# On Fractional Benney Type Systems

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Key words and phrases. Fractional Benney type systems, fractional Schrödinger equation, fractional porous medium equation, Cauchy problem.

#### **Abstract**

This paper introduces fractional type evolutionary equations modeling the interaction between short waves and long waves. We consider a fractional Benney type system, which is given by a fractional Schrödinger equation coupled with a fractional porous medium equation. Under the assumption of weak coupling or small initial data related to the fractional Schrödinger equation, it is proved the existence of weak solutions to the Cauchy problem.

### 1 Introduction

The main issue of this paper is to introduce and study the Cauchy problem for the following fractional Benney type systems

$$\begin{cases} i \partial_t u - (-\Delta)^s u = \alpha \ v \ u + \gamma \ |u|^2 u, & x \in \mathbb{R}, \ t > 0, \\ \partial_t v + (-\Delta)^{s/2} g(v) = \beta \ (-\Delta)^{s/2} |u|^2, & x \in \mathbb{R}, \ t > 0, \\ u(0, x) = u_0(x), \ v(0, x) = v_0(x), & x \in \mathbb{R}, \end{cases}$$
(1.1)

where  $\alpha$ ,  $\beta$  and  $\gamma$  are real constants. The complex value function u(t, x) is the unknown of the fractional Schrödinger equation, which describes the short wave, and the real value function v(t, x) is the unknown of the fractional porous medium equation, which describes the long wave. Here  $(-\Delta)^s$ , (0 < s < 1), denotes the

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usual fractional Laplacian in  $\mathbb{R}^n$ , which characterize nonlocal, long-range diffusion effects and can be defined by  $\mathcal{F}\{(-\Delta)^s f\}(\xi) = |\xi|^{2s} \mathcal{F}\{f\}(\xi)$ , where  $\mathcal{F}$  is the Fourier Transform. The function  $g \in C^1(\mathbb{R})$  is assumed to be nondecreasing, hence degenerated zones for the state variable v(t, x) are allowed. A particular case of (1.1), (e.g.  $g \equiv \text{constant}$ ), it has its own interest

$$\begin{cases} i \partial_t u - (-\Delta)^s u = \alpha \ v \ u + \gamma \ |u|^2 u, & x \in \mathbb{R}, \ t > 0, \\ \partial_t v = \beta \ (-\Delta)^{s/2} |u|^2, & x \in \mathbb{R}, \ t > 0, \\ u(0, x) = u_0(x), \ v(0, x) = v_0(x), & x \in \mathbb{R}. \end{cases}$$
(1.2)

The theory of evolutionary equations modeling the interaction between short waves and long waves goes back to Benney [3]. Indeed, in that paper Benney propose a general system (see equations (3.27), (3.28) in that paper), and we recall below the closer one studied by Bekiranov, Ogawa, Ponce [5], that is to say

$$\begin{cases} i \partial_t S - (-\Delta)S + i C_S \nabla S = \alpha S L + \gamma |S|^2 S, & x \in \mathbb{R}, \ t > 0, \\ \partial_t L + C_L \nabla L + \nu P(D_x) L + \lambda \nabla L^2 = \beta \nabla |S|^2, & x \in \mathbb{R}, \ t > 0, \end{cases}$$
(1.3)

where  $C_{S,\alpha}$ ,  $\gamma$ ,  $C_L$ ,  $\nu$ ,  $\lambda$  and  $\beta$  are real constants. Moreover,  $P(D_x)$  is a linear differential operator with constant coefficients. Applying a proper gauge transformation and a scaling of the variables, the system (1.3), when  $\nu = 0$ , is equivalent to

$$\begin{cases} i \partial_t u - (-\Delta)u = \alpha v u + \gamma |u|^2 u, \\ \partial_t v + C \nabla v = \beta \nabla |u|^2, \end{cases}$$
 (1.4)

where  $C = \pm 1$ . In fact, the authors in [5] claim that, the system (1.4) is the most typical case in the theory of wave interaction.

In particular, for s = 1 and g(v) = v, the system (1.1) recalls (1.4), since we have the following equivalence

$$\|(-\Delta)^{1/2}f\|_{L^2(\mathbb{R}^n)} = \|\nabla f\|_{L^2(\mathbb{R}^n)}, \text{ for each } f \in H^1(\mathbb{R}^n).$$

Then, one may roughly speaking interpret (1.1) as a generalization of (1.4). In particular, the system (1.1) makes sense for  $x \in \mathbb{R}^n$  and t > 0. Although, this is not exactly the case. Indeed, even if  $\nabla f$  and  $(-\Delta)^{1/2}f$  have the same  $L^2$ -norm they are different objects, that is, the former has local behavior and the other is nonlocal.

Therefore, we highlight the motivations to consider the fractional Benney type systems proposed in this paper, besides the multidimensional one. Indeed, the

short (transversal) wave described by the Schrödinger equation may represent a signal (wave packets), that is u(t, x) is a function that conveys information to control, for instance, some underwater equipment. This information propagates in a generalized medium, where long (longitudinal) waves are described by the porous medium equation. Here, the fractional Laplacian introduces the long-range interactions in both equations, which are coupled by the  $\alpha$ ,  $\beta$  constants. This discussion follows to applications in Synthetic Aperture Radar (see [2]), and atmospheric internal gravity waves (see [19], [25]), which represent complex anomalous systems and it seems better modeled by fractional Laplacians.

Last but not least, Benney in [3] also consider the following system (see equations (3.8), (3.9) in that paper)

$$\begin{cases} iS_t - (-\Delta)S + i C_g \nabla S = \alpha LS + \gamma |S|^2 S, & x \in \mathbb{R}, t > 0, \\ L_t + C_l \nabla L = \beta \nabla |S|^2, & x \in \mathbb{R}, t > 0. \end{cases}$$
(1.5)

In particular, when  $C_g = C_l$  long waves and short waves are resonant, and in this case Tsutsumi and Hatano in [23] proved that, the transformation:  $x \mapsto y = x - C_g t$  eliminates the first x-derivative terms in (1.5), hence we have

$$\begin{cases} iu_t - (-\Delta)u = \alpha \ vu + \gamma \ |u|^2 u, \\ v_t = \beta \ \nabla |u|^2, \end{cases}$$
 (1.6)

which resembles the fractional short wave and long wave system (1.2).

#### **Statement of the Main Result.**

The following definition tells us in which sense a pair (u(t, x), v(t, x)) is a weak solution to the Cauchy problem (1.1). Hereafter, we fix  $\gamma = 1$ , and without loss of generality g(0) = 0.

**Definition 1.1.** Given an initial data  $(u_0, v_0) \in H^s(\mathbb{R}) \times (L^2(\mathbb{R}) \cap L^\infty(\mathbb{R}))$  and any T > 0, a pair  $(u, v) \in L^\infty(0, T; H^s(\mathbb{R})) \times L^\infty(0, T; L^2(\mathbb{R}) \cap L^\infty(\mathbb{R}))$  is called a weak solution of the Cauchy problem (1.1), when it satisfies:

$$i\int_{0}^{T} \int_{\mathbb{R}} \left( u(t,x) \, \partial_{t} \overline{\varphi}(t,x) + (-\Delta)^{s/2} u(t,x) \, (-\Delta)^{s/2} \overline{\varphi}(t,x) \right) dx dt + i \int_{\mathbb{R}} u_{0}(x) \, \overline{\varphi}(0,x) dx$$

$$+ \alpha \int_{0}^{T} \int_{\mathbb{R}} v(t,x) \, u(t,x) \, \overline{\varphi}(t,x) dx dt + \int_{0}^{T} \int_{\mathbb{R}} |u(t,x)|^{2} \, u(t,x) \, \overline{\varphi}(t,x) dx dt = 0,$$

$$(1.7)$$

$$\int_{0}^{T} \int_{\mathbb{R}} v(t, x) \, \partial_{t} \psi(t, x) - g(v(t, x)) \, (-\Delta)^{s/2} \psi(t, x) dx dt + \int_{\mathbb{R}} v_{0}(x) \, \psi(0, x) dx + \beta \int_{0}^{T} \int_{\mathbb{R}} |u|^{2}(t, x) \, (-\Delta)^{s/2} \psi(t, x) dx dt = 0,$$
(1.8)

for each test function  $\varphi, \psi \in C_c^{\infty}((-\infty, T) \times \mathbb{R})$ , with  $\varphi$  being complex-valued and  $\psi$  real-valued.

Now, we state plainly the main result of this paper.

**Theorem 1.2** (Main Theorem). Let  $(u_0, v_0) \in H^s(\mathbb{R}) \times (L^2(\mathbb{R}) \cap L^{\infty}(\mathbb{R}))$ ,  $(\frac{1}{2} < s < 1)$ , and  $g \in C^1(\mathbb{R})$  satisfying

$$0 \le g'(\cdot) \le M < \infty$$
.

For any T > 0, there exist  $\alpha_0 > 0$ ,  $E_0 > 0$ , such that, if  $|\alpha| \le \alpha_0$  or  $||u_0||_{L^2(\mathbb{R})} \le E_0$ , then there exists a weak solution

$$(u, v) \in L^{\infty}(0, T; H^{s}(\mathbb{R})) \times L^{\infty}(0, T; L^{2}(\mathbb{R}) \cap L^{\infty}(\mathbb{R}))$$

of the Cauchy problem (1.1). Moreover, for a.a.  $t \in (0, T)$ 

$$||v(t)||_{L^{\infty}(\mathbb{R})} \le ||v_0||_{L^{\infty}(\mathbb{R})}. \tag{1.9}$$

Clearly, how lower are the  $\alpha, \beta$  constants less coupled are the equations in (1.1). In fact, the  $\alpha$  constant makes the difference concerning the global in time existence, (see Theorem 1.2). Another very important point is the energy input to the signal, i.e.  $||u_0||_{L^2}$ . As far as the information has to be sent, more energy is needed. Again, the statement of the Main Theorem shows that, the global in time solvability depends on the amount of energy given to the signal.

Finally, we recall that the fractional Schrödinger equation appears in the water wave models in [14]. In fact, the fractional Schrödinger equation was introduced in the theory related to fractional quantum mechanics associated to s-stable Lévy process (see for instance [15]). This field is developing fast, hence jointly with [14] we address the reader to the following papers [9], [10] and [12]. Moreover, the fractional porous medium equations has been widely studied in the last years. For instance, we address Vázquez [24] (and references there in), where is described the physical and mathematical background related to nonlinear diffusion equations involving nonlocal effects.

# 2 Notation and Background

In this section we fix the notations, and collect some preliminary results. First, let  $\Omega \subset \mathbb{R}^n$  be open set. We denote by  $dx, d\xi$ , etc. the Lebesgue measure on  $\Omega$  and by  $L^p(\Omega)$ ,  $p \in [1, +\infty)$ , the set of (real or complex) p-summable functions with respect to the Lebesgue measure. Moreover, we denote by  $\mathcal{F}\varphi(\xi) \equiv \widehat{\varphi}(\xi)$  the Fourier Transform of  $\varphi$ , which is an isometry in  $L^2(\mathbb{R}^n)$ .

### • The space $W^{s,p}(\Omega)$

The Sobolev space is denoted by  $W^{s,p}(\Omega)$ , where a real  $p \ge 1$  is the integrability index and a real  $s \ge 0$  is the smoothness index. More precisely, for  $s \in (0,1)$ ,  $p \in [1, +\infty)$ , the fractional Sobolev space of order s with Lebesgue exponent p is defined by

$$W^{s,p}(\Omega) := \Big\{ u \in L^p(\Omega) : \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} \, dx \, dy < +\infty \Big\},$$

endowed with norm

$$||u||_{W^{s,p}(\Omega)} = \left( \int_{\Omega} |u|^p dx + \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} dx dy \right)^{\frac{1}{p}}.$$

For s > 1 we write  $s = m + \sigma$ , where m is an integer and  $\sigma \in (0, 1)$ . In this case, the space  $W^{s,p}(\Omega)$  consists of those equivalence classes of functions  $u \in W^{m,p}(\Omega)$  whose distributional derivatives  $D^{\alpha}u$ , with  $|\alpha| = m$ , belong to  $W^{\sigma,p}(\Omega)$ , that is

$$W^{s,p}(\Omega) = \left\{ u \in W^{m,p}(\Omega) : \sum_{|\alpha|=m} ||D^{\alpha}u||_{W^{\sigma,p}(\Omega)} < \infty \right\},\,$$

which is a Banach space with respect to the norm

$$||u||_{W^{s,p}(\Omega)} = (||u||_{W^{m,p}(\Omega)}^p + \sum_{|u|=m} ||D^{\alpha}u||_{W^{\sigma,p}(\Omega)}^p)^{\frac{1}{p}}.$$

If s = m is an integer, then the space  $W^{s,p}(\Omega)$  coincides with the Sobolev space  $W^{m,p}(\Omega)$ . It is very interesting the case when p = 2, i.e.  $W^{s,2}(\Omega)$ , which is also a Hilbert space and we can consider the inner product

$$\langle u, v \rangle_{W^{s,2}(\Omega)} = \langle u, v \rangle + \int_{\Omega} \int_{\Omega} \frac{(u(x) - u(y))}{|x - y|^{\frac{n}{2} + s}} \frac{(v(x) - v(y))}{|x - y|^{\frac{n}{2} + s}} dx dy,$$

where  $\langle \cdot, \cdot \rangle$  is the inner product in  $L^2(\Omega)$ .

#### • The space $H^s(\mathbb{R}^n)$

Now, following Tartar [21] we take into account an alternative definition of the space  $H^s(\mathbb{R}^n) = W^{s,2}(\mathbb{R}^n)$  via Fourier Transform. Precisely, we may define

$$H^{s}(\mathbb{R}^{n}) := \left\{ u \in L^{2}(\mathbb{R}^{n}) : \int_{\mathbb{R}^{n}} (1 + |\xi|^{2s}) |\mathcal{F}u(\xi)|^{2} d\xi < \infty \right\}$$
 (2.10)

and we observe that the above definition, is valid also for any real  $s \ge 1$ . Moreover,  $H^s(\mathbb{R}^n)$  is a Hilbert space with the scalar product

$$(u,v)_{H^s(\mathbb{R}^n)} = \int_{\mathbb{R}^n} (1+|\xi|^2)^s \, \widehat{u}(\xi) \, \overline{\widehat{v}}(\xi) \, d\xi.$$

The equivalence of the above definitions is stated in the following

**Lemma 2.1.** Let 0 < s < 1. Then, the definitions of  $H^s(\mathbb{R}^n)$  and  $W^{s,2}(\mathbb{R}^n)$  are equivalent. In particular, for any  $u \in H^s(\mathbb{R}^n)$ 

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} \, dx \, dy = 2 \, C_{n,s}^{-1} \, \int_{\mathbb{R}^n} |\xi|^{2s} \, |\Im u(\xi)|^2 d\xi, \tag{2.11}$$

where

$$C_{n,s}^{-1} = \int_{\mathbb{R}^n} \frac{1 - \cos(\zeta_1)}{|\zeta|^{n+2s}} d\zeta.$$

One remarks that, for s > n/2, the Hilbert space  $H^s(\mathbb{R}^n)$  is an algebra (see [16]). Moreover, there exists a constant C = C(s) > 0, such that for any  $f, g \in H^s(\mathbb{R}^n)$ 

$$||f g||_{H^{s}(\mathbb{R}^{n})} \le C ||f||_{H^{s}(\mathbb{R}^{n})} ||g||_{H^{s}(\mathbb{R}^{n})}. \tag{2.12}$$

# **2.1** Fractional Laplacian operator in $\mathbb{R}^n$

The fractional Laplacian operator can be defined in  $\mathbb{R}^n$  by

$$(\widehat{-\Delta)^s} f(\xi) = |\xi|^{2s} \hat{f}(\xi), \qquad (0 < s < 1).$$
 (2.13)

Hence the fractional Laplacian is a pseudo-differential operator with principal symbol  $|\xi|^{2s}$ . The fractional Laplacian can be similarly described using singular integrals

$$(-\Delta)^{s} f(x) = C_{n,s} \text{ P.V.} \int_{\mathbb{R}^{n}} \frac{f(x) - f(\xi)}{|x - \xi|^{n+2s}} d\xi.$$
 (2.14)

Moreover, its inverse denoted by  $\mathcal{K}_s := (-\Delta)^{-s}$ , (0 < s < 1), is given by convolution with the Riesz kernel  $K_s(x) = C_{n,s} |x|^{2s-n}$ , that is,  $\mathcal{K}_s f = K_s * f$ .

It follows from (2.10), (2.11) and (2.13) that, there exist positive constants  $m_s$ ,  $M_s$ , such that, for each  $f \in H^s(\mathbb{R}^n)$ 

$$m_s(||f||_{L^2(\mathbb{R}^n)} + ||(-\Delta)^{s/2}f||_{L^2(\mathbb{R}^n)}) \le ||f||_{H^s(\mathbb{R}^n)} \le M_s(||f||_{L^2(\mathbb{R}^n)} + ||(-\Delta)^{s/2}f||_{L^2(\mathbb{R}^n)}).$$
(2.15)

#### • Bilinear form

In order to study the fractional diffusion term, it will be important to associate a bilinear form to the operator  $\mathcal{K}_s$  in the space  $H^s(\mathbb{R}^n)$ , 0 < s < 1, which is given for any pair  $v, w \in H^s(\mathbb{R}^n)$  by

$$\mathcal{B}_{s}(v,w) := C_{n,s} \iint_{\mathbb{R}^{2n}} (v(x) - v(y)) \frac{1}{|x - y|^{n+2s}} (w(x) - w(y)) \ dxdy. \tag{2.16}$$

The bilinear form  $\mathcal{B}_s$  were considered in [7] as an auxiliary tool in the study of regularity properties of solutions to the fractional type porous medium equation.

**Lemma 2.2** (See [7]). If v is given by v = G(w), with  $G' \ge 0$ , then,  $\mathcal{B}_s(v, w) \ge 0$ . Furthermore, for every  $v, w \in H^1(\mathbb{R}^n)$  we have the characterization

$$\mathcal{B}_{s}(v,w) = C \iint_{\mathbb{R}^{2n}} \nabla v(x) \frac{1}{|x-y|^{n-2+2s}} \nabla w(y) \, dx dy, \tag{2.17}$$

where C is a positive constant.

