STABILIZATION OF HIGHLY NONLINEAR HYBRID STOCHASTIC DIFFERENTIAL DELAY EQUATIONS WITH LÉVY NOISE BY DELAY FEEDBACK CONTROL*

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Abstract. This paper focuses on a class of highly nonlinear stochastic differential delay equations (SDDEs) driven by Lévy noise and Markovian chain, where the drift and diffusion coefficients satisfy more general polynomial growth condition (than the classical linear growth condition). Under the local Lipschitz condition, the existence-and-unique theorem of the solution to the highly nonlinear SDDE is established. The key aim is to investigate the stabilization problem by delay feedback controls. The key features include that the time delay in the given system is of time-varying and may not be differentiable while the time lag in the feedback control can also be of time-varying as long as it has a sufficiently small upper bound.

Key words. Highly non-linearity, Stochastic differential delay equation, Markov chain, Lévy noise, Exponential stability

AMS subject classifications. 60J60, 60J27, 93D15

1. Introduction.

Nonlinear stochastic differential delay equations (SDDEs) have been widely used to model many systems in aerospace, nuclear industry, artificial intelligence, modern military systems, financial systems and other fields. Stability and stabilization of SDDEs have been two of the most important research topics. There has already existed huge literature in the field of stability and stabilization of SDDEs. The classical and frequently imposed condition in the study of the stabilization by feedback control is that the diffusion and drift coefficients of the underlying SDDEs need to satisfy the linear growth condition (see, e.g., [3, 9, 10, 16, 17, 26, 28]). But this condition is too restrictive for many nonlinear SDDE systems in applications.

To meet the need of applications, several authors (see, e.g., [5, 7, 14, 21]) developed the stabilization theory for highly nonlinear SDDEs driven by Brownian motions and Markov chains, where the diffusion and drift coefficients only need to satisfy the polynomial growth condition. Their theory is hence applicable to many more practical SDDE systems. Nevertheless, their theory is only applicable to SDDE systems where the time delay is either constant and differentiable with its derivative being bounded by a positive number less than 1. This condition has been imposed only because of the mathematical technique used—the technique of time change but might not be a natural feature of SDDE systems in the real world. For example, piece-wise constant delays or sawtooth delays occur frequently in sampled-data controls or network-based controls (see, e.g., [1]) but they are not differentiable. It was in this spirit that a much weaker condition was recently established in [5] to replace the differentiability of the time delay. As demonstrated, their new results are applicable to a much wider class of SDDE systems in applications.

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Although Brownian motions have been widely used to model the system uncertainties which are affected by many independent factors with no-one playing a dominated role, while Markov chains to model the abrupt changes of system parameters or structures (see, e.g., [5, 7, 14, 16, 17, 18, 21, 23, 26]), they cannot model the random jumps of the system states. This can be seen clearly by the continuity of the solutions of SDDEs driven by Brownian motions and Markov chains. On the other hands, the states of many practical systems are indeed subject to random jumps due to unpredictable events, e.g., earthquake, storm, flood, bankrupt, war, pandemic. Lévy processes have been used to model such random jumps, as these processes have significant tail and peak pulse characteristics (see, e.g., [4, 13, 24, 25, 27, 29]). Naturally, stability of such-type SDDEs have also been studied. For example, Yin et al. in [25] were concerned with the stability of a class of switching jump-diffusion processes. Yuan et al. in [27] investigated sufficient conditions for stability of delay jump diffusion processes. Zhu in [29] focused on the pth moment and almost sure stability of a class of stochastic differential equations with Lévy noise.

It is noted that the aforementioned references [25, 27, 29] with Lévy noise all consider the stability of SDDEs satisfying the linear growth condition. From the perspective of practical applications, it is very necessary to study the stability and stabilization of highly nonlinear Markov-modulated SDDEs with Lévy noise. The main aim of this paper is to explore how a feedback control with time-varying delay can stabilize a given unstable highly nonlinear Markov-modulated SDDE with Lévy noise. The key contributions of this paper are as follows:

- This is the first paper on the stabilization by feedback controls for a class of SDDEs driven by the Lévy processes, in addition to Brownian motions and Markov chains, where the coefficients are highly nonlinear (i.e., do not satisfy the linear growth conditions).
- Notably, the time-varying delays in the given SDDE as well as in the feedback control need only to meet a much weaker condition than those imposed in most of the existing papers. For example, they are no-longer required to be differentiable. Different methods from those used, for example, in the proof of [5, Lemma 2.2], are developed to cope with the càdlàg property of the underlying solution as well as the general time-varying delays.
- This paper does not only establish a general existence-and-unique theorem on the global solution of the nonlinear SDDE driven by Lévy noise, but also obtains the finiteness and boundedness of the moments of the solution. These are not only generalisations of [5, Theorem 2.4 and 2.6], but will also form a foundation for further research in this area.

