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# MATHEMATICAL ANALYSIS OF A SUBSURFACE FLOW MODEL

SAFAA AL NAZER, CHRISTOPHE BOUREL, CAROLE ROSIER

ABSTRACT. The purpose of this article is the mathematical analysis of a new class of models to describe the flow in shallow aquifers, as alternatives to the 3d-Richards model. This type of models was introduced in a previous work and consists of the coupling of an almost 1d vertical flow in the upper part of the aquifer with a 2d horizontal flow in the lower part. These two regions being separated by a time-dependent interface, an unknown of the problem.

A result of existence of weak solutions is proved for a very general form of the hydraulic conductivity (anisotropic case). The strategy is based on the classical framework of parabolic equations in non-cylindrical domains. It also exploits the compressibility of the fluid to overcome the difficulty associated with the degeneracy in the time derivative term of Richards equation.

**Keywords.** Richards equation, quasilinear parabolic equations, global in time existence, free boundary problem, shallow aquifer.

## 1. INTRODUCTION

The purpose of this paper is to mathematically study a model describing the water flow in shallow aquifers. This model is applied in a hydrogeological context with emphasis on the exchange between surface and subsurface waters. A specific application is the description of the contamination of a water table by pollutants coming from the surface.

In this context, aquifers are classically considered as porous media in which a mixture of two fluids flows: water in a liquid phase and air in a gas phase. Although this situation can be described by a two-phase flow model (coupled system of PDEs), it is classical to take advantage of the practical physical and geometrical situation to obtain a simplification of the modeling. The main interest in practice is to deal with a model that is easier to handle, especially from a numerical point of view.

A usual simplification aims to approximate the 3d-Richards model by taking advantage of the particular geometry of the aquifer under consideration. The latter is assumed to be very large and shallow, as is the physical situation (for example in the context of the contamination of a water table by pollutants from an agricultural field). The assumption that the aquifer is shallow makes the problem asymmetric, with the vertical and horizontal components of the flow behaving very differently.

A classical hypothesis in this context is known as the Dupuit hypothesis (see [1]) which assumes that the flows are essentially orthogonal to the walls. This allows the problem to be simplified by vertically integrating the Richards equation in the saturated zone. This has led to the use of a family of 2d models developed since the 1960s (see for example the works of Jacob Bear, [2, 3]). The vertical integration approach is only valid for very focused length and time scales, the time scale is for example completely different from the typical duration of chemical reactions. (See once again [2] for empirical and qualitative arguments, see [4] for asymptotic calculations.)

To overcome these limitations, models have been proposed that retain the Dupuit-like structure to describe the flow in the water table (saturated part), and that couple it with many 1d-vertical Richards models to describe the recharge from the surface (unsaturated part). The aim is twofold: to have a good description of the flow in the vadose zone in order to facilitate the coupling of the overland flow with the subsurface flow; and also to obtain a more accurate velocity of the flow in this zone (which is crucial in the context of the reactive transport of contaminants from the surface). This type of strategy is used in a numerical setting for example in [5, 6, 7, 8]. More recently, this type of model has been proposed in [9]. The mathematical study of this class of models is particularly delicate because of the nonlinearities, the free boundary between each zone, the lack of control over the horizontal components of the pressure gradient and the difficulty arising from the coupling between the two zones, which is expressed in terms of the flux at the interface. We also have to deal with the classical difficulties in the Richards equations.

The aim of this paper is to propose a new model, physically very close to those given in [9], but for which the theoretical study is achievable. First, we take into account the low compressibility of the water in conservative laws modeling the dynamics of underground water. In this way it is possible to avoid the degeneracy and the non-linearity in the time derivative term of the Richards equation. Secondly, the horizontal conductivity in the capillary fringe of the aquifer is now assumed to be non-zero, allowing full estimates of the pressure gradient in this region. It should be emphasized that the whole problem, as those given in [9], is mass-conservative.

The previous transformations bring us back to the framework of quasilinear parabolic equations on a non-cylindrical domain. Indeed the Richards equation holds in the time dependent domain thanks to the Dirichlet boundary condition satisfied by the pressure at the interface. There are several methods to deal with free boundary problems. We choose here the framework of the auxiliary domain method introduced by Lions and Mignot in [10, 11], which is now possible since the model is described by parabolic equations.

One can understand this new model as being an approximation of the models introduced in [9], where the compressibility parameter appears only as a mathematical trick to simplify the study. Nevertheless, this model itself has a physical meaning since it approximates the original 3d-Richards equation in the same way as the models in [9] do. The strategy to formally justify the model is the same as that presented in [9].

The paper [12] also takes advantage of the two key assumptions. (Namely, slight compressibility of the aqueous phase and 'small' lateral flow in the unsaturated area.) But it also assumes an isotropic hydraulic conductivity and neglects the Robin condition for surface-water groundwater interaction on $\Gamma_{soil}$ unlike this study.

The document is organized as follows. The geometry of the aquifer is first described as well as the class of models considered in this article generalizing the model of [9]. The main result is given in Section 2. It concerns the global in time existence of the solution of the model described in Section 1. The proof of the Theorem is given in Section 3. It follows a fixed-point strategy to deal with difficulties related to nonlinearities and to the coupling of the two equations.

### 1.1. GEOMETRY, PHYSICAL PARAMETERS AND BOUNDARY CONDITIONS

For the three-dimensional description, we denote by $\boldsymbol{x} := (x, z)$, $x = (x_1, x_2) \in \mathbb{R}^2$, $z \in \mathbb{R}$, the usual coordinates. Moreover, we respectively denote by $\nabla$ and $\nabla\cdot$ the classical gradient and divergence operator for functions defined on $\mathbb{R}^3$. We introduce also the notation $\nabla'$ and $\nabla'\cdot$ defined for functions $f : \mathbb{R}^d \mapsto \mathbb{R}$ and $\boldsymbol{g} : \mathbb{R}^d \mapsto \mathbb{R}^d$ with $d \in \{2, 3\}$ by

$$\nabla' f = \begin{pmatrix} \partial_1 f \\ \partial_2 f \end{pmatrix}, \quad \nabla' \cdot \boldsymbol{g} = \partial_1 g_1 + \partial_2 g_2.$$

**Geometry** The aquifer is represented by a three-dimensional cylindrical domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$ given by

$$\Omega = \Omega_{2d} \times ]h_{\text{bot}}, h_{\text{soil}}[, \tag{1.1}$$

where $\Omega_{2d} \subset \mathbb{R}^{n-1}$ and $h_{\text{bot}}$, $h_{\text{soil}}$ are real number such that $h_{\text{bot}} < h_{\text{soil}}$. They represent respectively the lower and upper level of the aquifer. We also introduce a fixed level $h_{\text{bot}} < h_{\text{max}} < h_{\text{soil}}$.

We always denote by $\vec{\nu}$ the outward unit normal and $\vec{e}_3$ is the unitary vertical vector pointing up. We decompose the boundary $\partial\Omega$ of $\Omega$ in three zones (bottom, top and vertical)

$$\partial\Omega = \overline{\Gamma_{\text{bot}}} \sqcup \overline{\Gamma_{\text{soil}}} \sqcup \overline{\Gamma_{\text{ver}}} \ ,$$

with

$$\Gamma_{\text{bot}} := \Omega_{2d} \times \{h_{\text{bot}}\}, \quad \Gamma_{\text{soil}} := \Omega_{2d} \times \{h_{\text{soil}}\}, \quad \Gamma_{\text{ver}} := \partial\Omega_{2d} \times ]h_{\text{bot}}, h_{\text{soil}}[. \tag{1.2}$$

The description of the flow is divided into two sub-regions of $\Omega$ (possibly time-dependent) in each of which the flow exhibits different behavior. The definition of these zones is based on the function $h = h(t, x)$, which is an unknown in our problem. We then introduce, for a given function $h = h(t, x)$ such that $h_{\text{bot}} \leq h \leq h_{\text{soil}}$:

$$\Omega_t^- := \big\{(x, z) \in \Omega,\ z < h(x, t)\big\} \quad \text{and} \quad \Omega_t := \big\{(x, z) \in \Omega,\ z > h(x, t)\big\}, \tag{1.3}$$

$$\Gamma_t := \big\{(x, z) \in \Omega,\ z = h(x, t)\big\}. \tag{1.4}$$

**Richards hypothesis** The Richards model is moreover based on the assumption that the air pressure in the subsurface is equal to the atmospheric pressure, thus it is not an unknown of the problem. It is assumed that the moisture content and the relative conductivity of the soil are *functions* of the fluid pressure $P$, denoted by $\theta = \theta(P)$ and $\kappa = \kappa(P)$ respectively . The saturation pressure $P_s$ (which is a fixed real number) is introduced. The fully-saturated part of the soil corresponds to the region $\{\boldsymbol{x},\ P(\cdot, \boldsymbol{x}) > P_s\}$, while the partially-saturated part is $\{\boldsymbol{x},\ P_d < P(\cdot, \boldsymbol{x}) \leq P_s\}$. The dry part is defined by the set $\{\boldsymbol{x},\ P(\cdot, \boldsymbol{x}) \leq P_d\}$. The moisture content is such that

$$\theta = \begin{cases} \phi & \text{(saturated zone)} & \text{if} \quad P(\cdot, \boldsymbol{x}) > P_s, \\ \theta(P) & \text{(with } \theta_0 \leq \theta(P) \leq \phi \text{ and } \theta'(P) > 0) & \text{if} \quad P_d < P(\cdot, \boldsymbol{x}) \leq P_s, \\ \theta_0 & \text{(dry zone)} & \text{if} \quad P(\cdot, \boldsymbol{x}) \leq P_d, \end{cases} \tag{1.5}$$

where $\theta_0 > 0$ corresponds to a residual moisture content that is positive. The associated relative hydraulic mobility is then defined by

$$\kappa(P) = \begin{cases} 1 & \text{(saturated zone)} & \text{if} \quad P(\cdot, \boldsymbol{x}) > P_s, \\ \kappa(P) & \text{(with } 0 \leq \kappa(P) \leq 1 \text{ and } \kappa'(P) > 0) & \text{if} \quad P_d < P(\cdot, \boldsymbol{x}) \leq P_s, \\ 0 & \text{(dry zone)} & \text{if} \quad P(\cdot, \boldsymbol{x}) \leq P_d. \end{cases} \tag{1.6}$$

