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THE BROADWELL MODEL IN A BOX: STRONG L^1-CONVERGENCE TO EQUILIBRIUM

Leif Arkeryd [1] and Reinhard Illner

UNIVERSITÄT KAISERSLAUTERN
Fachbereich Mathematik
Arbeitsgruppe Technomathematik
Postfach 3049
W-6750 Kaiserslautern

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by

Leif Arkeryd [1] and Reinhard Illner [2,3]

[1] Department of Mathematics, Chalmers Institute of Technology, 41296 Gothenburg, Sweden

[2] Fachbereich Mathematik, Universitaet Kaiserslautern, 6750 Kaiserslautern, Germany

Abstract

The global solution of the one-dimensional Broadwell model in the interval [0,1], with reflecting boundary conditions at 0 and 1, is shown to converge strongly in $L^1[0,1]$ to the constant equilibrium solution.

1. Introduction. We are concerned with the initial boundary value problem

$$\begin{aligned}(\partial_t + \partial_x)v &= z^2 - vw \\ (\partial_t - \partial_x)w &= z^2 - vw \\ \partial_t z &= \frac{1}{2}(vw - z^2)\end{aligned} \tag{1}$$

$v = v(t,x)$, $w = w(t,x)$, $z = z(t,x)$, $t \in [0,\infty)$, $x \in [0,1]$, with initial conditions

$$\begin{aligned}v(0,x) &= v_0(x) \geq 0 \\ w(0,x) &= w_0(x) \geq 0 \\ z(0,x) &= z_0(x) \geq 0\end{aligned} \tag{2}$$

and boundary conditions

$$\begin{aligned}v(t,0) &= w(t,0) \\ v(t,1) &= w(t,1)\end{aligned} \tag{3}$$

for $t > 0$. We refer to (3) as "reflecting boundary conditions".

The equations (1) are known as "the Broadwell model in one space dimension". This model, introduced by Broadwell [1] in 1964, is one of the simplest discrete velocity models of the Boltzmann equation, and has found much attention since it was first introduced [2-6]. We refer to [6] for a careful introduction and for a very complete list of other references.

Here, we are interested in the asymptotic behavior of the global solutions to (1-3). We shall confine our attention to nonnegative bounded continuous initial values, but everything we prove generalizes to $v_0, w_0, z_0 \in L^\infty_+$. It is well-known that (1-3) is equivalent to looking for 2-periodic solutions of the pure initial value problem, where the data are extended to [-1,1] via

$$\begin{aligned} v_0(-x) &= w_0(x) \\ w_0(-x) &= v_0(x) \\ z_0(-x) &= z_0(x) \end{aligned} \tag{4}$$

and 2-periodically to $\Re$ otherwise (if the initial values v_0, w_0 and z_0 are continuous and satisfy $v_0(0) = w_0(0)$, $v_0(1) = w_0(1)$, then this continuation yields continuous functions; however, in terms of the well known **mild** solution concept discontinuities in the data can be handled without difficulty).

A global mild solution of (1-3) is a triple of functions

$$v, w, z : [0, \infty) \times [0, 1] \longrightarrow \Re_+$$

assuming the initial values at $t = 0$, satisfying the boundary conditions (3) for all $t \geq 0$, and such that for $(t, x) \in [0, \infty) \times (0, 1)$ and $x + \tau \in (0, 1)$, $x - \tau \in (0, 1)$

$$\begin{aligned} \frac{d}{d\tau}[v(t+\tau, x+\tau)] &= (z^2 - vw)(t+\tau, x+\tau) \\ \frac{d}{d\tau}[w(t+\tau, x-\tau)] &= (z^2 - vw)(t+\tau, x-\tau) \\ \frac{d}{d\tau}[z(t+\tau, x)] &= \frac{1}{2}(vw - z^2)(t+\tau, x). \end{aligned}$$

In short, we require differentiability of v, w and z along the characteristics.