The paper is organized as follows. In Section 2, we propose model, notations and assumptions. In Section 3, we give the conditions that the control function needs to meet. In Section 4, we show the sufficient conditions for exponential stability and almost surely exponential stability. In Section 5, we provide an example to show the effectiveness of the theoretical results. Conclusions are presented in the last section.

2. Model, notations and assumptions. Throughout this paper, unless otherwise specified, we use the following notations. A^T is the transpose of a vector or matrix A. $|x|$ denotes its Euclidean norm, where $x \in \mathbb{R}^d$ is a vector. For a matrix A, $|A| = \sqrt{\mathrm{trace}(A^T A)}$ denotes its trace norm. If A is a symmetric real-valued matrix ($A = A^T$), denote by $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ its smallest and largest eigenvalue, respectively. For $\lambda > 0$, denote by $D([-\lambda, 0]; \mathbb{R}^d)$ the family of càdlàg functions (i.e. one that is right-continuous with left limits) φ from $[-\lambda, 0] \to \mathbb{R}^d$ with the norm

$\|\varphi\| = \sup_{-\lambda\le u\le 0}|\varphi(u)|$. Denote by $D^b_{\mathcal{F}_0}([-\lambda,0];\mathbb{R}^d)$ the family of all bounded, $\mathcal{F}_0$-measurable, $D([-\lambda,0];\mathbb{R}^d)$-valued random variables. Denote by $C^{2,1}(\mathbb{R}^d\times S\times\mathbb{R}_+;\mathbb{R})$ the family of all real-valued functions $V(x,i,t)$ on $\mathbb{R}^d\times S\times\mathbb{R}_+$ which are continuously twice differentiable in x and once in t. For such a $C^{2,1}$-function V, we set $V_t = \frac{\partial V}{\partial t}$, $V_x = (\frac{\partial V}{\partial x_1},\cdots,\frac{\partial V}{\partial x_d})$ and $V_{xx} = (\frac{\partial^2 V}{\partial x_i\partial x_j})_{d\times d}$. For two real numbers a and b, $a\vee b = \max\{a,b\}$ and $a\wedge b = \min\{a,b\}$. I_A is the indicator function of A, where A is a subset of Ω; that is, $I_A(\omega)=1$ for $\omega\in A$ and $I_A(\omega)=0$ for $\omega\notin A$.

Let $B(t) = (B_1(t),\cdots,B_m(t))^T$ be an m-dimensional Brownian motion defined on the probability space $(\Omega,\mathcal{F},\mathbb{P})$ with its filtration $\{\mathcal{F}_t\}_{t\ge0}$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). For fixed $\omega\in\Omega$, $N(t,\cdot)(\omega)$ is a Poisson random measure defined on $\mathbb{R}_+\times\mathbb{R}^n_0$, where $\mathbb{R}^n_0 = \mathbb{R}^n - \{0\}$, and its compensated Poisson random measure is denoted by $\widetilde{N}(dt,dz) = N(dt,dz) - \vartheta(dz)dt$, where ϑ is a Lévy measure satisfying

$$\int_{\mathbb{R}^n_0}(1\wedge|z|^2)\vartheta(dz) < \infty. \tag{2.1}$$

Usually, the pair (B,N) is called a Lévy noise. It is easy to show from (2.1) that $\vartheta\big(\{z\in\mathbb{R}^n_0 : |z|\ge b\}\big) < \infty$ for any $b>0$ but we may not have $\vartheta\big(\{z\in\mathbb{R}^n_0 : |z| < b\}\big) < \infty$. That is, the Lévy measure might not be finite.

Let $\{r(t), t\ge0\}$ be a right-continuous Markov chain on the probability space taking values in a finite state space $S=\{1,2,\cdots,N\}$ with generator $\Gamma = (\gamma_{ij})_{N\times N}$ given by

$$\mathbb{P}\{r(t+\Delta)=j|r(t)=i\} = \begin{cases}\gamma_{ij}\Delta + o(\Delta) & i\ne j,\\ 1+\gamma_{ii}\Delta + o(\Delta) & i=j,\end{cases} \tag{2.2}$$

where $\Delta>0$ and $\gamma_{ij}\ge0$ is the transition rate from i to j if $i\ne j$ while $\gamma_{ii} = -\sum_{j\ne i}\gamma_{ij}$. In this paper, we assume that the Markov chain $r(\cdot)$, the Brownian motion $B(\cdot)$ and the Poisson random measure $N(\cdot,\cdot)$ are independent of each other.