**Permeability tensor $\boldsymbol{K}_0$** The soil transmission properties are characterized by the porosity function $\phi$ and the permeability tensor $\boldsymbol{K}_0(x,z)$. The matrix $\boldsymbol{K}_0$ is a $3\times 3$ symmetric positive definite tensor describing the conductivity of the *saturated* soil at the position $(x,z)\in\Omega$. We introduce $\boldsymbol{K}_{xx}\in\mathcal{M}_{22}(\mathbb{R})$, $K_{zz}\in\mathbb{R}^*$ and $\boldsymbol{K}_{xz}\in\mathcal{M}_{21}(\mathbb{R})$ such that

$$\boldsymbol{K}_0 = \begin{pmatrix} \boldsymbol{K}_{xx} & \boldsymbol{K}_{xz} \\ \boldsymbol{K}_{xz}^T & K_{zz} \end{pmatrix}. \tag{1.7}$$

**Fluid compressibility** The fluid is considered weakly compressible by assuming that pressure $P$ is related to the density $\rho$ as follows (cf. [13]):

$$\frac{d\rho}{\rho} = \alpha_p dP \Leftrightarrow \rho = \rho_0 e^{\alpha_p(P-P_0)}. \tag{1.8}$$

The real number $0<\alpha_p \ll 1$ represents the fluid compressibility coefficient and $P_0$ is the pressure of reference. In this work, we use a variant of the 3d-Richards equations taking into account (1.8) [12].

**Soil Compressibility** The effects of the rock compressibility are neglected in the model, the porosity of the medium $\phi$ does not depend on the pressure variations. We also assume that the soil is homogeneous and thus associated with a constant $\phi>0$.

**Boundary conditions** On the boundary $\Gamma_{\text{soil}}$, we consider a general Neumann condition

$$\boldsymbol{v}\cdot\vec{\nu} = F \quad \text{for } (t,x,z)\in(0,T)\times\Gamma_{\text{soil}}, \tag{1.9}$$

where $\boldsymbol{v}$ is the fluid velocity and $F$ is a source term. On the other hand, an impermeable bedrock is considered at the bottom of the aquifer $\Gamma_{\text{bot}}$. For the sake of simplicity, we also consider such an impermeable layer at the lateral boundary $\Gamma_{\text{ver}}$:

$$\boldsymbol{v}\cdot\vec{\nu} = 0 \quad \text{for } (t,x,z)\in(0,T)\times\Gamma_{\text{bot}}\cup\Gamma_{\text{ver}}. \tag{1.10}$$

### 1.2. A COUPLED PROBLEM APPROXIMATING THE FLOW IN SHALLOW AQUIFERS

After introducing some additional notation, we present the class of models that are the object of this article.

**Horizontal perturbation and averaged conductivity** We introduce

$$\boldsymbol{S} = \boldsymbol{K}_{xx} - \frac{1}{K_{zz}}\boldsymbol{K}_{xz}(\boldsymbol{K}_{xz}^T) \quad \text{and} \quad \boldsymbol{M}_0 = \begin{pmatrix} \boldsymbol{S} & 0 \\ 0 & 0 \end{pmatrix}. \tag{1.11}$$

The $2\times 2$ tensor $\boldsymbol{S}$ is the Schur complement of the block $K_{zz}$ in the tensor $\boldsymbol{K}_0$. It will act as an effective permeability tensor. Let $\boldsymbol{N}_0$ be a $2\times 2$ symmetric positive tensor. We introduce

$$\boldsymbol{B} = \begin{pmatrix} \boldsymbol{N}_0 & 0 \\ 0 & K_{zz} \end{pmatrix}, \quad \boldsymbol{G}_0 = \begin{pmatrix} \boldsymbol{N}_0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \boldsymbol{A}_0 = \begin{pmatrix} \boldsymbol{S}-\boldsymbol{N}_0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{1.12}$$

We also introduce the averaged conductivity tensors $\boldsymbol{K}$ and $\boldsymbol{J}$ defined in $(0,T)\times\Omega_{2d}$ for any function $H=H(t,x)$ and $h=h(t,x)$ by

$$\boldsymbol{K}(H)(t,x) = \int_{h_{\text{bot}}}^{h_{\text{soil}}} \kappa\big(\rho g(H(t,x)-z)\big)\,\boldsymbol{S}(x,z)\,dz, \tag{1.13}$$

$$\boldsymbol{J}(H)(t,x) = \boldsymbol{K}(H)(t,x) - \int_{h(t,x)}^{h_{\text{soil}}} \kappa\big(\rho g(H(t,x)-z)\big)\,\boldsymbol{N}_0(x,z)\,dz. \tag{1.14}$$

If the tensor $\boldsymbol{N}_0$ is zero as in [9], the first equation (1.15) is not precisely defined from a theoretical point of view. Indeed, as the horizontal hydraulic conductivity is zero, it is a 1d equation but defined in a 3d domain. There is therefore no control of the pressure gradient with respect to horizontal spatial variables. The idea is to now consider a small horizontal flow in the upper part of the aquifer by introducing $\boldsymbol{N}_0$ in order to obtain full estimates of the pressure gradient. $\boldsymbol{N}_0$ is assumed small enough to have $\boldsymbol{J}$ and $\boldsymbol{B}$ positive definite.

**Class of coupled models** We introduce the following family of models $\mathcal{M}_\delta = \mathcal{M}_\delta(\boldsymbol{N}_0, \alpha_p)$ defined for $0 \leq \delta \ll 1$, $\boldsymbol{N}_0$ a $2 \times 2$ positive tensor, $0 \leq \alpha_p \ll 1$. They consist in finding the fluid pressure $P$ such that:

- In $\Omega_t$, the following 3d-Richards equation holds

$$\begin{cases} \partial_t \theta(P) + \theta \alpha_p \partial_t P + \nabla \cdot \boldsymbol{q} = 0 & \text{for} \quad t \in ]0, T[, \quad (x,z) \in \Omega_t, \\ \boldsymbol{q} \cdot \vec{\nu} = F & \text{on} \quad (0,T) \times \Gamma_{\text{soil}}, \\ \boldsymbol{q} \cdot \vec{\nu} = 0 & \text{on} \quad (0,T) \times \Gamma_{\text{ver}}, \\ P\big(t, x, h(t,x)\big) = P_s & \text{in} \quad (0,T) \times \Omega_{2d}, \\ P(0,x,z) = P_0(x,z) & \text{in} \quad \Omega_0. \end{cases} \tag{1.15}$$

  The effective velocity $\boldsymbol{q}$ is given by

$$\boldsymbol{q} = -\kappa(P) \boldsymbol{B} \nabla \Big( \frac{P}{\rho g} + z \Big). \tag{1.16}$$

- In $\Omega_t^-$, the pressure $P$ satisfies for $t \in ]0, T[$ and $(x,z) \in \Omega_t^-$

$$P(t,x,z) = P_s + \rho g (h - z). \tag{1.17}$$

- The averaged hydraulic head satisfies

$$\rho g \alpha_p (h - h_{\text{bot}}) \partial_t H - \nabla' \cdot (\boldsymbol{J}(H) \nabla' H) = -\nabla' \cdot \left( \int_h^{h_{\text{soil}}} \boldsymbol{q} \, dz \right)$$
$$- \boldsymbol{q}_{|z = h_{\text{soil}}} \cdot \vec{\nu} - \int_h^{h_{\text{soil}}} \big( \partial_t \theta(P) + \theta(P) \alpha_p \partial_t P \big) \, dz, \quad \text{on} \quad (0,T) \times \Omega_{2d}, \tag{1.18}$$
$$\boldsymbol{J}(H) \nabla' H \cdot \vec{\nu} = 0 \quad \text{on } (0,T) \times \partial \Omega_{2d}, \qquad h(0,x) = h_0(x) \quad \text{in } \Omega_{2d}.$$

  where the averaged conductivity is given in (1.14).
- The depth of $\Gamma_t$, $h$, satisfies in $\Omega_{2d}$

$$h(t,x) = \max\Big\{ \min\Big\{ H(t,x) - \frac{P_s}{\rho g}, h_{\max} \Big\}, h_{\text{bot}} + \delta \Big\}. \tag{1.19}$$

The class of models $\mathcal{M}_\delta$ is a generalization of the family of models proposed in [9]. The parameter $\delta > 0$ is introduced to eliminate the time degeneracy in equation (1.18). It is possible to apply the same strategy as in [9] to justify that models $\mathcal{M}_\delta$ are good approximations of the compressible 3d-Richards model in shallow aquifers. Indeed, thanks to formal asymptotic expansions, we characterize the effective problems associated with the compressible 3d-Richards problem and with the problems $\mathcal{M}_\delta$ when the ratio *depth/horizontal length* of the aquifer is very small, and for different time scales (short, intermediate and long). More precisely we show that the two effective problems are the same in the long and intermediate time scale for all $\alpha_p \geq 0$ and in the short time scale for $\alpha_p = 0$ [14]. These asymptotic results hold only in the case where the solution $P$ is such that $h_{\text{bot}} + \delta < h < h_{\max}$ (corresponding to an aquifer which is neither empty nor overflowing). Moreover, the models $\mathcal{M}_\delta$ have the same coupled structure as those of [9]. They can be seen as the coupling of the two flows characterized by the effective models at the short and long time scales. The first one is a quasi-vertical* 1d-Richards problem in the upper part of the aquifer (see (1.15)). It mimics the behavior of the flow in the case of a short time scale. The second one is a 2d horizontal problem assuming an instantaneous vertical flow in the lower part of the aquifer (corresponding to the long time scale).