**Proposition 2.3.** Let  $v \in H^1(\mathbb{R}^n)$ ,  $G \in C^1(\mathbb{R})$  with  $G'(\cdot) \ge m > 0$ . Then

$$\int_{\mathbb{R}^n} (-\Delta)^{s/2} G(v) \ v \ dx \ge m \ C_{n,s}^{-1} \| (-\Delta)^{s/4} v \|_{L^2(\mathbb{R}^n)}^2.$$

*Proof.* It follows directly from (2.16), (2.17) and applying the intermediate value theorem.

### 2.2 Auxiliary kernels

#### • Unitary group for the Schrödinger equation

For each  $\varepsilon > 0$ , we consider the following Cauchy problem for  $u(t, x) \in \mathbb{C}$ , driven by the linear fractional perturbed Schrödinger equation

$$\begin{cases} i \partial_t u - (-\Delta)^s u - \varepsilon^a (-\Delta) u = 0, & x \in \mathbb{R}^n \ t \in \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}^n, \end{cases}$$
 (2.18)

where  $a \in \mathbb{R}$  is a fixed parameter chosen a posteriori. Applying the Fourier transform in the spatial variable, we have

$$\begin{cases} i \, \partial_t \widehat{u}(t,\xi) - |\xi|^{2s} \, \widehat{u}(t,\xi) - \varepsilon^a \, |\xi|^2 \, \widehat{u}(t,\xi) = 0, \ \xi \in \mathbb{R}^n \ t \in \mathbb{R}, \\ \widehat{u}(0,\xi) = \widehat{u}_0(\xi), \ \xi \in \mathbb{R}^n, \end{cases}$$

which solution is given by  $\widehat{u}(t,\xi) = e^{-i\left(|\xi|^{2s} + \varepsilon^a |\xi|^2\right)t}\widehat{u_0}(\xi)$ . Therefore, it follows that

$$u(t,x) = \mathcal{F}^{-1}\left\{e^{-i\left(|\xi|^{2s}+\varepsilon^a\ |\xi|^2\right)t}\mathcal{F}u_0(\xi)\right\}(x)$$

solves the Cauchy problem (2.18). For  $u_0 \in L^2(\mathbb{R}^n)$ ,  $(\mathcal{F}u_0 \in L^2(\mathbb{R}^n))$ , then

$$e^{-i\left(|\xi|^{2s}+\varepsilon^a|\xi|^2\right)t}\mathcal{F}u_0(\xi)\in L^2(\mathbb{R}^n).$$

Now, we define for each  $t \in \mathbb{R}$  the operator

$$u \mapsto U_{\varepsilon}(t)u := \mathcal{F}^{-1}e^{-i\left(|\xi|^{2s}+\varepsilon^{a}|\xi|^{2}\right)t}\mathcal{F}u, \tag{2.19}$$

which is bounded in  $L^2(\mathbb{R}^n)$  for each  $u \in L^2(\mathbb{R}^n)$ . Indeed, we have

$$\begin{aligned} ||U_{\varepsilon}(t)u||_{L^{2}(\mathbb{R}^{n})}^{2} &= \int_{\mathbb{R}^{n}} |U_{\varepsilon}(t)u(x)|^{2} dx = \int_{\mathbb{R}^{n}} |\widehat{U_{\varepsilon}(t)}u(\xi)|^{2} d\xi \\ &= \int_{\mathbb{R}^{n}} |e^{-i\left(|\xi|^{2s} + \varepsilon^{a} |\xi|^{2}\right)^{2}} \widehat{u}(\xi)|^{2} d\xi = \int_{\mathbb{R}^{n}} |\widehat{u}(\xi)|^{2} d\xi. \end{aligned}$$

Therefore, the family  $(U_{\varepsilon}(t))_{t\in\mathbb{R}}$  is a group of isometries in  $L^2(\mathbb{R}^n)$ .

One remarks that,  $H^s(\mathbb{R}^n)$ , (s > 0), is invariant by the isometry group  $(U_{\varepsilon}(t))_{t \in \mathbb{R}}$ . For each  $u \in H^s(\mathbb{R}^n)$ , we have

$$||U_{\varepsilon}(t)u||_{H^{s}(\mathbb{R}^{n})} = \int_{\mathbb{R}^{n}} (1 + |\xi|^{2})^{s} |\widehat{U_{\varepsilon}(t)}u(\xi)|^{2} d\xi$$
$$= \int_{\mathbb{R}^{n}} (1 + |\xi|^{2})^{s} |\widehat{u}(\xi)|^{2} d\xi = ||u||_{H^{s}(\mathbb{R}^{n})}.$$

Thus  $U_{\varepsilon}(t)(H^s(\mathbb{R}^n))$  is a closed subspace in  $H^s(\mathbb{R}^n)$  and, we have

$$H^{s}(\mathbb{R}^{n}) = U_{s}(t)(H^{s}(\mathbb{R}^{n})) \oplus (U_{s}(t)(H^{s}(\mathbb{R}^{n})))^{\perp}.$$

Moreover, since  $U_{\varepsilon}(t)$  is symmetric in  $H^{s}(\mathbb{R}^{n})$ 

$$\begin{split} (U_{\varepsilon}(t)u,w)_{H^{s}(\mathbb{R}^{n})} &= \int_{\mathbb{R}^{n}} (1+|\xi|^{2})^{s} \widehat{U_{\varepsilon}(t)} u(\xi) \, \widehat{w}(\xi) \, d\xi \\ &= \int_{\mathbb{R}^{n}} (1+|\xi|^{2})^{s} e^{-i \, \left(|\xi|^{2s} + \varepsilon^{a} \, |\xi|^{2}\right) t} \, \widehat{u}(\xi) \, \widehat{w}(\xi) \, d\xi \\ &= \int_{\mathbb{R}^{n}} (1+|\xi|^{2})^{s} \, \widehat{u}(\xi) \, \widehat{U_{\varepsilon}(t)} w(\xi) \, d\xi = (u,U_{\varepsilon}(t)w)_{H^{s}(\mathbb{R}^{n})} \end{split}$$

and also an isometry, it follows that  $(U_{\varepsilon}(t)(H^{s}(\mathbb{R}^{n})))^{\perp} = \{0\}.$ 

#### • Semigroups of contractions for the heat equation

For each  $\varepsilon > 0$ , we consider the following Cauchy problem for  $v(t, x) \in \mathbb{R}$ , driven by the linear Heat equation

$$\begin{cases} \partial_t v - \varepsilon^b \Delta v = 0, & x \in \mathbb{R}^n, \quad t > 0, \\ v(0, x) = v_0(x), & x \in \mathbb{R}^n, \end{cases}$$
 (2.20)

where  $b \in \mathbb{R}$  is a fixed parameter chosen a posteriori. Again, applying the Fourier transform in the spatial variable, we obtain

$$\begin{cases} \partial_t \widehat{v}(t,\xi) + \varepsilon^b |\xi|^2 \widehat{v}(t,\xi) = 0, \ \xi \in \mathbb{R}^n, \ t > 0, \\ \widehat{v}(0,\xi) = \widehat{v}_0(\xi), \ \xi \in \mathbb{R}^n, \end{cases}$$

which solution is given by  $\widehat{v}(t,\xi) = e^{-\varepsilon^b |\xi|^2 t} \widehat{v_0}(\xi)$ . Consequently,

$$v(t,x) = \mathcal{F}^{-1} \left\{ e^{-\varepsilon^b |\xi|^2 t} \, \mathcal{F} v_0(\xi) \right\} (x)$$

solves the Cauchy problem (2.20), and it is well known that, for  $v_0 \in L^2(\mathbb{R}^n)$ ,  $(\mathcal{F}v_0 \in L^2(\mathbb{R}^n))$ , it follows that  $e^{-\varepsilon^b |\xi|^2 t} \mathcal{F}v_0(\xi) \in L^2(\mathbb{R}^n)$ .

Similarly, we define for each t > 0 the operator

$$v \mapsto W_{\varepsilon}(t)v = \mathcal{F}^{-1}e^{-\varepsilon^b |\xi|^2 t} \mathcal{F}v. \tag{2.21}$$

The operator  $W_{\varepsilon}(t)$  is bounded in  $L^2(\mathbb{R}^n)$ , in fact the family  $\{W_{\varepsilon}(t)v\}_{t>0}$  is a semi-group of contractions. Indeed, for any t>0,  $\|W_{\varepsilon}(t)v_0\|_{L^2} \leq \|v_0\|_{L^2}$ , for any  $v\in L^2(\mathbb{R}^n)$ . Also in  $H^1(\mathbb{R}^n)$ , that is

$$||W_{\varepsilon}(t)v_{0}||_{H^{1}}^{2} = \int_{\mathbb{R}^{n}} (1+|\xi|^{2}) |\widehat{W_{\varepsilon}(t)v_{0}}(\xi)|^{2} d\xi = \int_{\mathbb{R}^{n}} (1+|\xi|^{2}) |e^{-\varepsilon^{b}|\xi|^{2}t} \widehat{v_{0}}(\xi)|^{2} d\xi \leq ||v_{0}||_{H^{1}}^{2}.$$

One recalls that, the Heat kernel has a regularity effect. Indeed, a refined estimate is given by the following

**Lemma 2.4.** For any  $v \in L^2(\mathbb{R})$ , there exists a constant C > 0 independent of t and v, such that for any t > 0

$$\|\partial_x W_{\varepsilon}(t)v\|_{L^2(\mathbb{R})} \le \frac{C}{\sqrt{t}} \|v\|_{L^2(\mathbb{R})}.$$
 (2.22)

*Proof.* From (2.21), we have

$$W_{\varepsilon}(t)v = \mathcal{F}^{-1}e^{-\varepsilon^{b}|\xi|^{2}t}\,\mathcal{F}v = \frac{1}{\sqrt{4\pi\varepsilon^{b}t}}e^{-\frac{x^{2}}{4\varepsilon^{b}t}}*v,$$

then

$$\partial_x W_\varepsilon(t) v = \frac{1}{\sqrt{4\pi\varepsilon^b t}} \frac{-2x}{4\varepsilon^b t} e^{-\frac{x^2}{4\varepsilon^b t}} * v.$$

Applying Young's inequality, it follows that

$$\|\partial_x W_{\varepsilon}(t)v\|_{L^2(\mathbb{R})} \leq \frac{2}{4\varepsilon^b t} \|\frac{x}{\sqrt{4\pi\varepsilon^b t}} e^{-\frac{x^2}{4\varepsilon^b t}}\|_{L^1(\mathbb{R})} \|v\|_{L^2(\mathbb{R})} = \frac{1}{\sqrt{\pi\varepsilon^b}} \frac{1}{t^{1/2}} \|v\|_{L^2(\mathbb{R})}.$$

### 2.3 Auxiliary inequalities

The next two auxiliary results will be used broadly in this paper.

**Proposition 2.5.** (Chain Rule) Let  $f \in H^s(\mathbb{R}^n)$ , 0 < s < 1,  $F \in C^1(\mathbb{C})$  with  $||F'||_{L^{\infty}(\mathbb{R})} \leq M$  for some M > 0. Then

$$\|(-\Delta)^{s/2}F(f)\|_{L^2(\mathbb{R}^n)} \le \|F'\|_{L^{\infty}(\mathbb{R})} \|(-\Delta)^{s/2}f\|_{L^2(\mathbb{R}^n)}$$
(2.23)

Now, we provide the following (sharp) result.

**Proposition 2.6.** Let  $f \in H^s(\mathbb{R})$ ,  $\frac{1}{2} < s < 1$ . Then,

$$||f||_{L^{\infty}(\mathbb{R})} \le \frac{2}{\sqrt{\pi(2s-1)}} ||f||_{L^{2}(\mathbb{R})}^{1-\frac{1}{2s}} ||(-\Delta)^{s/2}f||_{L^{2}(\mathbb{R})}^{\frac{1}{2s}}, \tag{2.24}$$

and

$$\|(-\Delta)^{s/2}|f|^2\|_{L^2(\mathbb{R})} \le 2\|f\|_{L^{\infty}(\mathbb{R})}\|(-\Delta)^{s/2}f\|_{L^2(\mathbb{R})}.$$
 (2.25)

*Proof.* 1. First, since s > 1/2, it follows from the well-known Embedding Theorem that,  $H^s$  is an algebra of functions. Moreover, a function  $f \in H^s(\mathbb{R})$  may be represented by a continuous function which vanishes at infinity. Let us show (2.24), hence applying the inverse Fourier transform, we have for each  $x \in \mathbb{R}$ 

$$|f(x)| = \left| \frac{1}{(2\pi)^{1/2}} \int_{\mathbb{R}} e^{ix\xi} \widehat{f}(\xi) \, d\xi \right| \le \frac{1}{(2\pi)^{1/2}} \int_{\mathbb{R}} |\widehat{f}(\xi)| \, d\xi$$
$$= \frac{1}{(2\pi)^{1/2}} \Big( \int_{|\xi| \le R} |\widehat{f}(\xi)| \, d\xi + \int_{|\xi| \ge R} \frac{|\xi|^s}{|\xi|^s} |\widehat{f}(\xi)| \, d\xi \Big),$$

where R>0 is any fixed real number. Then, applying the Cauchy-Schwartz inequality

$$|f(x)| \leq \frac{1}{(2\pi)^{1/2}} \Big( \int_{|\xi| \leq R} 1 \, d\xi \Big)^{1/2} \Big( \int_{|\xi| \leq R} |\widehat{f}(\xi)|^2 \, d\xi \Big)^{1/2}$$

$$+ \frac{1}{(2\pi)^{1/2}} \Big( \int_{|\xi| \geq R} \frac{1}{|\xi|^{2s}} \, d\xi \Big)^{1/2} \Big( \int_{|\xi| \geq R} |\xi|^{2s} \, |\widehat{f}(\xi)|^2 \, d\xi \Big)^{1/2}$$

$$\leq \frac{1}{(2\pi)^{1/2}} \Big( \sqrt{2} \, R^{1/2} \, ||f||_{L^2(\mathbb{R})} + \sqrt{\frac{2}{2s-1}} \, R^{\frac{1}{2}-s} \, ||(-\Delta)^{s/2} f||_{L^2(\mathbb{R})} \Big)$$

$$\leq \frac{1}{\sqrt{\pi} \sqrt{2s-1}} \Big( R^{1/2} ||f||_{L^2(\mathbb{R})} + R^{\frac{1}{2}-s} ||(-\Delta)^{s/2} f||_{L^2(\mathbb{R})} \Big).$$

$$(2.26)$$

Conveniently, we consider  $R = ||f||_{L^{2}(\mathbb{R})}^{-\frac{1}{s}} ||(-\Delta)^{s/2}f||_{L^{2}(\mathbb{R})}^{\frac{1}{s}}$  in (2.26) to obtain

$$|f(x)| \leq \frac{1}{\sqrt{\pi(2s-1)}} \Big( ||f||_{L^2(\mathbb{R})}^{1-\frac{1}{2s}} ||(-\Delta)^{s/2}f||_{L^2(\mathbb{R})}^{\frac{1}{2s}} + ||f||_{L^2(\mathbb{R})}^{1-\frac{1}{2s}} ||(-\Delta)^{s/2}f||_{L^2(\mathbb{R})}^{\frac{1}{2s}} \Big).$$

2. Now, we prove (2.25). Again, from (2.14) and the definition of the Fractional Laplacian, we obtain

$$\begin{aligned} &\|(-\Delta)^{s/2}|f|^2\|_{L^2(\mathbb{R})}^2 = \frac{C_{n,s}}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\|f|^2(x) - |f|^2(y)|^2}{|x - y|^{1+2s}} \, dx dy \\ &\leq C_{n,s} \left( \iint_{\mathbb{R} \times \mathbb{R}} \frac{|f(x)|(\overline{f}(x) - \overline{f}(y))|^2}{|x - y|^{1+2s}} \, dx dy + \iint_{\mathbb{R} \times \mathbb{R}} \frac{|\overline{f}(y)|(f(x) - f(y))|^2}{|x - y|^{1+2s}} \, dx dy \right) \\ &\leq 2 \|f\|_{L^{\infty}(\mathbb{R})} \|(-\Delta)^{s/2} f\|_{L^2(\mathbb{R})}. \end{aligned}$$

#### 2.4 Generalized Grownwall Lemma

We consider the following (see [11]),

**Theorem 2.7.** Let  $\eta(t)$  be a nonnegative function which satisfies the inequality

$$\eta(t) \le C + \int_{t_0}^t \left( a(\tau) \, \eta(\tau) + b(\tau) \, \eta^{\sigma}(\tau) \right) d\tau, \quad C \ge 0, \, \sigma \ge 0, \tag{2.27}$$

where a(t) and b(t) are continuous nonnegative functions for  $t \ge t_0$ .

1. For  $0 \le \sigma < 1$ ,

$$\eta(t) \le \left\{ C^{1-\sigma} \exp\left[ (1-\sigma) \int_{t_0}^t a(\tau) d\tau \right] + (1-\sigma) \int_{t_0}^t b(\tau) \exp\left[ (1-\sigma) \int_{\tau}^t a(r) dr \right] d\tau \right\}^{\frac{1}{1-\sigma}}.$$
(2.28)

2. For  $\sigma = 1$ ,

$$\eta(t) \le C \exp\Big\{ \int_{t_0}^t \left[ a(\tau) + b(\tau) \right] d\tau \Big\}. \tag{2.29}$$

3. For  $\sigma > 1$ , with the additional hypothesis

$$C < \left\{ \exp\left[ (1 - \sigma) \int_{t_0}^{t_0 + h} a(\tau) \, d\tau \right] \right\}^{\frac{1}{\sigma - 1}} \left\{ (\sigma - 1) \int_{t_0}^{t_0 + h} b(\tau) \, d\tau \right\}^{-\frac{1}{\sigma - 1}}, \tag{2.30}$$

we also get for  $t_0 \le t \le t_0 + h$ , for h > 0

$$\eta(t) \le C \left\{ \exp\left[ (1 - \sigma) \int_{t_0}^t a(\tau) \, d\tau \right] - C^{-1}(\sigma - 1) \int_{t_0}^t b(\tau) \exp\left[ (1 - \sigma) \int_{\tau}^t a(r) \, dr \right] d\tau \right\}^{\frac{1}{\sigma - 1}}.$$
(2.31)

### 2.5 Entropies

Following the scalar conservation laws theory, we say that a Lipschitz convex function  $\eta : \mathbb{R} \to \mathbb{R}$  is an entropy. The most important example is the family of Kružkov's entropies, that is

$$\eta_k(v) := |v - k|$$
, for each  $k \in \mathbb{R}$ .