In general, the SDDE with Markov switching, driven by the Lévy noise, has the form

$$\begin{aligned} dy(t) =& f(y(t^-),y((t-\delta_t)^-),r(t),t)dt + g(y(t^-),y((t-\delta_t)^-),r(t),t)dB(t)\\ &+\int_{0<|z|<c} h(y(t^-),y((t-\delta_t)^-),r(t),t,z)\widetilde{N}(dt,dz)\\ &+\int_{|z|\ge c} H(y(t^-),y((t-\delta_t)^-),r(t),t,z)N(dt,dz), \end{aligned} \tag{2.3}$$

where $y(t^-) = \lim_{s\uparrow t} y(s)$, $f:\mathbb{R}^d\times\mathbb{R}^d\times S\times\mathbb{R}_+\to\mathbb{R}^d$, $g:\mathbb{R}^d\times\mathbb{R}^d\times S\times\mathbb{R}_+\to\mathbb{R}^{d\times m}$, $h:\mathbb{R}^d\times\mathbb{R}^d\times S\times\mathbb{R}_+\times\mathbb{R}^n_0\to\mathbb{R}^d$ and $H:\mathbb{R}^d\times\mathbb{R}^d\times S\times\mathbb{R}_+\times\mathbb{R}^n_0\to\mathbb{R}^d$, the constant $c\in(0,\infty)$ allows us to specify what we mean by 'large' and 'small' jumps in specific applications, and δ_t is a time-varying function. Observe that the last integral term in (2.3) is a compound Poisson process, which can be handled easily by using interlacing (see, e.g., [2, pp. 112-115]) or by the methods developed in this paper on how to deal with small jumps. It hence makes sense to begin by omitting the large jumps term and concentrate on the study of the equation driven by continuous noise interspersed with small jumps (see, e.g., [2, pp. 302]). We will therefore concentrate on the study of the simplified SDDE with small jumps in the form

$$\begin{aligned} dx(t) =& f(x(t^-), x((t-\delta_t)^-), r(t), t)dt + g(x(t^-), x((t-\delta_t)^-), r(t), t)dB(t) \\ &+ \int_{0<|z|<c} h(x(t^-), x((t-\delta_t)^-), r(t), t, z)\widetilde{N}(dt, dz), \end{aligned} \tag{2.4}$$

with the initial data

$$\{x(t) : -\lambda \le t \le 0\} = \xi \in D^b_{\mathcal{F}_0}([-\lambda, 0]; \mathbb{R}^d) \text{ and } r(0) = i_0, \tag{2.5}$$

where $x(t^-) = \lim_{s\uparrow t} x(s)$ and the details of positive constant λ will be given in Assumption 2.1. Next we will state an assumption about δ_t and a useful lemma.

ASSUMPTION 2.1. [5] *The time-varying delay δ_t is a Borel measurable function from $\mathbb{R}_+$ to $[\lambda_1, \lambda]$, and has the property that*

$$\bar{\lambda} := \limsup_{\Delta\to 0^+} \Big(\sup_{s\ge -\lambda} \frac{\mu(M_{s,\Delta})}{\Delta} \Big) < \infty, \tag{2.6}$$

where λ_1 and λ are both positive constants with $\lambda_1 < \lambda$, $M_{s,\Delta} = \{t \in \mathbb{R}_+ : t - \delta_t \in [s, s+\Delta)\}$ and $\mu(\cdot)$ denotes the Lebesgue measure on $\mathbb{R}_+$.

It is worth noting that many time-varying delay functions in practice satisfy this assumption. For example, consider that δ_t is a Lipschitz continuous function with its Lipscitiz coefficient $\lambda_2 \in (0, 1)$. That is,

$$|\delta_t - \delta_s| \le \lambda_2(t-s), \ \forall\, 0 \le s < t < \infty. \tag{2.7}$$

For any $s \ge -\lambda$, let $r = \inf\{t \in M_{s,\Delta}\}$. It is easy to see that $r \in M_{s,\Delta}$, namely $s \le r - \delta_r < s + \Delta$. If $t \ge r + \Delta/(1-\lambda_2)$, then

$$t - \delta_t - s \ge t - \delta_t - (r - \delta_r) \ge t - r - |\delta_t - \delta_r| \ge (1-\lambda_2)(t-r) \ge \Delta.$$

Hence $t - \delta_t \ge s + \Delta$, i.e., $t \notin M_{s,\Delta}$. In other words, we get $M_{s,\Delta} \subset [r, r+\Delta/(1-\lambda_2))$, which implies $\mu(M_{s,\Delta})/\Delta \le 1/(1-\lambda_2)$. As this holds for arbitrary $s \ge -\lambda$ and $\Delta \in (0,1)$, Assumption 2.1 must hold with $\bar{\lambda} = 1/(1-\lambda_2)$. This, in particular, shows that many sawtooth delays (that occur frequently in sampled-data controls or network-based controls), e.g.,

$$\delta_t = \sum_{k=1}^{\infty} \big[(0.15 + 0.05(t-2k))I_{[2k,2k+1)}(t) + (0.25 - 0.05(t-2k))I_{[2k+1,2(k+1))}(t)\big],$$

satisfy Assumption 2.1.