A classical solution of (1-3) is a triple of functions v, w and z in $C^1([0, \infty) \times [0, 1])$ which satisfy (1-3) pointwise everywhere and which satisfy in addition the consistency boundary condition

$$\begin{aligned} \partial_x v(t, 0) &= -\partial_x w(t, 0) \\ \partial_x v(t, 1) &= -\partial_x w(t, 1) \end{aligned} \tag{5}$$

(which follows from (1) and (3) by subtracting the equations for v and w). Of course, if we look for classical solutions, we have to require that $v_0, w_0, z_0 \in C^1$ and that (3) and (5) hold for the data.

The following results are well known.

Theorem 1. *If v_0, w_0 and z_0 are nonnegative and continuous, then (1-3) has a global nonnegative continuous solution.*

Proof. See [2], [3] or [5].

Remark. It is not known whether the solution remains uniformly bounded for all times. The boundedness results due to Beale [3] and Bony [5] apply only to the pure initial value problem with initial values in $L^1_+ \cap L^\infty_+$.

Theorem 2. *The global solution given by Theorem 1 satisfies*

$$\frac{d}{dt}\int_0^1 (v(t,x) + w(t,x) + 4z(t,x))dx = 0 \tag{6}$$

(mass conservation)

$$\frac{d}{dt}\int_0^1 (v - w)(t,x)dx = 2v(t,0) - 2v(t,1) \tag{7}$$

(momentum transfer)

$$\begin{aligned} \frac{d}{dt}\int_0^1 (v + 2z)(t,x)dx &= v(t,0) - v(t,1) \\ \frac{d}{dt}\int_0^1 (w + 2z)(t,x)dx &= w(t,1) - w(t,0) \end{aligned} \tag{8}$$

and

$$\begin{aligned} \int_0^1 (v\ln v + w\ln w + 4z\ln z)(t,x)dx + \int_0^t\int_0^1 (vw - z^2)\ln\frac{vw}{z^2}(\tau,x)\,dxd\tau \\ = \int_0^1 (v_0\ln v_0 + w_0\ln w_0 + 4z_0\ln z_0)\,dx \end{aligned} \tag{9}$$

(H-Theorem).

Remark. Property (9) is the main ingredient for the proof of Theorem 3 below. Note that the integrand of the second term in (9) has only one sign:

$$(vw - z^2)\ln\frac{vw}{z^2} \geq 0.$$

It is easy to calculate the (unique) steady solution of (1) which is expected in the limit $t \to \infty$: By time independence, from $\partial_t z = 0$ we get $vw = z^2$, i.e. $\partial_x v = \partial_x w = 0$. The boundary conditions (3) imply that v, w, and z must all be equal to the same constant $a > 0$, and from the mass conservation law (6) $a = \frac{1}{6}\int_0^1 (v_0 + w_0 + 4z_0)\, dx$.

Our objective in this paper is to prove

Theorem 3. *Let $g(t,x)$ denote any of the three functions $v(t,x)$, $w(t,x)$ or $z(t,x)$ solving (1-3). Then*

$$\lim_{t\to\infty} \int_0^1 |g(t,x) - a|\, dx = 0. \tag{10}$$

The asymptotic behavior of solutions of (1-3) has long been a problem of interest. The question is intriguing because it is so easy *to guess* the right limit, but nontrivial to prove (10). Besides, a proof of (10) is expected to contain methodology which ought to be useful for more realistic kinetic models (whether our methodology will have that property is something which remains to be seen. The best result so far is due to Slemrod [6], who proved orbital stability for the Broadwell model. Specifically, he showed that there are traveling waves $\tilde{v}(t,x) = \bar{v}(x-t)$, $\tilde{w}(t,x) = \bar{w}(x+t)$ and a function $\tilde{z}(t,x)$ such that v, w and z approach $\tilde{v}, \tilde{w}$ and $\tilde{z}$ weak- $*$ in an appropriate Orlicz space.

The methods employed in [6] (a renormalization similar to the one used in [7], and compensated compactness arguments) did not suffice to prove that $\bar{v} = \bar{w} = \tilde{z} = a$.

