# Odd Hole Recognition in Graphs of Bounded Clique Size

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**Abstract**

In a graph $G$, an odd hole is an induced odd cycle of length at least five. A clique of $G$ is a set of pairwise adjacent vertices. In this paper we consider the class $\mathcal{C}_k$ of graphs whose cliques have a size bounded by a constant $k$. Given a graph $G$ in $\mathcal{C}_k$, we show how to recognize in polynomial time whether $G$ contains an odd hole.

**Keywords:** odd hole, recognition algorithm, cleaning, decomposition

# 1 Introduction

A *hole* is a graph induced by a cycle of length at least four. A hole is *odd* if it contains an odd number of vertices. Otherwise, it is even. Graph $G$ *contains* graph $H$ if $H$ is isomorphic to an induced subgraph of $G$. Chudnovsky, Cornuéjols, Liu, Seymour and Vušković recently proved that it is polynomial to test whether a graph contains an odd hole or its complement [2]. However, it is still an open problem to test whether a graph contains an odd hole. Bienstock [1] proved that it is $NP$-complete to test whether a graph contains an odd hole passing through a specific vertex. A *clique* is a set of pairwise adjacent vertices. The *clique number* of a graph is the size of its largest clique. In this paper, we show that it is polynomial to test whether a graph of bounded clique number contains an odd hole.

We use the same general strategy as in [2]. Let $H$ be an odd hole in a graph $G$. We say that $u \in V(G) \setminus V(H)$ is *H-minor* if its neighbors in $H$ lie in some 2-edge path of $H$. In particular, $u$ is $H$-minor if $u$ has no neighbor in $H$. A vertex $u \in V(G) \setminus V(H)$ is *H-major* if it is not $H$-minor. We say that $H$ is *clean* if $G$ contains no $H$-major vertex. A graph $G$ is *clean* if it is either odd-hole-free or it contains a clean shortest odd hole. As in [2] our approach for testing whether a graph $G$ of bounded clique number contains an odd hole consists of two steps:

(i) constructing in polynomial time a clean graph $G'$ that contains an odd hole if and only if $G$ does, or in some cases identifying an odd hole of $G$, and

(ii) checking whether the clean graph $G'$ contains an odd hole.

For step (ii), we can use the polynomial algorithms in [2]. The main result of this paper is a polynomial algorithm for step (i).

## 1.1 Notation

A *pyramid* $\Pi(xyz;u)$ is a graph induced by three chordless paths $P_1 = x, \ldots, u$, $P_2 = y, \ldots, u$ and $P_3 = z, \ldots, u$ having no common or adjacent intermediate vertices, such that at most one of the paths is of length 1 and the vertex set $\{x, y, z\}$ induces a clique of size 3. Furthermore, every two of the paths $P_1, P_2, P_3$ induce a hole. Since two of the three paths must have the same parity, one of these holes is odd. Therefore, every pyramid contains an odd hole.

A *wheel*, denoted by $(H, x)$, is a graph induced by a hole $H$ and a vertex $x \notin V(H)$ having at least three neighbors in $H$, say $x_1, \ldots, x_n$. Vertex $x$ is the *center* of the wheel. A subpath of $H$ connecting $x_i$ and $x_j$ is a *sector* if it contains no intermediate vertex $x_l$, $l \in \{1, \ldots, n\}$. A *short sector* is a sector of length 1, and a *long sector* is a sector of length at least 2. A wheel is *odd* if it contains an odd number of short sectors, and *even* otherwise. Each of the long sectors together with vertex $x$ induces a hole. If each of these holes is even and the wheel $(H, v)$ is odd then $H$ is an odd hole, since the wheel $(H, x)$ contains an odd number of short sectors. Therefore, every odd wheel contains an odd hole.

