0)	(0,0,3,0)			
42	(1,0,0,0)	(0,0,2,0)	(0,0,1,0)	(0,0,2,0)	(1,0,5,0)			(1,0,0,0)	
43	(1,0,0,0)	(0,1,0,0)	(1,0,0,0)	(0,0,0,1)	(2,1,0,1)	(0,1,0,1)			
44	(1,0,0,0)	(0,1,0,0)	(1,0,0,0)	(0,1,0,0)	(2,2,0,0)	(0,2,0,0)			
45	(0,2,0,0)	(0,1,0,0)	(0,0,1,0)	(0,0,0,1)	(0,3,1,1)		(0,0,1,1)		
46	(0,1,0,1)	(0,1,0,0)	(0,1,0,1)	(0,0,0,1)	(0,3,0,3)		(0,0,0,3)		
47	(0,0,3,0)	(0,0,1,1)	(0,0,1,0)	(0,0,0,1)	(0,0,5,2)			(0,0,0,2)	
48	(0,0,1,2)	(0,0,0,3)	(0,0,0,2)	(0,0,0,1)	(0,0,1,8)				(0,0,1,0)
49	(0,1,1,0)	(0,0,2,0)	(0,0,1,0)	(0,0,1,0)	(0,1,5,0)			(0,1,0,0)	
50	(1,0,0,0)	(0,1,0,0)	(0,1,0,0)	(0,1,0,0)	(1,3,0,0)		(1,0,0,0)		
51	(1,0,0,0)	(1,0,0,0)	(0,0,1,0)	(0,0,1,0)	(2,0,2,0)	(0,0,2,0)			
52	(0,0,2,0)	(0,0,1,1)	(0,0,1,0)	(0,0,1,0)	(0,0,5,1)			(0,0,0,1)	
53	(1,0,0,0)	(1,0,0,0)	(0,0,0,1)	(0,0,0,1)	(2,0,0,2)	(0,0,0,2)			
54	(0,0,0,2)	(0,0,0,3)	(0,0,0,1)	(0,0,0,1)	(0,0,0,7)				<b>(0,0,0,0)</b>

TABLE 19. A sequence witnessing that the function  $\tilde{h} = (4, 4, 4, 12)$  is 0-generating

$m$	$\tilde{h}^{[m]}(0)$	$\tilde{h}^{[m]}(1)$	$\tilde{h}^{[m]}(2)$	$\tilde{h}^{[m]}(3)$	$\sum_{i \in \mathbb{3}} \tilde{h}^{[m]}(i)$	$\tilde{h}^{\{m+1\}}(0)$	$\tilde{h}^{\{m+1\}}(1)$	$\tilde{h}^{\{m+1\}}(2)$	$\tilde{h}^{\{m+1\}}(3)$
0	(1,0,0,0)	(0,1,0,0)	(0,0,1,0)	(0,0,0,1)	(1,1,1,1)	(0,1,1,1)		(1,1,0,1)	
1	(1,0,0,0)	(0,1,0,0)	(1,1,0,1)	(0,0,0,1)	(2,2,0,2)		(2,0,0,2)		
2	(1,0,0,0)	(2,0,0,2)	(0,0,1,0)	(0,0,0,1)	(3,0,1,3)	(0,0,1,3)			
3	(0,1,1,1)	(0,1,0,0)	(0,0,1,0)	(0,0,0,1)	(0,2,2,2)			(0,2,0,2)	
4	(1,0,0,0)	(0,1,0,0)	(0,2,0,2)	(0,0,0,1)	(1,3,0,3)		(1,0,0,3)		
5	(1,0,0,0)	(1,0,0,3)	(0,0,1,0)	(0,0,0,1)	(2,0,1,4)				(2,0,1,0)
6	(1,0,0,0)	(0,1,0,0)	(0,0,1,0)	(2,0,1,0)	(3,1,2,0)	(0,1,2,0)			
7	(0,1,2,0)	(0,1,0,0)	(0,0,1,0)	(0,0,0,1)	(0,2,3,1)			(0,2,0,1)	
8	(0,0,1,3)	(0,1,0,0)	(0,0,1,0)	(0,0,0,1)	(0,1,2,4)			(0,1,0,4)	(0,1,2,0)
9	(1,0,0,0)	(0,1,0,0)	(0,0,1,0)	(0,1,2,0)	(1,2,3,0)			(1,2,0,0)	
10	(1,0,0,0)	(0,1,0,0)	(1,2,0,0)	(0,0,0,1)	(2,3,0,1)		(2,0,0,1)		
11	(1,0,0,0)	(0,1,0,0)	(0,1,0,4)	(0,0,0,1)	(1,2,0,5)				(1,2,0,0)
12	(1,0,0,0)	(0,1,0,0)	(0,0,1,0)	(1,2,0,0)	(2,3,1,0)		(2,0,1,0)		
13	(1,0,0,0)	(2,0,1,0)	(0,0,1,0)	(0,0,0,1)	(3,0,2,1)	(0,0,2,1)			
14	(1,0,0,0)	(2,0,0,1)	(0,0,1,0)	(0,0,0,1)	(3,0,1,2)	(0,0,1,2)			
15	(0,0,1,2)	(0,1,0,0)	(0,0,1,0)	(0,0,0,1)	(0,1,2,3)			(0,1,0,3)	
16	(0,0,1,2)	(0,1,0,0)	(0,2,0,1)	(0,0,0,1)	(0,3,1,4)		(0,0,1,4)		
17	(1,0,0,0)	(0,0,1,4)	(0,0,1,0)	(0,0,0,1)	(1,0,2,5)				(1,0,2,0)
18	(1,0,0,0)	(0,1,0,0)	(0,0,1,0)	(1,0,2,0)	(2,1,3,0)			(2,1,0,0)	