LEMMA 2.2. *Let Assumption 2.1 hold. Let φ be a càdlàg function from $[-\lambda, \infty)$ to $\mathbb{R}_+$ such that it has at most finite number of jumps during any finite time interval. Then, for any $T > 0$,*

$$\int_0^T \varphi(t-\delta_t)dt \le \bar{\lambda} \int_{-\lambda}^{T-\lambda_1} \varphi(t)dt. \tag{2.8}$$

Proof. This lemma is a generalisation of [5, Lemma 2.2], where φ was assumed to be continuous. The proof here is different from that in [5] as we need to deal with the càdlàg property. By Assumption 2.1, for any $\varepsilon > 0$, there is a positive number $\bar{\Delta}$ such that

$$\sup_{s \geq -\lambda} \frac{\mu(M_{s,\Delta})}{\Delta} \leq \bar{\lambda} + \varepsilon, \quad \forall \Delta \in (0, \bar{\Delta}). \tag{2.9}$$

Fix any $T > 0$. We may assume, without loss of any generality, that φ has only one jump at $T_1 \in (-\lambda, T - \lambda_1)$, as the case of multiple jumps can be proved in the same fashion. Noting that $-\lambda \leq t - \delta_t \leq T - \lambda_1$ for $t \in [0, T]$, we divide the interval $[-\lambda, T - \lambda_1]$ into three parts $[-\lambda, T_1)$, $(T_1, T - \lambda_1]$ plus a single value set $\{T_1\}$. Let n_1 and n_2 be a pair of arbitrarily large integers such that $\Delta_1 := (T_1 + \lambda)/n_1 < \bar{\Delta}$ and $\Delta_2 := (T - \lambda_1 - T_1)/n_2 < \bar{\Delta}$. Set $t^1_u = -\lambda + u\Delta_1$ for $u = 0, 1, \cdots, n_1$ and $t^2_v = T_1 + v\Delta_2$ for $v = 0, 1, \cdots, n_2$. By the definition of the Riemann-Lebesgue integral, we have

$$\begin{aligned} \int_0^T \varphi(t - \delta_t)dt = {} & \lim_{n_1 \to \infty} \sum_{u=0}^{n_1 - 1} \mu(M_{t^1_u, \Delta_1})\varphi(t^1_u) + \lim_{n_2 \to \infty} \sum_{v=0}^{n_2 - 1} \mu(M_{t^2_v, \Delta_2})\varphi(t^2_v) \\ & + \Big[\varphi(T_1) - \varphi(T_1^-)\Big]\mu(M_{T_1}), \end{aligned} \tag{2.10}$$

where $M_{T_1} = \{t \in [-\lambda, T - \lambda_1] : t - \delta_t = T_1\}$. Let $\Delta_3 \in (0, 0.5\bar{\Delta})$ be arbitrarily small so that $T_1 - \Delta_3 > -\lambda$. Then $M_{T_1} \subset M_{T_1 - \Delta_3, 2\Delta_3}$ and, by (2.9), $\mu(M_{T_1}) \leq 2(\bar{\lambda} + \varepsilon)\Delta_3$. As Δ_3 is arbitrary, we must have $\mu(M_{T_1}) = 0$. By (2.9), we also have $\mu(M_{t^1_u, \Delta_1}) \leq (\bar{\lambda} + \varepsilon)\Delta_1$ and $\mu(M_{t^2_v, \Delta_2}) \leq (\bar{\lambda} + \varepsilon)\Delta_2$. It then follows from (2.10) that

$$\begin{aligned} \int_0^T \varphi(t - \delta_t)dt & \leq \lim_{n_1 \to \infty} \sum_{u=0}^{n_1 - 1} (\bar{\lambda} + \varepsilon)\Delta_1\varphi(t^1_u) + \lim_{n_2 \to \infty} \sum_{v=0}^{n_2 - 1} (\bar{\lambda} + \varepsilon)\Delta_2\varphi(t^2_v) \\ & = (\bar{\lambda} + \varepsilon)\int_{-\lambda}^{T_1} \varphi(t)dt + (\bar{\lambda} + \varepsilon)\int_{T_1}^{T - \lambda_1} \varphi(t)dt \\ & = (\bar{\lambda} + \varepsilon)\int_{-\lambda}^{T - \lambda_1} \varphi(t)dt. \end{aligned} \tag{2.11}$$

Letting $\varepsilon \to 0$ yields the required assertion (2.8). ∎

Remark 2.3. [5, Lemma 2.2] is not applicable to our SDDE as it requires the continuity of φ while the solution here is càdlàg. That is why we need to establish our new Lemma 2.2. Moreover, the proof of [5, Lemma 2.2] relies entirely on the continuity of φ while our proof here needs to deal with the càdlàg property.

We need to impose some assumptions on the coefficients.