Then, we recall that any smooth entropy  $\eta(v)$ , which is linear at infinity, can be recovered by the family of Kružkov's entropies. Indeed, a straight calculation shows that

$$\eta(v) = \frac{1}{2} \int_{\mathbb{R}} \eta''(\xi) |v - \xi| d\xi,$$

modulo an additive constant. Symilarly, given  $g \in C^1(\mathbb{R})$  and  $q : \mathbb{R} \to \mathbb{R}$ , such that,  $q' = \eta' g'$ , then

$$q(v) = \frac{1}{2} \int_{\mathbb{R}} \eta''(\xi) |g(v) - g(\xi)| d\xi.$$

Under the above conditions,  $(\eta, q)$  is called here an entropy pair.

Now, we consider the following

**Lemma 2.8.** Let v be a real  $H^1(\mathbb{R})$  function,  $g \in C^1(\mathbb{R})$  satisfying

$$0 \le g'(\cdot) \le M < \infty$$
,

and  $s \in (0, 1)$ . Then, for each  $k \in \mathbb{R}$  fixed, and each  $x \in \mathbb{R}$ ,

$$(-\Delta)^{s}|g(v(x)) - g(k)| = sgn(v(x) - k)(-\Delta)^{s}g(v(x)) - R_{k}(x), \tag{2.32}$$

where the non-negative remainder function  $R_k(\cdot)$  is given by

$$R_{k}(x) := \begin{cases} 2 C_{1,s} \int_{\{v(y) < k\}} \frac{g(k) - g(v(y))}{|x - y|^{1 + 2s}} dy, & \{v(x) > k\}, \\ 2 C_{1,s} \int_{\{v(y) > k\}} \frac{g(v(y)) - g(k)}{|x - y|^{1 + 2s}} dy, & \{v(x) < k\}. \end{cases}$$
(2.33)

*Proof.* Since the function g is non-decreasing, it follows that

$$sgn(v(x) - k)(g(v(x) - g(k))) = |g(v(x) - g(k))|.$$

Therefore, we have

$$(-\Delta)^{s}|g(v(x)) - g(k)| = (-\Delta)^{s} ((\operatorname{sgn}(v(x) - k)(g(v(x)) - g(k)))$$

$$= \operatorname{sgn}(v(x) - k)(-\Delta)^{s} g(v(x)) + (g(v(x)) - g(k))(-\Delta)^{s} \operatorname{sgn}(v(x) - k)$$

$$- C_{1,s} \int_{\mathbb{R}} \frac{(\operatorname{sgn}(v(x) - k) - \operatorname{sgn}(v(y) - k))}{|x - y|^{1 + 2s}} ((g(v(x)) - g(v(y)))) dy$$

$$= \operatorname{sgn}(v(x) - k)(-\Delta)^{s} g(v(x))$$

$$+ (g(v(x)) - g(k))C_{1,s} \int_{\mathbb{R}} \frac{\operatorname{sgn}(v(x) - k) - \operatorname{sgn}(v(y) - k)}{|x - y|^{1 + 2s}} dy$$

$$- C_{1,s} \int_{\mathbb{R}} (g(v(x)) - g(v(y))) \frac{\operatorname{sgn}(v(x) - k) - \operatorname{sgn}(v(y) - k)}{|x - y|^{1 + 2s}} dy$$

$$= \operatorname{sgn}(v(x) - k)(-\Delta)^{s} g(v(x))$$

$$+ C_{1,s} \int_{\mathbb{R}} (g(v(y)) - g(k)) \frac{\operatorname{sgn}(v(x) - k) - \operatorname{sgn}(v(y) - k)}{|x - y|^{1 + 2s}} dy,$$

where we have used that  $\{x \in \mathbb{R} : v(x) > k\}$  and  $\{x \in \mathbb{R} : v(x) < k\}$  are open sets, since v is continuous.

# 3 On a Perturbed System

In order to show the solvability of the Cauchy problem (1.1), we perturbe both equations  $(1.1)_1$ , and  $(1.1)_2$ , adding Laplacian terms with different velocities of perturbation. Specifically, let a, b > 0 be fixed parameters and for each  $\varepsilon \in (0, 1)$ , we consider the following system posed in  $(0, T) \times \mathbb{R}$ ,

$$\begin{cases} i \partial_{t} u^{\varepsilon} - (-\Delta)^{s} u^{\varepsilon} + \varepsilon^{a} \Delta u^{\varepsilon} = \alpha v^{\varepsilon} u^{\varepsilon} + |u^{\varepsilon}|^{2} u^{\varepsilon}, \\ \partial_{t} v^{\varepsilon} - \varepsilon^{b} \Delta v^{\varepsilon} = \beta (-\Delta)^{s/2} (|u^{\varepsilon}|^{2}) - (-\Delta)^{s/2} g_{\varepsilon}(v^{\varepsilon}), \\ u^{\varepsilon}(0, x) = u_{0}^{\varepsilon}(x), \quad v^{\varepsilon}(0, x) = v_{0}^{\varepsilon}(x), \end{cases}$$
(3.34)

where T > 0 is a real number, conveniently  $g_{\varepsilon}(v) := g(v) + \varepsilon v$ , and the pair  $(u_0^{\varepsilon}, v_0^{\varepsilon}) \in H^1(\mathbb{R}) \times H^1(\mathbb{R})$  is an approaching sequence converging strongly to  $(u_0, v_0)$  in  $H^s(\mathbb{R}) \times (L^2(\mathbb{R}) \cap L^{\infty}(\mathbb{R}))$ ,  $(\|v_0^{\varepsilon}\|_{L^{\infty}(\mathbb{R})} \le \|v_0\|_{L^{\infty}(\mathbb{R})})$ . First, we show (local in time) existence and uniqueness of mild solution to (3.34). Then, we derive a

priori important estimates, which enable us to extend the local in time solution. Moreover, we stress that these a priori estimates will be also important to show that the family  $\{(u^{\varepsilon}, v^{\varepsilon})\}$  of solution to (3.34) is relatively compact.

### 3.1 Existence and uniqueness

The following definition tell us in which sense the pair  $(u^{\varepsilon}, v^{\varepsilon})$  is a solution of the Cauchy problem (3.34).

**Definition 3.1.** The pair  $(u^{\varepsilon}, v^{\varepsilon}) \in C([0, T]; H^1(\mathbb{R})) \times C([0, T]; H^1(\mathbb{R}))$  is called a mild solution of (3.34) if satisfies the following integral equations

$$\begin{cases} u^{\varepsilon}(t) = U_{\varepsilon}(t) u_{0}^{\varepsilon} - i \int_{0}^{t} U_{\varepsilon}(t - t') \left( \alpha v^{\varepsilon}(t') u^{\varepsilon}(t') + |u^{\varepsilon}(t')|^{2} u^{\varepsilon}(t') \right) dt', \\ v^{\varepsilon}(t) = W_{\varepsilon}(t) v_{0}^{\varepsilon} + \int_{0}^{t} W_{\varepsilon}(t - t') \left( \beta (-\Delta)^{s/2} |u^{\varepsilon}(t')|^{2} - (-\Delta)^{s/2} g_{\varepsilon}(v^{\varepsilon}) \right) dt', \end{cases}$$
(3.35)

where  $U_{\varepsilon}(t)$ ,  $W_{\varepsilon}(t)$  are given respectively by (2.19) and (2.21).

We are going to apply the Banach Fixed Point Theorem to show the local-intime existence of solutions as defined above. To begin, we consider the following lemma (we put  $\varepsilon = 1$  for simplicity with obvious notation).

**Lemma 3.2.** Let  $\frac{1}{2} < s < 1$ ,  $g \in C^1(\mathbb{R})$ , satisfying  $1 \le g'(\cdot) \le M < \infty$ , (g(0) = 0). For T > 0, let  $(\tilde{u}, \tilde{v}) \in C([0, T]; H^1(\mathbb{R})) \times C([0, T]; H^1(\mathbb{R}))$ , then for each  $(u_0, v_0) \in H^1(\mathbb{R}) \times H^1(\mathbb{R})$  the Cauchy problem (decoupled system)

$$\begin{cases} \partial_{t}u + i (-\Delta)^{s}u - i \Delta u = -i \alpha \tilde{v} \tilde{u} - i |\tilde{u}|^{2} \tilde{u}, & x \in \mathbb{R} \ t > 0, \\ \partial_{t}v - \Delta v = \beta (-\Delta)^{s/2} (|\tilde{u}|^{2}) - (-\Delta)^{s/2} g(\tilde{v}), & x \in \mathbb{R} \ t > 0, \end{cases}$$
(3.36)  
$$u(0, x) = u_{0}(x), \ v(0, x) = v_{0}(x),$$

admits a unique mild solution  $(u, v) \in C([0, T]; H^1(\mathbb{R})) \times C([0, T]; H^1(\mathbb{R}))$ .

*Proof.* First, we define for each  $t \in (0, T)$ 

$$F(t) := -i \ \alpha \ \tilde{v}(t) \ \tilde{u}(t) - i \ |\tilde{u}|^2(t) \ \tilde{u}(t), \quad G(t) := \beta(-\Delta)^{s/2}(|\tilde{u}|^2)(t) - (-\Delta)^{s/2}g(\tilde{v})(t).$$

Claim 1: The complex value function  $F \in C([0, T]; L^2(\mathbb{R}))$ .

Proof of Claim: Indeed, for all  $t \in [0, T]$ ,  $|\tilde{u}|^2(t)$   $\tilde{u}(t) \in H^1(\mathbb{R})$ ,  $\tilde{u}(t)$   $\tilde{v}(t) \in H^1(\mathbb{R})$ . Then, for h sufficiently small

$$F(t+h) - F(t) = i \alpha \left( \tilde{v}(t) \tilde{u}(t) - \tilde{v}(t+h) \tilde{u}(t+h) \right)$$
$$+ i \left( |\tilde{u}|^2(t) \tilde{u}(t) - |\tilde{u}|^2(t+h) \tilde{u}(t+h) \right) = i \alpha I_1 + i I_2,$$

with obvious notation. A simple algebraic computation shows that

$$\lim_{h\to 0} ||I_1||_{L^2(\mathbb{R})} = 0, \quad \text{and} \quad \lim_{h\to 0} ||I_2||_{L^2(\mathbb{R})} = 0,$$

from which the claim is proved.

Claim 2: The real value function  $G \in C([0, T]; L^2(\mathbb{R}))$ .

Proof of Claim: We observe that  $(-\Delta)^{s/2}(|\tilde{u}|^2)(t) \in L^2(\mathbb{R})$ , for each  $t \in (0,T)$ . Also from the assumptions for the function g, that is  $g \in C^1(\mathbb{R})$ , g(0) = 0 and  $|g'(v)| \leq M$ ,  $(\forall v \in \mathbb{R})$ , it follows that  $(-\Delta)^{s/2}g(\tilde{v})(t) \in L^2(\mathbb{R})$ . Now, for h sufficiently small, we have

$$G(t+h) - G(t) = \beta \left( (-\Delta)^{s/2} (|\tilde{u}|^2)(t+h) - (-\Delta)^{s/2} (|\tilde{u}|^2)(t) \right)$$
$$- \left( (-\Delta)^{s/2} g(\tilde{v})(t+h) - (-\Delta)^{s/2} g(\tilde{v})(t) \right) = \beta J_1 - J_2,$$

with obvious notation. Then, from (2.15) and the embedding theorem

$$||J_1||_{L^2(\mathbb{R})}^2 \le |||\tilde{u}|^2(t+h) - |\tilde{u}|^2(t)||_{H^s(\mathbb{R})}^2 \le |||\tilde{u}|^2(t+h) - |\tilde{u}|^2(t)||_{H^1(\mathbb{R})}^2.$$

Analogously, we have

$$\begin{split} \|J_{2}\|_{L^{2}(\mathbb{R})}^{2} &\leq \|g(\tilde{v})(t+h) - g(\tilde{v})(t)\|_{H^{s}(\mathbb{R})}^{2} \leq \|g(\tilde{v})(t+h) - g(\tilde{v})(t)\|_{H^{1}(\mathbb{R})}^{2} \\ &= \|g(\tilde{v})(t+h) - g(\tilde{v})(t)\|_{L^{2}(\mathbb{R})}^{2} + \|\partial_{x}g(\tilde{v})(t+h) - \partial_{x}g(\tilde{v})(t)\|_{L^{2}(\mathbb{R})}^{2} \\ &\leq M^{2} \Big( \int_{\mathbb{R}} |\tilde{v}(t+h,x) - \tilde{v}(t,x)|^{2} dx + \int_{\mathbb{R}} |\partial_{x}\tilde{v}(t+h,x) - \partial_{x}\tilde{v}(t,x)|^{2} dx \Big) \\ &\leq 2M^{2} \|\tilde{v}(t+h) - \tilde{v}(t)\|_{H^{1}(\mathbb{R})}^{2}. \end{split}$$

Then, passing to the limit as  $h \to 0$ , the claim is proved.

Finally, since  $F, G \in C([0, T]; L^2(\mathbb{R}))$  applying Lemma 4.15 and Corollary 4.12 in [8], there exists a unique solution  $(u, v) \in C([0, T]; H^1(\mathbb{R})) \times C([0, T]; H^1(\mathbb{R}))$  given by

$$u(t) = U(t) u_0 - i \int_0^t U(t - t') \left( \alpha \widetilde{v}(t') \widetilde{u}(t') + |\widetilde{u}(t')|^2 \widetilde{u}(t') \right) dt',$$

$$v(t) = W(t) v_0 + \beta \int_0^t W(t - t') \left( \beta \left( -\Delta \right)^{s/2} (|\widetilde{u}(t')|^2) - (-\Delta)^{s/2} g(\widetilde{v})(t') \right) dt',$$
(3.37)

where  $U(t) \equiv U_{\varepsilon=1}(t)$ ,  $W(t) \equiv W_{\varepsilon=1}(t)$  are given respectively by (2.19), and (2.21).

**Proposition 3.3.** Let  $\frac{1}{2} < s < 1$ ,  $g \in C^1(\mathbb{R})$ ,  $0 < m \le g'(\cdot) \le M < \infty$ , (g(0) = 0). Then, for any  $(u_0^{\varepsilon}, v_0^{\varepsilon}) \in H^1(\mathbb{R}) \times H^1(\mathbb{R})$ , there exists T > 0 such that, the Cauchy problem (3.34) has a unique mild solution.

*Proof.* 1. Hereupon, we denote by  $X_T$  the Banach space  $C([0, T]; H^1(\mathbb{R}))$ , where T > 0 is chosen a posteriori. For  $R > 2 \max\{||u_0^{\varepsilon}||_{H^1(\mathbb{R})}, ||v_0^{\varepsilon}||_{H^1(\mathbb{R})}\}$ , we define

$$B_R^T := \{ f \in X_T : ||f||_{L^{\infty}(0,T;H^1(\mathbb{R}))} \le R \},$$

and the mapping  $\Phi: B_R^T \times B_R^T \to X_T \times X_T$ ,  $(\tilde{u}, \tilde{v}) \mapsto (u^{\varepsilon}, v^{\varepsilon}) \equiv \Phi(\tilde{u}, \tilde{v})$ , where  $(u^{\varepsilon}, v^{\varepsilon})$  is the unique mild solution of the Cauchy problem (3.36) (for each  $\varepsilon > 0$  fixed). Then, from (3.37) we have for any  $t \in [0, T]$ 

$$\begin{split} &\Phi_1(\tilde{u},\tilde{v}) \equiv u^\varepsilon(t) = U_\varepsilon(t) u_0^\varepsilon - i \int_0^t U_\varepsilon(t-t') \left(\alpha \ \tilde{v}(t') \ \tilde{u}(t') + \ |\tilde{u}(t')|^2 \ \tilde{u}(t')\right) dt', \\ &\Phi_2(\tilde{u},\tilde{v}) \equiv v^\varepsilon(t) = W_\varepsilon(t) v_0^\varepsilon + \int_0^t W_\varepsilon(t-t') \left(\beta \ (-\Delta)^{s/2} (|\tilde{u}(t')|^2) - (-\Delta)^{s/2} g(\tilde{v})(t')\right) dt'. \end{split}$$

2. First, we show that  $(\Phi_1(\tilde{u}, \tilde{v}), \Phi_2(\tilde{u}, \tilde{v})) \in B_R^T \times B_R^T$ . Indeed, since for each  $t \in [0, T]$ ,  $||U_{\varepsilon}(t)u_0^{\varepsilon}||_{H^1(\mathbb{R})} = ||u_0^{\varepsilon}||_{H^1(\mathbb{R})}$ , then

$$||U_{\varepsilon}(\cdot)u_0^{\varepsilon}||_{L^{\infty}(0,T;H^1(\mathbb{R}))} = ||u_0^{\varepsilon}||_{H^1(\mathbb{R})}.$$

Moreover, we have

$$\begin{split} \| \int_0^t U_{\varepsilon}(t-t') \left( \alpha \ \tilde{v}(t') \ \tilde{u}(t') + \ |\tilde{u}(t')|^2 \ \tilde{u}(t') \right) dt' \|_{H^1(\mathbb{R})} \\ & \leq \int_0^t \| (\alpha \ \tilde{v}(t') \ \tilde{u}(t') + \ |\tilde{u}(t')|^2 \ \tilde{u}(t')) \|_{H^1(\mathbb{R})} \ dt' \\ & \leq \int_0^t C \left( |\alpha| \ \|\tilde{v}(t')\|_{H^1(\mathbb{R})} \ \|\tilde{u}(t')\|_{H^1(\mathbb{R})} + \ \|\tilde{u}(t')|^2 \|_{H^1(\mathbb{R})} \ \|\tilde{u}(t')\|_{H^1(\mathbb{R})} \right) dt' \\ & = C \|\alpha\| \int_0^t \|\tilde{v}(t')\|_{H^1(\mathbb{R})} \|\tilde{u}(t')\|_{H^1(\mathbb{R})} \ dt' + C \int_0^t \|\tilde{u}(t')\|_{H^1(\mathbb{R})}^3 \ dt' \\ & \leq 2 \max\{|\alpha|, R\} \ C \ R^2 \ T, \end{split}$$

where we have used (2.12). Consequently, for T satisfying

$$T < \frac{1}{4\max\{|\alpha|, R\} C R},\tag{3.38}$$

$$||\Phi_1(\tilde{u},\tilde{v})||_{L^\infty(0,T;H^1(\mathbb{R}))} \leq ||u_0^\varepsilon||_{H^1(\mathbb{R})} + 2\max\{|\alpha|,R\} \ C \ R^2 \ T < \frac{R}{2} + \frac{R}{2} = R.$$