In a graph $G$, a *jewel* is a sequence $v_1, \cdots, v_5$, $P$ such that $v_1, \cdots, v_5$ are distinct vertices, $v_1v_2$, $v_2v_3$, $v_3v_4$, $v_4v_5$, $v_5v_1$ are edges, $v_1v_3$, $v_2v_4$, $v_1v_4$ are nonedges, and $P$ is a path of $G$ between $v_1$ and $v_4$ such that $v_2$, $v_3$, $v_5$ have no neighbors in $V(P) \setminus \{v_1, v_4\}$. Clearly a jewel either contains an odd wheel or a 5-hole, so if there is a jewel in a graph $G$ then there is an odd hole in $G$.

Note that there is an $O(|V(G)|^9)$ algorithm to test whether a graph $G$ contains a pyramid and an $O(|V(G)|^6)$ algorithm to test whether a graph $G$ contains a jewel as shown in [2]. In fact the algorithm for testing for pyramids in general is due to Chudnovsky and Seymour.

# 2 Cleaning

In this section, we show how to clean a graph $G$ of bounded clique number. The cleaning algorithm produces a polynomial family of induced subgraphs of $G$ such that if $G$ contains a smallest odd hole $H^*$, then one of the graphs produced by the cleaning algorithm, say $G'$, contains $H^*$ and $H^*$ is clean in $G'$.

## 2.1 Vertices with at most Three Neighbors in $H^*$

**Lemma 1** *Suppose that $G$ does not contain a pyramid. If a vertex $u \notin V(H^*)$ has a neighbor but no more than three neighbors in $H^*$ then $u$ is $H^*$-minor.*

*Proof:* If $u$ has one neighbor in $H^*$ then $u$ is $H^*$-minor. Now suppose that $u$ has two neighbors in $H^*$, say $u_1$ and $u_2$. Let $P_1$ and $P_2$ be the two $u_1u_2$-subpaths of $H^*$. Since $H^*$ is odd, $P_1$ and $P_2$ have different parity, say $P_1$ is odd. If $P_1$ is of length 1 then $u$ is $H^*$-minor. Otherwise, $V(P_1) \cup \{u\}$ induces an odd hole. Since this hole cannot be shorter than $H^*$, $P_2$ is of length 2, and hence $u$ is $H^*$-minor.

Now assume that $u$ has three neighbors in $H^*$, and let $P_1, P_2$ and $P_3$ be the three sectors of the wheel $(H^*, u)$. If exactly one of the sectors is short then $V(H^*) \cup \{u\}$ induces a pyramid. If two of the sectors are short then $u$ is $H^*$-minor. Finally suppose that all three sectors are long. Since $H^*$ is odd, at least one of the sectors, say $P_1$, is odd. Then $V(P_1) \cup \{u\}$ induces an odd hole shorter than $H^*$, a contradiction. □

## 2.2 Vertices with More Than Three Neighbors in $H^*$

Let $H^*$ be a shortest odd hole in $G$. Let $S(H^*)$ be the set of $H^*$-major vertices that have four or more neighbors in $H^*$. Note that, for any $u \in S(H^*)$, every long sector of the wheel $(H^*, u)$ is of even length since $H^*$ is a smallest odd hole of $G$; hence, $(H^*, u)$ contains an odd number of short sectors.

Let $S \subseteq V(G)$. We say that vertex $x \in V(G) \setminus S$ is *$S$-complete* if $x$ is adjacent to every vertex in $S$. We say that an edge $xy$ is *$S$-complete* if both vertices $x$ and $y$ are $S$-complete.

**Lemma 2** *Suppose that $G$ does not contain a jewel. If $u, v \in S(H^*)$ are not adjacent then an odd number of edges of $H^*$ are $\{u, v\}$-complete.*

*Proof:* Let $u$ and $v$ be nonadjacent vertices of $S(H^*)$. Suppose that an even number of edges of $H^*$ are $\{u, v\}$-complete. Then some long sector $P$ of the wheel $(H^*, u)$ contains an odd number of short sectors of $(H^*, v)$. Let $u_1$ and $u_2$ be the endvertices of $P$. Let $P'$ be the subpath of $H^*$ induced by $(V(H^*) \setminus V(P)) \cup \{u_1, u_2\}$. Note that $P'$ must be of length at least four, since otherwise