19	(1,0,0,0)	(0,1,0,0)	(2,1,0,0)	(0,0,0,1)	(3,2,0,1)	(0,2,0,1)			
20	(0,2,0,1)	(0,1,0,0)	(0,0,1,0)	(0,0,0,1)	(0,3,1,2)		(0,0,1,2)		
21	(0,0,1,2)	(0,1,0,0)	(0,1,0,3)	(0,0,0,1)	(0,2,1,6)				(0,2,1,0)
22	(1,0,0,0)	(0,1,0,0)	(0,0,1,0)	(0,2,1,0)	(1,3,2,0)		(1,0,2,0)		
23	(1,0,0,0)	(1,0,2,0)	(0,0,1,0)	(0,0,0,1)	(2,0,3,1)			(2,0,0,1)	
24	(1,0,0,0)	(0,1,0,0)	(2,0,0,1)	(0,0,0,1)	(3,1,0,2)	(0,1,0,2)			
25	(0,0,1,2)	(0,0,1,2)	(0,0,1,0)	(0,0,0,1)	(0,0,3,5)			(0,0,0,5)	
26	(0,1,0,2)	(0,1,0,0)	(0,0,0,5)	(0,0,0,1)	(0,2,0,8)				(0,2,0,0)
27	(1,0,0,0)	(0,1,0,0)	(0,0,1,0)	(0,2,0,0)	(1,3,1,0)		(1,0,1,0)		
28	(0,0,1,2)	(0,0,1,2)	(0,0,0,5)	(0,0,0,1)	(0,0,2,10)				(0,0,2,0)
29	(1,0,0,0)	(0,1,0,0)	(0,0,1,0)	(0,0,2,0)	(1,1,3,0)			(1,1,0,0)	
30	(0,0,2,1)	(0,1,0,0)	(0,0,1,0)	(0,0,0,1)	(0,1,3,2)			(0,1,0,2)	
31	(0,1,0,2)	(0,1,0,0)	(0,1,0,2)	(0,0,0,1)	(0,3,0,5)		(0,0,0,5)		
32	(1,0,0,0)	(0,0,0,5)	(0,0,0,5)	(0,0,0,1)	(1,0,0,11)				(1,0,0,0)
33	(1,0,0,0)	(0,1,0,0)	(1,1,0,0)	(1,0,0,0)	(3,2,0,0)	(0,2,0,0)			
34	(1,0,0,0)	(1,0,1,0)	(0,0,1,0)	(1,0,0,0)	(3,0,2,0)	(0,0,2,0)			
35	(0,0,2,0)	(0,1,0,0)	(0,0,1,0)	(0,0,0,1)	(0,1,3,1)			(0,1,0,1)	
36	(0,2,0,0)	(0,1,0,0)	(0,0,1,0)	(0,0,0,1)	(0,3,1,1)		(0,0,1,1)		
37	(0,1,0,2)	(0,1,0,0)	(0,1,0,1)	(0,0,0,1)	(0,3,0,4)		(0,0,0,4)		
38	(0,0,1,2)	(0,0,1,1)	(0,0,1,0)	(0,0,0,1)	(0,0,3,4)			(0,0,0,4)	
39	(0,1,0,2)	(0,0,0,4)	(0,0,0,4)	(0,0,0,1)	(0,1,0,11)				(0,1,0,0)
40	(1,0,0,0)	(0,1,0,0)	(0,1,0,1)	(0,1,0,0)	(1,3,0,1)		(1,0,0,1)		
41	(1,0,0,0)	(0,1,0,0)	(1,1,0,0)	(0,1,0,0)	(2,3,0,0)		(2,0,0,0)		
42	(1,0,0,0)	(2,0,0,0)	(0,0,1,0)	(0,0,0,1)	(3,0,1,1)	(0,0,1,1)			
43	(0,0,1,1)	(0,0,1,1)	(0,0,1,0)	(0,0,0,1)	(0,0,3,3)			(0,0,0,3)	
44	(0,0,1,1)	(0,0,0,4)	(0,0,0,3)	(0,0,0,1)	(0,0,1,9)				(0,0,1,0)
45	(1,0,0,0)	(0,0,1,1)	(0,0,1,0)	(0,0,1,0)	(1,0,3,1)			(1,0,0,1)	
46	(1,0,0,0)	(0,1,0,0)	(1,0,0,1)	(1,0,0,0)	(3,1,0,1)	(0,1,0,1)			
47	(1,0,0,0)	(1,0,0,1)	(1,0,0,1)	(0,0,0,1)	(3,0,0,3)	(0,0,0,3)			
48	(0,1,0,1)	(0,1,0,0)	(0,1,0,1)	(0,0,0,1)	(0,3,0,3)		(0,0,0,3)		
49	(0,0,0,3)	(0,0,0,3)	(0,0,0,3)	(0,0,0,1)	(0,0,0,10)				<b>(0,0,0,0)</b>

TABLE 20. A sequence witnessing that the function  $\hbar = (4, 4, 5, 9)$  is 0-generating

$m$	$\hbar^{[m]}(0)$	$\hbar^{[m]}(1)$	$\hbar^{[m]}(2)$	$\hbar^{[m]}(3)$	$\sum_{i \in \mathbb{3}} \hbar^{[m]}(i)$	$\hbar^{\{m+1\}}(0)$	$\hbar^{\{m+1\}}(1)$	$\hbar^{\{m+1\}}(2)$	$\hbar^{\{m+1\}}(3)$
0	(1,0,0,0)	(0,1,0,0)	(0,0,1,0)	(0,0,0,1)	(1,1,1,1)	(0,1,1,1)			
1	(0,1,1,1)	(0,1,0,0)	(0,0,1,0)	(0,0,0,1)	(0,2,2,2)		(0,0,2,2)	(0,2,0,2)	(0,2,2,0)