Our proof is based on the following elementary observation. From the mass conservation law,

$$\int_0^1 (v(T,x) + w(T,x))\, dx - \int_0^1 (v_0(x) + w_0(x))\, dx = 2\int_0^T \int_0^1 (z^2 - vw)\, dxdt. \tag{11}$$

As the left hand side of (11) is a priori bounded by (6), the integral

$$\int_0^T \int_0^1 (z^2 - vw)\, dxdt$$

is bounded uniformly in T. Our job would be easy if we could find such bounds on

$$\int_0^T \int_0^1 |z^2 - vw| dxdt = \int_0^T \int_0^1 (z^2 - vw)\mathrm{sgn}(z^2 - vw)\, dxdt \tag{12}$$

because then the integrals along characteristics over the right hand sides of (1) would a.e. be convergent in the L^1-sense. Now observe that by (9), there is a $C > 0$ such that

$$\int_0^T \int_0^1 (vw - z^2) \ln \frac{vw}{z^2} dxdt \le C. \tag{13}$$

$\ln \frac{vw}{z^2}$ always has the same sign as $(vw - z^2)$, i.e. $\ln \frac{vw}{z^2}$ plays a part like $\mathrm{sgn}(z^2 - vw)$ in (12). We shall combine this observation with the renormalization trick to show that for large enough t, v, w and z change very slowly along their characteristics, except on a set of small measure. Then, a simple geometrical argument and the boundary conditions yield the proof of Theorem 3.

2. Some Lemmata, and the Proof of Theorem 3.

Lemma 1. *Let $\{f_n\}$ be a sequence of positive measurable functions on $[0,1]$ such that $f_n \to f$ in measure as $n \to \infty$, $\int_0^1 f dx \le C$ and such that for all $\epsilon > 0$ there is a $\delta > 0$ with $\int_M f_n\, dx \le \epsilon$ if $\lambda(M) \le \delta$ (i.e. the sequence $\{f_n\}$ is weakly relatively compact in L^1). Then $f_n \to f$ strongly in L^1.*

Proof. Let $\epsilon > 0$. First, we can choose a $\delta > 0$ such that for all n

$$\int_M f_n\, dx + \int_M f\, dx < \epsilon/2$$

if $\lambda(M) < \delta$. Then, there is an N_0 such that for all $n \ge N_0$

$$\lambda\{x; |f_n(x) - f(x)| \ge \epsilon/2\} < \delta.$$

Therefore,

$$\int |(f_n - f)(x)|\, dx = \int_{\{x; |f_n - f| < \epsilon/2\}} |(f_n - f)(x)|\, dx + \epsilon/2 < \epsilon$$

for $n \ge N_0$.

Lemma 2. *For every $\epsilon > 0$, there is a $\delta > 0$ such that for all $t > 0$ and all M with $\lambda(M) < \delta$*

$$\int_M v(t,x)\, dx + \int_M w(t,x)\, dx + \int_M z(t,x)\, dx < \epsilon$$

(i.e. the functions $v(t,\cdot)$, $w(t,\cdot)$ and $z(t,\cdot)$, $t \ge 0$, form a weakly relatively compact set in L^1).

Proof. This follows from the H-Theorem. Let H_0 denote the initial value of the H-functional, then

$$\int v \ln v + \int w \ln w + 4 \int z \ln z \le H_0,$$

and

$$\int_M v(t,x)\,dx = \int_{M\cap\{v\geq e^m\}} v + \int_{M\cap\{v<e^m\}} v$$
$$\leq \frac{1}{m}\Big(\int_{v\geq 1} v\ln v + \int_{w\geq 1} w\ln w + 4\int_{z\geq 1} z\ln z\Big) + e^m\delta$$
$$\leq \frac{1}{m}(H_0 + \frac{6}{e}) + e^m\delta.$$

Now, given ϵ, choose m such that $\frac{1}{m}(H_0+\frac{6}{e}) < \epsilon/2$, and then choose δ such that $\delta e^m < \epsilon/2$.