Similarly, we estimate  $\|\Phi_2(\tilde{u}, \tilde{v})\|_{L^{\infty}(0,T;H^1(\mathbb{R}))}$ . Applying (2.22), it follows that

$$\begin{split} \| \int_{0}^{t} W_{\varepsilon}(t-t') \left( \beta \left( -\Delta \right)^{s/2} (|\tilde{u}(t')|^{2}) - (-\Delta)^{s/2} g(\tilde{v})(t') \right) dt' \|_{H^{1}(\mathbb{R})} \\ & \leq \int_{0}^{t} \left( 1 + \frac{C}{(t-t')^{1/2}} \right) \| \beta \left( -\Delta \right)^{s/2} (|\tilde{u}(t')|^{2}) - (-\Delta)^{s/2} g(\tilde{v})(t') \|_{L^{2}(\mathbb{R})} dt' \\ & \leq \int_{0}^{t} |\beta| \left( 1 + \frac{C}{(t-t')^{1/2}} \right) \| (-\Delta)^{s/2} (|\tilde{u}(t')|^{2}) \|_{L^{2}(\mathbb{R})} dt' \\ & \leq \int_{0}^{t} \left( 1 + \frac{C}{(t-t')^{1/2}} \right) \| (-\Delta)^{s/2} g(\tilde{v})(t') \|_{L^{2}(\mathbb{R})} dt' = I_{1} + I_{2}, \end{split}$$

with obvious notation. To follow, we have

$$I_{1} \leq \frac{1}{m_{s}} \int_{0}^{t} |\beta| \left(1 + \frac{C}{(t - t')^{1/2}}\right) \|\tilde{u}(t')\|_{H^{s}(\mathbb{R})}^{2} dt'$$

$$\leq \frac{|\beta|}{m_{s}} \int_{0}^{t} \left(1 + \frac{C}{(t - t')^{1/2}}\right) \|\tilde{u}(t')\|_{H^{1}(\mathbb{R})}^{2} dt' < \frac{|\beta|}{m_{s}} R^{2} (T + 2C\sqrt{T}),$$

and

$$I_{2} \leq \frac{1}{m_{s}} \int_{0}^{t} \left(1 + \frac{C}{(t - t')^{1/2}}\right) \|g(\tilde{v})(t')\|_{H^{s}(\mathbb{R})} dt'$$

$$\leq \frac{1}{m_{s}} \int_{0}^{t} \left(1 + \frac{C}{(t - t')^{1/2}}\right) \left(\|g(\tilde{v})(t')\|_{L^{2}(\mathbb{R})}^{2} + \|\partial_{x}g(\tilde{v})(t')\|_{L^{2}(\mathbb{R})}^{2}\right)^{1/2} dt'$$

$$\leq \frac{M}{m_{s}} \int_{0}^{t} \left(1 + \frac{C}{(t - t')^{1/2}}\right) \|\tilde{v})(t')\|_{H^{1}(\mathbb{R})} dt' < \frac{M}{m_{s}} R \left(T + 2C\sqrt{T}\right).$$

Consequently, for T satisfying

$$T < \min\{\frac{m_s}{8 \max\{|\beta|R, M\}}, \frac{(m_s)^2}{64C^2(\max\{|\beta|R, M\})^2}\},\tag{3.39}$$

$$\|\Phi_2(\tilde{u},\tilde{v})\|_{L^\infty(0,T;H^1(\mathbb{R}))} \leq \|v_0^\varepsilon\|_{H^1(\mathbb{R})} + \frac{R}{m_s}(|\beta|R+M)(T+\sqrt{T}) < \frac{R}{2} + \frac{R}{2} = R.$$

3. Now, we show that  $\Phi$  is a contraction on  $B_R^T \times B_R^T$ . Let  $(\tilde{u}_i, \tilde{v}_i) \in B_R^T \times B_R^T$ , (i = 1, 2), then we have

$$\begin{split} \|\Phi_{1}(\tilde{u}_{1},\tilde{v}_{1}) - \Phi_{1}(\tilde{u}_{2},\tilde{v}_{2})\|_{H^{1}(\mathbb{R})} \\ &\leq |\alpha| \int_{0}^{t} \|U_{\varepsilon}(t-t') \left(\tilde{v}_{2}(t') \ \tilde{u}_{2}(t') - \tilde{v}_{1}(t') \ \tilde{u}_{1}(t')\right)\|_{H^{1}(\mathbb{R})} dt' \\ &+ \int_{0}^{t} \|U_{\varepsilon}(t-t') (|\tilde{u}_{2}(t')|^{2} \ \tilde{u}_{2}(t') - |\tilde{u}_{1}(t')|^{2} \ \tilde{u}_{1}(t'))\|_{H^{1}(\mathbb{R})} dt' \\ &\leq |\alpha| \int_{0}^{t} \|\tilde{v}_{2}(t') \ \tilde{u}_{2}(t') - \tilde{v}_{1}(t') \ \tilde{u}_{1}(t')\|_{H^{1}(\mathbb{R})} dt' \\ &+ \int_{0}^{t} \||\tilde{u}_{2}(t')|^{2} \ \tilde{u}_{2}(t') - |\tilde{u}_{1}(t')|^{2} \ \tilde{u}_{1}(t')\|_{H^{1}(\mathbb{R})} dt' = |\alpha| \ J_{1} + J_{2}. \end{split}$$

Applying (2.12) we obtain

$$\begin{aligned} |\alpha|J_{1} &\leq C |\alpha| \int_{0}^{t} \|\tilde{v}_{2}(t')\|_{H^{1}(\mathbb{R})} \|\tilde{u}_{2}(t') - \tilde{u}_{1}(t')\|_{H^{1}(\mathbb{R})} dt' \\ &+ C |\alpha| \int_{0}^{t} \|\tilde{u}_{1}(t')\|_{H^{1}(\mathbb{R})} \|\tilde{v}_{2}(t') - \tilde{v}_{1}(t')\|_{H^{1}(\mathbb{R})} dt' \\ &\leq C |\alpha| R T (\|\tilde{u}_{2} - \tilde{u}_{1}\|_{L^{\infty}(0,T;H^{1}(\mathbb{R}))} + \|\tilde{v}_{2} - \tilde{v}_{1}\|_{L^{\infty}(0,T;H^{1}(\mathbb{R}))}). \end{aligned}$$

$$(3.41)$$

Similarly, we also have

$$J_{2} \leq C \int_{0}^{t} (\|\tilde{u}_{2}(t')\|_{H^{1}(\mathbb{R})}^{2} \|\tilde{u}_{2}(t') - \tilde{u}_{1}(t')\|_{H^{1}(\mathbb{R})} + \|\tilde{u}_{1}(t')\|_{H^{1}(\mathbb{R})} \|\tilde{u}_{2}(t')\|^{2} - |\tilde{u}_{1}(t')|^{2}\|_{H^{1}(\mathbb{R})} dt'$$

$$\leq 3C R^{2} \int_{0}^{t} \|\tilde{u}_{2}(t') - \tilde{u}_{1}(t')\|_{H^{1}(\mathbb{R})} dt' \leq 3C R^{2} T \|\tilde{u}_{2} - \tilde{u}_{1}\|_{L^{\infty}(0,T;H^{1}(\mathbb{R}))}.$$

$$(3.42)$$

Therefore, from (3.40)–(3.42), it follows that

$$\begin{split} \|\Phi_{1}(\tilde{u}_{1}, \tilde{v}_{1}) - \Phi_{1}(\tilde{u}_{2}, \tilde{v}_{2})\|_{H^{1}(\mathbb{R})} \\ & \leq C \, R \, \max\{|\alpha|, 3R\} \, T \, \big( \|\tilde{u}_{1} - \tilde{u}_{2}\|_{L^{\infty}(0, T; H^{1}(\mathbb{R}))} + \|\tilde{v}_{1} - \tilde{v}_{2}\|_{L^{\infty}(0, T; H^{1}(\mathbb{R}))} \big). \end{split}$$

To this end, we have

$$\begin{split} \|\Phi_{2}(\tilde{u}_{1},\tilde{v}_{1}) - \Phi_{2}(\tilde{u}_{2},\tilde{v}_{2})\|_{H^{1}(\mathbb{R})} \\ &\leq \int_{0}^{t} |\beta| \, \|W_{\varepsilon}(t-t')((-\Delta)^{s/2}|\tilde{u}_{1}(t')|^{2} - (-\Delta)^{s/2}|\tilde{u}_{2}(t')|^{2})\|_{H^{1}(\mathbb{R})} \, dt' \\ &+ \int_{0}^{t} \|W_{\varepsilon}(t-t')((-\Delta)^{s/2}g(\tilde{v}_{2})(t') - (-\Delta)^{s/2}g(\tilde{v}_{1})(t'))\|_{H^{1}(\mathbb{R})} \, dt' \\ &\leq \int_{0}^{t} |\beta| \, \left(1 + \frac{C}{(t-t')^{1/2}}\right) \|(-\Delta)^{s/2}(|\tilde{u}_{1}(t')|^{2} - |\tilde{u}_{2}(t')|^{2})\|_{L^{2}(\mathbb{R})} \, dt' \\ &+ \int_{0}^{t} \left(1 + \frac{C}{(t-t')^{1/2}}\right) \|(-\Delta)^{s/2}(g(\tilde{v}_{2})(t') - g(\tilde{v}_{1})(t'))\|_{L^{2}(\mathbb{R})} \, dt' \\ &= K_{1} + K_{2}, \end{split}$$

$$(3.43)$$

where we have used (2.22), and obvious notation. Applying (2.15), we obtain

$$K_{1} \leq \frac{|\beta|}{m_{s}} \int_{0}^{t} \left(1 + \frac{C}{(t - t')^{1/2}}\right) \| |\tilde{u}_{1}(t')|^{2} - \tilde{u}_{2}(t')|^{2} \|_{H^{s}(\mathbb{R})} dt'$$

$$\leq \frac{|\beta|}{m_{s}} \int_{0}^{t} \left(1 + \frac{C}{(t - t')^{1/2}}\right) \left( \|\tilde{u}_{1}(t')\|_{H^{1}(\mathbb{R})} \|\tilde{u}_{1}(t') - \tilde{u}_{2}(t')\|_{H^{1}(\mathbb{R})} + \|\tilde{u}_{2}(t')\|_{H^{1}(\mathbb{R})} \|\tilde{u}_{1}(t') - \tilde{u}_{2}(t')\|_{H^{1}(\mathbb{R})} \right) dt'$$

$$\leq 2R \frac{|\beta|}{m_{s}} \left(T + C\sqrt{T}\right) \|\tilde{u}_{1} - \tilde{u}_{2}\|_{L^{\infty}(0,T;H^{1}(\mathbb{R}))}, \tag{3.44}$$

and

$$K_{2} \leq \frac{1}{m_{s}} \int_{0}^{t} \left(1 + \frac{C}{(t - t')^{1/2}}\right) \|g(\tilde{v}_{2})(t') - g(\tilde{v}_{1})(t')\|_{H^{s}(\mathbb{R})} dt'$$

$$\leq \frac{1}{m_{s}} \int_{0}^{t} \left(1 + \frac{C}{(t - t')^{1/2}}\right) \left(\|g(\tilde{v}_{2})(t') - g(\tilde{v}_{1})(t')\|_{L^{2}(\mathbb{R})}^{2} + \|\partial_{x}g(\tilde{v}_{2})(t') - \partial_{x}g(\tilde{v}_{1})(t')\|_{L^{2}(\mathbb{R})}^{2}\right)^{1/2} dt'$$

$$\leq \frac{M}{m_{s}} \left(T + C\sqrt{T}\right) \|\tilde{v}_{1} - \tilde{v}_{2}\|_{L^{\infty}(0,T;H^{1}(\mathbb{R}))}.$$

$$(3.45)$$

Consequently, from (3.43)–(3.45) we obtain

$$\|\Phi_2(\tilde{u}_1,\tilde{v}_1) - \Phi_2(\tilde{u}_2,\tilde{v}_2)\|_{H^1(\mathbb{R})}$$

$$\leq \frac{\max\{2R |\beta|, M\}}{m_s} (T + \sqrt{T}) (\|\tilde{u}_1 - \tilde{u}_2\|_{L^{\infty}(0,T;H^1(\mathbb{R}))} + \|\tilde{v}_1 - \tilde{v}_2\|_{L^{\infty}(0,T;H^1(\mathbb{R}))}).$$

4. Finally, from items (2) and (3) there exists a T > 0, sufficiently small, such that  $\Phi: B_R^T \times B_R^T \to B_R^T \times B_R^T$  is a (strict) contraction. Hence we can apply the Banach Fixed Point Theorem and obtain a unique (local in time) solution  $(u^{\varepsilon}, v^{\varepsilon})$  of the Cauchy problem (3.34).

### 3.2 A priori estimates

For each  $\varepsilon > 0$ , let  $(u^{\varepsilon}, v^{\varepsilon})$  be the unique solution for the Cauchy problem (3.34), and recall that, the sequences  $\{u_0^{\varepsilon}\}$  and  $\{v_0^{\varepsilon}\}$  are uniformly bounded in  $H^1(\mathbb{R})$  with respect to  $\varepsilon > 0$  fixed.

**Lemma 3.4** (First estimate). Let  $\frac{1}{2} < s < 1$ . Then, for each  $t \in (0, T)$ 

$$\frac{d}{dt} \int_{\mathbb{R}} |u^{\varepsilon}(t,x)|^{2} dx = 0, \tag{3.46}$$

$$\frac{d}{dt} \Big( \int_{\mathbb{R}} |(-\Delta)^{s/2} u^{\varepsilon}(t,x)|^{2} dx + \varepsilon^{a} \int_{\mathbb{R}} |\partial_{x} u^{\varepsilon}(t,x)|^{2} dx + \frac{1}{2} \int_{\mathbb{R}} |u^{\varepsilon}(t,x)|^{4} dx + \alpha \int_{\mathbb{R}} v^{\varepsilon}(t,x) |u^{\varepsilon}(t,x)|^{2} dx \Big) = \alpha \beta \int_{\mathbb{R}} (-\Delta)^{s/2} (|u^{\varepsilon}(t,x)|^{2}) |u^{\varepsilon}(t,x)|^{2} dx + \alpha \int_{\mathbb{R}} |u^{\varepsilon}(t,x)|^{2} dx + \alpha \int_{\mathbb{R}} |u^{\varepsilon}(t,x)|^{2} dx \Big) = \alpha \beta \int_{\mathbb{R}} (-\Delta)^{s/2} (|u^{\varepsilon}(t,x)|^{2}) |u^{\varepsilon}(t,x)|^{2} dx + \alpha \int_{\mathbb{R}} |u^{\varepsilon}(t,x)|^{2} (-\Delta)^{s/2} g_{\varepsilon}(v^{\varepsilon}(t,x)) dx - \alpha \varepsilon^{b} \int_{\mathbb{R}} \partial_{x} |u^{\varepsilon}(t,x)|^{2} \partial_{x} v^{\varepsilon}(t,x) dx, \tag{3.47}$$

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} |v^{\varepsilon}(t, x)|^{2} dx + \int_{\mathbb{R}} (-\Delta)^{s/2} g(v^{\varepsilon})(t, x) v^{\varepsilon}(t, x) dx 
+ \varepsilon^{b} \int_{\mathbb{R}} |\partial_{x} v^{\varepsilon}(t, x)|^{2} dx = \beta \int_{\mathbb{R}} (-\Delta)^{s/2} (|u^{\varepsilon}(t, x)|^{2}) v^{\varepsilon}(t, x) dx.$$
(3.48)

- *Proof.* 1. First, by approximating the initial data in  $H^1(\mathbb{R})$  by functions in  $C_c^{\infty}(\mathbb{R})$ , and a standard limit argument, we can assume that  $(u^{\varepsilon}, v^{\varepsilon})$  satisfies the Cauchy problem (3.34), (at least almost everywhere), and we are allowed to make the computations below. Indeed, since  $H^s(\mathbb{R})$  is an algebra for any s > 1/2, we may follow the same strategy developed in the previous section, and for 0 < T' < T, we obtain  $(u^{\varepsilon}, v^{\varepsilon}) \in (C([0, T']; H^k(\mathbb{R})) \cap C^1([0, T']; H^{k-2}(\mathbb{R})))^2$ , for each integer k > 2.
- 2. To follow, multiplying equation  $(3.34)_1$  by  $\overline{u^{\varepsilon}}(t,x)$  and integrating in  $\mathbb{R}$ , we have

$$\begin{split} i \int_{\mathbb{R}} \partial_t u^{\varepsilon}(t,x) \, \overline{u^{\varepsilon}}(t,x) \, dx &- \int_{\mathbb{R}} |(-\Delta)^{s/2} u^{\varepsilon}(t,x)|^2 \, dx - \varepsilon^a \int_{\mathbb{R}} |\partial_x u^{\varepsilon}(t,x)|^2 \, dx \\ &= \alpha \int_{\mathbb{R}} v^{\varepsilon}(t,x) \, |u^{\varepsilon}(t,x)|^2 \, dx + \int_{\mathbb{R}} |u^{\varepsilon}(t,x)|^4 \, dx. \end{split}$$

Therefore, taking the imaginary part of the above equation, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}|u^{\varepsilon}(t,x)|^2\ dx = \operatorname{Re}\int_{\mathbb{R}}\partial_t u^{\varepsilon}(t,x)\ \overline{u^{\varepsilon}}(t,x)\ dx = 0.$$

3. Now, let us multiply equation  $(3.34)_1$  by  $\partial_t \overline{u^{\varepsilon}}(t, x)$ , and integrate in  $\mathbb{R}$  to obtain

$$\begin{split} i \int_{\mathbb{R}} \partial_t u^{\varepsilon}(t,x) \, \partial_t \overline{u^{\varepsilon}}(t,x) \, dx &- \int_{\mathbb{R}} (-\Delta)^s u^{\varepsilon}(t,x) \, \partial_t \overline{u^{\varepsilon}}(t,x) \, dx \\ &+ \varepsilon^a \int_{\mathbb{R}} \Delta u^{\varepsilon}(t,x) \, \partial_t \overline{u^{\varepsilon}}(t,x) \, dx \\ &= \alpha \int_{\mathbb{R}} v^{\varepsilon}(t,x) \, u^{\varepsilon}(t,x) \, \partial_t \overline{u^{\varepsilon}}(t,x) \, dx + \int_{\mathbb{R}} |u^{\varepsilon}(t,x)|^2 \, u^{\varepsilon}(t,x) \, \partial_t \overline{u^{\varepsilon}}(t,x) \, dx. \end{split}$$