$(H^*, u)$ is a jewel, a contradiction. If $P$ contains three or more neighbors of $v$, then the vertex set $V(P) \cup \{u, v\}$ induces an odd wheel with center $v$, and hence contains an odd hole shorter than $H^*$, contradicting our choice of $H^*$. Otherwise, let $v_1$ and $v_2$ be the two neighbors of $v$ in $P$. Vertex $v$ cannot have exactly four neighbors in $H^*$, say $v_1, v_2, v_3, v_4$, such that both $v_3u_1$ and $v_4u_2$ are edges, because otherwise the vertex set $(V(H^*) \setminus V(P)) \cup \{v\}$ induces a shorter odd hole than $H^*$, since $P$ is even and $P'$ is of length at least four. Therefore, there exist vertices $u_3, v_3 \in V(H^*) \setminus V(P)$, the neighbors of $u$ and $v$ respectively, such that $u$ and $v$ have no other neighbors on $u_3v_3$-subpath of $H^*$, call it $Q$, and vertices $u_3$ and $v_3$ are not adjacent to $u_1$ or $u_2$. But now the vertex set $V(Q) \cup V(P) \cup \{u, v\}$ induces a pyramid $\Pi(v_1v_2v; u)$, and hence contains an odd hole smaller than $H^*$, contradicting our choice of $H^*$. □

**Lemma 3** *Suppose that $G$ does not contain a jewel. If $A \subseteq S(H^*)$ is a stable set, then an odd number of edges of $H^*$ are $A$-complete.*

*Proof:* Let $A \subseteq S(H^*)$ be a stable set and suppose that an even number of edges of $H^*$ are $A$-complete. Let $A'$ be a smallest subset of $A$ with the property that an even number of edges of $H^*$ are $A'$-complete. Note that by Lemma 2, $|A'| \geq 3$. Let $s_1, \ldots, s_m$ be the vertices of $H^*$ adjacent to at least one vertex in $A'$, encountered in that order when traversing $H^*$ clockwise. For $i \in [m]$, let $S_i$ be the $s_is_{i+1}$-subpath of $H^*$ (indices taken modulo $m$), that does not contain any intermediate vertex $s_j$, $j \in [m]$.

**Claim** For every $i \in [m]$, $S_i$ is either an edge or of even length.

*Proof of Claim:* If there is a vertex $x \in A'$ adjacent to both $s_i$ and $s_{i+1}$, then $S_i$ is a sector of the wheel $(H^*, x)$ and hence the result holds. Otherwise, let $x_1$ and $x_2$ be vertices of $A'$ such that $x_1$ is adjacent to $s_i$ and $x_2$ is adjacent to $s_{i+1}$. By Lemma 2 let $y_1y_2$ be an edge of $H^*$ such that $x_1$ and $x_2$ are adjacent to both $y_1$ and $y_2$. W.l.o.g. $y_1$ is not adjacent to either $s_i$ or $s_{i+1}$. Then the vertex set $V(S_i) \cup \{x_1, x_2, y_1\}$ induces a hole, and since this hole is smaller than $H^*$ it must be even, hence $S_i$ is of even length. This completes the proof of the claim. ◇

For $C \subseteq A'$, let $\delta_C$ denote the number of edges of $H^*$ that are $C$-complete. Let $\delta$ be the number of paths in $S_1, \ldots, S_m$ of length one. Then

$$\delta = \sum_{i=1}^{|A'|} (-1)^{i+1} \sum_{C \subseteq A', |C|=i} \delta_C$$

By the choice of $A'$, for every $C \subseteq A'$ such that $C \neq A'$, $\delta_C$ is odd. Hence the parity of $\delta$ is equal to the parity of

$$\sum_{i=1}^{|A'|-1} \binom{|A'|}{i} + \delta_{A'}$$

which is itself equal to the parity of $\delta_{A'}$ since

$$\sum_{i=1}^{|A'|-1} \binom{|A'|}{i} = 2^{|A'|} - 2$$

By Claim and because $H^*$ is odd, $\delta$ is odd. Hence $\delta_{A'}$ must be odd as well, contradicting the choice of $A'$. □