In view of Lemma 1 and 2, Theorem 3 will be proved if we can show that for every sequence $t_N \to \infty$ $v(t_N,x)$, $w(t_N,x)$ and $z(t_N,x)$ converge to a in measure. To this end, we next introduce the "renormalized solution concept", which is for our problem completely equivalent to the mild (or classical) solution concept respectively, depending on the regularity of the initial values. Let D_+, D_- and D be shorthands for $\partial_t+\partial_x$, $\partial_t-\partial_x$ and ∂_t respectively, then (v,w,z) is called a renormalized mild solution of (1-3) if

$$\begin{aligned} D_+\ln(1+v) &= C_+ := \frac{z^2-vw}{1+v} \\ D_-\ln(1+w) &= C_- := \frac{z^2-vw}{1+w} \\ D\ln(1+z) &= C_0 := \frac{1}{2}\cdot\frac{vw-z^2}{1+z} \end{aligned} \tag{14}$$

and if the initial and boundary conditions hold.

In the sequel, $\tilde{C}(v,w,z)$ will be an abbreviation for any of the three right hand sides in (14). Also, let $C>0$ be any positive constant (sufficiently large; $C>3$ is enough). Then we have

Lemma 3. *For every sequence $t_N \to \infty$ there is a sequence $\epsilon_N \searrow 0$, depending only on the initial values and on C, such that*

$$\int_{t_N}^{t_N+C}\int_0^1 |\tilde{C}(v,w,z)|\,dxdt \leq \epsilon_N. \tag{15}$$

Proof. From (13), there is a sequence $a_N \searrow 0$ such that

$$\int_{t_N}^{\infty}\int_0^1 (z^2-vw)\ln\frac{z^2}{vw}\,dxdt \leq a_N.$$

Let $I_N = [t_N, t_N + C]$ and consider $\tilde{C}(v,w,z) = \frac{z^2 - vw}{1+w}$ (the other two equations are dealt with in exactly the same way). Also, let $M = \{(t,x) \in I_N \times [0,1];\ z^2 \le 2vw\}$. We split

$$\int_{I_N}\int_0^1 \frac{|z^2 - vw|}{1+w} = \int_M \ldots + \int_{M^c} \ldots =: I + II.$$

The integral II is easily estimated: On M^c, $z^2 > 2vw$, hence

$$II \le \frac{1}{\ln 2}\int_{M^c}(z^2 - vw)\ln\frac{z^2}{vw}\,dxdt \le \frac{a_N}{\ln 2}.$$

To estimate I, let $\{h_N\}$ be a sequence with $h_N \searrow 0$ (later, h_N will be chosen as a suitable function of a_N), and define

$$\begin{aligned} M_1 &= \{(t,x) \in M;\ vw > h_N,\ z^2 \ge vw(1+h_N) \text{ or } z^2 \le vw(1-h_N)\} \\ M_2 &= \{(t,x) \in M;\ vw \le h_N\} \\ M_3 &= \{(t,x) \in M;\ vw(1-h_N) \le z^2 \le vw(1+h_N)\} \end{aligned}$$

Clearly, $M_1 \cup M_2 \cup M_3 = M$.

On M_1, the imposed conditions entail that $|z^2 - vw| \ge h_N^2$ and $|\frac{z^2}{vw} - 1| \ge h_N$. For large enough N, the latter implies $|\ln\frac{z^2}{vw}| \ge \frac{1}{2}h_N$, i.e. we have an estimate

$$\frac{1}{2}h_N^3\int_{M_1} dxdt \le \int_{M_1}(z^2 - vw)\ln\frac{z^2}{vw}\,dxdt \le a_N,$$

which means that

$$\lambda^2(M_1) \le \frac{2a_N}{h_N^3}. \tag{16}$$

Therefore, if we choose h_N such that $a_N/h_N^3 \to 0$ as $N \to \infty$, it follows from Lemma 2 that

$$\int_{M_1}\left|\frac{z^2 - vw}{1+w}\right|dxdt \le \int_{M_1} v\,dxdt = o(1). \tag{17}$$

On M_2, the conditions $z^2 \le 2vw$ and $vw \le h_N$ yield $z^2 \le 2h_N$, hence

$$\int_{M_2}\frac{|z^2 - vw|}{1+w}\,dxdt \le C \cdot h_N. \tag{18}$$

On M_3, the condition $|z^2 - vw| \le vw \cdot h_N$ leads to

$$\int_{M_3}\frac{|z^2 - vw|}{1+w}\,dxdt \le h_N\int_{M_3}\frac{vw}{1+w} \le h_N\int_{M_3} v,$$