## 2. MATHEMATICAL FRAMEWORK AND MAIN RESULTS

As the problem ($\mathcal{M}_\delta$) is a free boundary problem, we define the general framework of parabolic equations in non cylindrical domains, introduced by Lions in [10] and Mignot in [11].

### 2.1. NOTATION AND AUXILIARY RESULTS

For any $T>0$, let $\mathcal{O}_T$ be the open domain of $\mathbb{R}^+\times\Omega$ defined by

$$\mathcal{O}_T=\big\{(t,x,z)\in(0,T)\times\Omega,\ h(t,x)<z\big\},$$

where $h$ is the position of the interface $\Gamma_t$. We set

$$\Omega_t=\big\{(x,z)\in\Omega,\ z\in]h(t,x),h_{\text{soil}}[\big\},\quad \mathcal{O}_T^c=\big((0,T)\times\Omega\big)\setminus\mathcal{O}_T,$$

$$\Gamma=\partial\mathcal{O}_T \text{ (boundary of } \mathcal{O}_T),\quad \Gamma'=\Gamma\setminus\Omega_0 \text{ (lateral boundary of } \mathcal{O}_T).$$

We define

$$H^{0,1}(\mathcal{O}_T)=\{u\in L^2(\mathcal{O}_T),\ \nabla u\in L^2(\mathcal{O}_T)\times L^2(\mathcal{O}_T)\}.$$

It is a Hilbert space endowed with the norm

$$\|u\|^2_{H^{0,1}(\mathcal{O}_T)}=\int_{\mathcal{O}_T}|u|^2dxdt+\int_{\mathcal{O}_T}|\nabla u|^2dxdt.$$

$F(\mathcal{O}_T)$ denotes the closure in $H^{0,1}(\mathcal{O}_T)$ of functions of $\mathcal{D}(\bar{\mathcal{O}}_T)$ vanishing in a neighborhood of $\Gamma_t$ and $F'(\mathcal{O}_T)$ its topological dual. We also introduce

$$\mathcal{B}(\mathcal{O}_T)=\Big\{u\in F(\mathcal{O}_T)|\ \frac{du}{dt}\in F'(\mathcal{O}_T)\Big\},$$

endowed with the Hilbertian norms $\|\cdot\|_{\mathcal{B}(\mathcal{O}_T)}=\big(\|\cdot\|^2_{F(\mathcal{O}_T)}+\|\partial_t\cdot\|^2_{F'(\mathcal{O}_T)}\big)^{1/2}$.

Finally, $B_0(\mathcal{O}_T)$ (resp. $B_T(\mathcal{O}_T)$) is the closure in $\mathcal{B}(\mathcal{O}_T)$ of functions of $\mathcal{B}(\mathcal{O}_T)$ vanishing in a neighborhood of $t=0$ (resp. t=T). We now give some auxiliary results proved in [10].

**Lemma 2.1.** *If $\mathcal{O}_T$ is sufficiently regular, we have*

(1) $H^{0,1}(\mathcal{O}_T)=L^2([0,T];H^1(\Omega_t))$ *where*

$$L^2(0,T;H^1(\Omega_t))=\Big\{u,\ u(t,.)\in H^1(\Omega_t),\ t\in[0,T]\ \textit{a.e.}\quad \textit{and}\ \|u\|_{H^{0,1}(\mathcal{O}_T)}=\int_0^T\|u\|^2_{H^1(\Omega_t)}<+\infty\Big\}.$$

*A similar result holds for $F(\mathcal{O}_T)$.*

(2) *For $u\in F(\mathcal{O}_T)$, we can define $\gamma(u)$, the trace of $u$ on $\Gamma'$ in $L^2(\Gamma')$. Moreover $u\in F(\mathcal{O}_T)\Longleftrightarrow\gamma(u)=0$ on $\Gamma_t$.*

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*As the tensor $\boldsymbol{N}_0$ is assumed to be small.

*(3) Let $u \in \mathscr{B}(\mathcal{O}_T)$, thus $u \in B_T(\mathcal{O}_T) \Longleftrightarrow u(T,.) = 0$.*
*(4) $\forall u, v \in \mathscr{B}(\mathcal{O}_s)$, we have*

$$\langle \frac{\partial u}{\partial t}, v\rangle_{F'(\mathcal{O}_s),F(\mathcal{O}_s)} + \langle \frac{\partial v}{\partial t}, u\rangle_{F'(\mathcal{O}_s),F(\mathcal{O}_s)} = (u(s,.), v(s,.))_{L^2(\Omega_s)} - (u(0,.), v(0,.))_{L^2(\Omega_0)}. \quad (2.1)$$

For the sake of brevity, we will write $H^1(\Omega) = W^{1,2}(\Omega)$ and

$$V(\Omega) = H^1_{0,\Gamma_{\rm bot}}(\Omega) = \{u \in H^1(\Omega),\, u = 0 \text{ on } \Gamma_{\rm bot}\}, \quad V'(\Omega) = (H^1_{0,\Gamma_{\rm bot}}(\Omega))'.$$

For any $T > 0$, let $W_0(0,T,\Omega)$ denote the space

$$W_0(0,T,\Omega) := \{\omega \in L^2(0,T;V(\Omega)),\, \partial_t\omega \in L^2(0,T;V'(\Omega))\},$$

endowed with the Hilbertian norm $\|\cdot\|^2_{W_0(0,T,\Omega)} = \|\cdot\|^2_{L^2(0,T;V(\Omega))} + \|\partial_t\cdot\|^2_{L^2(0,T;V'(\Omega))}$. In the same way, we introduce the space

$$W(0,T,\Omega_{2d}) := \{\omega \in L^2(0,T;H^1(\Omega_{2d})),\, \partial_t\omega \in L^2(0,T;(H^1(\Omega_{2d}))')\},$$

endowed with the Hilbertian norm

$$\|\cdot\|^2_{W(0,T,\Omega_{2d})} = \big(\|\cdot\|^2_{L^2(0,T;H^1(\Omega_{2d}))} + \|\partial_t\cdot\|^2_{L^2(0,T;(H^1(\Omega_{2d}))')}\big).$$

### 2.2. Mathematical assumptions and transformation of the initial problem

We aim to give an existence result of physically admissible weak solutions for the model $(\mathscr{M}_\delta)$ completed by initial and boundary conditions.

Let us first look at the mathematical assumptions. We start with the properties of the porous structure.

- We have

$$\theta \in \mathscr{C}^1(\mathbb{R}), \quad 0 < \theta_- := \phi s_0 \le \theta(x) \le \theta_+, \quad \theta'(x) \ge 0 \quad \forall x \in \mathbb{R}, \quad (2.2)$$

$$\kappa \in \mathscr{C}(\mathbb{R}), \quad 0 < \kappa_- \le \kappa(x) \le \kappa_+ \quad \forall x \in \mathbb{R}. \quad (2.3)$$

- It is also assumed that the tensors $\boldsymbol{B}$ and $\boldsymbol{J}$ are bounded and uniformly elliptic. More precisely, there exist a couple of positive real numbers, $0 < K^- \le K^+$, such that

$$K^-|\xi|^2 \le \boldsymbol{B}\xi\cdot\xi \le K^+|\xi|^2, \ \forall \xi \in \mathbb{R}^3 \setminus \{0\}, \quad (2.4)$$

  and

$$K^-|\xi|^2 \le \boldsymbol{J}\xi\cdot\xi \le K^+|\xi|^2, \ \forall \xi \in \mathbb{R}^2 \setminus \{0\}. \quad (2.5)$$

- Finally we assume that the source term $F$ is a given function belonging to the space $L^2(0,T,L^2(\Omega_{2d}))$.
- The functions $H_0 \in L^2(\Omega_{2d})$ and $P_0 \in H^2(\Omega)$ satisfy the compatibility condition

$$P_0\big(x, h(0,x)\big) = P_s \quad \text{in} \quad \Omega_0,$$

  where $h$ is given by (1.19).
- As discussed in Subsection 1.2, we also assume that $\delta > 0$, $\alpha_p > 0$, and that $\boldsymbol{N}_0$ is a positive definite tensor as well as $\boldsymbol{J}$ and $\boldsymbol{B}_0$.

Before stating the main result of this work, we will transform the original problem as in [15] and bring us back to the framework introduced in [11].