Then, writing  $u^{\varepsilon} = u_1^{\varepsilon} + iu_2^{\varepsilon}$  and integrating by parts, it follows that

$$\begin{split} i \int_{\mathbb{R}} |\partial_t u^\varepsilon(t,x)|^2 \, dx &- \int_{\mathbb{R}} (-\Delta)^{s/2} u^\varepsilon(t,x) \, \partial_t \overline{(-\Delta)^{s/2} u^\varepsilon}(t,x) \, dx \\ &- \varepsilon^a \int_{\mathbb{R}} \partial_x u^\varepsilon(t,x) \, \partial_t \overline{\partial_x u^\varepsilon}(t,x) \, dx \\ &= \alpha \int_{\mathbb{R}} v^\varepsilon(t,x) \, (u_1^\varepsilon(t,x) \partial_t u_1^\varepsilon(t,x) + u_2^\varepsilon(t,x) \partial_t u_2^\varepsilon(t,x)) dx \\ &+ i \, \alpha \int_{\mathbb{R}} v^\varepsilon(t,x) \, (u_2^\varepsilon(t,x) \partial_t u_1^\varepsilon(t,x) - u_1^\varepsilon(t,x) \partial_t u_2^\varepsilon(t,x)) \, dx \\ &+ \frac{1}{2} \int_{\mathbb{R}} (u^\varepsilon(t,x))^2 \, \partial_t (\overline{(u^\varepsilon)^2}(t,x)) \, dx. \end{split}$$

Taking the real part we have

$$\frac{d}{dt} \left[ \int_{\mathbb{R}} |(-\Delta)^{s/2} u^{\varepsilon}(t,x)|^{2} dx + \varepsilon^{a} \int_{\mathbb{R}} |\partial_{x} u^{\varepsilon}(t,x)|^{2} dx + \frac{1}{2} \int_{\mathbb{R}} |u^{\varepsilon}(t,x)|^{4} dx \right] 
+ \alpha \int_{\mathbb{R}} v^{\varepsilon}(t,x) |u^{\varepsilon}(t,x)|^{2} dx = \alpha \int_{\mathbb{R}} |u^{\varepsilon}(t,x)|^{2} \partial_{t} v^{\varepsilon}(t,x) dx.$$
(3.49)

The right-hand side of the above equation is computed by multiplying  $(3.34)_2$  by  $\alpha |u^{\varepsilon}(t,x)|^2$  and integrating in  $\mathbb{R}$ , that is to say

$$\alpha \int_{\mathbb{R}} |u^{\varepsilon}(t,x)|^{2} \, \partial_{t} v^{\varepsilon}(t,x) \, dx = \alpha \, \beta \int_{\mathbb{R}} (-\Delta)^{s/2} (|u^{\varepsilon}|^{2})(t,x) \, |u^{\varepsilon}(t,x)|^{2} \, dx$$

$$- \alpha \int_{\mathbb{R}} |u^{\varepsilon}(t,x)|^{2} \, (-\Delta)^{s/2} g_{\varepsilon}(v^{\varepsilon})(t,x) \, dx - \alpha \, \varepsilon^{b} \int_{\mathbb{R}} \partial_{x} |u^{\varepsilon}(t,x)|^{2} \, \partial_{x} v^{\varepsilon}(t,x) \, dx,$$

and replacing it in (3.49), we obtain

$$\begin{split} &\frac{d}{dt}\bigg[\int_{\mathbb{R}}|(-\Delta)^{s/2}u^{\varepsilon}(t,x)|^{2}~dx+\varepsilon^{a}\int_{\mathbb{R}}|\partial_{x}u^{\varepsilon}(t,x)|^{2}~dx+\frac{1}{2}\int_{\mathbb{R}}|u^{\varepsilon}(t,x)|^{4}~dx\\ &+\alpha\int_{\mathbb{R}}v^{\varepsilon}(t,x)~|u^{\varepsilon}(t,x)|^{2}~dx\bigg]=\alpha~\beta~\int_{\mathbb{R}}(-\Delta)^{s/2}(|u^{\varepsilon}|^{2})(t,x)~|u^{\varepsilon}(t,x)|^{2}~dx\\ &-\alpha\int_{\mathbb{R}}|u^{\varepsilon}(t,x)|^{2}~(-\Delta)^{s/2}g_{\varepsilon}(v^{\varepsilon})(t,x)~dx-\alpha~\varepsilon^{b}\int_{\mathbb{R}}\partial_{x}|u^{\varepsilon}|^{2}(t,x)~\partial_{x}v^{\varepsilon}(t,x)~dx. \end{split}$$

3. Finally, equation (3.48) follows directly by multiplying (3.34)<sub>2</sub> by  $v^{\varepsilon}(t, x)$  and integrating in  $\mathbb{R}$ . Indeed, we have

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}|v^{\varepsilon}(t,x)|^{2} dx + \int_{\mathbb{R}}(-\Delta)^{s/2}g_{\varepsilon}(v^{\varepsilon})(t,x) v^{\varepsilon}(t,x) dx + \varepsilon^{b}\int_{\mathbb{R}}|\partial_{x}v^{\varepsilon}(t,x)|^{2} dx 
= \beta \int_{\mathbb{R}}(-\Delta)^{s/2}(|u^{\varepsilon}|^{2})(t,x) v^{\varepsilon}(t,x) dx.$$

Now we pass to the second estimate.

**Theorem 3.5** (Second estimate). Let  $\frac{1}{2} < s < 1$ , and  $g \in C^1(\mathbb{R})$  satisfying  $0 < \varepsilon \le g'_{\varepsilon}(\cdot) \le M$ .

Then for any T > 0, there exist  $\alpha_0 > 0$  and  $E_0 > 0$ , such that, for each  $t \in (0, T)$ 

$$\int_{\mathbb{R}} |(-\Delta)^{s/2} u^{\varepsilon}(t,x)|^{2} dx + \varepsilon^{4} \int_{\mathbb{R}} |\partial_{x} u^{\varepsilon}(t,x)|^{2} dx + \frac{1}{4} \int_{\mathbb{R}} |u^{\varepsilon}(t,x)|^{4} dx \leq h(t), \quad (3.50)$$

$$\int_{\mathbb{R}} |v^{\varepsilon}(t,x)|^{2} dx \leq e^{T} ||v^{\varepsilon}_{0}||_{L^{2}(\mathbb{R})}^{2} + \frac{16|\beta|^{2} e^{T}}{\pi (2s-1)} ||u^{\varepsilon}_{0}||_{L^{2}(\mathbb{R})}^{2-\frac{1}{s}} \int_{0}^{t} h(\tau)^{1+\frac{1}{2s}} d\tau \equiv H(t),$$

$$C_{1,s}^{-1} \int_{0}^{t} ||\varepsilon^{1/2} (-\Delta)^{s/4} v^{\varepsilon}(\tau)||_{L^{2}(\mathbb{R})}^{2} d\tau + \int_{0}^{t} ||\varepsilon^{7/2} \nabla v^{\varepsilon}(\tau)||_{L^{2}(\mathbb{R})}^{2} d\tau$$

$$\leq \frac{||v^{\varepsilon}_{0}||_{L^{2}(\mathbb{R})}^{2}}{2} + \frac{8|\beta|^{2}}{\pi (2s-1)} ||u^{\varepsilon}_{0}||_{L^{2}(\mathbb{R})}^{2-\frac{1}{s}} \int_{0}^{t} h(\tau)^{2+\frac{1}{s}} d\tau + \frac{1}{2} \int_{0}^{t} H^{2}(\tau) d\tau,$$
(3.52)

for  $|\alpha| \le \alpha_0$  or  $||u_0||_{L^2(\mathbb{R})} \le E_0$ , where h is a continuous positive function (independent of  $\varepsilon$ ).

*Proof.* 1. First, from Proposition 2.3

$$\int_{\mathbb{R}} (-\Delta)^{s/2} g_{\varepsilon}(v^{\varepsilon})(t,x) \ v^{\varepsilon}(t,x) \ dx \ge \varepsilon \ C_{1,s}^{-1} \left\| (-\Delta)^{s/4} v^{\varepsilon}(t) \right\|_{L^{2}(\mathbb{R})}^{2}.$$

From the above inequality and equation (3.48), it follows that

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} |v^{\varepsilon}(t,x)|^{2} dx + \varepsilon C_{1,s}^{-1} \int_{\mathbb{R}} |(-\Delta)^{s/4} v^{\varepsilon}(t,x)|^{2} dx + \varepsilon^{b} \int_{\mathbb{R}} |\partial_{x} v^{\varepsilon}(t,x)|^{2} dx 
\leq \beta \int_{\mathbb{R}} (-\Delta)^{s/4} |u^{\varepsilon}(t,x)|^{2} (-\Delta)^{s/4} v^{\varepsilon}(t,x) dx 
\leq \frac{C_{1,s} \beta^{2}}{2\varepsilon} \int_{\mathbb{R}} |(-\Delta)^{s/4} |u^{\varepsilon}|^{2} (t,x)|^{2} dx + \frac{\varepsilon C_{1,s}^{-1}}{2} \int_{\mathbb{R}} |(-\Delta)^{s/4} v^{\varepsilon}(t,x)|^{2} dx,$$

where we have used Young's inequality. Then, integrating from 0 to t > 0,

$$\int_{\mathbb{R}} |v^{\varepsilon}(t,x)|^{2} dx + \varepsilon C_{1,s}^{-1} \int_{0}^{t} \|(-\Delta)^{s/4} v^{\varepsilon}(\tau)\|_{L^{2}(\mathbb{R})}^{2} d\tau + 2 \varepsilon^{b} \int_{0}^{t} \|\partial_{x} v^{\varepsilon}(\tau)\|_{L^{2}(\mathbb{R})}^{2} d\tau 
\leq \int_{\mathbb{R}} |v_{0}^{\varepsilon}(x)|^{2} dx + \frac{C_{1,s} \beta^{2}}{\varepsilon} \int_{0}^{t} \|(-\Delta)^{s/4} |u^{\varepsilon}|^{2} (\tau)\|_{L^{2}(\mathbb{R})}^{2} d\tau.$$
(3.53)

2. Now, applying Proposition 2.6 and equation (3.46), we have

$$\begin{split} \|(-\Delta)^{s/2}|u^{\varepsilon}|^{2}(t)\|_{L^{2}(\mathbb{R})} &\leq 2\|u^{\varepsilon}(t)\|_{L^{\infty}(\mathbb{R})} \|(-\Delta)^{s/2}u^{\varepsilon}(t)\|_{L^{2}(\mathbb{R})} \\ &\leq \frac{4}{\sqrt{\pi(2s-1)}} \|u_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})}^{1-\frac{1}{2s}} \|(-\Delta)^{s/2}u^{\varepsilon}(t)\|_{L^{2}(\mathbb{R})}^{1+\frac{1}{2s}}. \end{split}$$
(3.54)

Then, we obtain from (3.53) and (3.54)

$$\int_{\mathbb{R}} |v^{\varepsilon}(t,x)|^{2} dx + \varepsilon C_{1,s}^{-1} \int_{0}^{t} \|(-\Delta)^{s/4} v^{\varepsilon}(\tau)\|_{L^{2}(\mathbb{R})}^{2} d\tau + 2 \varepsilon^{b} \int_{0}^{t} \|\partial_{x} v^{\varepsilon}(\tau)\|_{L^{2}(\mathbb{R})}^{2} d\tau 
\leq \|v_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})}^{2} + \frac{16C_{1,s} \beta^{2}}{\varepsilon \pi (2s-1)} \|u_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})}^{2-\frac{1}{s}} \int_{0}^{t} \|(-\Delta)^{s/2} u^{\varepsilon}(\tau)\|_{L^{2}(\mathbb{R})}^{2+\frac{1}{s}} d\tau.$$
(3.55)

Similarly, we obtain

$$\int_{\mathbb{R}} |v^{\varepsilon}(t,x)|^{2} dx + 2\varepsilon C_{1,s}^{-1} \int_{0}^{t} \int_{\mathbb{R}} |(-\Delta)^{s/4} v^{\varepsilon}(t,x)|^{2} dx d\tau + 2\varepsilon^{b} \int_{0}^{t} \int_{\mathbb{R}} |\partial_{x} v^{\varepsilon}(t,x)|^{2} dx d\tau$$

$$\leq ||v_{0}^{\varepsilon}||_{L^{2}(\mathbb{R})}^{2} + |\beta|^{2} \int_{0}^{t} \int_{\mathbb{R}} |(-\Delta)^{s/2} |u^{\varepsilon}(t,x)|^{2} |dx + \int_{0}^{t} \int_{\mathbb{R}} |v^{\varepsilon}(t,x)|^{2} dx,$$

and applying Gronwall's Lemma

$$\int_{\mathbb{R}} |v^{\varepsilon}(t,x)|^2 dx \leq e^T ||v^{\varepsilon}_0||^2_{L^2(\mathbb{R})} + |\beta|^2 e^T \int_0^t ||(-\Delta)^{s/2}|u^{\varepsilon}(\tau)|^2 ||^2_{L^2(\mathbb{R})} d\tau.$$

Therefore, from the above inequality and (3.54), we have

$$\int_{\mathbb{R}} |v^{\varepsilon}(t,x)|^{2} dx \leq e^{T} ||v_{0}^{\varepsilon}||_{L^{2}(\mathbb{R})}^{2} + \frac{16|\beta|^{2} e^{T}}{\pi (2s-1)} ||u_{0}^{\varepsilon}||_{L^{2}(\mathbb{R})}^{2-\frac{1}{s}} \int_{0}^{t} ||(-\Delta)^{s/2} u^{\varepsilon}(\tau)||_{L^{2}(\mathbb{R})}^{2+\frac{1}{s}} d\tau, \tag{3.56}$$

and

$$\varepsilon C_{1,s}^{-1} \int_{0}^{t} \int_{\mathbb{R}} |(-\Delta)^{s/4} v^{\varepsilon}(t,x)|^{2} dx d\tau + \varepsilon^{b} \int_{0}^{t} \int_{\mathbb{R}} |\partial_{x} v^{\varepsilon}(t,x)|^{2} dx d\tau \leq \frac{\|v_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})}^{2}}{2} \\
+ \frac{8 |\beta|^{2}}{\pi (2s-1)} \|u_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})}^{2-\frac{1}{s}} \int_{0}^{t} \|(-\Delta)^{s/2} u^{\varepsilon}(\tau)\|_{L^{2}(\mathbb{R})}^{2+\frac{1}{s}} d\tau + \frac{1}{2} \int_{0}^{t} \|v^{\varepsilon}(\tau)\|_{L^{2}(\mathbb{R})}^{2} d\tau. \tag{3.57}$$

Now, from equations (3.47) it follows that

$$\frac{d}{dt} \left[ \int_{\mathbb{R}} |(-\Delta)^{s/2} u^{\varepsilon}(t,x)|^{2} dx + \varepsilon^{a} \int_{\mathbb{R}} |\partial_{x} u^{\varepsilon}(t,x)|^{2} dx \right] 
+ \frac{1}{2} \int_{\mathbb{R}} |u^{\varepsilon}(t,x)|^{4} dx + \alpha \int_{\mathbb{R}} v^{\varepsilon}(t,x) |u^{\varepsilon}(t,x)|^{2} dx \right] 
\leq |\alpha| \int_{\mathbb{R}} |(-\Delta)^{s/2} |u^{\varepsilon}|^{2}(t,x) g_{\varepsilon}(v^{\varepsilon})(t,x)| dx 
+ |\alpha| |\beta| \int_{\mathbb{R}} |(-\Delta)^{s/2} (|u^{\varepsilon}|^{2})(t,x)| |u^{\varepsilon}(t,x)|^{2} dx 
+ |\alpha| |\varepsilon^{b} \int_{\mathbb{R}} |\partial_{x} |u^{\varepsilon}(t,x)|^{2} |\partial_{x} v^{\varepsilon}(t,x)| dx =: |\alpha| E + |\alpha| |\beta| F + |\alpha| |\varepsilon^{b} G,$$
(3.58)

with obvious notation. Again, from Proposition 2.6 and equation (3.46), we may write:

$$(i) \ E \leq \|g_{\varepsilon}(v^{\varepsilon})(t)\|_{L^{2}(\mathbb{R})} \|(-\Delta)^{s/2}|u^{\varepsilon}|^{2}(t)\|_{L^{2}(\mathbb{R})}$$

$$\leq \|g'_{\varepsilon}\|_{L^{\infty}(\mathbb{R})} \|v^{\varepsilon}(t)\|_{L^{2}(\mathbb{R})} 2 \|u^{\varepsilon}(t)\|_{L^{\infty}(\mathbb{R})} \|(-\Delta)^{s/2}u^{\varepsilon}(t)\|_{L^{2}(\mathbb{R})}$$

$$\leq \frac{4}{\sqrt{\pi(2s-1)}} \|g'_{\varepsilon}\|_{L^{\infty}(\mathbb{R})} \|u^{\varepsilon}_{0}\|_{L^{2}(\mathbb{R})}^{1-\frac{1}{2s}} \|v^{\varepsilon}(t)\|_{L^{2}(\mathbb{R})} \|(-\Delta)^{s/2}u^{\varepsilon}(t)\|_{L^{2}(\mathbb{R})}^{1+\frac{1}{2s}}.$$

$$(ii) \ F \leq \|u^{\varepsilon}(t)\|_{L^{\infty}(\mathbb{R})} \int_{\mathbb{R}} |u^{\varepsilon}(t,x)| |(-\Delta)^{s/2}(|u^{\varepsilon}|^{2})(t,x)| dx$$

$$\leq \|u^{\varepsilon}(t)\|_{L^{\infty}(\mathbb{R})} \|u^{\varepsilon}(t)\|_{L^{2}(\mathbb{R})} \|(-\Delta)^{s/2}|u^{\varepsilon}|^{2}(t)\|_{L^{2}(\mathbb{R})}$$

$$\leq \frac{4}{\sqrt{\pi(2s-1)}} \|u^{\varepsilon}_{0}\|_{L^{2}(\mathbb{R})}^{2-\frac{1}{2s}} \|(-\Delta)^{s/2}u^{\varepsilon}(t)\|_{L^{2}(\mathbb{R})}^{\frac{1}{2s}} \|u^{\varepsilon}(t)\|_{L^{\infty}(\mathbb{R})} \|(-\Delta)^{s/2}u^{\varepsilon}(t)\|_{L^{2}(\mathbb{R})}^{2}$$

$$\leq \frac{8}{\pi(2s-1)} \|u^{\varepsilon}_{0}\|_{L^{2}(\mathbb{R})}^{3-\frac{1}{s}} \|(-\Delta)^{s/2}u^{\varepsilon}(t)\|_{L^{2}(\mathbb{R})}^{1+\frac{1}{s}}.$$

$$(iii) G \leq \int_{\mathbb{R}} \left( |\partial_{x} u^{\varepsilon}(t, x)| |\overline{u^{\varepsilon}}(t, x)| + |u^{\varepsilon}(t, x)| |\partial_{x} \overline{u^{\varepsilon}}(t, x)| \right) |\partial_{x} v^{\varepsilon}(t, x)| dx$$

$$\leq 2||u^{\varepsilon}(t)||_{L^{\infty}(\mathbb{R})} \int_{\mathbb{R}} |\partial_{x} u^{\varepsilon}(t, x)| |\partial_{x} v^{\varepsilon}(t, x)| dx$$

$$\leq 2||u^{\varepsilon}(t)||_{L^{\infty}(\mathbb{R})} ||\partial_{x} v^{\varepsilon}(t)||_{L^{2}(\mathbb{R})} ||\partial_{x} u^{\varepsilon}(t)||_{L^{2}(\mathbb{R})}$$

$$\leq \frac{4}{\sqrt{\pi}} ||u^{\varepsilon}(t)||_{L^{2}(\mathbb{R})}^{1/2} ||\partial_{x} u^{\varepsilon}(t)||_{L^{2}(\mathbb{R})}^{1/2} ||\partial_{x} v^{\varepsilon}(t)||_{L^{2}(\mathbb{R})} ||\partial_{x} u^{\varepsilon}(t)||_{L^{2}(\mathbb{R})}$$

$$\leq \frac{4}{\sqrt{\pi}} ||u^{\varepsilon}(t)||_{L^{2}(\mathbb{R})}^{1/2} ||\partial_{x} v^{\varepsilon}(t)||_{L^{2}(\mathbb{R})} ||\partial_{x} u^{\varepsilon}(t)||_{L^{2}(\mathbb{R})}^{3/2}.$$