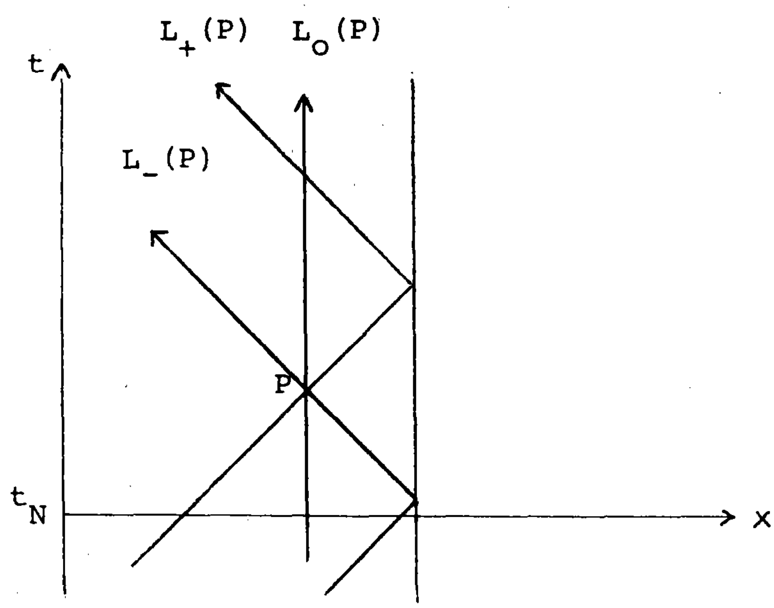
and from mass conservation it follows that

$$\int_{M_3} \frac{|z^2 - vw|}{1+w}\, dx dt \le C h_N. \tag{19}$$

Choosing $h_N = a_N^{1/6}$ (say), we see that $a_N/h_N^3 \to 0$ as $N \to \infty$ and that $h_N \to 0$. This completes the proof.

Now, given t_N, we will denote by P any point $(t,x) \in I_N \times [0,1]$, and by $L_+(P)$, $L_-(P)$ and $L_0(P)$ the characteristics in $I_N \times [0,1]$ associated with D_+, D_- and D which pass through P. We extend L_+ and L_- by the reflecting boundary conditions (see Figure 1).

Figure 1.



By $\int_{L_i(P)} C_i\, ds$, $i = +,-,0$, we denote the integrals of the r.h.s. of (14) along the corresponding characteristic.

Corollary 4. *Let $t \in I_N$ and $P = (t,x)$. There is a constant $c > 0$ such that*

$$\lambda\{x; \max_{i=+,-,0} \int_{L_i(P)} |C_i|\, ds > \sqrt{\epsilon_N}\} \le c \cdot \sqrt{\epsilon_N}.$$

Proof. Straightforward from Lemma 3, by Fubini's theorem.

Consequently, $\int_{L_i(P)} |C_i|\, ds$ will be smaller than $\sqrt{\epsilon_N}$, except on a set of small measure. This implies

Corollary 5. *Except on a set of measure* $\le c\sqrt{\epsilon_N}$, $\ln(1+v)$ ($\ln(1+w)$, $\ln(1+z)$ *respectively) vary less than* $\sqrt{\epsilon_N}$ *along their characteristics in* $I_N \times [0,1]$.

Lemma 6. *There is a constant* $C_1 > 0$ *such that for all* $t \ge 0$ *and all* $N \in \mathbf{N}$

$$\lambda\{x; z(t,x) \ge N\} \le C_1/(N \ln N).$$

Proof. $N \cdot \lambda\{x; z(t,x) \ge N\} \le \int_{z \ge N} z\, dx \le \frac{1}{\ln N}(H_0 + \frac{6}{e})$.