The assumptions (2.2)-(2.3) allow to eliminate the non-linearity in time of Equation (1.15), namely they are sufficient to define the primitive function $\mathscr{P}$ such that

$$\mathscr{P}(P) = \theta(P) + \alpha_p \int^P \theta(s)\, ds.$$

A direct calculation gives $\mathscr{P}'(P) = \theta'(P) + \alpha_p \theta(P) \geq \alpha_p \theta_- > 0$. Indeed, by previous hypothesis, we have $\theta'(P) \geq 0$ and $\theta(P) > \phi s_0$.
As $\mathscr{P}$ is a bijective application, the existence of $p$ such that

$$p = \mathscr{P}(P)$$

is equivalent to the existence of solution $P$ of the original Richards problem. The transformation $\mathscr{P}$ of Equation (1.15) is

$$\partial_t p - \frac{1}{\rho g} \nabla \cdot \Big( \frac{\kappa(\mathscr{P}^{-1}(p))}{(\theta' + \alpha_p \theta)(\mathscr{P}^{-1}(p))} \boldsymbol{B} \nabla p \Big) - \nabla \cdot \Big( \kappa(\mathscr{P}^{-1}(p)) \boldsymbol{B} \vec{e}_3 \Big) = 0.$$

Finally, we introduce the notation

$$\tau(p) = \frac{1}{\rho g} \frac{\kappa(\mathscr{P}^{-1}(p))}{(\theta' + \alpha_p \theta)(\mathscr{P}^{-1}(p))}.$$

Note that, due to the hypotheses (2.2)-(2.3), there exist two positive reals $\tau_-$ and $\tau_+$ such that

$$0 < \tau_- := \frac{\kappa_-}{\rho g \alpha_p \theta_+} \leq \tau(p) \leq \tau_+ := \frac{\kappa_+}{\rho g \alpha_p \theta_-}. \tag{2.6}$$

Let $\delta > 0$ and $d = h_{\max} - h_{\text{bot}} > 0$, we introduce the function $T_l : \mathbb{R} \mapsto \mathbb{R}$ defined by

$$T_l(H) = h - h_{\text{bot}} = \max\Big\{\min\Big\{H - \frac{P_s}{\rho g}, h_{\max}\Big\}, h_{\text{bot}} + \delta\Big\} - h_{\text{bot}}.$$

Furthermore, the hypothesis $\delta > 0$ is sufficient to define the primitive function $\mathscr{T}$ so that

$$u = \mathscr{T}(H) = \begin{cases} \delta\Big(H - \dfrac{P_s}{\rho g} - h_{\text{bot}}\Big) - \dfrac{\delta^2}{2} & \text{if} \quad H - \dfrac{P_s}{\rho g} \leq h_{\text{bot}} + \delta, \\ \dfrac{1}{2}\Big(H - \dfrac{P_s}{\rho g} - h_{\text{bot}}\Big)^2 & \text{if} \quad H - \dfrac{P_s}{\rho g} \in [h_{\text{bot}} + \delta, h_{\max}], \\ d\Big(H - \dfrac{P_s}{\rho g} - h_{\text{bot}}\Big) - \dfrac{d^2}{2} & \text{if} \quad H - \dfrac{P_s}{\rho g} \geq h_{\max}. \end{cases} \tag{2.7}$$

A direct calculation gives $\mathscr{T}'(H) = T_l(H) \geq \delta > 0$. As $\mathscr{T}$ is a bijective application, the existence of $u$ such that $u = \mathscr{T}(H)$ is equivalent to the existence of solution $H$ of the original Equation (1.18). Moreover, we have

$$H = \mathscr{T}^{-1}(u) = \begin{cases} \dfrac{u}{\delta} + \dfrac{\delta}{2} + h_{\text{bot}} + \dfrac{P_s}{\rho g} & \text{if} \quad u \leq \dfrac{\delta^2}{2}, \\ \sqrt{2u} + h_{\text{bot}} + \dfrac{P_s}{\rho g} & \text{if} \quad \dfrac{\delta^2}{2} \leq u \leq \dfrac{d^2}{2}, \\ \dfrac{u}{d} + \dfrac{d}{2} + h_{\text{bot}} + \dfrac{P_s}{\rho g} & \text{if} \quad u \geq \dfrac{d^2}{2}, \end{cases}$$

and
$$\frac{1}{d} \le \|\big(\mathcal{T}^{-1}\big)'\|_\infty \le \frac{1}{\delta}. \tag{2.8}$$

Notice that the function $h$ is expressed in the new variables as $h = T_l(\mathcal{T}^{-1}(u)) + h_{\text{bot}}$. The equation (1.18) becomes

$$S_0\partial_t u - \nabla' \cdot \big(\mathbf{J}(\mathcal{T}^{-1}(u))\nabla'\mathcal{T}^{-1}(u)\big) = -\int_h^{h_{\text{soil}}} \partial_t p\, dz - \nabla' \cdot \Big(\int_h^{h_{\text{soil}}} \boldsymbol{q}\, dz\Big) - \boldsymbol{q}_{|z=h_{\text{soil}}} \cdot \vec{\nu} \quad \text{in } (0,T)\times\Omega_{2d}.$$

We introduce the notation
$$\mathbf{L}(u) = \frac{1}{\mathcal{T}'(\mathcal{T}^{-1}(u))} \boldsymbol{J}(\mathcal{T}^{-1}(u)),$$
to get $\boldsymbol{J}(\mathcal{T}^{-1}(u))\nabla'\mathcal{T}^{-1}(u) = \boldsymbol{L}(u)\nabla' u$. Note that, due to Equations (2.5)-(2.8), the two positive real numbers $L_- = \delta K^-$ and $L_+ = d K^+$ are such that for all $\xi \in \mathbb{R}^2$
$$L^-|\xi|^2 \le \boldsymbol{L}(u)\xi\cdot\xi \le L^+|\xi|^2. \tag{2.9}$$

**Remark 2.2.** *In the case where the Robin condition $\alpha P + \beta \boldsymbol{q}\cdot\vec{\nu} = F$ is considered at the upper boundary $z = h_{\text{soil}}$ in (1.15), the flux $\boldsymbol{q}_{|z=h^+_{\text{soil}}} \cdot \vec{\nu}$ can be replaced by $\frac{1}{\beta}(F - \alpha P_{|\Gamma_{\text{soil}}})$. This makes it possible to take into account exchanges between surface water and groundwater. Since $h_{\text{soil}}$ is assumed to be constant, the unit normal vector $\vec{\nu}$ corresponds to $\vec{e}_3$. The term $F - \alpha P_{|\Gamma_{\text{soil}}}$ appears during the proofs of Lemmas 3.1 and 3.2 because of the flux $\boldsymbol{q}_{|z=h^+_{\text{soil}}} \cdot \vec{\nu}$ in (2.10). Considering the Robin condition, the $L^2$-norm in $(0,T)\times\Omega_{2d}$ of the term $\alpha P_{|\Gamma_{\text{soil}}}$ is required, which is possible thanks to the norm of $p$ in $L^2(0,T,H^1(\Omega))$. The final result of Lemma 3.1 is not changed by the presence of $\alpha P_{|\Gamma_{\text{soil}}}$. However it is necessary to assume that $\alpha$ is sufficiently small (compared to $\tau_- K^-$) in the proof of Lemma 3.2. Finally, the case of the Dirichlet condition (i.e. when $\beta = 0$) cannot be considered. Indeed, the proof of Lemma 3.1 requires estimates of the flux $\boldsymbol{q}_{|z=h^+_{\text{soil}}} \cdot \vec{\nu}$, that is to say estimates of $p$ in $H^2(\Omega)$.*

We are led to consider the following problem completed by the boundary and initial conditions:

$$S_0\partial_t u - \nabla' \cdot \big(\boldsymbol{L}(u)\nabla' u\big) = -\int_{h_{\text{bot}}+T_l(\mathcal{T}^{-1}(u(t,x)))}^{h_{\text{soil}}} \partial_t p\, dz - \nabla_x \cdot \Big(\int_{h_{\text{bot}}+T_l(\mathcal{T}^{-1}(u(t,x)))}^{h_{\text{soil}}} \boldsymbol{q}\, dz\Big) - \boldsymbol{q}_{|z=h_{\text{soil}}} \cdot \vec{\nu} \quad \text{in } (0,T)\times\Omega_{2d}, \tag{2.10}$$

$$\boldsymbol{L}(u)\nabla' u\cdot\vec{\nu} = 0 \quad \text{on } (0,T)\times\partial\Omega_{2d}, \qquad u(0,x) = \mathcal{T}\big(H_0(x)\big) \quad \text{in } \Omega_{2d}, \tag{2.11}$$

$$\partial_t p - \nabla\cdot\big(\tau(p)\,\boldsymbol{B}\nabla p\big) - \nabla\cdot\Big(\kappa(\mathcal{P}^{-1}(p))\boldsymbol{B}\vec{e}_3\Big) = 0 \quad \text{in } \mathcal{O}_T, \tag{2.12}$$

$$p|_{\Gamma_t} = \mathcal{P}(P_s) \quad \text{in } (0,T), \qquad \Big(\tau(p)\,\boldsymbol{B}\nabla p + \kappa(\mathcal{P}^{-1}(p))\boldsymbol{B}\vec{e}_3\Big)\cdot\vec{\nu} = 0 \quad \text{on } (0,T)\times\Gamma_{\text{ver}},$$
$$-\Big(\tau(p)\,\boldsymbol{B}\nabla p + \kappa(\mathcal{P}^{-1}(p))\boldsymbol{B}\vec{e}_3\Big)\cdot\vec{\nu} = F \quad \text{on } (0,T)\times\Gamma_{\text{soil}},$$
$$p(0,x,z) = \mathcal{P}(P_0)(x,z) \quad \text{in } \Omega_0. \tag{2.13}$$

**Remark 2.3.** *Let $p \in W(0,T;\Omega)$ such that $p = 0$ in $\mathcal{O}_T^c$. We need to clarify the meaning of the term $\int_{h(t,x)}^{h_{soil}(x)} \partial_t p\, dz$ ($h = T_l(\mathcal{T}^{-1}(u)(t,x))$):*

$$\int_{h(t,x)}^{h_{soil}(x)} \partial_t p\, dz = \int_{h_{\rm bot}}^{h_{soil}(x)} \chi_{z\geq h(t,x)}\, \partial_t p\, dz$$

*is the function of $(H^1(\Omega_{2d}))'$ such that $\forall\, v \in H^1(\Omega_{2d}) \subset H^1(\Omega)$, for $\eta_0 > 0$ small enough*

$$\begin{aligned}\langle \int_{h(t,x)}^{h_{soil}} \partial_t p dz,\, v\rangle_{H^1(\Omega_{2d})',H^1(\Omega_{2d})} &\\ = \langle \int_{h_{\rm bot}}^{h_{soil}} \rho_{\eta_0} * \chi_{\{z\geq(h_{\rm bot}+\delta/2)\}}\, \partial_t p dz,\, v\rangle_{H^1(\Omega_{2d})',H^1(\Omega_{2d})} &\\ = \langle \partial_t p, \underbrace{\rho_{\eta_0} * \chi_{\{z\geq(h_{\rm bot}+\delta/2)\}}\, v}_{\in V(\Omega)}\rangle_{V'(\Omega),V(\Omega)}, &\end{aligned}$$