2	(1,0,0,0)	(0,1,0,0)	(0,0,1,0)	(0,2,2,0)	(1,3,3,0)		(1,0,3,0)		
3	(1,0,0,0)	(0,1,0,0)	(0,2,0,2)	(0,0,0,1)	(1,3,0,3)		(1,0,0,3)		
4	(1,0,0,0)	(1,0,3,0)	(0,0,1,0)	(0,0,0,1)	(2,0,4,1)			(2,0,0,1)	
5	(1,0,0,0)	(0,1,0,0)	(2,0,0,1)	(0,0,0,1)	(3,1,0,2)	(0,1,0,2)			
6	(1,0,0,0)	(0,0,2,2)	(0,0,1,0)	(0,0,0,1)	(1,0,3,3)			(1,0,0,3)	(1,0,3,0)
7	(1,0,0,0)	(0,1,0,0)	(0,0,1,0)	(1,0,3,0)	(2,1,4,0)			(2,1,0,0)	
8	(1,0,0,0)	(0,1,0,0)	(2,1,0,0)	(0,0,0,1)	(3,2,0,1)	(0,2,0,1)			
9	(1,0,0,0)	(0,1,0,0)	(1,0,0,3)	(0,0,0,1)	(2,1,0,4)				(2,1,0,0)
10	(1,0,0,0)	(0,1,0,0)	(0,0,1,0)	(2,1,0,0)	(3,2,1,0)	(0,2,1,0)			
11	(0,2,1,0)	(0,1,0,0)	(0,0,1,0)	(0,0,0,1)	(0,3,2,1)		(0,0,2,1)		
12	(1,0,0,0)	(0,0,2,1)	(0,0,1,0)	(0,0,0,1)	(1,0,3,2)			(1,0,0,2)	
13	(0,2,0,1)	(0,1,0,0)	(0,0,1,0)	(0,0,0,1)	(0,3,1,2)		(0,0,1,2)		
14	(0,1,0,2)	(0,1,0,0)	(1,0,0,2)	(0,0,0,1)	(1,2,0,5)				(1,2,0,0)
15	(1,0,0,0)	(0,1,0,0)	(0,0,1,0)	(1,2,0,0)	(2,3,1,0)		(2,0,1,0)		
16	(1,0,0,0)	(2,0,1,0)	(0,0,1,0)	(0,0,0,1)	(3,0,2,1)	(0,0,2,1)			
17	(0,0,2,1)	(0,1,0,0)	(0,0,1,0)	(0,0,0,1)	(0,1,3,2)			(0,1,0,2)	(0,1,3,0)
18	(1,0,0,0)	(0,1,0,0)	(0,0,1,0)	(0,1,3,0)	(1,2,4,0)			(1,2,0,0)	
19	(1,0,0,0)	(0,1,0,0)	(1,2,0,0)	(0,0,0,1)	(2,3,0,1)		(2,0,0,1)		
20	(1,0,0,0)	(2,0,0,1)	(0,0,1,0)	(0,0,0,1)	(3,0,1,2)	(0,0,1,2)			
21	(0,0,1,2)	(0,0,1,2)	(0,0,1,0)	(0,0,0,1)	(0,0,3,5)			(0,0,0,5)	(0,0,3,0)
22	(1,0,0,0)	(0,1,0,0)	(0,0,1,0)	(0,0,3,0)	(1,1,4,0)			(1,1,0,0)	
23	(1,0,0,0)	(0,0,1,2)	(0,0,0,5)	(0,0,0,1)	(1,0,1,8)				(1,0,1,0)
24	(0,0,1,2)	(0,1,0,0)	(0,0,0,5)	(0,0,0,1)	(0,1,1,8)				(0,1,1,0)
25	(0,0,1,2)	(0,0,2,1)	(0,0,1,0)	(0,0,0,1)	(0,0,4,4)			(0,0,0,4)	
26	(1,0,0,0)	(1,0,0,3)	(0,0,0,4)	(0,0,0,1)	(2,0,0,8)				(2,0,0,0)
27	(1,0,0,0)	(0,1,0,0)	(0,0,1,0)	(2,0,0,0)	(3,1,1,0)	(0,1,1,0)			
28	(0,1,1,0)	(0,1,0,0)	(0,0,1,0)	(0,1,1,0)	(0,3,3,0)		(0,0,3,0)		
29	(1,0,0,0)	(0,0,3,0)	(0,0,1,0)	(0,0,0,1)	(1,0,4,1)			(1,0,0,1)	
30	(0,1,0,2)	(0,1,0,0)	(0,1,0,2)	(0,0,0,1)	(0,3,0,5)		(0,0,0,5)		
31	(0,0,1,2)	(0,0,0,5)	(0,0,1,0)	(0,0,0,1)	(0,0,2,8)				(0,0,2,0)
32	(0,1,1,0)	(0,1,0,0)	(0,0,1,0)	(0,0,2,0)	(0,2,4,0)			(0,2,0,0)	
33	(1,0,0,0)	(0,1,0,0)	(0,2,0,0)	(0,0,0,1)	(1,3,0,1)		(1,0,0,1)		
34	(1,0,0,0)	(1,0,0,1)	(1,0,0,1)	(0,0,0,1)	(3,0,0,3)	(0,0,0,3)			
35	(0,0,0,3)	(0,1,0,0)	(0,1,0,2)	(0,0,0,1)	(0,2,0,6)				(0,2,0,0)
36	(1,0,0,0)	(0,1,0,0)	(0,0,1,0)	(0,2,0,0)	(1,3,1,0)		(1,0,1,0)		
37	(1,0,0,0)	(1,0,1,0)	(0,0,1,0)	(0,0,2,0)	(2,0,4,0)			(2,0,0,0)	
38	(1,0,0,0)	(0,1,0,0)	(2,0,0,0)	(0,0,0,1)	(3,1,0,1)	(0,1,0,1)			
39	(1,0,0,0)	(1,0,1,0)	(0,0,1,0)	(1,0,1,0)	(3,0,3,0)	(0,0,3,0)			