Let $0 < p < 1/2$ and define $M_* = \{x; \int_{t_N}^{t_N+C} \frac{|vw-z^2|}{1+z}(\tau,x)\, d\tau \le \sqrt{\epsilon_N}\}$. By corollary 4, $\lambda(M_*) \ge 1 - c\sqrt{\epsilon_N}$. In addition, let

$$M_*^1 = \{(t,x); z \le \epsilon_N^{-p}\} \cap (M_* \times I_N).$$

From Lemma 6,

$$\lambda^2(M_*^1) \ge C(1 - c\sqrt{\epsilon_N}) - C_1 C \frac{\epsilon_N^p}{\ln \epsilon_N^p} = C - o(1),$$

i.e. M_*^1 is a set of approximately full measure in $I_N \times [0,1]$. From the previous considerations we have

$$\int_{M_*^1} |vw - z^2|\, dx dt \le \frac{2}{\epsilon_N^p} \int_{M_*^1} \frac{|vw - z^2|}{1+z}\, dx dt \le 2\epsilon_N^{\frac{1}{2}-p}.$$

We use this to prove

Lemma 7. *For* $q < \frac{1}{2} - p$, $|(vw - z^2)(t,x)| \le \epsilon_N^q$ *on* $I_N \times [0,1]$, *with the exception of a set of Lebesgue measure less than*

$$2\epsilon_N^{\frac{1}{2}-p-q} + O(\sqrt{\epsilon_N}) + O(\frac{\epsilon_N^p}{\ln \epsilon_N^p}) = o(1).$$

Proof. Note that

$$\begin{aligned} \epsilon_N^q \cdot \int_{M_*^1 \cap \{|vw-z^2| \ge \epsilon_N^q\}} dx dt &\le \int_{M_*^1} |vw - z^2|\, dx dt \\ &\le 2\epsilon_N^{\frac{1}{2}-p}. \end{aligned}$$

This immediately implies

$$\lambda^2(M^1_* \cap \{|vw - z^2| \geq \epsilon_N^q\}) \leq 2\epsilon_N^{\frac{1}{2}-p-q},$$

and the rest of the Lemma follows from the previous estimates.

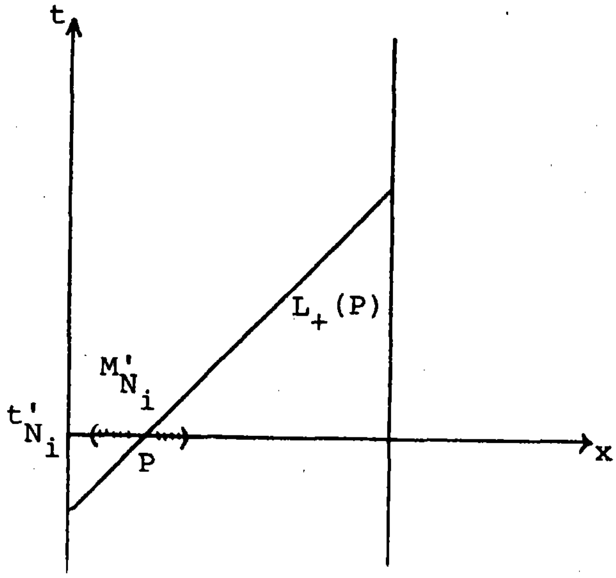
Finally, we use the properties which we have proved to show that $v(t_N, x)$ cannot be close to zero except on sets $M \subset [0,1]$ of measure $o(1)$. This is a consequence of the mass conservation and

Lemma 8. *Suppose that there is a $\delta > 0$ such that for a subsequence $\{N_i\}$ of the integers there are times $t'_{N_i} \in I_{N_i}$ and measurable sets $M'_{N_i} \subset [0,1]$ with $\lambda(M'_{N_i}) > \delta$ and $\sup_{x \in M'_{N_i}} v(t'_{N_i}, x) = o(1)$. Then $v(t,x)$, $w(t,x)$, $z(t,x) = o(1)$ on $I_{N_i} \times [0,1]$ except on sets of measure $o(1)$.*

Proof. We give only a sketch which can be detailed along the lines of our previous reasoning. Here and in the sequel, we use $\approx$ as an abbreviation for "equality up to order o(1) as $N \to \infty$".