**Theorem 4** *Suppose that $G$ does not contain a jewel. Let $A$ be a stable set of $S(H^*)$ and let $x_1x_2$ be an edge of $H^*$ such that every vertex of $A$ is adjacent to both $x_1$ and $x_2$ (such an edge exists by Lemma 3). Let $B$ be the set of vertices of $S(H^*)$ that have no neighbor in $\{x_1, x_2\}$, and have both a neighbor and a nonneighbor in $A$. Then there exists an edge $y_1y_2$ of $H^*$ such that $y_1$ is $A$-complete and every vertex of $B$ has a neighbor in $\{y_1, y_2\}$.*

*Proof:* If $B = \emptyset$ then the result is trivially true, so we may assume that $B \neq \emptyset$. This implies that $H^*$ is of length greater than 5.

**Claim 1** For every $u \in B$, an edge of $H^*$ is $(A \cup \{u\})$-complete.

*Proof of Claim 1:* Let $A_1$ be the neighbors of $u$ in $A$ and $A_2 = A \setminus A_1$. By Lemma 3, there is an edge $u_1u_2$ of $H^*$ such that every vertex of $A_2 \cup \{u\}$ is adjacent to both $u_1$ and $u_2$. Since $u$ has no neighbor in $\{x_1, x_2\}$, every vertex of $A_1$ must be adjacent to both $u_1$ and $u_2$, else there is a 5-hole. This completes the proof of Claim 1. $\diamond$

**Claim 2** If $X$ is a stable set of $B$, then there exists an edge $z_1z_2$ of $H^*$ such that $z_1$ is $A$-complete and every vertex of $X$ has a neighbor in $\{z_1, z_2\}$.

*Proof of Claim 2:* We consider the following two cases.

*Case 1* There is a vertex in $A$ that is not adjacent to any vertex in $X$.

Let $A_1 \subseteq A$ be such that $A_1 \cup X$ is a maximal stable set. By Lemma 3, an edge of $H^*$ is $(A_1 \cup X)$-complete, say $u_1u_2$. Let $w \in A \setminus A_1$. Note that $w$ is adjacent to some $x \in X$. If $w$ is not adjacent to $u_1$ or $u_2$, then there is a 5-hole in the graph induced by $\{x, y, w, u_1, u_2, x_1, x_2\}$, where $y \in A_1$. So every vertex of $A \setminus A_1$ is adjacent to both $u_1$ and $u_2$.

*Case 2* Every vertex of $A$ is adjacent to some vertex in $X$.

By Claim 1 and Case 1, we may assume w.l.o.g. that $|X| > 1$ and for every proper subset of $X$ the result holds. Let $w \in A$ be such that $|N(w) \cap X|$ is minimum. Let $Z = N(w) \cap X$ and note that $|Z| < |X|$. By our assumption, there exists an edge $y_1y_2$ of $H^*$ such that $y_1$ is $A$-complete and every vertex of $X \setminus Z$ has a neighbor in $\{y_1, y_2\}$. By Lemma 3 an edge of $H^*$ is $X$-complete, say edge $y_3y_4$.

We may assume that vertices $y_1, y_2, y_3, y_4$ are all distinct and $y_1y_3$ and $y_1y_4$ are not edges, since otherwise the result trivially holds. Also w.l.o.g. $y_2y_4$ is not an edge.

Suppose that $wy_4$ is not an edge. We may assume that some $z \in Z$ is not adjacent to $y_1$, since otherwise the edge $y_1y_2$ satisfies the claim. If some $v \in X \setminus Z$ is adjacent to $y_1$, then $\{y_1, v, w, z, y_4\}$ induces a 5-hole. So for every $v \in X \setminus Z$, $vy_1$ is not an edge, and hence $vy_2$ is an edge. If $w$ is adjacent to $y_2$, then $\{y_2, w, v, z, y_4\}$ induces a 5-hole. So $w$ is not adjacent to $y_2$. By Lemma 2 an edge of $H^*$ is $\{v, w\}$-complete. Hence there is a vertex $u$ of $H^*$ adjacent to both $v$ and $w$, but with no neighbor in $\{y_1, y_2\}$. Then $\{y_1, y_2, u, v, w\}$ induces a 5-hole.