Assumption 2.4. *Both coefficients f and g are locally Lipschitz continuous, and there exist positive constants p, q, α_1, α_2, α_3 with $p \wedge q > 2$ such that*

$$x^T f(x, y, i, t) + \frac{q - 1}{2}|g(x, y, i, t)|^2 \leq \alpha_1(|x|^2 + |y|^2) - \alpha_2|x|^p + \alpha_3|y|^p, \tag{2.12}$$

for all $(x, y, i, t) \in \mathbb{R}^d \times \mathbb{R}^d \times S \times \mathbb{R}_+$.

Assumption 2.5. *For any positive real number R, there exists a constant χ_R such that*

$$\int_{0<|z|<c} |h(x, y, i, t, z) - h(\bar{x}, \bar{y}, i, t, z)|\vartheta(dz) \leq \chi_R(|x - \bar{x}| + |y - \bar{y}|) \tag{2.13}$$

for all $x, \bar{x}, y, \bar{y} \in \mathbb{R}^d$ *with* $|x| \vee |\bar{x}| \vee |y| \vee |\bar{y}| \le R$ *and* $(i,t) \in S \times \mathbb{R}_+$*. There are also constants* $L > 0$ *and* $\alpha \ge 1$ *such that for all* $(x, y, i, t, z) \in \mathbb{R}^d \times \mathbb{R}^d \times S \times \mathbb{R}_+ \times \mathbb{R}_0^n$ *and* $0 < |z| < c$*,*

$$|h(x, y, i, t, z)| \le L|z|^\alpha(|x| + |y|). \tag{2.14}$$

Remark 2.6. It is quite standard to derive from $\int_{\mathbb{R}_0^n}(1 \wedge |z|^2)\vartheta(dz) < \infty$ that $\int_{0<|z|<c}|z|^r\vartheta(dz) < \infty$ for $r \ge 2$.

Condition (2.14) forces that $h(0, 0, i, t, z) \equiv 0$, which is naturally required for the stability purpose in this paper. The following two lemmas show the existence and uniqueness of the global solution and the finiteness of the moments.

Lemma 2.7. *Under Assumptions 2.1, 2.4 and 2.5, the SDDE (2.4) with the initial data (2.5) has a unique global solution* $x(t)$ *on* $[-\lambda, \infty)$ *and the solution has the properties that for all* $t \ge 0$

$$\mathbb{E}|x(t)|^q < \infty \tag{2.15}$$

and

$$\mathbb{E}\int_0^t |x(s)|^{p+q-2}ds < \infty. \tag{2.16}$$

Proof. To make the proof more understandable, we divide the whole proof into three steps.

Step 1. We claim that we can find two positive numbers β_1 and β_2 such that

$$\int_{0<|z|<c}\Big[|x + h(x, y, i, t, z)|^q - |x|^q - q|x|^{q-2}x^T h(x, y, i, t, z)\Big]\vartheta(dz) \le \beta_1|x|^q + \beta_2|y|^q. \tag{2.17}$$

To show this, we construct a function $F(s) = |x + sh_t(z)|^q$ for $s \ge 0$, where we use $h_t(z) := h(x, y, i, t, z)$ to simplify notation. By using the mean value theorem, there exists a constant $\xi_1 \in (0, 1)$ such that

$$\begin{aligned} F(1) - F(0) &= |x + h_t(z)|^q - |x|^q \\ &= q|x + \xi_1 h_t(z)|^{q-2}(x + \xi_1 h_t(z))^T h_t(z). \end{aligned} \tag{2.18}$$

Then construct a function $G(v) = q|x + v\xi_1 h_t(z)|^{q-2}(x + v\xi_1 h_t(z))^T h_t(z)$ for $v \ge 0$. Similarly, it can be shown that there exists a constant $\xi_2 \in (0, 1)$ such that

$$\begin{aligned} G(1) - G(0) &= q|x + \xi_1 h_t(z)|^{q-2}(x + \xi_1 h_t(z))^T h_t(z) - q|x|^{q-2}x^T h_t(z) \\ &\le \xi_1\Big\{q(q-1)|x + \xi_1\xi_2 h_t(z)|^{q-2}|h_t(z)|^2\Big\}. \end{aligned} \tag{2.19}$$

These imply

$$\begin{aligned} &|x + h(x, y, i, t, z)|^q - |x|^q - q|x|^{q-2}x^T h(x, y, i, t, z) \\ &\le \xi_1 q(q-1)(|x| + |h(x, y, i, t, z)|)^{q-2}|h(x, y, i, t, z)|^2 \\ &\le 2^{q-2}\xi_1 q(q-1)\Big(|x|^{q-2}|h(x, y, i, t, z)|^2 + |h(x, y, i, t, z)|^q\Big). \end{aligned} \tag{2.20}$$

Using (2.14) and the Young inequality, we can get

$$\begin{aligned}|x|^{q-2}|h(x,y,i,t,z)|^2 &\le 2L^2|z|^{2\alpha}(|x|^q+|x|^{q-2}|y|^2)\\ &\le 2L^2|z|^{2\alpha}\left(\frac{2(q-1)}{q}|x|^q+\frac{2}{q}|y|^q\right)\end{aligned}\tag{2.21}$$

and

$$\begin{aligned}|h(x,y,i,t,z)|^q &\le L^q|z|^{q\alpha}(|x|+|y|)^q\\ &\le 2^{q-1}L^q|z|^{q\alpha}(|x|^q+|y|^q).\end{aligned}\tag{2.22}$$

Substituting (2.20)-(2.22) into the left-hand-side terms of (2.17) and using Remark 2.6, we obtain (2.17) as claimed.