By the previous considerations, we can find a point $P \in M'_{N_i}$ such that $v(P) \approx 0$ and such that v varies slowly along the characteristic $L_+(P)$ (see Figure 2).

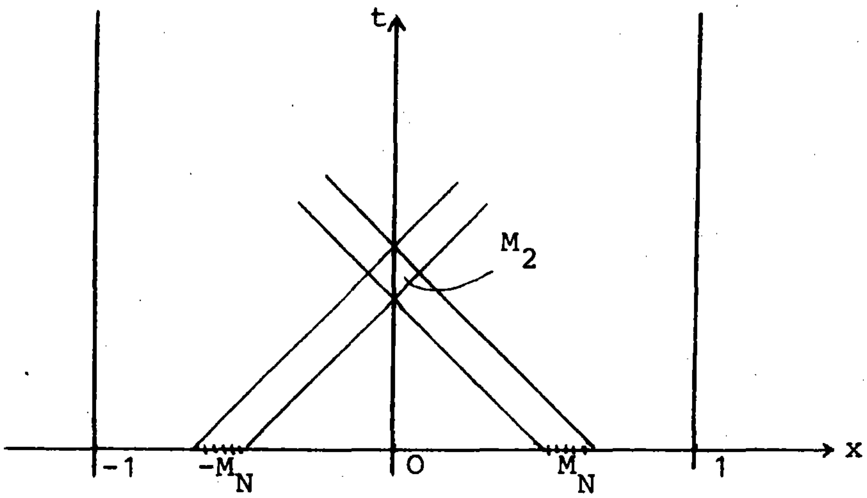
Figure 2.



Hence $v \approx 0$ on $L_+(P)$, and because $z^2 \approx vw$ except on a set of measure $o(1)$, it follows that $z \approx 0$ on $L_+(P)$ with the exception of a small set ("small set" is used here and in the sequel in the obvious sense). As z varies slowly along most of the characteristics L_0, we conclude that $z \approx 0$ in $I_{N_i} \times [0,1]$, with the exception of a small set.

We use a contradiction argument to show that also $v \approx 0$ and $w \approx 0$ on $I_{N_i} \times [0,1]$ except on small sets. Otherwise, there would be a subsequence of the t_{N_i} (denoted for simplicity by t_N) and sets $M_N \subset [0,1]$ with $\lambda(M_N) \not\to 0$ and, say, $w(t_N, x) \not\to 0$ for $x \in M_N$. Because the boundary value problem is equivalent to the periodic pure initial value problem with data extended by (4), $v(t_N, x) \not\approx 0$ for $x \in -M_N$ (see Figure 3).

Figure 3.



As v and w vary slowly along (most of) their forward characteristics, it follows that $vw \not\approx 0$ for $(t,x) \in M_2$, where M_2 is a set of macroscopic (i.e. $\not\approx 0$) two-dimensional measure. On the other hand, we already know that $z \approx 0$ on (most of) M_2 , and $z^2 \approx vw$ on (most of) M_2. This contradicts $vw \not\approx 0$, and the assertion of the Lemma follows.

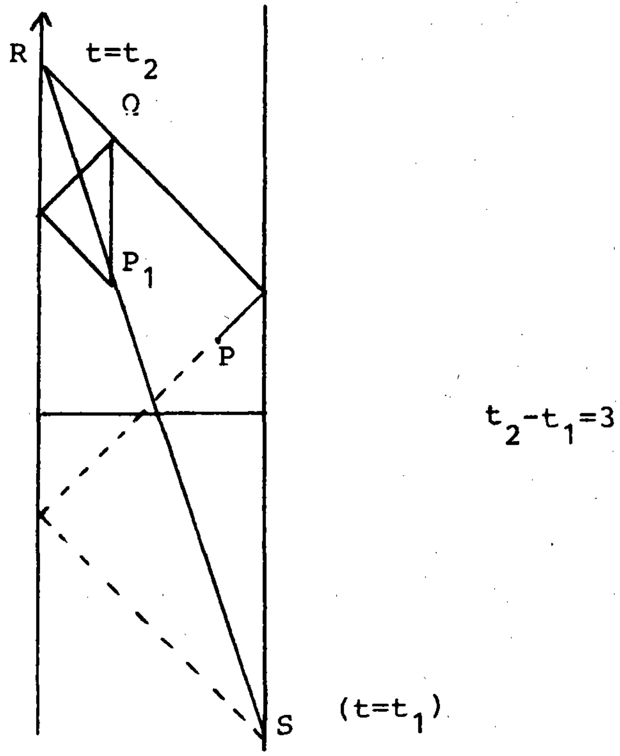
Corollary 9. *Except on sets of measure $o(1)$ in $I_N \times [0,1]$, $v(t,x) \not\approx 0$.*