Therefore $wy_4$ is an edge. We now show that $y_4$ is $A$-complete. Let $w' \in A$ and assume $w'y_4$ is not an edge. By the choice of $w$ and by the above argument, there is a vertex $v \in X \setminus Z$ adjacent to $w'$. But then the graph induced by $\{w, w', x_1, x_2, v, y_4\}$ contains a 5-hole. This completes the proof of Claim 2. $\diamond$

**Claim 3** For every edge $v_1v_2$ in $G(B)$, there exists $v \in A$ that is adjacent to neither $v_1$ nor $v_2$.

*Proof of Claim 3:* Let $A_1$ be the set of neighbors of $v_1$ in $A$, and $A_2 = A \setminus A_1$. Suppose the claim does not hold. Then $v_2$ is universal for $A_2$. Let $w_1$ be a vertex of $A_1$ that $v_2$ is not adjacent to. Then $v_1, v_2, w_2, x_2, w_1, u_1$, where $w_2 \in A_2$, is a 5-hole. This completes the proof of Claim 3. $\diamond$

By Claim 1, we may assume that for every proper subset $B'$ of $B$, the statement holds. By Claim 2 we may assume that $B$ is not a stable set. Let $v_1v_2$ be an edge of $G(B)$. By Claim 3, let $v$ be a vertex of $A$ that is adjacent to neither $v_1$ nor $v_2$. Let $y_1y_2$ be an edge of $H^*$ such that $y_1$ is $A$-complete and all vertices of $B \setminus v_2$ have a neighbor in $\{y_1, y_2\}$. Let $y_3y_4$ be an edge of $H^*$ such that $y_3$ is $A$-complete and all vertices of $B \setminus v_1$ have a neighbor in $\{y_3, y_4\}$. Then the theorem follows from the following claim.

**Claim 4** $v_1$ has a neighbor in $\{y_3, y_4\}$, or $v_2$ has a neighbor in $\{y_1, y_2\}$.

*Proof of Cliam 4:* Suppose the claim does not hold. $v_1$ has no neighbor in $\{y_3, y_4\}$ and $v_2$ has no neighbor in $\{y_1, y_2\}$.

If a vertex of $\{y_1, y_2\}$ coincides with a vertex of $\{y_3, y_4\}$, then $\{y_1, y_2, y_3, y_4, v_1, v_2\}$ induces a 5-hole. Therefore, vertices $y_1, y_2, y_3, y_4$ are all distinct.

We now show that $v$ and $v_1$ must have a common neighbor in $\{y_1, y_2\}$. Assume not. Then $vy_1$ and $v_1y_2$ are edges, and $vy_2$ and $v_1y_1$ are not. By Lemma 2, there is an edge of $H^*$ that is $\{v, v_1\}$-complete, and hence there is a vertex $u$ of $H^*$ adjacent to both $v$ and $v_1$, but with no neighbor in $\{y_1, y_2\}$. Then $\{y_1, y_2, v, v_1, u\}$ induces a 5-hole. Therefore, $v$ and $v_1$ have a common neighbor $y$ in $\{y_1, y_2\}$, and similarly $v$ and $v_2$ have a common neighbor $y'$ in $\{y_3, y_4\}$. If $yy'$ is not an edge, then $\{y, y', v, v_1, v_2\}$ induces a 5-hole. Therefore, $yy'$ is an edge.