Step 2. Fix $T>0$ arbitrarily. Since almost every sample path of $r(\cdot)$ is a right-continuous step function with a finite number of simple jumps on $[0,T]$, there is a sequence $\{\varsigma_v\}_{v\ge 0}$ of stopping times such that for almost every $\omega\in\Omega$ there is a finite $\bar v=\bar v(\omega)$ for $0=\varsigma_0<\varsigma_1<\cdots<\varsigma_{\bar v}=T$ and $\varsigma_v=T$ if $v>\bar v$, and $r(\cdot)$ is a random constant on every interval $[\varsigma_u,\varsigma_{u+1})$, namely $r(t)=r(\varsigma_u)$ on $\varsigma_u\le t<\varsigma_{u+1}$ for all $u\ge 0$. For each integer $k\ge 1$ and $(x,y,i,t,z)\in\mathbb{R}^d\times\mathbb{R}^d\times S\times\mathbb{R}_+\times\mathbb{R}_0^n$, define the truncation functions

$$f_k(x,y,i,t)=f\left(\frac{|x|\wedge k}{|x|}x,\frac{|y|\wedge k}{|y|}y,i,t\right),$$

$g_k(x,y,i,t)$ and $h_k(x,y,i,t,z)$ similarly, where we set $((|x|\wedge k)/|x|)x=0$ when $x=0$. When $t\in[\varsigma_u,\varsigma_{u+1})$, by the similar method (see, e.g., [20, Theorem 3.3]), we can see that the equation

$$\begin{aligned}dx_k(t)=&f_k(x_k(t^-),x_k((t-\delta_t)^-),r(\varsigma_u),t)dt+g_k(x_k(t^-),x_k((t-\delta_t)^-),r(\varsigma_u),t)dB(t)\\ &+\int_{0<|z|<c}h_k(x_k(t^-),x_k((t-\delta_t)^-),r(\varsigma_u),t,z)\widetilde N(dt,dz),\end{aligned}$$

has a unique solution whenever $r(\varsigma_u)$ and $x_k(t)$ on $t\in[\varsigma_u-\lambda,\varsigma_u]$ are known. By induction, we therefore see that there is a unique solution $x_k(t)$ to the equation

$$\begin{aligned}dx_k(t)=&f_k(x_k(t^-),x_k((t-\delta_t)^-),r(t),t)dt+g_k(x_k(t^-),x_k((t-\delta_t)^-),r(t),t)dB(t)\\ &+\int_{0<|z|<c}h_k(x_k(t^-),x_k((t-\delta_t)^-),r(t),t,z)\widetilde N(dt,dz),\end{aligned}\tag{2.23}$$

on $t\in[0,T]$ with initial data $x_k(t)=\xi(t)$ on $t\in[-\lambda,0]$. Now we introduce a notation: if $\varpi(t)$, $t\ge-\lambda$ is a predictable process such that $\varpi(t)=\xi(t)$ on $-\lambda\le t\le 0$, define the stopping time

$$\rho_k(\varpi):=\inf\{t\in[0,T]:|\varpi(t)|\vee|\varpi(t-\delta_t)|\ge k\},$$

and set $\inf\emptyset=\infty$ in this paper. Following the method in the proof of [15, Theorem 2.2, pp. 95-97], we obtain that

$$\rho_k(x_k)\le\rho_k(x_{k+1})$$

and

$$x_k(t)=x_{k+1}(t)\text{ whenever }0\le t<\rho_k(x_k).\tag{2.24}$$

Set $e_k = \rho_k(x_k)$ and $e_\infty = \lim_{k\to\infty} e_k$. Define a local process $x(t)$, $t \in [-\lambda, e_\infty)$ as follows: $x(t) = \xi(t)$ on $t \in [-\lambda, 0]$ and if $e_{k-1} < e_k$,

$$x(t) = x_k(t),\ t \in [e_{k-1}, e_k),\ k \geq 1,$$

where $e_0 = 0$. If $e_{k-1} = e_k$, set $x(e_k) = x(e_{k-1})$. It follows from (2.24) that