Replacing in equation (3.58)

$$\begin{split} &\frac{d}{dt}\Big(\int_{\mathbb{R}}|(-\Delta)^{s/2}u^{\varepsilon}(t,x)|^{2}~dx+\varepsilon^{a}\int_{\mathbb{R}}|\partial_{x}u^{\varepsilon}(t,x)|^{2}~dx\\ &+\frac{1}{2}\int_{\mathbb{R}}|u^{\varepsilon}(t,x)|^{4}~dx+\alpha\int_{\mathbb{R}}v^{\varepsilon}(t,x)~|u^{\varepsilon}(t,x)|^{2}~dx\Big)\\ &\leq |\alpha|\frac{4}{\sqrt{\pi(2s-1)}}~\|g_{\varepsilon}'\|_{L^{\infty}(\mathbb{R})}~\|u_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})}^{1-\frac{1}{2s}}~\|v^{\varepsilon}(t)\|_{L^{2}(\mathbb{R})}\|(-\Delta)^{s/2}u^{\varepsilon}(t)\|_{L^{2}(\mathbb{R})}^{1+\frac{1}{2s}}\\ &+|\alpha|~|\beta|\frac{8}{\pi(2s-1)}~\|u_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})}^{3-\frac{1}{s}}~\|(-\Delta)^{s/2}u^{\varepsilon}(t)\|_{L^{2}(\mathbb{R})}^{1+\frac{1}{s}}\\ &+\frac{4}{\sqrt{\pi}}|\alpha|~\varepsilon^{b}~\|u_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})}^{1/2}~\|\partial_{x}v^{\varepsilon}(t)\|_{L^{2}(\mathbb{R})}~\|\partial_{x}u^{\varepsilon}(t)\|_{L^{2}(\mathbb{R})}^{3/2}, \end{split}$$

and integrating from 0 to t > 0

$$\begin{split} &\int_{\mathbb{R}} |(-\Delta)^{s/2} u^{\varepsilon}(t,x)|^{2} \ dx + \varepsilon^{a} \int_{\mathbb{R}} |\partial_{x} u^{\varepsilon}(t,x)|^{2} \ dx + \frac{1}{2} \int_{\mathbb{R}} |u^{\varepsilon}(t,x)|^{4} \ dx \\ &\leq \left\| (-\Delta)^{s/2} u_{0}^{\varepsilon} \right\|_{L^{2}(\mathbb{R})}^{2} + \varepsilon^{a} \left\| \partial_{x} u_{0}^{\varepsilon} \right\|_{L^{2}(\mathbb{R})}^{2} + \frac{1}{2} \left\| u_{0}^{\varepsilon} \right\|_{L^{4}(\mathbb{R})}^{4} + \int_{\mathbb{R}} |v_{0}^{\varepsilon}(x)| \ |u_{0}^{\varepsilon}(x)|^{2} \ dx \\ &+ \frac{4 \ |\alpha|}{\sqrt{\pi (2s-1)}} \left\| g_{\varepsilon}^{\prime} \right\|_{L^{\infty}(\mathbb{R})} \left\| u_{0}^{\varepsilon} \right\|_{L^{2}(\mathbb{R})}^{1-\frac{1}{2s}} \int_{0}^{t} \left\| v^{\varepsilon}(\tau) \right\|_{L^{2}(\mathbb{R})} \left\| (-\Delta)^{s/2} u^{\varepsilon}(\tau) \right\|_{L^{2}(\mathbb{R})}^{1+\frac{1}{2s}} \ d\tau \\ &+ \frac{8 \ |\alpha| \ |\beta|}{\pi (2s-1)} \left\| u_{0}^{\varepsilon} \right\|_{L^{2}(\mathbb{R})}^{3-\frac{1}{s}} \int_{0}^{t} \left\| (-\Delta)^{s/2} u^{\varepsilon}(\tau) \right\|_{L^{2}(\mathbb{R})}^{1+\frac{1}{s}} \ d\tau \\ &+ \frac{4}{\sqrt{\pi}} |\alpha| \ \varepsilon^{b} \left\| u_{0}^{\varepsilon} \right\|_{L^{2}(\mathbb{R})}^{1/2} \int_{0}^{t} \left\| \partial_{x} v^{\varepsilon}(\tau) \right\|_{L^{2}(\mathbb{R})} \left\| \partial_{x} u^{\varepsilon}(\tau) \right\|_{L^{2}(\mathbb{R})}^{3/2} \ d\tau \\ &+ |\alpha| \int_{\mathbb{R}} |v^{\varepsilon}(t,x)| \ |u^{\varepsilon}(t,x)|^{2} \ dx. \end{split}$$

Then, we have

$$\begin{split} &\int_{\mathbb{R}} |(-\Delta)^{s/2} u^{\varepsilon}(t,x)|^{2} \ dx + \varepsilon^{a} \int_{\mathbb{R}} |\partial_{x} u^{\varepsilon}(t,x)|^{2} \ dx + \frac{1}{2} \int_{\mathbb{R}} |u^{\varepsilon}(t,x)|^{4} \ dx \\ &\leq \|(-\Delta)^{s/2} u_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})}^{2} + \varepsilon^{a} \|\partial_{x} u_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})}^{2} + \frac{1}{2} \|u_{0}^{\varepsilon}\|_{L^{4}(\mathbb{R})}^{4} + \|u_{0}^{\varepsilon}\|_{L^{\infty}(\mathbb{R})} \int_{\mathbb{R}} |v_{0}^{\varepsilon}(x)| \ |u_{0}^{\varepsilon}(x)| \ dx \\ &+ \frac{4 \ |\alpha|}{\sqrt{\pi(2s-1)}} \|g_{\varepsilon}'\|_{L^{\infty}(\mathbb{R})} \|u_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})}^{1-\frac{1}{2s}} \int_{0}^{t} \|v^{\varepsilon}(\tau)\|_{L^{2}(\mathbb{R})} \|(-\Delta)^{s/2} u^{\varepsilon}(\tau)\|_{L^{2}(\mathbb{R})}^{1+\frac{1}{2s}} \ d\tau \\ &+ \frac{8 \ |\alpha| \ |\beta|}{\pi(2s-1)} \|u_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})}^{3-\frac{1}{s}} \int_{0}^{t} \|(-\Delta)^{s/2} u^{\varepsilon}(\tau)\|_{L^{2}(\mathbb{R})}^{1+\frac{1}{s}} \ d\tau \\ &+ \frac{4}{\sqrt{\pi}} |\alpha| \ \varepsilon^{b} \ \|u_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})}^{1/2} \int_{0}^{t} \|\partial_{x} v^{\varepsilon}(\tau)\|_{L^{2}(\mathbb{R})} \|\partial_{x} u^{\varepsilon}(\tau)\|_{L^{2}(\mathbb{R})}^{3/2} \ d\tau \\ &+ \int_{\mathbb{R}} \left(\sqrt{2} \ |\alpha| \ |v^{\varepsilon}(t,x)|\right) \left(\frac{|u^{\varepsilon}(t,x)|^{2}}{\sqrt{2}}\right) dx, \end{split}$$

from which follows that

$$\int_{\mathbb{R}} |(-\Delta)^{s/2} u^{\varepsilon}(t,x)|^{2} dx + \varepsilon^{a} \int_{\mathbb{R}} |\partial_{x} u^{\varepsilon}(t,x)|^{2} dx + \frac{1}{4} \int_{\mathbb{R}} |u^{\varepsilon}(t,x)|^{4} dx 
\leq \|(-\Delta)^{s/2} u_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})}^{2} + \varepsilon^{a} \|\partial_{x} u_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})}^{2} + \frac{1}{2} \|u_{0}^{\varepsilon}\|_{L^{4}(\mathbb{R})}^{4} + \|u_{0}^{\varepsilon}\|_{L^{\infty}(\mathbb{R})} \|v_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})} \|u_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})}^{2} 
+ \frac{4 |\alpha|}{\sqrt{\pi}(2s-1)} \|g_{\varepsilon}'\|_{L^{\infty}(\mathbb{R})} \|u_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})}^{1-\frac{1}{2s}} \int_{0}^{t} \|v^{\varepsilon}(\tau)\|_{L^{2}(\mathbb{R})} \|(-\Delta)^{s/2} u^{\varepsilon}(\tau)\|_{L^{2}(\mathbb{R})}^{1+\frac{1}{2s}} d\tau 
+ \frac{8 |\alpha| |\beta|}{\pi(2s-1)} \|u_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})}^{3-\frac{1}{s}} \int_{0}^{t} \|(-\Delta)^{s/2} u^{\varepsilon}(\tau)\|_{L^{2}(\mathbb{R})}^{1+\frac{1}{s}} d\tau 
+ \frac{4}{\sqrt{\pi}} |\alpha| \varepsilon^{b} \|u_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})}^{1/2} \int_{0}^{t} \|\partial_{x} v^{\varepsilon}(\tau)\|_{L^{2}(\mathbb{R})} \|\partial_{x} u^{\varepsilon}(\tau)\|_{L^{2}(\mathbb{R})}^{3/2} d\tau + |\alpha|^{2} \int_{\mathbb{R}} |v^{\varepsilon}(t,x)|^{2} dx.$$
(3.59)

3. Now, replacing (3.56) in (3.59), we have

$$\begin{split} &\int_{\mathbb{R}} |(-\Delta)^{s/2} u^{\varepsilon}(t,x)|^{2} \ dx + \varepsilon^{a} \int_{\mathbb{R}} |\partial_{x} u^{\varepsilon}(t,x)|^{2} \ dx + \frac{1}{4} \int_{\mathbb{R}} |u^{\varepsilon}(t,x)|^{4} \ dx \\ &\leq \|(-\Delta)^{s/2} u_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})}^{2} + \varepsilon^{a} \|\partial_{x} u_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})}^{2} + \frac{1}{2} \|u_{0}^{\varepsilon}\|_{L^{4}(\mathbb{R})}^{4} + \|u_{0}^{\varepsilon}\|_{L^{\infty}(\mathbb{R})} \|v_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})} \|u_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})}^{2} \\ &+ \frac{4 \|\alpha\|}{\sqrt{\pi(2s-1)}} \|g_{\varepsilon}'\|_{L^{\infty}(\mathbb{R})} \|u_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})}^{1-\frac{1}{2s}} \int_{0}^{t} \|v^{\varepsilon}(\tau)\|_{L^{2}(\mathbb{R})} \|(-\Delta)^{s/2} u^{\varepsilon}(\tau)\|_{L^{2}(\mathbb{R})}^{1+\frac{1}{2s}} d\tau \\ &+ \frac{8 \|\alpha\| \|\beta\|}{\pi(2s-1)} \|u_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})}^{3-\frac{1}{s}} \int_{0}^{t} \|(-\Delta)^{s/2} u^{\varepsilon}(\tau)\|_{L^{2}(\mathbb{R})}^{1+\frac{1}{s}} d\tau \\ &+ \frac{4}{\sqrt{\pi}} |\alpha| \|\varepsilon^{b}\| \|u_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})}^{1/2} \int_{0}^{t} \|\partial_{x} v^{\varepsilon}(\tau)\|_{L^{2}(\mathbb{R})} \|\partial_{x} u^{\varepsilon}(\tau)\|_{L^{2}(\mathbb{R})}^{3/2} d\tau \\ &+ |\alpha|^{2} e^{T} \|v_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})}^{2} + \frac{16 |\alpha|^{2} \beta^{2} e^{T}}{\pi(2s-1)} \|u_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})}^{2-\frac{1}{s}} \int_{0}^{t} \|(-\Delta)^{s/2} u^{\varepsilon}(\tau)\|_{L^{2}(\mathbb{R})}^{2+\frac{1}{s}} d\tau \end{split}$$

or conveniently we write

$$\begin{split} 1 + \int_{\mathbb{R}} |(-\Delta)^{s/2} u^{\varepsilon}(t,x)|^{2} \, dx + \varepsilon^{a} \int_{\mathbb{R}} |\partial_{x} u^{\varepsilon}(t,x)|^{2} \, dx + \frac{1}{4} \int_{\mathbb{R}} |u^{\varepsilon}(t,x)|^{4} \, dx &\leq \theta(t) := 1 \\ + \|(-\Delta)^{s/2} u_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})}^{2} + \varepsilon^{a} \|\partial_{x} u_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})}^{2} + \frac{1}{2} \|u_{0}^{\varepsilon}\|_{L^{4}(\mathbb{R})}^{4} \\ + \|u_{0}^{\varepsilon}\|_{L^{\infty}(\mathbb{R})} \|v_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})} \|u_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})}^{2} + |\alpha|^{2} e^{T} \|v_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})}^{2} \\ + \frac{4 |\alpha|}{\sqrt{\pi(2s-1)}} \|g_{\varepsilon}'\|_{L^{\infty}(\mathbb{R})} \|u_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})}^{1-\frac{1}{2s}} \int_{0}^{t} \|v^{\varepsilon}(\tau)\|_{L^{2}(\mathbb{R})} \|(-\Delta)^{s/2} u^{\varepsilon}(\tau)\|_{L^{2}(\mathbb{R})}^{1+\frac{1}{2s}} \, d\tau \\ + \frac{8 |\alpha| |\beta|}{\pi(2s-1)} \|u_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})}^{3-\frac{1}{s}} \int_{0}^{t} \|(-\Delta)^{s/2} u^{\varepsilon}(\tau)\|_{L^{2}(\mathbb{R})}^{1+\frac{1}{s}} \, d\tau \\ + \frac{4}{\sqrt{\pi}} |\alpha| \varepsilon^{b} \|u_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})}^{1/2} \int_{0}^{t} \|\partial_{x} v^{\varepsilon}(\tau)\|_{L^{2}(\mathbb{R})} \|\partial_{x} u^{\varepsilon}(\tau)\|_{L^{2}(\mathbb{R})}^{3/2} \, d\tau \\ + \frac{16 |\alpha|^{2} \beta^{2} e^{T}}{\pi(2s-1)} \|u_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})}^{2-\frac{1}{s}} \int_{0}^{t} \|(-\Delta)^{s/2} u^{\varepsilon}(\tau)\|_{L^{2}(\mathbb{R})}^{2+\frac{1}{s}} \, d\tau. \end{split}$$

$$(3.60)$$

From the above definition, we have

$$\begin{split} \theta'(t) & \leq \frac{4 \, |\alpha|}{\sqrt{\pi (2s-1)}} \, \|g_{\varepsilon}'\|_{L^{\infty}(\mathbb{R})} \, \|u_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})}^{1-\frac{1}{2s}} \, \|v^{\varepsilon}(t) \, \|_{L^{2}(\mathbb{R})} \, \theta(t)^{\frac{1}{2}+\frac{1}{4s}} \\ & + \frac{8 \, |\alpha| \, |\beta|}{\pi (2s-1)} \, \|u_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})}^{3-\frac{1}{s}} \, \theta(t)^{\frac{1}{2}+\frac{1}{2s}} \\ & + \frac{4 |\alpha| \, \varepsilon^{b}}{\sqrt{\pi} \varepsilon^{3a/4}} \, \|u_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})}^{1/2} \, \|\partial_{x} v^{\varepsilon}(t)\|_{L^{2}(\mathbb{R})} \, \theta(t)^{3/4} + \frac{16 |\alpha|^{2} \beta^{2} e^{T}}{\pi (2s-1)} \|u_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})}^{2-\frac{1}{s}} \, \theta(t)^{1+\frac{1}{2s}}, \end{split}$$

where we have used (3.60). Since 1/2 < s < 1, then

$$\frac{3}{4} < \frac{1}{2} + \frac{1}{4s} < 1, \quad 1 < \frac{1}{2} + \frac{1}{2s} < \frac{3}{2},$$

and consequently dividing the above inequality by  $\theta(t)^{\frac{1}{2} + \frac{1}{4s}}$ , we obtain

$$\frac{1}{\frac{1}{2} - \frac{1}{4s}} \left[ \theta(t)^{\frac{1}{2} - \frac{1}{4s}} \right]' e^{t} \leq \frac{4 |\alpha|}{\sqrt{\pi (2s - 1)}} \|g_{\varepsilon}'\|_{L^{\infty}(\mathbb{R})} \|u_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})}^{1 - \frac{1}{2s}} \|v^{\varepsilon}(t)\|_{L^{2}(\mathbb{R})} e^{t} 
+ \frac{8 |\alpha| |\beta|}{\pi (2s - 1)} \|u_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})}^{3 - \frac{1}{s}} \theta(t)^{\frac{1}{4s}} e^{t} 
+ \frac{4}{\sqrt{\pi}} |\alpha| \varepsilon^{b} \varepsilon^{-3a/4} \|u_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})}^{1/2} \|\partial_{x} v^{\varepsilon}(t)\|_{L^{2}(\mathbb{R})} e^{t} + \frac{16 |\alpha|^{2} \beta^{2} e^{T}}{\pi (2s - 1)} \|u_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})}^{2 - \frac{1}{s}} \theta(t)^{\frac{1}{2} + \frac{1}{4s}} e^{t},$$