40	(0,0,3,0)	(0,1,0,0)	(0,0,1,0)	(0,0,0,1)	(0,1,4,1)			(0,1,0,1)	
41	(0,1,0,1)	(0,1,0,0)	(0,1,0,1)	(0,0,0,1)	(0,3,0,3)		(0,0,0,3)		
42	(1,0,0,0)	(0,0,0,3)	(0,0,0,4)	(0,0,0,1)	(1,0,0,8)				(1,0,0,0)
43	(1,0,0,0)	(0,1,0,0)	(0,2,0,0)	(1,0,0,0)	(2,3,0,0)		(2,0,0,0)		
44	(1,0,0,0)	(0,1,0,0)	(1,1,0,0)	(1,0,0,0)	(3,2,0,0)	(0,2,0,0)			
45	(1,0,0,0)	(2,0,0,0)	(0,0,1,0)	(0,0,0,1)	(3,0,1,1)	(0,0,1,1)			
46	(1,0,0,0)	(1,0,1,0)	(0,0,1,0)	(1,0,0,0)	(3,0,2,0)	(0,0,2,0)			
47	(0,0,2,0)	(0,0,1,2)	(0,0,1,0)	(0,0,0,1)	(0,0,4,3)			(0,0,0,3)	
48	(0,0,1,1)	(0,0,0,3)	(0,0,0,3)	(0,0,0,1)	(0,0,1,8)				(0,0,1,0)
49	(0,0,2,0)	(0,1,0,0)	(0,0,1,0)	(0,0,1,0)	(0,1,4,0)			(0,1,0,0)	
50	(0,2,0,0)	(0,1,0,0)	(0,0,1,0)	(0,0,0,1)	(0,3,1,1)		(0,0,1,1)		
51	(0,0,2,0)	(0,0,1,1)	(0,0,1,0)	(0,0,0,1)	(0,0,4,2)			(0,0,0,2)	
52	(0,2,0,0)	(0,1,0,0)	(0,0,1,0)	(0,0,1,0)	(0,3,2,0)		(0,0,2,0)		
53	(1,0,0,0)	(0,0,2,0)	(0,0,1,0)	(0,0,1,0)	(1,0,4,0)			(1,0,0,0)	
54	(1,0,0,0)	(0,1,0,0)	(1,0,0,0)	(1,0,0,0)	(3,1,0,0)	(0,1,0,0)			
55	(0,1,0,0)	(0,1,0,0)	(0,1,0,0)	(0,0,0,1)	(0,3,0,1)		(0,0,0,1)		
56	(0,0,0,3)	(0,0,0,1)	(0,0,0,2)	(0,0,0,1)	(0,0,0,7)				<b>(0,0,0,0)</b>

TABLE 21. A sequence witnessing that the function  $\hbar = (4, 5, 5, 7)$  is 0-generating

$m$	$\hbar^{[m]}(0)$	$\hbar^{[m]}(1)$	$\hbar^{[m]}(2)$	$\hbar^{[m]}(3)$	$\sum_{i \in \mathbb{3}} \hbar^{[m]}(i)$	$\hbar^{\{m+1\}}(0)$	$\hbar^{\{m+1\}}(1)$	$\hbar^{\{m+1\}}(2)$	$\hbar^{\{m+1\}}(3)$
0	(1,0,0,0)	(0,1,0,0)	(0,0,1,0)	(0,0,0,1)	(1,1,1,1)	(0,1,1,1)			
1	(0,1,1,1)	(0,1,0,0)	(0,0,1,0)	(0,0,0,1)	(0,2,2,2)		(0,0,2,2)	(0,2,0,2)	(0,2,2,0)
2	(1,0,0,0)	(0,1,0,0)	(0,0,1,0)	(0,2,2,0)	(1,3,3,0)		(1,0,3,0)		
3	(1,0,0,0)	(0,1,0,0)	(0,2,0,2)	(0,0,0,1)	(1,3,0,3)		(1,0,0,3)		
4	(1,0,0,0)	(1,0,0,3)	(0,0,1,0)	(0,0,0,1)	(2,0,1,4)				(2,0,1,0)
5	(1,0,0,0)	(0,1,0,0)	(0,0,1,0)	(2,0,1,0)	(3,1,2,0)	(0,1,2,0)			
6	(1,0,0,0)	(1,0,3,0)	(0,0,1,0)	(0,0,0,1)	(2,0,4,1)			(2,0,0,1)	
7	(1,0,0,0)	(0,1,0,0)	(2,0,0,1)	(0,0,0,1)	(3,1,0,2)	(0,1,0,2)			
8	(1,0,0,0)	(0,0,2,2)	(0,0,1,0)	(0,0,0,1)	(1,0,3,3)			(1,0,0,3)	(1,0,3,0)
9	(1,0,0,0)	(0,1,0,0)	(0,0,1,0)	(1,0,3,0)	(2,1,4,0)			(2,1,0,0)	
10	(1,0,0,0)	(0,1,0,0)	(2,1,0,0)	(0,0,0,1)	(3,2,0,1)	(0,2,0,1)			
11	(1,0,0,0)	(0,1,0,0)	(1,0,0,3)	(0,0,0,1)	(2,1,0,4)				(2,1,0,0)
12	(1,0,0,0)	(0,1,0,0)	(0,0,1,0)	(2,1,0,0)	(3,2,1,0)	(0,2,1,0)			
13	(0,2,1,0)	(0,1,0,0)	(0,0,1,0)	(0,0,0,1)	(0,3,2,1)		(0,0,2,1)		
14	(1,0,0,0)	(0,0,2,1)	(0,0,1,0)	(0,0,0,1)	(1,0,3,2)			(1,0,0,2)	
15	(0,2,0,1)	(0,1,0,0)	(0,0,1,0)	(0,0,0,1)	(0,3,1,2)		(0,0,1,2)		