$$x(t) = x_k(t) \text{ whenever } 0 < t < e_k.$$

So for every $k \geq 1$,

$$\begin{aligned} x((t\wedge e_k)^-) = x_k((t\wedge e_k)^-) = &\int_0^{(t\wedge e_k)^-} f_k(x_k(s^-), x_k((s-\delta_s)^-), r(s), s)ds \\ &+ \int_0^{(t\wedge e_k)^-} g_k(x_k(s^-), x_k((s-\delta_s)^-), r(s), s)dB(s) \\ &+ \int_0^{(t\wedge e_k)^-}\int_{0<|z|<c} h_k(x_k(s^-), x_k((s-\delta_s)^-), r(s), s, z)\widetilde{N}(ds, dz) + x(0) \\ = &\int_0^{(t\wedge e_k)^-} f(x(s^-), x((s-\delta_s)^-), r(s), s)ds + \int_0^{(t\wedge e_k)^-} g(x(s^-), x((s-\delta_s)^-), r(s), s)dB(s) \\ &+ \int_0^{(t\wedge e_k)^-}\int_{0<|z|<c} h(x(s^-), x((s-\delta_s)^-), r(s), s, z)\widetilde{N}(ds, dz) + x(0) \end{aligned}$$

for any $t \in [0, T]$. It is also easy to see that if $e_\infty < T$, and then

$$\limsup_{t\to e_\infty} |x(t)| = \limsup_{k\to\infty} |x(e_k^-)| = \limsup_{k\to\infty} |x_k(e_k^-)| = \infty.$$

Hence $\{x(t) : -\lambda \leq t < e_\infty\}$ is a maximal local solution on $[-\lambda, T]$. By the standard method (see, e.g., [19, Theorem 3.15, pp. 91-92]), the uniqueness can be proved. Letting $T \to \infty$, so we see that the hybrid SDDE (2.4) with the initial data (2.5) has a unique maximal local solution $x(t)$ on $[-\lambda, e_\infty)$, where e_∞ is the explosion time. We need to show $e_\infty = \infty$ a.s. Next, we define the stopping time

$$\sigma_\kappa = e_\infty \wedge \inf\{t \in [0, e_\infty) : |x(t)| \geq \kappa\}$$

for each integer $\kappa \geq \|\xi\|$. Because σ_κ is non-decreasing, it has a limit and we set $\sigma_\infty = \lim_{\kappa\to\infty} \sigma_\kappa$. So it is obvious to see that $\sigma_\infty \leq e_\infty$ a.s.

Step 3. Restrict $t \in [0, \lambda_1]$, so $x(t-\delta_t) = \xi(t-\delta_t)$ is already known because $-\lambda \leq t-\delta_t \leq 0$. By the generalised Itô formula (see, e.g., [27] or Lemma 2.10 below), Assumption 2.4 and (2.17), we get

$$\begin{aligned} \mathbb{E}|x(t\wedge\sigma_\kappa)|^q - |\xi(0)|^q \leq &\mathbb{E}\int_0^{t\wedge\sigma_\kappa} q|x(s^-)|^{q-2}\Big[\alpha_1(|x(s^-)|^2 + |x((s-\delta_s)^-)|^2) \\ &- \alpha_2|x(s^-)|^p + \alpha_3|x((s-\delta_s)^-)|^p\Big]ds \\ &+ \mathbb{E}\int_0^{t\wedge\sigma_\kappa} \Big(\beta_1|x(s^-)|^q + \beta_2|x((s-\delta_s)^-)|^q\Big)ds. \end{aligned} \tag{2.25}$$

An easy application of the Young inequality to $|x(s^-)|^{q-2}|x((s-\delta_s)^-)|^2$ and $\alpha_3|x(s^-)|^{q-2}|x((s-\delta_s)^-)|^p$ shows that (2.25) can be written as

$$\mathbb{E}|x(t\wedge\sigma_\kappa)|^q + 0.5q\alpha_2\mathbb{E}\int_0^{t\wedge\sigma_\kappa}|x(s^-)|^{p+q-2}ds$$
$$\le |\xi(0)|^q + \alpha_5 + (2q\alpha_1+\beta_1)\mathbb{E}\int_0^{t\wedge\sigma_\kappa}|x(s^-)|^q ds$$
$$= |\xi(0)|^q + \alpha_5 + (2q\alpha_1+\beta_1)\mathbb{E}\int_0^{t\wedge\sigma_\kappa}|x(s)|^q ds, \tag{2.26}$$

where $\alpha_5 = \int_0^{\lambda_1}[(2q\alpha_1+\beta_2)|x((s-\delta_s)^-)|^q + q\alpha_4|x((s-\delta_s)^-)|^{p+q-2}]ds$ is finite clearly, and

$$\alpha_4 = \frac{p}{p+q-2}\alpha_3^{\frac{p+q-2}{p}}\Big(\frac{2(q-2)}{\alpha_2(p+q-2)}\Big)^{\frac{q-2}{p+q-2}}.$$

Please note the last equality in (2.26) holds because the solution $x(t)$ has, almost surely, at most finite number of jumps during any finite time interval (see, e.g., [2]). This property will be used frequently in this paper and we will not explicitly state it unless it is necessary. The remaining proof is the same as in that of [5, Theorem 2.4] and is so omitted. □

Remark 2.8. Lemma 2.7 states an existence-and-unique theorem in the case of Lévy noise which is more general than Theorem 2.4 in [5]. In addition, the discontinuity of the local solutions makes it difficult to splice the local solutions into the global solution.