where we have multiplied the inequality by  $e^t$ . Then, integrating from 0 to t > 0

$$\begin{split} & \int_{0}^{t} \left[ \theta(\tau)^{\frac{1}{2} - \frac{1}{4s}} \right]' e^{\tau} d\tau \leq \frac{|\alpha|(4s - 2)}{2s \sqrt{\pi(2s - 1)}} \left\| g_{\varepsilon}' \right\|_{L^{\infty}(\mathbb{R})} \left\| u_{0}^{\varepsilon} \right\|_{L^{2}(\mathbb{R})}^{1 - \frac{1}{2s}} \int_{0}^{t} \left\| v^{\varepsilon}(\tau) \right\|_{L^{2}(\mathbb{R})} e^{\tau} d\tau \\ & + \frac{2 |\alpha| |\beta|}{s\pi} \left\| u_{0}^{\varepsilon} \right\|_{L^{2}(\mathbb{R})}^{3 - \frac{1}{s}} \int_{0}^{t} \theta(\tau)^{\frac{1}{4s}} e^{\tau} d\tau + \frac{4 |\alpha|^{2} \beta^{2} e^{T}}{s\pi} \left\| u_{0}^{\varepsilon} \right\|_{L^{2}(\mathbb{R})}^{2 - \frac{1}{s}} \int_{0}^{t} \theta(\tau)^{\frac{1}{2} + \frac{1}{4s}} e^{\tau} d\tau \\ & + \frac{|\alpha| \varepsilon^{b} \varepsilon^{-3a/4} (2s - 1)}{\sqrt{\pi} s} \left\| u_{0}^{\varepsilon} \right\|_{L^{2}(\mathbb{R})}^{1/2} \int_{0}^{t} \left\| \partial_{x} v^{\varepsilon}(\tau) \right\|_{L^{2}(\mathbb{R})} e^{\tau} d\tau \end{split}$$

and integrating by parts in the left hand side

$$\begin{split} &\theta(t)^{\frac{1}{2} - \frac{1}{4s}} e^{t} \leq \theta(0)^{\frac{1}{2} - \frac{1}{4s}} + \int_{0}^{t} \theta(\tau)^{\frac{1}{2} - \frac{1}{4s}} e^{\tau} d\tau \\ &+ \frac{|\alpha| (2s - 1)}{s \sqrt{2\pi (2s - 1)}} \|g_{\varepsilon}'\|_{L^{\infty}(\mathbb{R})} \|u_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})}^{1 - \frac{1}{2s}} (e^{2t} - 1)^{1/2} \\ &\times (e^{T} \|v_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})}^{2} + \frac{16\beta^{2}e^{T}}{\pi (2s - 1)} \|u_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})}^{2 - \frac{1}{s}} \int_{0}^{t} \|(-\Delta)^{s/2} u^{\varepsilon}(\tau)\|_{L^{2}(\mathbb{R})}^{2 + \frac{1}{s}} d\tau)^{1/2} \\ &+ \frac{2|\alpha| |\beta|}{s\pi} \|u_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})}^{3 - \frac{1}{s}} \int_{0}^{t} \theta(\tau)^{\frac{1}{4s}} e^{\tau} d\tau \\ &+ \frac{|\alpha| \varepsilon^{b} \varepsilon^{-3a/4} (2s - 1)}{\sqrt{2\pi} s} \|u_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})}^{1/2} (e^{2t} - 1)^{1/2} \\ &\times \left(\frac{\|v_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})}^{2}}{2\varepsilon^{b}} + \frac{8C_{1,s} \beta^{2}}{\varepsilon^{b+1}\pi (2s - 1)} \|u_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})}^{2 - \frac{1}{s}} \int_{0}^{t} \|(-\Delta)^{s/2} u^{\varepsilon}(\tau)\|_{L^{2}(\mathbb{R})}^{2 + \frac{1}{s}} d\tau\right)^{1/2} \\ &+ \frac{4|\alpha|^{2}\beta^{2}e^{T}}{s\pi} \|u_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})}^{2 - \frac{1}{s}} \int_{0}^{t} \theta(\tau)^{\frac{1}{2} + \frac{1}{4s}} e^{\tau} d\tau, \end{split}$$

where we have used Holder's inequality and equations (3.52)-(3.56).

4. The goal now is to apply the Generalized Gronwall Lemma (Section 2.4). We observe that

$$\theta(0) = \left[1 + \|(-\Delta)^{s/2} u_0^{\varepsilon}\|_{L^2(\mathbb{R})}^2 + \varepsilon^a \|\partial_x u_0^{\varepsilon}\|_{L^2(\mathbb{R})}^2 + \frac{1}{2} \|u_0^{\varepsilon}\|_{L^4(\mathbb{R})}^4 + \|u_0^{\varepsilon}\|_{L^\infty(\mathbb{R})} \|v_0^{\varepsilon}\|_{L^2(\mathbb{R})} \|u_0^{\varepsilon}\|_{L^2(\mathbb{R})} + |\alpha|^2 e^T \|v_0^{\varepsilon}\|_{L^2(\mathbb{R})}^2\right],$$

hence from that and taking the square in equation (3.61), we have

$$\begin{split} &\theta(t)^{1-\frac{1}{2s}} \ e^{2t} \leq 2^{6} \bigg[ 1 + \|(-\Delta)^{s/2} u_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})}^{2} + \varepsilon^{a} \ \|\partial_{x} u_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})}^{2} + \frac{1}{2} \|u_{0}^{\varepsilon}\|_{L^{4}(\mathbb{R})}^{4} \\ &+ \|u_{0}^{\varepsilon}\|_{L^{\infty}(\mathbb{R})} \|v_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})} \|u_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})} + |\alpha|^{2} e^{T} \|v_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})}^{2} \bigg]^{1-\frac{1}{2s}} \\ &+ 2^{6} t^{2} \bigg( \int_{0}^{t} \theta(\tau)^{\frac{1}{2}-\frac{1}{4s}} \ e^{\tau} \ d\tau \bigg)^{2} + \frac{2^{5} |\alpha|^{2} \ (2s-1)}{s^{2} \pi} \ \|g_{\varepsilon}'\|_{L^{\infty}(\mathbb{R})}^{2} \|u_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})}^{2-\frac{1}{s}} \ e^{2t} \\ &\times \bigg( e^{T} \|v_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})}^{2} + \frac{16\beta^{2} e^{T}}{\pi (2s-1)} \|u_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})}^{2-\frac{1}{s}} \int_{0}^{t} \|(-\Delta)^{s/2} u^{\varepsilon}(\tau)\|_{L^{2}(\mathbb{R})}^{2+\frac{1}{s}} \ d\tau \bigg) \\ &+ \frac{2^{8} |\alpha|^{2} \|\beta|^{2} t^{2}}{s^{2} \pi^{2}} \|u_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})}^{6-\frac{2}{s}} \bigg( \int_{0}^{t} \theta(\tau)^{\frac{1}{4s}} e^{\tau} \ d\tau \bigg)^{2} \\ &+ \frac{2^{5} |\alpha|^{2} \varepsilon^{2b} \varepsilon^{-3a/2} \ (2s-1)^{2}}{\pi s^{2}} \|u_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})}^{2-\frac{1}{s}} \int_{0}^{t} \|(-\Delta)^{s/2} u^{\varepsilon}(\tau)\|_{L^{2}(\mathbb{R})}^{2+\frac{1}{s}} \ d\tau \bigg) \\ &+ \frac{2^{10} |\alpha|^{4} |\beta|^{4} e^{2T}}{s^{2} \pi^{2}} \|u_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})}^{4-\frac{2}{s}} t^{2} \bigg( \int_{0}^{t} \theta(\tau)^{\frac{1}{2}+\frac{1}{4s}} e^{\tau} \ d\tau \bigg)^{2}. \end{split}$$

Then, we apply Jesen's inequality to obtain

$$\begin{split} &\theta(t)^{1-\frac{1}{2s}} \ e^{2t} \leq 2^{6} \bigg[ 1 + \|(-\Delta)^{s/2} u_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})}^{2} + \varepsilon^{a} \ \|\partial_{x} u_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})}^{2} + \frac{1}{2} \|u_{0}^{\varepsilon}\|_{L^{4}(\mathbb{R})}^{4} \\ &+ \|u_{0}^{\varepsilon}\|_{L^{\infty}(\mathbb{R})} \|v_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})} \|u_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})} + |\alpha|^{2} e^{T} \|v_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})}^{2} \bigg]^{1-\frac{1}{2s}} \\ &+ 2^{6} T \int_{0}^{t} \theta(\tau)^{1-\frac{1}{2s}} \ e^{2\tau} \ d\tau + \frac{2^{5} |\alpha|^{2} \ (2s-1)}{s^{2} \pi} \ \|g_{\varepsilon}'\|_{L^{\infty}(\mathbb{R})}^{2} \ \|u_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})}^{2-\frac{1}{s}} \ e^{2T} \\ &\times \left(e^{T} \|v_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})}^{2} + \frac{16\beta^{2} e^{T}}{\pi (2s-1)} \|u_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})}^{2-\frac{1}{s}} \int_{0}^{t} \|(-\Delta)^{s/2} u^{\varepsilon}(\tau)\|_{L^{2}(\mathbb{R})}^{2+\frac{1}{s}} \ d\tau \right) \\ &+ \frac{2^{8} |\alpha|^{2} \ |\beta|^{2} T}{s^{2} \pi^{2}} \ \|u_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})}^{6-\frac{2}{s}} \int_{0}^{t} \theta(\tau)^{\frac{1}{2s}} e^{2\tau} \ d\tau \\ &+ \frac{2^{5} |\alpha|^{2} \ \varepsilon^{2b} \ \varepsilon^{-3a/2} \ (2s-1)^{2}}{\pi s^{2}} \ \|u_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})}^{2-\frac{1}{s}} \int_{0}^{t} \|(-\Delta)^{s/2} u^{\varepsilon}(\tau)\|_{L^{2}(\mathbb{R})}^{2+\frac{1}{s}} \ d\tau \right) \\ &+ \frac{2^{10} |\alpha|^{4} |\beta|^{4} e^{2T}}{s^{2} \pi^{2}} \|u_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})}^{4-\frac{2}{s}} T \int_{0}^{t} \theta(\tau)^{1+\frac{1}{2s}} e^{2\tau} \ d\tau. \end{split}$$

Moreover, after an algebraic manipulation and using that  $e^t > 1$  for any t > 0, we may write

$$\theta(t)^{1-\frac{1}{2s}} e^{2t} \le C + C_1 \int_0^t \theta(\tau)^{1-\frac{1}{2s}} e^{2\tau} d\tau + C_2 \int_0^t \theta(\tau)^{\frac{1}{2s}} e^{2\tau} d\tau + C_3 \int_0^t \theta(\tau)^{1+\frac{1}{2s}} e^{2\tau} d\tau,$$
(3.62)

where

$$\begin{split} C &:= 2^{6} (1 + \|(-\Delta)^{s/2} u_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})}^{2} + \|\partial_{x} u_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})}^{2} + \frac{1}{2} \|u_{0}^{\varepsilon}\|_{L^{4}(\mathbb{R})}^{4} \\ &+ \|u_{0}^{\varepsilon}\|_{L^{\infty}(\mathbb{R})} \|v_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})} \|u_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})}^{2} + |\alpha|^{2} e^{T} \|v_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})}^{2})^{1 - \frac{1}{2s}} \\ &+ \frac{2^{5} |\alpha|^{2} (2s - 1)}{s^{2} \pi} \|g_{\varepsilon}'\|_{L^{\infty}(\mathbb{R})}^{2} \|u_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})}^{2 - \frac{1}{s}} \|v_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})}^{2} e^{3T} \\ &+ \frac{2^{4} |\alpha|^{2} \varepsilon^{b} \varepsilon^{-3a/2} (2s - 1)^{2}}{\pi s^{2}} \|u_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})}^{2} \|v_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})}^{2} e^{2T}, \\ C_{1} &:= 2^{6} T, \qquad C_{2} := \frac{2^{8} |\alpha|^{2} |\beta|^{2} T}{s^{2} \pi^{2}} \|u_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})}^{6 - \frac{2}{s}}, \\ C_{3} &:= \frac{2^{9} |\alpha|^{2} |\beta|^{2}}{s^{2} \pi^{2}} \|g_{\varepsilon}'\|_{L^{\infty}(\mathbb{R})}^{2} \|u_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})}^{4 - \frac{2}{s}} e^{3T} \\ &+ \frac{2^{8} C_{1,s} |\alpha|^{2} |\beta|^{2} \varepsilon^{b - 1} \varepsilon^{-3a/2} (2s - 1)}{\pi^{2} s^{2}} \|u_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})}^{3 - \frac{1}{s}} e^{2T} + \frac{2^{10} |\alpha|^{4} |\beta|^{4} e^{2T}}{\pi^{2} s^{2}} \|u_{0}^{\varepsilon}\|_{L^{2}(\mathbb{R})}^{4 - \frac{2}{s}} T. \end{split}$$

Therefore, taking a = 4 and b = 7 the above positive constants C,  $C_1$ ,  $C_2$  and  $C_3$  are independent of  $\varepsilon > 0$ . Now, since

$$(1 - \frac{1}{2s})(\frac{2s+1}{2s-1}) = 1 + \frac{1}{2s}$$
, and  $(1 - \frac{1}{2s})(\frac{1}{2s-1}) = \frac{1}{2s}$ ,

then we have from (3.62)

$$\theta(t)^{1-\frac{1}{2s}} e^{2t} \le C + C_1 \int_0^t \theta(\tau)^{1-\frac{1}{2s}} e^{2\tau} d\tau$$

$$+ C_2 \int_0^t (\theta(\tau)^{1-\frac{1}{2s}})^{\frac{1}{2s-1}} e^{2\tau} d\tau + C_3 \int_0^t (\theta(\tau)^{1-\frac{1}{2s}})^{\frac{2s+1}{2s-1}} e^{2\tau} d\tau.$$

For each 1/2 < s < 1 we have

$$1 < \frac{1}{2s-1} < \frac{2s+1}{2s-1}$$

therefore from the above inequality we may write

$$\begin{split} \theta(t)^{1-\frac{1}{2s}} \; e^{2t} &\leq C + C_1 \, \int_0^t \theta(\tau)^{1-\frac{1}{2s}} \; e^{2\tau} \; d\tau \\ &\quad + C_2 \, \int_0^t \left(\theta(\tau)^{1-\frac{1}{2s}} \; e^{2\tau}\right)^{\frac{2s+1}{2s-1}} \; d\tau + C_3 \, \int_0^t \left(\theta(\tau)^{1-\frac{1}{2s}} e^{2\tau}\right)^{\frac{2s+1}{2s-1}} \; d\tau \end{split}$$

or defining  $\eta(t) := \theta(t)^{1-\frac{1}{2s}} e^{2t}$ 

$$\eta(t) \le C + \int_0^t \left[ C_1 \eta(\tau) + \left( C_2 + C_3 \right) (\eta(\tau))^{\frac{2s+1}{2s-1}} \right] d\tau.$$
(3.63)

Therefore, applying the Generalized Grownwall Lemma, more precisely (2.31) with  $\sigma = \frac{2s+1}{2s-1} > 1$ , we must have for each  $s \in (1/2, 1)$ 

$$C < \left\{ \exp\left[ \left( 1 - \frac{2s+1}{2s-1} \right) \int_0^T C_1 d\tau \right] \right\}^{\frac{1}{\frac{2s+1}{2s-1}-1}} \left\{ \left( \frac{2s+1}{2s-1} - 1 \right) \int_0^T \left( C_2 + C_3 \right) d\tau \right\}^{-\frac{1}{\frac{2s+1}{2s-1}-1}}$$

$$= \frac{(2s-1)^{\frac{2s-1}{2}} \exp\left[ -C_1 T \right]}{\left\{ 2\left( C_2 + C_3 \right) T \right\}^{\frac{2s-1}{2}}}$$

or equivalently

$$C\left(C_2 + C_3\right)^{\frac{2s-1}{2}} \exp[64T^2] T^{\frac{2s-1}{2}} \le \left(\frac{2s-1}{2}\right)^{\frac{2s-1}{2}}.$$
 (3.64)

One remarks that

$$\lim_{s \to \frac{1}{2}} \left( \frac{2s - 1}{2} \right)^{\frac{2s - 1}{2}} = 1.$$

Hence for any  $s \in (1/2, 1)$  fixed, there exists  $\alpha_0 > 0$  and  $E_0 > 0$ , such that condition (3.64) is satisfied when  $||u_0||_{L^2(\mathbb{R})} \le E_0$ , or  $|\alpha| \le \alpha_0$ . In fact, if there is no coupling, that is  $\alpha = 0$  ( $C_2 = C_3 = 0$ ), then condition (3.64) is trivially satisfied. Consequently, we have

$$\eta(t) \le C \left\{ \exp\left[\left(1 - \frac{2s+1}{2s-1}\right) \int_{0}^{t} C_{1} d\tau\right] - C^{-1} \left(\frac{2s+1}{2s-1} - 1\right) \int_{0}^{t} \left(C_{2} + C_{3}\right) \exp\left[\left(1 - \frac{2s+1}{2s-1}\right) \int_{\tau}^{t} C_{1} d\tau\right] d\tau\right\}^{\frac{1}{\frac{2s+1}{2s-1}-1}}$$

$$= C \left\{ \exp\left[\frac{2C_{1}}{1-2s}t\right] - C^{-1} \frac{2}{2s-1} \left(C_{2} + C_{3}\right) \int_{0}^{t} \exp\left[\frac{2C_{1}}{1-2s}(t-\tau)\right] d\tau\right\}^{\frac{2s-1}{2}}$$

$$= C \left\{ \exp\left[\frac{2C_{1}}{1-2s}t\right] - \frac{C^{-1}\left(C_{2} + C_{3}\right)}{C_{1}} \left(1 - \exp\left[\frac{2C_{1}}{1-2s}t\right]\right)\right\}^{\frac{2s-1}{2}},$$

from which follows the proof of the theorem.