16	(0,1,0,2)	(0,1,0,0)	(0,0,1,0)	(0,0,0,1)	(0,2,1,3)				(0,2,1,0)
17	(0,1,0,2)	(0,1,0,0)	(0,2,0,2)	(0,0,0,1)	(0,4,0,5)		(0,0,0,5)		
18	(1,0,0,0)	(0,0,0,5)	(0,0,1,0)	(0,0,0,1)	(1,0,1,6)				(1,0,1,0)
19	(0,1,0,2)	(0,0,1,2)	(0,0,1,0)	(0,0,0,1)	(0,1,2,5)				(0,1,2,0)
20	(0,1,2,0)	(0,0,1,2)	(0,0,1,0)	(0,0,0,1)	(0,1,4,3)			(0,1,0,3)	
21	(0,1,0,2)	(0,1,0,0)	(0,1,0,3)	(0,0,0,1)	(0,3,0,6)				(0,3,0,0)
22	(1,0,0,0)	(0,1,0,0)	(0,0,1,0)	(0,3,0,0)	(1,4,1,0)		(1,0,1,0)		
23	(1,0,0,0)	(1,0,1,0)	(0,0,1,0)	(1,0,1,0)	(3,0,3,0)	(0,0,3,0)			
24	(0,0,3,0)	(0,1,0,0)	(0,0,1,0)	(0,0,0,1)	(0,1,4,1)			(0,1,0,1)	
25	(0,2,0,1)	(0,1,0,0)	(0,1,0,1)	(0,0,0,1)	(0,4,0,3)		(0,0,0,3)		
26	(1,0,0,0)	(0,0,0,3)	(1,0,0,2)	(0,0,0,1)	(2,0,0,6)				(2,0,0,0)
27	(1,0,0,0)	(0,1,0,0)	(0,0,1,0)	(2,0,0,0)	(3,1,1,0)	(0,1,1,0)			
28	(0,1,1,0)	(0,1,0,0)	(0,0,1,0)	(0,1,2,0)	(0,3,4,0)			(0,3,0,0)	
29	(1,0,0,0)	(0,1,0,0)	(0,3,0,0)	(0,0,0,1)	(1,4,0,1)		(1,0,0,1)		
30	(0,1,1,0)	(0,1,0,0)	(0,0,1,0)	(0,2,1,0)	(0,4,3,0)		(0,0,3,0)		
31	(1,0,0,0)	(0,0,3,0)	(0,0,1,0)	(0,0,0,1)	(1,0,4,1)			(1,0,0,1)	
32	(1,0,0,0)	(1,0,0,1)	(1,0,0,1)	(0,0,0,1)	(3,0,0,3)	(0,0,0,3)			
33	(0,0,0,3)	(0,1,0,0)	(0,0,1,0)	(0,0,0,1)	(0,1,1,4)				(0,1,1,0)
34	(0,0,0,3)	(0,1,0,0)	(0,1,0,1)	(0,0,0,1)	(0,2,0,5)				(0,2,0,0)
35	(0,1,1,0)	(0,1,0,0)	(0,0,1,0)	(0,2,0,0)	(0,4,2,0)		(0,0,2,0)		
36	(1,0,0,0)	(0,0,2,0)	(0,0,1,0)	(0,1,1,0)	(1,1,4,0)			(1,1,0,0)	
37	(1,0,0,0)	(0,1,0,0)	(1,1,0,0)	(0,2,0,0)	(2,4,0,0)		(2,0,0,0)		
38	(1,0,0,0)	(2,0,0,0)	(0,0,1,0)	(0,0,0,1)	(3,0,1,1)	(0,0,1,1)			
39	(1,0,0,0)	(0,0,2,0)	(0,0,1,0)	(1,0,1,0)	(2,0,4,0)			(2,0,0,0)	
40	(1,0,0,0)	(0,1,0,0)	(2,0,0,0)	(0,0,0,1)	(3,1,0,1)	(0,1,0,1)			
41	(0,1,0,1)	(0,1,0,0)	(0,0,1,0)	(0,2,0,0)	(0,4,1,1)		(0,0,1,1)		
42	(0,0,1,1)	(0,0,1,1)	(0,0,1,0)	(0,0,0,1)	(0,0,3,3)			(0,0,0,3)	
43	(0,0,1,1)	(0,0,1,1)	(0,0,0,3)	(0,0,0,1)	(0,0,2,6)				(0,0,2,0)
44	(0,0,1,1)	(0,0,2,0)	(0,0,1,0)	(0,0,0,1)	(0,0,4,2)			(0,0,0,2)	
45	(1,0,0,0)	(0,0,0,3)	(0,0,0,2)	(0,0,0,1)	(1,0,0,6)				(1,0,0,0)
46	(1,0,0,0)	(0,1,0,0)	(1,1,0,0)	(1,0,0,0)	(3,2,0,0)	(0,2,0,0)			
47	(1,0,0,0)	(1,0,1,0)	(0,0,1,0)	(1,0,0,0)	(3,0,2,0)	(0,0,2,0)			
48	(0,0,2,0)	(0,1,0,0)	(0,0,1,0)	(0,1,1,0)	(0,2,4,0)			(0,2,0,0)	
49	(0,2,0,0)	(0,1,0,0)	(0,1,0,1)	(0,0,0,1)	(0,4,0,2)		(0,0,0,2)		
50	(0,1,0,1)	(0,0,0,2)	(0,0,0,2)	(0,0,0,1)	(0,1,0,6)				(0,1,0,0)
51	(1,0,0,0)	(0,1,0,0)	(0,2,0,0)	(0,1,0,0)	(1,4,0,0)		(1,0,0,0)		
52	(1,0,0,0)	(1,0,0,0)	(0,0,1,0)	(1,0,0,0)	(3,0,1,0)	(0,0,1,0)			
53	(0,0,1,0)	(0,1,0,0)	(0,0,1,0)	(0,0,2,0)	(0,1,4,0)			(0,1,0,0)	