*where $\rho \in C^\infty(\mathbb{R})$, $\rho \geq 0$, is supported in the unit ball and satisfies $\int_{\mathbb{R}} \rho(\mathbf{x}) d\mathbf{x} = 1$. We set $\rho_{\eta_0}(x) = \rho(x/\eta_0)/\eta_0$, where $\eta_0$ is chosen such that Supp($\rho_{\eta_0} * \chi_{\{z\geq(h_{\rm bot}+\delta/2)\}}) \subset \{z,\, z \geq (h_{\rm bot} + \delta/4)\}$ and $\rho_{\eta_0} * \chi_{\{z\geq(h_{\rm bot}+\delta/2)\}} = 1$ if $z \geq (h_{\rm bot} + 3\delta/4)$.*

The boundary condition at the interface $\Gamma_t$ of the system (2.12)-(2.13) is classically reduced to homogeneous Dirichlet boundary condition. Namely, setting $\bar p = p - \mathcal{P}(P_s)$, since $\mathcal{P}(P_s)$ is a constant, the system (2.12)-(2.13) becomes

$$\begin{cases} \partial_t \bar p - \nabla\cdot\big(\bar\tau(\bar p))\boldsymbol{B}\nabla\bar p\big) - \nabla\cdot\big(\bar\kappa(\bar p)\boldsymbol{B}\vec e_3\big) = 0 \quad \text{in} \quad \mathcal{O}_T, \\ \bar p|_{\Gamma_t} = 0 \quad \text{in} \quad (0,T), \qquad \Big(\bar\tau(\bar p)\boldsymbol{B}\nabla p + \bar\kappa(\bar p)\boldsymbol{B}\vec e_3\Big)\cdot\vec\nu = 0 \quad \text{on} \quad (0,T)\times\Gamma_{ver}, \\ -\Big(\bar\tau(\bar p)\boldsymbol{B}\nabla p + \bar\kappa(\bar p)\boldsymbol{B}\vec e_3\Big)\cdot\vec\nu = F \quad \text{on} \quad (0,T)\times\Gamma_{\rm soil}, \\ \bar p(0,x,z) = \mathcal{P}(P_0)(x,z) - \mathcal{P}(P_s) \quad \text{in} \quad \Omega_0, \end{cases}$$

where $\bar\tau(\bar p) = \tau\big(\bar p + \mathcal{P}(P_s)\big)$ and $\bar\kappa(\bar p) = \kappa\Big(\mathcal{P}^{-1}\big(\bar p + \mathcal{P}(P_s)\big)\Big)$. Note that just by renaming the functions $\tau$ and $\kappa$, we return to the case $\mathcal{P}(P_s) = 0$ on $\Gamma_t$. So, from now on the subscript "$\bar{\cdot}$" is omitted in the previous system and in the original system: (2.12)-(2.13) with $\mathcal{P}(P_s) = 0$ is considered.

To solve the difficulties related to the free boundary, the method of auxiliary domains introduced in [11] is used. For this purpose, the function $p$ is extended by zero outside the variable domain $\Omega_t$. Thus, we consider the following definition of weak solutions associated with the system (2.10)-(2.13):

**Definition 2.4.** *We call weak solution of the problem* (2.10)-(2.13) *any solution* $(u,p)$ *with* $u \in W(0,T,\Omega_{2d})$ *and* $p \in W_0(0,T,\Omega)$ *such that for all* $(\phi_1,\phi_2) \in L^2(0,T;H^1(\Omega_{2d})) \times L^2(0,T;V(\Omega))$

$$\begin{aligned} S_0\int_0^T \langle \frac{\partial u}{\partial t},\phi_1\rangle_{H^1(\Omega_{2d})',H^1(\Omega_{2d})}\, dt + \int_0^T\int_{\Omega_{2d}} \big(\boldsymbol{L}(u)\nabla' u\cdot\nabla'\phi_1\big) dx\, dt &\\ = \int_0^T \langle \partial_t p, \rho_{\eta_0} * \chi_{\{z\geq(h_{\rm bot}+\delta/2)\}}\phi_1\rangle_{H^1(\Omega_{2d})',H^1(\Omega_{2d})}\, dt &\\ + \int_0^T\int_{\Omega_{2d}} \Big(\Big(F\phi_1 + \int_{h(t,x)}^{h_{\rm soil}} \boldsymbol{q}\, dz\cdot\nabla'\phi_1\Big) dx\, dt & \end{aligned}$$

$$u(0,x) = (h_0(x) - h_{\rm bot})^2 \quad in\, \Omega_{2d},$$

$$\langle \partial_t p, \phi_2 \rangle_{F'(\mathcal{O}_s),F(\mathcal{O}_s)} + \int_0^T \Big( \int_{\Omega_t} \big( \tau(p)\, \tilde{\boldsymbol{B}} \nabla p + \kappa(\mathscr{P}^{-1}(p)) \tilde{\boldsymbol{B}}\, \vec{e}_3 \big) \cdot \nabla \phi_2 \big)\, dx \Big)\, dt$$
$$= \int_0^T \int_{\Omega_{2d}} F\, \phi_{2|_{\Gamma_{\rm soil}}}\, dx\, dt \quad (2.14)$$

$$p = 0 \; in\, \mathcal{O}_T^c, \quad p(0,x,z) = \mathscr{P}(P_0)(x,z) \; in\, \Omega_0,$$

*where* $\tilde{\boldsymbol{B}} = \boldsymbol{B}$ *in* $\mathcal{O}_T$ *and* $\tilde{\boldsymbol{B}} = 0$ *in* $\mathcal{O}_T^c$.

We now give the definition of a weak solution of problem (1.15)-(1.18), essentially following Knabner and Otto [16, 17].

**Definition 2.5.** *Any couple* $(H,P) \in L^2(0,T,H^1(\Omega_{2d})) \times L^2(0,T,H^1(\Omega))$ *is called weak solution of problem* (1.15)-(1.18) *if*

- $\partial_t \mathscr{T}(H) \in L^2(0,T,H^1(\Omega_{2d})')$ *and* $\forall \phi_1 \in L^2(0,T;H^1(\Omega_{2d}))$

$$S_0 \int_0^T \langle \partial_t \mathscr{T}(H), \phi_1 \rangle_{H^1(\Omega_{2d})',H^1(\Omega_{2d})},\, dt + \int_0^T \int_{\Omega_{2d}} \big( \boldsymbol{J}(H) \nabla' H \cdot \nabla' \phi_1 \big) dx\, dt$$
$$= \langle \partial_t \mathscr{P}(P^+), \rho_{\eta_0} * \chi_{\{z \geq (h_{\rm bot}+\delta/2)\}} \phi_1 \rangle_{V'(\Omega),V(\Omega)}$$
$$+ \int_0^T \int_{\Omega_{2d}} \Big( F \phi_1 + \int_{h(t,x)}^{h_{\rm soil}} \boldsymbol{q}\, dz \cdot \nabla' \phi_1 \Big) dx\, dt \quad (2.15)$$

$$\boldsymbol{q} = -\kappa(P^+) \boldsymbol{B} \nabla \left( \frac{P^+}{\rho g} + z \right), \quad \mathscr{T}(H)(0,x) = (h_0(x) - h_{\rm bot})^2 \; in\, \Omega_{2d}.$$

- *The interface h is defined thanks to* (1.18).
- *In* $\Omega_t$, $P = P^+$ *where* $P^+ \in P_s + L^2(0,T,V(\Omega))$, $\partial_t \mathscr{P}(P^+) \in L^2(0,T,V'(\Omega))$ *and for all* $\phi_2 \in L^2(0,T;V(\Omega))$

$$\langle \partial_t \mathscr{P}(P^+), \phi_2 \rangle_{F'(\mathcal{O}_s),F(\mathcal{O}_s)} + \int_0^T \int_{\Omega_t} \kappa(P^+) \big( \frac{1}{\rho g} \tilde{\boldsymbol{B}} \nabla P^+ + \tilde{\boldsymbol{B}}\, \vec{e}_3 \big) \cdot \nabla \phi_2\, dx\, dt$$
$$= \int_0^T \int_{\Omega_{2d}} F\, \phi_{2|_{\Gamma_{\rm soil}}}\, dx\, dt, \quad (2.16)$$

$$P^+ = P_s \; in\, \mathcal{O}_T^c, \quad P^+(0,x,z) = P_0(x,z) \; in\, \Omega_0,$$

  *where* $\tilde{\boldsymbol{B}} = B$ *in* $\mathcal{O}_T$ *and* $\tilde{\boldsymbol{B}} = 0$ *in* $\mathcal{O}_T^c$.

- *In* $\Omega_t^-$, $P = P^-$ *where* $P^- \in L^2(0,T,H^1(\Omega))$ *is defined thanks to* (1.17).

## 2.3. MAIN RESULTS

**Theorem 2.6.** *Assuming hypotheses stated in Subsection 2.2, then system* (2.10) - (2.13) *admits a weak solution* $(u,p)$ *satisfying*

- $u \in L^2(0,T;H^1(\Omega_{2d}))$ *and* $\partial_t u \in L^2(0,T;(H^1(\Omega_{2d}))')$,
- $p \in L^2(0,T;V(\Omega))$ *and* $\partial_t p \in L^2(0,T;V'(\Omega))$.