Finally, we establish a maximum principle for the solution  $v^{\varepsilon}$  of  $(3.34)_2$ .

**Proposition 3.6** (Maximum Principle). Let  $(u^{\varepsilon}, v^{\varepsilon})$  be the unique solution of (3.34). Then,  $v^{\varepsilon}$  satisfies

$$\sup_{(0,T)\times\mathbb{R}} |v^{\varepsilon}| \le ||v_0||_{L^{\infty}(\mathbb{R})}. \tag{3.65}$$

*Proof.* For  $\varepsilon > 0$  fixed, let us define  $w := v^{\varepsilon} - ||v_0||_{L^{\infty}(\mathbb{R})}$ , and we will show that,  $w^+ = \max\{w, 0\} = 0$ . Clearly  $w^+ \ge 0$ , then we assume by contradiction that,  $w^+ > 0$ . Therefore, there exists  $\mu > 0$ , such that  $w^+ \ge \mu$ . Since  $w^+(t) \in H^1(\mathbb{R})$  for each  $t \in (0, T)$ , we can use  $w^+$  as a test function for equation (3.34)<sub>2</sub>, (similar to equation (4.69)), that is to say

$$\frac{d}{dt} \int_{\mathbb{R}} |w^{+}(t)|^{2} dx + \int_{\mathbb{R}} g_{\varepsilon}(v^{\varepsilon}(t)) (-\Delta)^{s/2} w^{+}(t) dx$$

$$= \beta \int_{\mathbb{R}} |u^{\varepsilon}(t)|^{2} (-\Delta)^{s/2} w^{+}(t) dx - \int_{\mathbb{R}} \varepsilon^{7} |\nabla w^{+}(t)|^{2} dx.$$

Now, we consider the following estimate

$$\begin{split} g_{\varepsilon}(v^{\varepsilon}) \, (-\Delta)^{s/2} w^{+} &+ \varepsilon^{7} \, |\nabla w^{+}|^{2} - \beta \, |u^{\varepsilon}|^{2} \, (-\Delta)^{s/2} w^{+} \\ &\geq -M \, |w^{+}| \, |(-\Delta)^{1/2} w^{+}| + \varepsilon^{7} \, |\nabla w^{+}|^{2} - \frac{|\beta| \, ||u^{\varepsilon}||_{L^{\infty}(\mathbb{R})}^{2}}{\mu} |(-\Delta)^{1/2} w^{+}| \\ &\geq -\frac{\left(M \, \mu + |\beta| \, ||u^{\varepsilon}||_{L^{\infty}(\mathbb{R})}^{2}\right)^{2}}{2 \, \varepsilon^{7} u^{2}} |w^{+}|^{2} - \varepsilon^{7} \, |(-\Delta)^{1/2} w^{+}|^{2} + \varepsilon^{7} \, |\nabla w^{+}|^{2}, \end{split}$$

where we have used Young's inequality. Consequently, since  $w^+(0) = 0$  we have

$$\int_{\mathbb{R}} |w^+(t)|^2 dx \le \frac{\left(M\mu + |\beta| \|u^\varepsilon\|_{L^\infty(\mathbb{R})}^2\right)^2}{2\varepsilon^7\mu^2} \int_0^t \int_{\mathbb{R}} |w^+(\tau, x)|^2 dx d\tau,$$

which applying the Gronwall's lemma implies a contradiction. Therefore, we have

$$w^{+}(t) = (v^{\varepsilon}(t) - ||v_{0}||_{L^{\infty}(\mathbb{R})})^{+} \equiv 0.$$
(3.66)

A similar argument can also show that

$$(-v^{\varepsilon}(t) - ||v_0||_{L^{\infty}(\mathbb{R})})^+ \equiv 0. \tag{3.67}$$

Nonetheless, the equations (3.66) and (3.67) mean nothing other than

$$||v^{\varepsilon}(t)||_{L^{\infty}(\mathbb{R})} \leq ||v_0||_{L^{\infty}(\mathbb{R})}$$
 for each  $t \in (0, T)$ .

### 4 Existence of Weak Solutions

The main issue of this section is to show the solvability of the Cauchy problem (1.1), that is, we prove Theorem 1.2 (Main Theorem). More precisely, from the equivalence of mild solutions (when it exists) and weak solutions, we obtain a weak formulation from (3.35), see Lemma 4.1, and the goal is to pass to the limit as  $\varepsilon \to 0^+$  to show a solution of the Cauchy problem (1.1) in the sense of Definition 1.1. We apply the Aubin-Lions Theorem to show that the family  $\{u^{\varepsilon}\}$  is relatively compact in  $L^2$ . The similar result for the family  $\{v^{\varepsilon}\}$  does not follow analogously, since (1.1)<sub>2</sub> degenerates. Hence we apply Tartar's methodology in [20], (see also [18]), adapted to our context of fractional porous media equation.

First, we have the following

**Lemma 4.1.** Let  $\alpha_0 > 0$ ,  $E_0 > 0$  be given by Theorem 3.5, such that,  $|\alpha| \le \alpha_0$  or  $||u_0||_{L^2(\mathbb{R})} \le E_0$ . Then, the unique mild solution  $(u^{\varepsilon}, v^{\varepsilon})$  of (3.34) satisfies,

$$i\int_{0}^{T} \int_{\mathbb{R}} \left( u^{\varepsilon}(t,x) \, \partial_{t}\overline{\varphi}(t,x) + (-\Delta)^{s/2}u^{\varepsilon}(t,x) \, (-\Delta)^{s/2}\overline{\varphi}(t,x) \right) dxdt + i\int_{\mathbb{R}} u_{0}^{\varepsilon}(x) \, \overline{\varphi}(0,x) dx$$

$$-\varepsilon^{a} \int_{0}^{T} \int_{\mathbb{R}} u^{\varepsilon}(t,x) \, \Delta\overline{\varphi} \, dxdt + \alpha \int_{0}^{T} \int_{\mathbb{R}} v^{\varepsilon}(t,x) \, u^{\varepsilon}(t,x) \, \overline{\varphi}(t,x) \, dxdt$$

$$+ \int_{0}^{T} \int_{\mathbb{R}} |u^{\varepsilon}(t,x)|^{2} \, u^{\varepsilon}(t,x) \, \overline{\varphi}(t,x) \, dxdt = 0,$$

$$\int_{0}^{T} \int_{\mathbb{R}} v^{\varepsilon}(t,x) \, \partial_{t}\psi(t,x) - g_{\varepsilon}(v^{\varepsilon}(t,x)) \, (-\Delta)^{s/2}\psi(t,x) \, dxdt + \int_{\mathbb{R}} v_{0}^{\varepsilon}(x) \, \psi(0,x) \, dx$$

$$+ \varepsilon^{7} \int_{0}^{T} \int_{\mathbb{R}} v^{\varepsilon}(t,x) \, \Delta\psi(t,x) \, dxdt + \beta \int_{0}^{T} \int_{\mathbb{R}} |u^{\varepsilon}|^{2}(t,x) \, (-\Delta)^{s/2}\psi(t,x) \, dtdx = 0$$

$$(4.69)$$

for each test functions  $\varphi, \psi \in C_c^{\infty}((-\infty, T) \times \mathbb{R})$ , with  $\varphi$  being complex-valued and  $\psi$  real-valued.

Moreover, there exists a positive constant C independent of  $\varepsilon > 0$ , such that

$$\int_0^T \|\partial_t u^{\varepsilon}(t)\|_{H^{-1}(\mathbb{R})}^2 dt \le C, \qquad \int_0^T \|\partial_t v^{\varepsilon}(t)\|_{H^{-1}(\mathbb{R})}^2 dt \le C. \tag{4.70}$$

*Proof.* Equations (4.68), (4.69) are obtained from (3.35), that is, applying the equivalence between mild solutions and weak solutions, (see Ball [1], p. 371),

which are obtained via functional analysis arguments. Similarly, the inequalities in equation (4.70) are obtained from the weak formulation, i.e. equations (4.68) and (4.69), applying standard functional analysis results, the uniform boundedness of  $u_0^{\varepsilon}$ ,  $v_0^{\varepsilon}$ , and also the uniform estimates from Lemma 3.4 and Theorem 3.5.

### 4.1 Proof of main theorem

Now, we are ready to show the main result of this article.

- **Proof.** 1. First, under the conditions of Lemma 4.1, for each  $\varepsilon > 0$ , let  $(u^{\varepsilon}, v^{\varepsilon}) \in C([0,T); H^1(\mathbb{R})) \times C([0,T); H^1(\mathbb{R}))$  be the unique mild solution of (3.34), satisfying (3.35). Then, the pair $(u^{\varepsilon}(t,x), v^{\varepsilon}(t,x))$  satisfies the equations (4.68) and (4.69). To obtain (1.7), (1.8) we pass to the limit respectively in (4.68) and (4.69) as  $\varepsilon \to 0^+$ . Therefore, we need to show strong convergence, which implies a.e. convergence (along subsequences) of the sequences  $\{u^{\varepsilon}\}_{\varepsilon>0}$ , and  $\{v^{\varepsilon}\}_{\varepsilon>0}$ .
- 2. Let us show that the family  $\{u^{\varepsilon}\}_{\varepsilon>0}$  is relatively compact. From (3.46), (3.50), it follows that  $\{u^{\varepsilon}\}_{\varepsilon>0}$  is (uniformly) bounded in  $L^{\infty}(0,T;H^{s}(\mathbb{R}))$ , hence it is possible to select a subsequence, still denoted by  $\{u^{\varepsilon}\}_{\varepsilon>0}$ , which converges weakly- $\star$  to u in  $L^{\infty}(0,T;H^{s}(\mathbb{R}))$ . Applying the Rellich's Theorem, for any compact set  $K \subset \mathbb{R}$ , the embedding of  $H^{s}(K)$  in  $L^{2}(K)$  is compact. Therefore, since the sequence  $\{u^{\varepsilon}\}_{\varepsilon>0}$  is uniformly bounded in  $L^{2}(0,T;H^{-1}(\mathbb{R}))$ , we apply the Aubin-Lions Theorem and obtain (along a suitable subsequence) that  $u^{\varepsilon}$  converges strongly to u in  $L^{2}(0,T;L^{2}(K))$ , and thus

$$u^{\varepsilon}(t, x) \to u(t, x)$$
 as  $\varepsilon \to 0$  almost everywhere in  $(0, T) \times \mathbb{R}$ . (4.71)

3. Now, we show that the family  $\{v^{\varepsilon}\}_{{\varepsilon}>0}$  is relatively compact. First, we multiply equation  $(3.34)_2$  by  $\eta'_k(v^{\varepsilon})$ , (see Section 2.5), and applying a standard procedure (e.g. the theory of scalar conservation laws), we obtain in distribution sense

$$\begin{split} \partial_t \eta_k(v^{\varepsilon}) + (-\Delta)^{s/2} |g_{\varepsilon}(v^{\varepsilon}) - g_{\varepsilon}(k)| &= \beta \, \eta_k'(v^{\varepsilon}) (-\Delta)^{s/2} |u^{\varepsilon}|^2 + \varepsilon^7 \Delta \eta_k(v^{\varepsilon}) \\ &- \varepsilon^7 |\nabla v^{\varepsilon}|^2 \eta_k''(v^{\varepsilon}) - R_k^{\varepsilon}, \end{split} \tag{4.72}$$

where we have used (2.32) and obvious notation. Let  $\eta$  be any smooth (say  $C^2$ ) entropy (which is linear at infinity, i.e.  $\eta''(\cdot) \in C_0(\mathbb{R})$ ). Then, multiplying equation (4.72) by  $\eta''(k)$  and integrating in  $\mathbb{R}$  with respect to k, we obtain in the sense of distributions

$$\partial_{t}\eta(v^{\varepsilon}) + (-\Delta)^{s/2}q(v^{\varepsilon}) = \beta \eta'(v^{\varepsilon})(-\Delta)^{s/2}|u^{\varepsilon}|^{2} + \varepsilon^{7}\Delta\eta(v^{\varepsilon}) - \varepsilon (-\Delta)^{s/2}\eta(v^{\varepsilon}) - \varepsilon^{7}|\nabla v^{\varepsilon}|^{2}\eta''(v^{\varepsilon}) - \mathcal{R}^{\varepsilon},$$

$$(4.73)$$

where the function q satisfies  $q' = \eta' g'$ , (recall that  $g_{\varepsilon}(v^{\varepsilon}) = g(v^{\varepsilon}) + \varepsilon v^{\varepsilon}$ ), and

$$\mathcal{R}^{\varepsilon} := \frac{1}{2} \int_{\mathbb{R}} \eta''(k) \, R_k^{\varepsilon} \, dk.$$

From (3.56), (3.65) it follows that the family  $\{v^{\varepsilon}\}_{\varepsilon>0}$  is (uniformly) bounded in  $L^{\infty}(0,T;L^{2}(\mathbb{R})\cap L^{\infty}(\mathbb{R}))$ , hence it is possible to select a subsequence, still denoted by  $\{v^{\varepsilon}\}_{\varepsilon>0}$ , which converges weakly- $\star$  to v in  $L^{\infty}(0,T;L^{2}(\mathbb{R})\cap L^{\infty}(\mathbb{R}))$ . Moreover, we show that for any entropy pair  $(\eta,q)$ ,

$$\partial_t \eta(v^{\varepsilon}) + (-\Delta)^{s/2} q(v^{\varepsilon}) \in \left\{ \text{compact set of } H^{-1}_{\text{loc}}((0, \infty) \times \mathbb{R}) \right\}. \tag{4.74}$$

Indeed, we first observe that the left hand side of (4.73) is uniformly bounded in  $W^{-1,\infty}_{loc}((0,\infty)\times\mathbb{R})$ . From equation (3.52) the terms  $\varepsilon^7\Delta\eta(v^\varepsilon)$ ,  $\varepsilon(-\Delta)^{s/2}\eta(v^\varepsilon)$  are compact in  $H^{-1}_{loc}((0,\infty)\times\mathbb{R})$ , let us show the former the second one is similar. Let  $K\subset (0,T)\times\mathbb{R}$  be any compact set, and  $\phi\in C^\infty_c((0,T)\times\mathbb{R})$ . Then, we have

$$\begin{split} |\langle \varepsilon^7 \Delta \eta(v^{\varepsilon}), \phi \rangle| &\leq \int_0^T \int_{\mathbb{R}} |\varepsilon^7 \nabla \eta(v^{\varepsilon}) \cdot \nabla \phi| \, dx dt \\ &\leq \varepsilon^{7/2} \bigg( \int_0^T \int_{\mathbb{R}} |\varepsilon^{7/2} \nabla \eta(v^{\varepsilon})|^2 \, dx \, dt \bigg)^{1/2} ||\nabla \phi||_{L^2(K)} \leq \varepsilon^{7/2} C \, ||\nabla \phi||_{L^2(K)}, \end{split}$$

where C>0 does not depend on  $\varepsilon$ , and we have used that  $\eta$  is linear at infinity. Then, taking the supremum with respect to the set  $W=\{\phi\in H^1: ||\phi||_{H^1}\leq 1\}$  and passing to the limit as  $\varepsilon\to 0^+$ , the family  $\{\varepsilon^7\Delta\eta(v^\varepsilon)\}$  converges to zero in  $H^{-1}_{\rm loc}$ .

From equation equation (3.52) the family  $\{\varepsilon^7 | \nabla v^\varepsilon|^2 \eta''(v^\varepsilon)\}$  is uniformly bounded in  $L^1_{\text{loc}}((0,\infty)\times\mathbb{R})$ , and thus in the space of Radon measures  $\mathfrak{M}_{\text{loc}}((0,\infty)\times\mathbb{R})$ . Hence compact in  $W^{-1,q}_{\text{loc}}((0,\infty)\times\mathbb{R})$ , for  $1\leq q<3/2$ . Similarly, from (3.50), (3.54), the family  $\{\eta'(v^\varepsilon)(-\Delta)^{s/2}|u^\varepsilon|^2\}$  is uniformly bounded in  $L^1_{\text{loc}}((0,\infty)\times\mathbb{R})$ , and hence compact in  $W^{-1,q}_{\text{loc}}((0,\infty)\times\mathbb{R})$ , for  $1\leq q<3/2$ . Finally, we consider the following

<u>Claim:</u> The family  $\{\mathcal{R}^{\varepsilon}\}$  is compact in  $W_{\text{loc}}^{-1,q}((0,\infty)\times\mathbb{R})$ , for  $1\leq q<3/2$ .

Consequently, we have  $\partial_t \eta(v^{\varepsilon}) + (-\Delta)^{s/2} q(v^{\varepsilon})$  in

$$\left\{\text{bounded set of }W^{-1,\infty}_{\mathrm{loc}}((0,\infty)\times\mathbb{R})\right\}\cap\left\{\text{compact set of }W^{-1,q}_{\mathrm{loc}}((0,\infty)\times\mathbb{R})\right\},$$

and due to a well known interpolation argument, (see Lemma 3.12 in [18]), it follows (4.74).

Proof of Claim: It is enough to show that, the family  $\{\mathcal{R}^{\varepsilon}\}$  is uniformly bounded in  $\mathcal{M}_{loc}((0,\infty)\times\mathbb{R})$ . To this end, we observe that the left hand side of (4.73) is also uniformly bounded in  $H_{loc}^{-1}((0,\infty)\times\mathbb{R})$ , and jointly with the others terms clearly shows that,  $\mathcal{R}^{\varepsilon}$  is a uniformly bounded distribution ( in  $\mathcal{D}'$ ), which is positive by definition, hence from a well known result a Radon measure.

Now, from (4.74) we may apply the Tartar's method in [20], (see also [18]), which implies the compactness of the sequence  $\{v^{\varepsilon}\}$  in  $L^1_{loc}((0, \infty) \times \mathbb{R})$ . Thus along a suitable subsequence

$$v^{\varepsilon}(t, x) \to v(t, x)$$
 as  $\varepsilon \to 0$  almost everywhere in  $(0, T) \times \mathbb{R}$ . (4.75)

4. Finally, from (4.71), (4.75) and due to a standard diagonalization procedure, we apply the Dominated Convergence Theorem to pass to the limit as  $\varepsilon \to 0$  in the equations (4.68) and (4.69), which togheter the Definition 1.1 gives the solvability of the Cauchy problem (1.1). Moreover, inequality (1.9) follows from (3.65).  $\Box$ 

# Acknowledgements

Conflict of Interest: Author Wladimir Neves has received research grants from CNPq through the grant 308064/2019-4, and also by FAPERJ (Cientista do Nosso Estado) through the grant E-26/201.139/2021.

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