Let $a, y, y', b$ be the subpath of $H^*$ induced by $\{y_1, y_2, y_3, y_4\}$. Then w.l.o.g. $vy, vy', v_1y, v_2y'$ are edges and $v_2a, v_2y, v_1y', v_1b$ are not. Also not both $v$ and $v_1$ can be adjacent to $a$, and not both $v$ and $v_2$ can be adjacent to $b$; otherwise, there is a 5-hole $v, a, v_1, v_2, y'$ or $v, b, v_2, v_1, y$.

Let $z_2$ be the neighbor of $v_2$ in $H^*$ that is closest to $a$ in $H^* \setminus \{y, y'\}$. Note that $z_2 \neq b$ since $v_2$ is a major vertex. Let $P_2$ be the $az_2$-subpath of $H^*$ that does not contain $y$.

Suppose $v$ does not have a neighbor in $P_2$. By Lemma 2, some edge of $H^*$ is $\{v, v_2\}$-complete, and hence there is a vertex $u$ adjacent to both $v$ and $v_2$ but with no neighbor in $P_2$. Note that $u \neq b$. But then $P_2 \cup \{y, y', v, v_2, u\}$ induces a pyramid $\Pi(vyy', v_2)$, and hence there is an odd hole shorter than $H^*$, a contradiction. Therefore $v$ must have a neighbor in $P_2$.

We now show that $a$ is the unique neighbor of $v$ in $P_2$. Let $v'$ be the neighbor of $v$ in $P_2$ that is closest to $z_2$. Assume that $v' \neq a$. Let $P'$ be the $v'z_2$-subpath of $P_2$. If $v_1$ has no neighbor in $P'$, then the graph induced by $S = P' \cup \{y, y', v, v_1, v_2\}$ is a pyramid $\Pi(vyy', v_2)$. If $v_1$ has a neighbor in $P' \setminus z_2$, then the graph induced by $S$ contains a pyramid $\Pi(vyy', v_1)$. So $v_1$ is adjacent to $z_2$. Since the graph induced by $P_2 \cup \{y, y', v_1, v_2\}$ cannot be an odd wheel with center $v_1$, $v_1$ must have a neighbor in $P_2 \setminus P'$. By a similar argument as above we may conclude that $a$ and $z_2$ are the only neighbors of $v_1$ in $P_2$. Then $v$ cannot be adjacent to $a$. Let $v''$ be the neighbor of $v$ closest to $a$ in $P_2$. Note that $v'' \neq z_2$ since otherwise $P_2 \cup \{y, y', v_2, v\}$ induces an odd wheel with center $v$. Let $P''$ denote the $av''$-subpath of $P_2$. By Lemma 2, some edge of $H^*$ is $\{v, v_1\}$-complete, and hence there is a vertex $u$ of $H^*$ adjacent to both $v$ and $v_1$, but with no neighbor in $P''$. But then the graph induced by $P'' \cup \{y, v, v_1, u\}$ is a pyramid $\Pi(ayv_1, v)$. Therefore $a$ is the unique neighbor of $v$ in $P_2$.

Then $v_1$ cannot be adjacent to $a$. Suppose $v_1$ has a neighbor in $P_2$. By Lemma 2, there exists a vertex $u$ of $H^*$ adjacent to both $v$ and $v_1$, but with no neighbor in $P_2$. Then the graph induced by $P_2 \cup \{y, v, v_1, u\}$ contains a pyramid $\Pi(ayv, v_1)$. Therefore, $v_1$ has no neighbor in $P_2$.

Let $z_1$ be the neighbor of $v_1$ in $H^*$ that is closest to $b$ in $H^* \setminus \{y, y'\}$. Let $P_1$ be the $bz_1$-subpath of $H^*$ that does not contain $y$. By symmetry, $b$ is the unique neighbor of $v$ in $P_1$ and $v_2$ has no neighbor in $P_1$. Since $P_2, a, y, y'$ is a sector of wheel $(H^*, v_2)$, $P_2$ must be even, and similarly $P_1$ is even. Note that $z_1z_2$ cannot be an edge, since by Lemma 2, some edge of $H^* \setminus (P_1 \cup P_2 \cup \{y, y'\})$ is $\{v, v_1\}$-complete. But then $P_1 \cup P_2 \cup \{v, v_1, v_2\}$ induces an odd hole shorter than $H^*$, a contradiction. $\square$

### 2.3 Cleaning Algorithm

In this section, we present our cleaning algorithm for the class of graphs of bounded clique number. The running time depends on the clique number.