54	(1,0,0,0)	(1,0,0,0)	(0,1,0,0)	(1,0,0,0)	(3,1,0,0)	(0,1,0,0)			
55	(0,1,0,0)	(0,1,0,0)	(0,0,1,0)	(0,1,0,0)	(0,3,1,0)		(0,0,1,0)		
56	(1,0,0,0)	(0,0,1,0)	(0,0,1,0)	(0,0,2,0)	(1,0,4,0)			(1,0,0,0)	
57	(1,0,0,0)	(1,0,0,0)	(1,0,0,0)	(0,0,0,1)	(3,0,0,1)	(0,0,0,1)			
58	(0,0,0,1)	(0,0,0,2)	(0,0,0,2)	(0,0,0,1)	(0,0,0,6)				<b>(0,0,0,0)</b>



TABLE 22. A sequence witnessing that the function  $\hbar = (5, 5, 5, 6)$  is 0-generating

$m$	$\hbar^{[m]}(0)$	$\hbar^{[m]}(1)$	$\hbar^{[m]}(2)$	$\hbar^{[m]}(3)$	$\sum_{i \in \mathbb{3}} \hbar^{[m]}(i)$	$\hbar^{\{m+1\}}(0)$	$\hbar^{\{m+1\}}(1)$	$\hbar^{\{m+1\}}(2)$	$\hbar^{\{m+1\}}(3)$
0	(1,0,0,0)	(0,1,0,0)	(0,0,1,0)	(0,0,0,1)	(1,1,1,1)			(1,1,0,1)	(1,1,1,0)
1	(1,0,0,0)	(0,1,0,0)	(0,0,1,0)	(1,1,1,0)	(2,2,2,0)		(2,0,2,0)		
2	(1,0,0,0)	(0,1,0,0)	(1,1,0,1)	(0,0,0,1)	(2,2,0,2)		(2,0,0,2)		
3	(1,0,0,0)	(2,0,0,2)	(0,0,1,0)	(0,0,0,1)	(3,0,1,3)				(3,0,1,0)
4	(1,0,0,0)	(0,1,0,0)	(0,0,1,0)	(3,0,1,0)	(4,1,2,0)	(0,1,2,0)			
5	(1,0,0,0)	(2,0,2,0)	(0,0,1,0)	(0,0,0,1)	(3,0,3,1)			(3,0,0,1)	
6	(1,0,0,0)	(0,1,0,0)	(3,0,0,1)	(0,0,0,1)	(4,1,0,2)	(0,1,0,2)			
7	(0,1,0,2)	(0,1,0,0)	(0,0,1,0)	(0,0,0,1)	(0,2,1,3)				(0,2,1,0)
8	(1,0,0,0)	(0,1,0,0)	(0,0,1,0)	(0,2,1,0)	(1,3,2,0)			(1,3,0,0)	
9	(1,0,0,0)	(0,1,0,0)	(1,3,0,0)	(0,0,0,1)	(2,4,0,1)		(2,0,0,1)		
10	(0,1,0,2)	(0,1,0,0)	(0,0,1,0)	(0,2,1,0)	(0,4,2,2)		(0,0,2,2)		
11	(0,1,0,2)	(0,0,2,2)	(0,0,1,0)	(0,0,0,1)	(0,1,3,5)				(0,1,3,0)
12	(1,0,0,0)	(0,1,0,0)	(0,0,1,0)	(0,1,3,0)	(1,2,4,0)			(1,2,0,0)	
13	(0,1,2,0)	(0,1,0,0)	(0,0,1,0)	(0,0,0,1)	(0,2,3,1)		(0,0,3,1)	(0,2,0,1)	
14	(1,0,0,0)	(0,1,0,0)	(0,2,0,1)	(0,0,0,1)	(1,3,0,2)		(1,0,0,2)		
15	(1,0,0,0)	(0,0,3,1)	(0,0,1,0)	(0,0,0,1)	(1,0,4,2)			(1,0,0,2)	
16	(1,0,0,0)	(1,0,0,2)	(1,0,0,2)	(0,0,0,1)	(3,0,0,5)				(3,0,0,0)
17	(1,0,0,0)	(0,1,0,0)	(0,0,1,0)	(3,0,0,0)	(4,1,1,0)	(0,1,1,0)			
18	(0,1,1,0)	(0,1,0,0)	(0,0,1,0)	(0,0,0,1)	(0,2,2,1)		(0,0,2,1)	(0,2,0,1)	
19	(0,1,1,0)	(0,1,0,0)	(0,0,1,0)	(0,2,1,0)	(0,4,3,0)		(0,0,3,0)		
20	(1,0,0,0)	(0,0,3,0)	(0,0,1,0)	(0,0,0,1)	(1,0,4,1)			(1,0,0,1)	
21	(1,0,0,0)	(2,0,0,1)	(1,0,0,1)	(0,0,0,1)	(4,0,0,3)	(0,0,0,3)			
22	(0,1,1,0)	(0,1,0,0)	(0,2,0,1)	(0,0,0,1)	(0,4,1,2)		(0,0,1,2)		
23	(0,0,0,3)	(0,1,0,0)	(0,0,1,0)	(0,0,0,1)	(0,1,1,4)				(0,1,1,0)
24	(0,1,1,0)	(0,1,0,0)	(0,0,1,0)	(0,1,1,0)	(0,3,3,0)			(0,3,0,0)	
25	(1,0,0,0)	(0,1,0,0)	(0,3,0,0)	(0,0,0,1)	(1,4,0,1)		(1,0,0,1)		
26	(0,0,0,3)	(0,1,0,0)	(1,0,0,1)	(0,0,0,1)	(1,1,0,5)				(1,1,0,0)
27	(1,0,0,0)	(0,1,0,0)	(1,2,0,0)	(1,1,0,0)	(3,4,0,0)		(3,0,0,0)		