Proof. This follows from mass conservation and Lemma 8.

We collect the relevant information which we have obtained so far. On $I_N \times [0,1]$ (with the exception of a set of measure $o(1)$), $\ln(1+v)$, $\ln(1+w)$ and $\ln(1+z)$ (and hence v, w and z) vary less that $\sqrt{\epsilon_N}$ (less than $\exp\sqrt{\epsilon_N}-1$) along their characteristics. For small enough $q > 0$, $|(vw - z^2)(t,x)| \leq \epsilon_N^q$ except on a set of 2-dimensional Lebesgue measure $o(1)$. With these observations, we are ready for the

Proof of Theorem 3. Consider a point $P \in I_N \times [0,1]$ (see Figure 4).

Figure 4.



On the forward characteristic $L_+(P)$, choose a point Q such that $|(z^2 - vw)(Q)| \leq \epsilon_N^q$ (our Lemmata imply that for all P except those in a set of measure o(1) there is a set of small one-dimensional measure of points on $L_+(P)$ such that $|z^2 - vw| \leq \epsilon_N^q$ outside this set). The other two backward characteristics from Q intersect in a point P_1 along the line $\overline{SR}$. Again we can assume that $|(z^2 - vw)(P_1)| \leq \epsilon_N^q$ except for P_1 in a set of small measure, and that the corresponding dependent variables v, w and z vary less than $e^{\sqrt{\epsilon_N}} - 1$ along the characteristics connecting P_1 and Q. We find

$$z(Q) \approx z(P_1)$$
$$z^2(Q) \approx vw(Q)$$
$$z^2(P_1) \approx vw(P_1)$$
$$v(Q) \approx w(P_1)$$
$$w(Q) \approx v(P).$$

It follows that $vw(Q) \approx vw(P_1)$, and, from Lemma 6, because v, w and z can be assumed to be bounded except on a set of small measure, this implies $v(Q)v(P) \approx v(Q)v(P_1)$, i.e.

$v(P_1) \approx v(P)$ because, by Corollary 9, $v(Q)$ can be assumed to be positively bounded below uniformly over N except on small sets.

Summarizing, we have proved that there is a $P \in I_N \times [0,1]$ such that

$$v(P_*) \approx v(P)$$

for P_* on the line $\overline{SR}$, with the exception of a set of one-dimensional measure $o(1)$ on $\overline{SR}$. In other words, v is close to a constant on $\overline{SR}$, except on a set of small measure (see Figure 4). As v varies less than $e^{\sqrt{\epsilon_N}} - 1$ along most L_+-characteristics through $\overline{SR}$, v is close to a constant in $(t_1, t_2) \times [0,1]$ (except on a set of small measure), where $t_2 - t_1 = 3$. From the boundary conditions it follows then that w is close to the same constant C_2 except on a set of small measure in $(t_1, t_2) \times [0,1]$, and from $z^2 - vw \approx 0$ also $z \approx C_2$. By mass conservation, it follows that $C_2 = a$. Using once more that v, w and z vary slowly along most characteristics, we observe that for every sequence $t_N \to \infty$

$$(v(t_N, \cdot), w(t_N, \cdot), z(t_N, \cdot)) \to (a, a, a)$$

in measure. In view of the remarks after Lemma 2, this completes the proof of Theorem 3.

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[3] permanent address: Department of Mathematics and Statistics, University of Victoria, P.O. Box 3045, Victoria, B.C. V8W 3P4, Canada