As a consequence of Theorem 2.6, we claim the following result

**Theorem 2.7.** *Let us assume the hypotheses given in Subsection 2.2. Let* $\delta \in ]0,d[$. *Then, the model* $\mathcal{M}_\delta$ *admits a weak solution* $(H,P)$ *with* $H \in L^2(0,T,H^1(\Omega_{2d}))$ *and* $P \in L^2(0,T,H^1(\Omega))$.

## 3. PROOF OF THEOREM 2.6

Let us outline the global strategy for proving Theorem 2.6. The problem is a strongly coupled non-linear system, so we apply a fixed-point approach to solve it in two steps. First, we decouple the system and we establish an existence and uniqueness result for each decoupled and linearized problem. The decoupled problem characterizing $p$ is solved by considering a penalized problem. Then, we establish compactness results which allow to prove the global existence in time of the initial problem by applying the Schauder fixed-point theorem.

### 3.1. FIXED POINT ARGUMENT

We now construct the framework to apply the Schauder fixed-point theorem (see [18, 19]). We introduce two convex subsets $W_1$ and $W_2$ with $W_1 \times W_2 \subset W(0,T,\Omega_{2d}) \times W(0,T,\Omega)$, namely

$$W_1 := \big\{u \in W(0,T,\Omega_{2d});\ u(0) = u_0,\ \|u\|_{L^2(0,T;H^1(\Omega_{2d}))} \le C_u \quad \text{and } \|u\|_{L^2(0,T;(H^1(\Omega_{2d}))')} \le C'_u\big\}, \tag{3.1}$$

$$W_2 := \big\{p \in W(0,T,\Omega);\ p(0) = p_0,\ \|p\|_{L^2(0,T;H^1(\Omega))} \le C_p \quad \text{and } \|p\|_{L^2(0,T;(H^1(\Omega))')} \le C'_p\big\}, \tag{3.2}$$

where the constants $(C_p, C'_p)$ and $(C_u, C'_u)$ are characterized thereafter (see (3.6), (3.7), (3.20) and (3.23)). For all $(\bar u, \bar p) \in W_1 \times W_2$ and $H_0 \in L^2(\Omega_{2d})$ we consider the following linearized problem of finding the weak solution $u$ of

$$S_0\partial_t u - \nabla'\cdot\big(\boldsymbol{L}(\bar u)\nabla' u\big) = -\int_{h_{\rm bot}}^{h_{\rm soil}} \partial_t \bar p\, dz - \nabla'\cdot\Big(\int_{\bar h}^{h_{\rm soil}} \bar{\boldsymbol{q}}\, dz\Big) - F \quad \text{in} \quad (0,T)\times\Omega_{2d}, \tag{3.3}$$

$$\boldsymbol{L}(\bar u)\nabla' u\cdot\vec\nu = 0 \quad \text{on} \quad (0,T)\times\partial\Omega_{2d}, \qquad u(0,x) = \mathscr{T}(H_0)(x) \quad \text{in} \quad \Omega_{2d}, \tag{3.4}$$

where $\bar{\boldsymbol{q}} = -\kappa\big(\mathscr{P}^{-1}(\bar p)\big)\boldsymbol{B}\nabla\Big(\frac{\mathscr{P}^{-1}(\bar p)}{\rho g} + z\Big)$ and $\bar h(t,x) := h_{\rm bot} + T_l\Big(\mathscr{T}^{-1}\big(\bar u(t,x)\big)\Big)$.

**Lemma 3.1.** *Let us assume the hypotheses given in Subsection 2.2. Let* $H_0 \in L^2(\Omega_{2d})$ *and* $(\bar u, \bar p) \in W_1 \times W_2$. *There exists a unique weak solution* $u \in W(0,T,\Omega_{2d})$ *of* (3.3)-(3.4) *such that*

$$\|u\|_{L^2(0,T;H^1(\Omega_{2d}))} \le C_u \quad \textit{and} \quad \|u\|_{L^2(0,T;(H^1(\Omega_{2d}))')} \le C'_u,$$

*where* $C_u$ *and* $C'_u$ *are the constants introduced in the definition of* $W_1$.

*Proof.* It follows from the classical textbook [20] (pp. 178-179) that for any function $\bar u \in W(0,T,\Omega_{2d})$ and $\bar h$ such that $\bar h(t,x) := T_l(\mathscr{T}^{-1}(\bar u))$ there exists a solution $u \in W(0,T,\Omega_{2d})$ of the parabolic problem with smooth coefficients

$$S_0\partial_t u - \nabla'\cdot\big(\boldsymbol{L}(\bar u)\nabla' u\big) = -\int_{\bar h(t,x)}^{h_{\rm soil}} \partial_t \bar p\, dz - \nabla_x\cdot\big(\int_{\bar h(t,x)}^{h_{\rm soil}} \bar{\boldsymbol{q}}\, dz\big) - F \quad \text{in} \quad (0,T)\times\Omega_{2d}, \tag{3.5}$$

$$\nabla u\cdot\vec\nu = 0 \quad \text{on} \quad (0,T)\times\partial\Omega_{2d}, \qquad u(0,x) = \mathscr{T}(H_0)(x) \quad \text{in} \quad \Omega_{2d}.$$

The usual energy estimates give

$$\|u(t,\cdot)\|_{L^2(\Omega_{2d})} \le e^{\frac{(1+L^-/2)T}{S_0}}\Big(\|u_0\|_{L^2(\Omega_{2d})} + \frac{1}{S_0}\|F\|^2_{L^2((0,T)\times\Omega_{2d})} + \frac{4T}{S_0L^-}(\kappa_+K^+)^2|\Omega| + \frac{4}{S_0L^-}(\kappa_+K^+)^2\frac{\|\nabla\bar p\|^2_{L^2(\Omega_T)}}{\rho^2g^2} + \frac{2d}{S_0L^-}\|\partial_t\bar p\|^2_{L^2(0,T,(H^1(\Omega))')}\Big),$$

then

$$\|u\|^2_{L^2(0,T;H^1(\Omega_{2d}))} \le C_1(T,S_0,u_0,K^+,L^-,\kappa^+,d)(1+\|\overline{p}\|^2_{W(0,T,\Omega)})$$
$$= C_u(T,S_0,u_0,K^+,L^-,\kappa^+,d,C_p,C'_p). \quad (3.6)$$

On the other hand

$$\|\partial_t u\|_{L^2(0,T;H^1(\Omega_{2d})')} = \sup_{\|v\|_{L^2(0,T;H^1(\Omega_{2d}))}\le 1} \Big|\int_0^T \langle \partial_t u, v\rangle_{H^1(\Omega_{2d})',H^1(\Omega_{2d})} dt\Big|$$
$$\le \frac{2}{S_0}\Big(\frac{K^+}{\delta} C_u + d\,C'_p + \kappa_+ K^+\big(\frac{C_p}{\rho g} + |\Omega|^{1/2}\big) + \|F\|_{L^2((0,T)\times\Omega_{2d})}\Big) := C'_u. \quad (3.7)$$

The uniqueness of the solution is obvious. Indeed, if $u_1$ and $u_2$ are two solutions of (3.3)-(3.4), then $u = u_1 - u_2$ satisfies

$$S_0 \partial_t u - \nabla' \cdot \big(\boldsymbol{L}(\bar{u})\nabla' u\big) = 0 \quad \text{in} \quad (0,T)\times\Omega_{2d},$$
$$\boldsymbol{L}(\bar{u})\nabla' u \cdot \vec{v} = 0 \quad \text{on} \quad (0,T)\times\partial\Omega_{2d}, \qquad u(0,x) = 0 \quad \text{in} \quad \Omega_{2d}.$$

Following the previous computations, we deduce from Gronwall's lemma that $u = 0$ a.e. in $(0,T)\times\Omega_{2d}$. This concludes the proof of Lemma 3.1.

□

The results given in Lemma 2.1 require to have regular non-cylindrical domains with in particular sufficiently regular boundaries (of class $\mathscr{C}^1$ by pieces as mentioned by Mignot [11]). As we cannot guarantee as much regularity at the interface $h$ (which is in $W(0,T,\Omega_x)$), we use a regularization process. So we regularize $h$ by using convolution in space. Let $\psi \in C^\infty(\mathbb{R}^2)$, $\psi \ge 0$, with support in the unit ball such that $\int_{\mathbb{R}^2} \psi(x)dx = 1$. For $\eta > 0$ small enough, we set $\psi_\eta(x) = \psi(x/\eta)/\eta^2$. We extend $h$ by zero outside $\Omega_{2d}$, so we have $h \in C([0,T];L^2(\mathbb{R}^2)) \cap W(0,T,\mathbb{R}^2)$. Hence we define $\tilde{h}$ by the convolution product with respect to the space variable

$$\tilde{h} = \psi_\eta * h.$$

Its restriction to $\Omega_{2d}$ is denoted in the same way. It fulfils $\tilde{h}(t,.) \in C^\infty(\bar{\Omega}_x), \forall t \in [0,T]$, and as $\eta \to 0$, we have

$$\tilde{h} \to h \text{ strongly in } C([0,T];L^2(\Omega_{2d})) \cap L^2(0,T,H^1(\Omega_{2d})).$$

In Equations (2.12)-(2.13), we replace $h$ by $\tilde{h}$ (the substitution appears in the space integration domain $\Omega_t$).