28	(1,0,0,0)	(3,0,0,0)	(0,0,1,0)	(0,0,0,1)	(4,0,1,1)	(0,0,1,1)			
29	(0,0,1,1)	(0,0,1,2)	(0,0,1,0)	(0,0,0,1)	(0,0,3,4)				(0,0,3,0)
30	(1,0,0,0)	(0,1,0,0)	(0,0,1,0)	(0,0,3,0)	(1,1,4,0)			(1,1,0,0)	
31	(0,0,1,1)	(0,0,2,1)	(0,0,1,0)	(0,0,0,1)	(0,0,4,3)			(0,0,0,3)	
32	(1,0,0,0)	(1,0,0,1)	(0,0,0,3)	(0,0,0,1)	(2,0,0,5)				(2,0,0,0)
33	(1,0,0,0)	(0,1,0,0)	(1,0,0,1)	(2,0,0,0)	(4,1,0,1)	(0,1,0,1)			
34	(0,1,0,1)	(0,1,0,0)	(0,0,0,3)	(0,0,0,1)	(0,2,0,5)				(0,2,0,0)
35	(1,0,0,0)	(0,1,0,0)	(0,0,1,0)	(0,2,0,0)	(1,3,1,0)		(1,0,1,0)		
36	(1,0,0,0)	(0,1,0,0)	(1,1,0,0)	(0,2,0,0)	(2,4,0,0)		(2,0,0,0)		
37	(1,0,0,0)	(2,0,0,0)	(1,0,0,1)	(0,0,0,1)	(4,0,0,2)	(0,0,0,2)			
38	(1,0,0,0)	(1,0,1,0)	(0,0,1,0)	(2,0,0,0)	(4,0,2,0)	(0,0,2,0)			
39	(0,0,2,0)	(0,1,0,0)	(0,0,1,0)	(0,1,1,0)	(0,2,4,0)			(0,2,0,0)	
40	(0,0,0,2)	(0,0,1,2)	(0,0,1,0)	(0,0,0,1)	(0,0,2,5)				(0,0,2,0)
41	(1,0,0,0)	(1,0,1,0)	(0,0,1,0)	(0,0,2,0)	(2,0,4,0)			(2,0,0,0)	
42	(0,1,0,1)	(0,1,0,0)	(0,0,1,0)	(0,2,0,0)	(0,4,1,1)		(0,0,1,1)		
43	(0,0,2,0)	(0,0,1,1)	(0,0,1,0)	(0,0,0,1)	(0,0,4,2)			(0,0,0,2)	
44	(0,0,0,2)	(0,1,0,0)	(0,0,0,2)	(0,0,0,1)	(0,1,0,5)				(0,1,0,0)
45	(1,0,0,0)	(0,1,0,0)	(0,2,0,0)	(0,1,0,0)	(1,4,0,0)		(1,0,0,0)		
46	(1,0,0,0)	(1,0,0,0)	(2,0,0,0)	(0,1,0,0)	(4,1,0,0)	(0,1,0,0)			
47	(0,1,0,0)	(0,1,0,0)	(0,0,1,0)	(0,1,0,0)	(0,3,1,0)		(0,0,1,0)		
48	(1,0,0,0)	(0,0,1,0)	(0,0,1,0)	(0,0,2,0)	(1,0,4,0)			(1,0,0,0)	
49	(1,0,0,0)	(1,0,0,0)	(1,0,0,0)	(1,0,0,0)	(4,0,0,0)	<b>(0,0,0,0)</b>			

TABLE 23. A sequence witnessing that the function  $\hbar = (4, 5, 6, 6)$  is 0-generating

$m$	$\hbar^{[m]}(0)$	$\hbar^{[m]}(1)$	$\hbar^{[m]}(2)$	$\hbar^{[m]}(3)$	$\sum_{i \in \mathbb{3}} \hbar^{[m]}(i)$	$\hbar^{\{m+1\}}(0)$	$\hbar^{\{m+1\}}(1)$	$\hbar^{\{m+1\}}(2)$	$\hbar^{\{m+1\}}(3)$
0	(1,0,0,0)	(0,1,0,0)	(0,0,1,0)	(0,0,0,1)	(1,1,1,1)	(0,1,1,1)			
1	(0,1,1,1)	(0,1,0,0)	(0,0,1,0)	(0,0,0,1)	(0,2,2,2)		(0,0,2,2)	(0,2,0,2)	(0,2,2,0)
2	(1,0,0,0)	(0,1,0,0)	(0,0,1,0)	(0,2,2,0)	(1,3,3,0)		(1,0,3,0)		
3	(1,0,0,0)	(0,1,0,0)	(0,2,0,2)	(0,0,0,1)	(1,3,0,3)		(1,0,0,3)		
4	(1,0,0,0)	(1,0,0,3)	(0,0,1,0)	(0,0,0,1)	(2,0,1,4)				(2,0,1,0)
5	(1,0,0,0)	(0,1,0,0)	(0,0,1,0)	(2,0,1,0)	(3,1,2,0)	(0,1,2,0)			
6	(1,0,0,0)	(1,0,3,0)	(0,0,1,0)	(0,0,0,1)	(2,0,4,1)			(2,0,0,1)	
7	(1,0,0,0)	(0,1,0,0)	(2,0,0,1)	(0,0,0,1)	(3,1,0,2)	(0,1,0,2)			
8	(1,0,0,0)	(0,0,2,2)	(0,0,1,0)	(0,0,0,1)	(1,0,3,3)			(1,0,0,3)	(1,0,3,0)
9	(1,0,0,0)	(0,1,0,0)	(0,0,1,0)	(1,0,3,0)	(2,1,4,0)			(2,1,0,0)	
10	(1,0,0,0)	(0,1,0,0)	(2,1,0,0)	(0,0,0,1)	(3,2,0,1)	(0,2,0,1)			