**Input:** A graph $G$ of bounded clique number $k$.

**Output:** Either an odd hole or a family $\mathcal{F}$ of induced subgraphs of $G$ that satisfies the following properties:

(1) $G$ contains an odd hole if and only if some graph of $\mathcal{F}$ contains a clean shortest odd hole.

(2) $|\mathcal{F}|$ is $O(|V(G)|^{8k})$.

**Step 1:** Check whether $G$ contains a jewel or a pyramid (by algorithms in [2]). If it does, output an odd hole and stop. Otherwise, set $\mathcal{F}_1 = \{G\}$ and $\mathcal{F}_2 = \emptyset$.

**Step 2:** Repeat the following $k$ times. For each graph $F \in \mathcal{F}_1$ and every $(P_1, P_2)$ where $P_1 = x_0, x_1, x_2, x_3$ and $P_2 = y_0, y_1, y_2, y_3$ are two chordless paths of $F$, add to $\mathcal{F}_2$ the graph obtained from $F$ by removing the vertex set $(N(x_1) \cup N(x_2) \cup N(y_1) \cup N(y_2)) \setminus (V(P_1) \cup V(P_2))$. Set $\mathcal{F}_1 = \mathcal{F}_2$ and $\mathcal{F}_2 = \emptyset$.

**Step 3:** Set $\mathcal{F} = \mathcal{F}_1$.

**Theorem 5** *This algorithm produces the desired output, and its running time is $O(|V(G)|^{8k})$.*

*Proof:* Suppose that the algorithm does not output an odd hole. Suppose $G$ contains a shortest odd hole $H^*$. By Step 1 $G$ contains no jewel and no pyramid. Now we show how Step 2 generates a graph in $\mathcal{F}_1$ that contains $H^*$ and $H^*$ is clean in it.

By Lemma 1, $S(H^*)$ is the set of all $H^*$-major vertices. Let $A$ be a maximal stable set of $S(H^*)$. We follow the notation in Theorem 4. Let $P_1 = x_0, x_1, x_2, x_3$ and $P_2 = y_0, y_1, y_2, y_3$ such that $x_1x_2$ and $y_1y_2$ satisfy the conditions stated in Theorem 4. Let $S'(H^*)$ denote the set of vertices of $S(H^*)$ that have no neighbor in $\{x_1, x_2\}$, and are $A$-complete. Let $G'$ be the graph obtained from $G$ by removing $(N(x_1) \cup N(x_2) \cup N(y_1) \cup N(y_2)) \setminus (V(P_1) \cup V(P_2))$. Then $G'$ contains $H^*$ and the set of major nodes for $H^*$ in $G'$ is contained in $S'(H^*)$. The clique number of the graph induced by $S'(H^*)$ is one less than the clique number of the graph induced by $S(H^*)$. Hence, by the fact that the clique number of $G$ is bounded by $k$, Theorem 4 implies that, when the $k$ iterations of Step 2 are completed, some graph $F \in \mathcal{F}_1$ contains $H^*$ and $H^*$ is clean in $F$. Hence (1) holds.

$O(|V(G)|^{8k})$ graphs are created in Step 2. Hence, (2) holds. The running time of Step 1 is $O(|V(G)|^9)$ as discussed in [2]. The running time of Steps 2 is $O(|V(G)|^{8k})$. Therefore, the overall running time is $O(|V(G)|^{8k})$. □

In [2] polynomial time algorithms with following specifications are obtained.

**Input:** A connected clean graph $G$.

**Output:** ODD-HOLE-FREE when $G$ is odd-hole-free, and NOT ODD-HOLE-FREE otherwise.

The above two algorithms imply that it is polynomial to test whether a graph of bounded clique number contains an odd hole.

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