For all $P_0 \in H^2(\Omega)$, $(\bar{u},\bar{p}) \in W_1 \times W_2$ and $\tilde{h} \in L^2(0,T;C^\infty(\bar{\Omega}_x))$, we consider the following linearized and regularized problem: find $p_\eta \in W(0,T,\Omega)$ such that $\forall \phi \in L^2(0,T;V(\Omega))$

$$\langle \partial_t p_\eta, \phi\rangle_{F',F} + \int_0^T \Big(\int_\Omega \big(\tau(\bar{p})\boldsymbol{B}\nabla p_\eta + \kappa(\mathscr{P}^{-1}(\bar{p}))\boldsymbol{B}\,\vec{e}_3\big)\cdot\nabla\phi\Big)dt = \int_0^T\int_{\Omega_{2d}} F\,\phi\,dx\,dt, \quad (3.8)$$

$$p_\eta = 0 \quad \text{in } \mathscr{O}^c_T, \quad \text{and} \quad p_\eta(0,x,z) = \mathscr{P}(P_0)(x,z) \quad \text{in } \Omega_0. \quad (3.9)$$

**Lemma 3.2.** *Let us assume the hypotheses given in Subsection 2.2. Let $(\bar{u},\bar{p}) \in W_1 \times W_2$ and $P_0 \in H^2(\Omega)$. We introduce $\tilde{h} = \psi_\eta * h$ where $h = T_l(\mathscr{T}^{-1}(u))$ with $u \in W(0,T,\Omega_{2d})$ given by Lemma 3.1. For any $\eta > 0$, there exists a unique function $p_\eta$ in $W(0,T,\Omega)$ solution of* (3.8)-(3.9). *It satisfies the uniform estimates*

$$\|p_\eta\|_{L^2(0,T;H^1(\Omega))} \le C_p \quad \text{and} \quad \|p_\eta\|_{L^2(0,T;(H^1(\Omega))')} \le C'_p, \quad (3.10)$$

*where $C_p$ and $C'_p$ depend only on the data of the original problem* (2.12)-(2.13).

Let us admit for the moment this Lemma, the proof of which is postponed in Section 3.2. From now on we omit the subscript $\eta$ in $p_\eta$ (and then in $u_\eta$).

Let $(\bar u, \bar p) \in W_1 \times W_2$, Lemma 3.1 and Lemma 3.2 allow to define an application $\mathscr{F}$ such that:

$$W(0,T,\Omega_{2d}) \times W(0,T,\Omega) \to W(0,T,\Omega_{2d}) \times W(0,T,\Omega)$$
$$(\bar u, \bar p) \to \mathscr{F}(\bar u, \bar p) = (u,p). \tag{3.11}$$

The end of the proof of Theorem 2.6 consists of showing that $\mathscr{F}$ admits a fixed-point and one pass to the limit $\eta \to 0$. We now prove the existence of a fixed-point of $\mathscr{F}$ in an appropriate subset.

**Lemma 3.3.** *Let $\mathscr{F}$ be the map defined in* (3.11) *and $W_1$, $W_2$ given in* (3.1)-(3.2). *We have*

- *the subset $\mathscr{C} = W_1 \times W_2 \subset W(0,T,\Omega_{2d}) \times W(0,T,\Omega)$ is non-empty, (strongly) closed, convex, bounded and satisfies $\mathscr{F}(\mathscr{C}) \subset \mathscr{C}$,*
- *the map $\mathscr{F}$ is weakly sequentially continuous in $W(0,T,\Omega_{2d}) \times W(0,T,\Omega)$,*
- *there exists $(u,p) \in W_1 \times W_2$ such that $\mathscr{F}(u,p) = (u,p)$.*

*Proof.* We set $\mathscr{C} = W_1 \times W_2$ where $W_1$ and $W_2$ are defined in (3.1)-(3.2). The first point of Lemma 3.3 is obvious thanks to Lemma 3.1 and Lemma 3.2. Indeed $\mathscr{C}$ is clearly a non-empty closed convex set in $W(0,T,\Omega_{2d}) \times W(0,T,\Omega)$.

Regarding the second point of Lemma 3.3, we first note that $\mathscr{C}$ is compact for the weak topology. $\mathscr{F}$ maps $W_1 \times W_2$ into itself. Now let $(v_n)_{n\ge 0} = (\bar u_n, \bar p_n)_{n \ge 0}$ be an arbitrary sequence in $\mathscr{C}$ that is weakly convergent in $W(0,T,\Omega_{2d}) \times W(0,T,\Omega)$, and let $v = (\bar u, \bar p)$ be its weak limit. We want to show that

$$\mathscr{F}(v_n) \rightharpoonup \mathscr{F}(v) \quad \text{in } W(0,T,\Omega_{2d}) \times W(0,T,\Omega) \text{ as } n \to \infty.$$

Since $\mathscr{F}(v_n) \in W_1 \times W_2$ and $W_1 \times W_2$ is weakly compact, it suffices to show that there exits a subsequence $(v'_n)$ of $(v_n)$ such that $\mathscr{F}(v'_n) \rightharpoonup \mathscr{F}(v)$. Extracting a subsequence if necessary, we can assume without loss of generality that $\mathscr{F}(v_n) \rightharpoonup w$ in $W(0,T,\Omega_{2d}) \times W(0,T,\Omega)$ as $n \to \infty$ for some $w = (u,p) \in W_1 \times W_2$, and we need show that $w$ and $\mathscr{F}(v)$ agree. If we set $w_n = \mathscr{F}(v_n)$ ($w_n = (u_n, p_n)$), it follows from Aubin's Lemma that

$$w_n \to w \quad \text{in} \quad L^2((0,T)\times\Omega_{2d}) \times L^2((0,T)\times\Omega) \quad \text{and} \quad w_n(t,x) \to w(t,x) \quad \text{a.e.;}$$
$$v_n \to v \quad \text{in} \quad L^2((0,T)\times\Omega_{2d}) \times L^2((0,T)\times\Omega) \quad \text{and} \quad v_n(t,x) \to v(t,x) \quad \text{a.e.;}$$
$$\partial_t w_n \rightharpoonup \partial_t w \quad \text{in} \quad L^2(0,T;H^1(\Omega_{2d})') \times L^2(0,T;H^1(\Omega)')$$
$$\nabla w_n \rightharpoonup \nabla w \quad \text{weakly in} \quad L^2((0,T)\times\Omega_{2d}) \times L^2((0,T)\times\Omega).$$

Thanks to Lebesgue's theorem (and the properties of functions $\kappa, \tau$ and $T_l$) we get that $w = \mathscr{F}(v)$ (as $w(0,.) = (u(0,.), p(0,.)) = (u_0, p_0)$ because $w \in \mathscr{C}$) and the proof that $\mathscr{F}|_{\mathscr{C}}$ is weakly sequentially continuous is complete.

The existence of $(u,p) \in W_1 \times W_2$ such that $\mathscr{F}(u,p) = (u,p)$ follows from Schauder's theorem [19]. The proof of Lemma 3.3 is complete.

□

We now summarize all the results obtained so far. We can associate with any real number $\eta > 0$ the fixed-point $(u_\eta, p_\eta) \in W_1 \times W_2$ of $\mathscr{F}_\eta$. It is a solution of the system:

$$\partial_t p_\eta - \nabla\cdot\big(\tau(p_\eta)\boldsymbol{B}\nabla p_\eta\big) - \nabla\cdot\Big(\kappa(\mathscr{P}^{-1}(p_\eta))\boldsymbol{B}\,\vec e_3\Big) = 0 \quad \text{in } \mathscr{O}_{T,\eta}, \tag{3.12}$$

$$p_{\eta|_{\Gamma_t}} = \mathscr{P}(P_s) \quad \text{in } (0,T), \qquad \nabla\big(\mathscr{P}^{-1}(p_\eta) + \rho g z\big)\cdot\vec{\nu} = 0 \quad \text{on } (0,T)\times\Gamma_{\text{ver}},$$

$$-\nabla\big(\mathscr{P}^{-1}(p_\eta) + \rho g z\big)\cdot\vec{\nu} = F \quad \text{on } (0,T)\times\Gamma_{\text{soil}}, \qquad p_\eta(0,\cdot,\cdot) = \mathscr{P}(P_0) \quad \text{in } \Omega_0, \tag{3.13}$$

$$S_0\partial_t u_\eta - \nabla'\cdot\big(\boldsymbol{L}(u_\eta)\nabla' u_\eta\big) = -\int_{h_\eta(t,x)}^{h_{\text{soil}}} \partial_t p_\eta\, dz - \nabla_x\cdot\Big(\int_{h_\eta(t,x)}^{h_{\text{soil}}} \boldsymbol{q}_\eta\, dz\Big) - F \quad \text{in } (0,T)\times\Omega_{2d}, \tag{3.14}$$

$$\boldsymbol{L}(u_\eta)\nabla' u_\eta\cdot\vec{\nu} = 0 \quad \text{on } (0,T)\times\partial\Omega_{2d}, \quad u_\eta(0,x) = (h_0(x) - h_{\text{bot}})^2 \quad \text{in } \Omega_{2d}. \tag{3.15}$$

We can obtain similar estimates for $(u_\eta, p_\eta)$ to those derived in Lemma 3.1 and 3.2. We thus assert the existence of limit functions (extracting a subsequence if necessary) $(u,p)$ with $u \in W(0,T,\Omega_{2d})$ and $p \in W(0,T,\Omega)$ such that

$$\begin{aligned}
(u_\eta, p_\eta) &\to (u,p) && \text{in } L^2((0,T)\times\Omega_{2d})\times L^2((0,T)\times\Omega)\\
(u_\eta(t,x), p_\eta(t,x)) &\to (u(t,x), p(t,x)) && \text{a.e in } ((0,T)\times\Omega_{2d})\times((0,T)\times\Omega)\\
\tilde{h}(t,x) = \psi_\eta * h(t,x) &\to h(t,x), && \text{a.e in } (0,T)\times\Omega_{2d}\\
(\partial_t u_\eta, \partial_t p_\eta) &\rightharpoonup (\partial_t u, \partial_t p) && \text{in } L^2(0,T;H^1(\Omega_{2d})')\times L^2(0,T;H^1(\Omega)')\\
(\nabla u_\eta, \nabla p_\eta) &\rightharpoonup (\nabla u, \nabla p) && \text{weakly in } L^2((0,T)\times\Omega_{2d})\times L^2((0,T)\times\Omega).
\end{aligned}$$

Letting $\eta \to 0$ in weak formulations resulting from (3.12)-(3.15), we prove the existence of a weak solution $(u,p)$ of problem (2.10)-(2.13). This concludes the proof of Theorem 2.6. □

### 3.2. PROOF OF LEMMA 3.2

Again, we omit the subscript $\eta$ in $p_\eta$. The proof of Lemma 3.2 is done by introducing a penalized problem and by passing to the limit to return to the linearized problem (3.8)-(3.9).