11	(1,0,0,0)	(0,1,0,0)	(1,0,0,3)	(0,0,0,1)	(2,1,0,4)				(2,1,0,0)
12	(1,0,0,0)	(0,1,0,0)	(0,0,1,0)	(2,1,0,0)	(3,2,1,0)	(0,2,1,0)			
13	(0,2,0,1)	(0,1,0,0)	(0,0,1,0)	(0,0,0,1)	(0,3,1,2)		(0,0,1,2)		
14	(0,1,0,2)	(0,0,1,2)	(0,0,1,0)	(0,0,0,1)	(0,1,2,5)				(0,1,2,0)
15	(0,2,1,0)	(0,1,0,0)	(0,0,1,0)	(0,1,2,0)	(0,4,4,0)		(0,0,4,0)		
16	(1,0,0,0)	(0,0,4,0)	(0,0,1,0)	(0,0,0,1)	(1,0,5,1)			(1,0,0,1)	
17	(0,1,2,0)	(0,1,0,0)	(0,0,1,0)	(0,0,0,1)	(0,2,3,1)			(0,2,0,1)	
18	(0,1,0,2)	(0,1,0,0)	(0,2,0,1)	(0,0,0,1)	(0,4,0,4)		(0,0,0,4)		
19	(1,0,0,0)	(0,0,0,4)	(0,0,1,0)	(0,0,0,1)	(1,0,1,5)				(1,0,1,0)
20	(0,1,2,0)	(0,1,0,0)	(0,0,1,0)	(0,1,2,0)	(0,3,5,0)			(0,3,0,0)	
21	(1,0,0,0)	(0,1,0,0)	(0,3,0,0)	(0,0,0,1)	(1,4,0,1)		(1,0,0,1)		
22	(1,0,0,0)	(1,0,0,1)	(1,0,0,1)	(0,0,0,1)	(3,0,0,3)	(0,0,0,3)			
23	(0,0,0,3)	(0,1,0,0)	(0,0,1,0)	(0,0,0,1)	(0,1,1,4)				(0,1,1,0)
24	(0,0,0,3)	(0,1,0,0)	(1,0,0,1)	(0,0,0,1)	(1,1,0,5)				(1,1,0,0)
25	(0,0,0,3)	(0,1,0,0)	(0,2,0,1)	(0,0,0,1)	(0,3,0,5)				(0,3,0,0)
26	(1,0,0,0)	(0,1,0,0)	(0,0,1,0)	(0,3,0,0)	(1,4,1,0)		(1,0,1,0)		
27	(1,0,0,0)	(1,0,1,0)	(0,0,1,0)	(1,0,1,0)	(3,0,3,0)	(0,0,3,0)			
28	(0,0,3,0)	(0,1,0,0)	(0,0,1,0)	(0,0,0,1)	(0,1,4,1)			(0,1,0,1)	
29	(0,0,3,0)	(0,1,0,0)	(0,0,1,0)	(0,1,1,0)	(0,2,5,0)			(0,2,0,0)	
30	(1,0,0,0)	(0,1,0,0)	(0,2,0,0)	(1,1,0,0)	(2,4,0,0)		(2,0,0,0)		
31	(1,0,0,0)	(2,0,0,0)	(0,0,1,0)	(0,0,0,1)	(3,0,1,1)	(0,0,1,1)			
32	(0,0,1,1)	(0,0,1,2)	(0,0,1,0)	(0,0,0,1)	(0,0,3,4)				(0,0,3,0)
33	(1,0,0,0)	(1,0,1,0)	(0,0,1,0)	(0,0,3,0)	(2,0,5,0)			(2,0,0,0)	
34	(1,0,0,0)	(0,1,0,0)	(2,0,0,0)	(0,0,0,1)	(3,1,0,1)	(0,1,0,1)			
35	(0,1,0,1)	(0,1,0,0)	(0,2,0,0)	(0,0,0,1)	(0,4,0,2)		(0,0,0,2)		
36	(1,0,0,0)	(0,0,0,2)	(1,0,0,1)	(0,0,0,1)	(2,0,0,4)				(2,0,0,0)
37	(1,0,0,0)	(0,1,0,0)	(0,0,1,0)	(2,0,0,0)	(3,1,1,0)	(0,1,1,0)			
38	(0,1,0,1)	(0,0,0,2)	(0,1,0,1)	(0,0,0,1)	(0,2,0,5)				(0,2,0,0)
39	(0,1,1,0)	(0,1,0,0)	(0,0,1,0)	(0,2,0,0)	(0,4,2,0)		(0,0,2,0)		
40	(0,0,1,1)	(0,0,2,0)	(0,0,1,0)	(0,0,0,1)	(0,0,4,2)			(0,0,0,2)	
41	(1,0,0,0)	(0,0,0,2)	(0,0,0,2)	(0,0,0,1)	(1,0,0,5)				(1,0,0,0)
42	(1,0,0,0)	(1,0,1,0)	(0,0,1,0)	(1,0,0,0)	(3,0,2,0)	(0,0,2,0)			
43	(0,0,2,0)	(0,0,2,0)	(0,0,1,0)	(0,0,0,1)	(0,0,5,1)			(0,0,0,1)	
44	(1,0,0,0)	(2,0,0,0)	(0,0,0,1)	(0,0,0,1)	(3,0,0,2)	(0,0,0,2)			
45	(0,0,0,2)	(0,1,0,0)	(0,0,0,1)	(0,0,0,1)	(0,1,0,4)				(0,1,0,0)
46	(0,1,0,1)	(0,1,0,0)	(0,2,0,0)	(0,1,0,0)	(0,4,0,1)		(0,0,0,1)		
47	(0,0,0,2)	(0,0,0,1)	(0,0,0,1)	(0,0,0,1)	(0,0,0,5)				<b>(0,0,0,0)</b>

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