We thus consider the weak solution $p$ of the linearized problem (3.8)-(3.9). So we look for $p \in W_0(0,T,\Omega)$ such that, $\forall\phi \in L^2(0,T;V(\Omega))$,

$$\langle \partial_t p, \phi\rangle + \int_0^T \Big(\int_\Omega \big(\tau(\bar{p})\tilde{\boldsymbol{B}}\nabla p + \kappa(\mathscr{P}^{-1}(\bar{p}))\tilde{\boldsymbol{B}}\,\vec{e}_3\big)\cdot\nabla\phi\big)dx\Big)dt = \int_0^T\int_{\Omega_{2d}} F\, dx\, dt.$$

First we note that the solution of the system (3.8)-(3.9) is unique. Indeed, if $p_1$ and $p_2$ are two solutions of (3.8)-(3.9), then $\Theta = p_1 - p_2$ satisfies

$$\langle \partial_t\Theta, \phi\rangle_{F',F} + \int_0^T \Big(\int_{\Omega_t} \big(\tau(\bar{p})\boldsymbol{B}\nabla\Theta\cdot\nabla\phi\big)dx\Big)dt = 0.$$

Then, taking $\phi = \Theta$ and using the fourth point of Lemma 2.1, we conclude that

$$\frac{1}{2}\int_{\Omega_T} \Theta^2(T,x)\, dx\, dt + \tau_- K^- \int_0^T\int_{\Omega_t} |\nabla\Theta|^2\, dx\, dt \le 0,$$

because $\Theta(0,\cdot) = 0$. From this equality we deduce that $\Theta = 0$ a.e. in $(0,T)\times\Omega$ (as $\Theta = 0$ on the interface $\Gamma_T$). We will define a family of approximated problems which are linear parabolic problems in the cylindrical domain $(0,T)\times\Omega$, and whose the solution, restricted to the set $\Omega_T$, will converge to the solution $p$ of the linearized equation (3.8).

**Step 1. Penalized problems** Let $\epsilon > 0$, we now consider the following penalized problem on $\Omega$: find $p_\epsilon \in W_0(0,T,\Omega)$ s.t. $\forall \phi \in L^2(0,T;\mathscr{D}(\bar{\Omega}))$ vanishing in a neighborhood of $\Gamma_{\text{bot}}$

$$\langle \partial_t p_\epsilon, \phi\rangle + \int_0^T \Big(\int_\Omega \big(\tau(\bar p)\tilde{\boldsymbol{B}}\nabla p_\epsilon + \kappa(\mathscr{P}^{-1}(\bar p))\tilde{\boldsymbol{B}}\vec{e}_3\big)\cdot\nabla\phi\big)\Big)\,dx\,dt \\ + \int_0^T\int_{\Omega_{2d}} F\,\phi_{|\Gamma_{\text{soil}}}\,dx\,dt + \int_{\mathscr{O}_T^c} \nabla p_\epsilon\cdot\nabla\phi\,dx\,dt + \frac{1}{\epsilon}\int_{\mathscr{O}_T^c} p_\epsilon\phi\,dx\,dt = 0, \tag{3.16}$$

$$p_\epsilon(0,x,z) = \mathscr{P}(P_0)(x,z) \quad \text{in } \Omega_0 \quad \text{and} \quad p_\epsilon(0,x,z) = 0 \quad \text{in } \Omega\setminus\Omega_0. \tag{3.17}$$

We want to state that the penalized system (3.16)-(3.17) admits a unique solution $p_\epsilon$ which tends to the solution of problem (3.8)-(3.9) when $\epsilon \to 0$. Equation (3.16) can be written as

$$\langle \partial_t p_\epsilon, \phi\rangle + \underbrace{\int_0^T\int_\Omega \tau(\bar p)\,\tilde{\boldsymbol{B}}\nabla p_\epsilon\cdot\nabla\phi\,dx\,dt + \int_{\mathscr{O}_T^c}\nabla p_\epsilon\cdot\nabla\phi\,dx\,dt + \frac{1}{\epsilon}\int_{\mathscr{O}_T^c} p_\epsilon\phi\,dx\,dt}_{A_\epsilon(p_\epsilon,\phi)} \\ = \underbrace{-\int_0^T \Big(\int_\Omega \big(\kappa(\mathscr{P}^{-1}(\bar p))\tilde{\boldsymbol{B}}\vec{e}_3\cdot\nabla\phi\big)\Big)\,dx - \int_{\Omega_{2d}} F\,\phi_{|\Gamma_{\text{soil}}}\,dx\Big)\,dt}_{L_\epsilon(\phi)} \quad \forall\phi\in W_0(0,T,\Omega). \tag{3.18}$$

By (2.6) we find that the coefficients of $A_\epsilon$ are in $L^\infty((0,T)\times\Omega)$. We also have

$$A_\epsilon(p,p) \geq \inf\Big\{1, \tau_- K^-, \frac{1}{\epsilon}\Big\}\,\|p\|_{L^2(0,T,H^1(\Omega))}, \quad \forall p\in L^2(0,T;V(\Omega)).$$

We check directly that $L_\epsilon$ is a linear form on $L^2(0,T;V(\Omega))$. Thus we deduce the existence and uniqueness of a solution for the system (3.16)-(3.17).

**Step 2. Limit when $\epsilon \to 0$** We first derive some uniform estimates with respect to $\epsilon$ (and also $\eta$). By multiplying Equation (3.18) by $p_\epsilon$ and by integrating by parts over $\Omega$, we obtain for all $s \leq T$,

$$\langle \partial_t p_\epsilon, p_\epsilon\rangle + \underbrace{\int_0^s\int_\Omega \tau(\bar p)\tilde{\boldsymbol{B}}|\nabla p_\epsilon|^2\,dx\,dt + \int_{\mathscr{O}_s^c}|\nabla p_\epsilon|^2\,dx\,dt + \frac{1}{\epsilon}\int_{\mathscr{O}_s^c} p_\epsilon^2\,dx\,dt}_{I_1} \\ = \underbrace{-\int_0^s\Big(\int_\Omega \kappa(\mathscr{P}^{-1}(\bar p))\tilde{\boldsymbol{B}}\vec{e}_3\cdot\nabla p_\epsilon\,dx - \int_{\Omega_{2d}} F\,p_{\epsilon|\Gamma_{\text{soil}}}\,dx\Big)dt}_{I_2}.$$

Thanks to Lemma 2.1 and Gronwall's Lemma, we deduce that there exists a constant $C_p$ which only depends on the data such that

$$\|p_\epsilon\|^2_{L^2([0,T];V(\Omega))} \quad \leq \quad C_p. \tag{3.19}$$

More precisely, we have

$$\|\nabla p_\epsilon\|^2_{L^2([0,T]\times\Omega)} \quad \leq \quad \Big(\min\big(1, \frac{K^-\tau_-}{2}\big)\Big)^{-1} C_0\big(1 + \frac{\tau_- K^- T}{2} e^{\frac{\tau_- K^- T}{2}}\big) := C_p, \tag{3.20}$$

where

$$C_0 = \int_\Omega p_0^2\,dx + \frac{2}{\tau_- K^-}\big(T|\Omega|\big(\kappa_+ K^+\big)^2 + \|F\|^2_{L^2([0,T],L^2(\Omega_{2d}))}\big).$$

So the sequence $\{p_\epsilon\}$ is bounded in $L^2(0,T;V(\Omega))$ and the sequence $\{\frac{1}{\sqrt{\epsilon}}p_\epsilon\}$ is bounded in $L^2(\mathscr{O}_T^c)$. Following the proof given in [12], we can deduce that there exists $r \in L^2(0,T;V(\Omega))$ such that

$$p_\epsilon \rightharpoonup r \quad \text{weakly in} \quad L^2((0,T)\times\Omega), \tag{3.21}$$

$$\nabla p_\epsilon \rightharpoonup \nabla p \quad \text{weakly in} \quad L^2((0,T)\times\Omega). \tag{3.22}$$

Furthermore $r|_{\mathscr{O}_T} \in F(\mathscr{O}_T)$ and

$$D_t(r|_{\mathscr{O}_T}) \in F'(\mathscr{O}_T) \quad \text{and} \quad D_t(p_\epsilon|_{\mathscr{O}_T}) \rightharpoonup D_t(r|_{\mathscr{O}_T}) \quad \text{in} \quad F'(\mathscr{O}_T).$$

Thus $r|\mathscr{O}_T$ is the unique solution of (3.8)-(3.9), and the limit of $p_\epsilon|_{\mathscr{O}_T}$ being independent of the chosen subsequence, the whole sequence converges towards $r|\mathscr{O}_T$. Moreover, we obtain the first part of (3.10) for the solution $r \in L^2(0,T;V(\Omega))$ of the system (3.8)-(3.9) in the same way as for estimate (3.19) obtained for $p_\epsilon$. Finally, as was done in Lemma 3.1, we deduce from the first inequality of (3.10) that

$$\|\partial_t r\|^2_{L^2(0,T;V(\Omega)')} \quad \leq \quad C'_p(C_p, C_0, \tau_-, K^-), \tag{3.23}$$

where $C'_p$ depends on the data and on $C_p$. This concludes the proof of Lemma 3.2. □

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