LEMMA 2.9. *Let Assumptions 2.1, 2.4 and 2.5 hold with* $\bar\alpha_1 > \bar\alpha_2\bar\lambda$*, where*

$$\bar\alpha_1 = q\alpha_2 - \frac{\alpha_3 q(q-2)}{p+q-2}, \quad \bar\alpha_2 = \frac{\alpha_3 qp}{p+q-2}. \tag{2.27}$$

Then the solution of the SDDE (2.4) with the initial data (2.5) has the properties that

$$\sup_{0\le t<\infty}\mathbb{E}|x(t)|^q < \infty \tag{2.28}$$

and

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t \mathbb{E}|x(s)|^{p+q-2}ds < \infty. \tag{2.29}$$

Proof. By the Itô formula, Assumption 2.4 and (2.17), it is easy to show

$$e^{\varepsilon_1 t}\mathbb{E}|x(t)|^q - |\xi(0)|^q \le \mathbb{E}\int_0^t e^{\varepsilon_1 s}\Big[q|x(s^-)|^{q-2}\big[\alpha_1(|x(s^-)|^2+|x((s-\delta_s)^-)|^2)$$
$$-\alpha_2|x(s^-)|^p + \alpha_3|x((s-\delta_s)^-)|^p\big] + \varepsilon_1|x(s^-)|^q\Big]ds$$
$$+\mathbb{E}\int_0^t e^{\varepsilon_1 s}\Big(\beta_1|x(s^-)|^q + \beta_2|x((s-\delta_s)^-)|^q\Big)ds, \tag{2.30}$$

where $\varepsilon_1 > 0$ is the unique root to the equation $\bar\alpha_1 - \varepsilon_1 = \bar\lambda(\bar\alpha_2+\varepsilon_1)e^{\varepsilon_1\lambda}$. By the Young inequality we get

$$e^{\varepsilon_1 t}\mathbb{E}|x(t)|^q - |\xi(0)|^q \le \mathbb{E}\int_0^t e^{\varepsilon_1 s}\Big(\bar\alpha_3|x(s^-)|^q + \bar\alpha_4|x((s-\delta_s)^-)|^q$$
$$-\bar\alpha_1|x(s^-)|^{p+q-2} + \bar\alpha_2|x((s-\delta_s)^-)|^{p+q-2}\Big)ds, \tag{2.31}$$

where $\bar{\alpha}_3 = \varepsilon_1 + 2\alpha_1(q-1) + \beta_1$ and $\bar{\alpha}_4 = 2\alpha_1 + \beta_2$. The remaining proof is the same as in that of [5, Theorem 2.6] and is hence omitted. ◻

To close the section, we cite the generalised Itô formula from [27] as a lemma, which show how a function $V : \mathbb{R}^d \times S \times \mathbb{R}_+ \to \mathbb{R}$ maps the paired process $(x(t), r(t))$ into a new Itô process $V(x(t), r(t), t)$.

LEMMA 2.10. [27] *Let $V \in C^{2,1}(\mathbb{R}^d \times S \times \mathbb{R}_+; \mathbb{R})$. Then $V(x(t), r(t), t)$ is an Itô process of the form*

$$V(x(t), r(t), t) = V(x(0), r(0), 0) + \int_0^t \mathcal{L}V(x(s^-), x((s-\delta_s)^-), r(s), s)ds + M(t), \tag{2.32}$$

where $\mathcal{L}V$ is a mapping from $\mathbb{R}^d \times \mathbb{R}^d \times S \times \mathbb{R}_+$ to $\mathbb{R}$ defined by

$$\begin{aligned}\mathcal{L}V(x, y, i, t) &= V_t(x, i, t) + V_x(x, i, t)f(x, y, i, t) + \sum_{j=1}^N \gamma_{ij} V(x, j, t) \\ &+ \int_{0<|z|<c} \Big\{ V(x+h(x, y, i, t, z), i, t) - V(x, i, t) - V_x(x, i, t)h(x, y, i, t, z) \Big\} \vartheta(dz) \\ &+ \frac{1}{2}\text{trace}[g^T(x, y, i, t)V_{xx}(x, i, t)g(x, y, i, t)],\end{aligned} \tag{2.33}$$

while

$$\begin{aligned}M(t) = &\int_0^t V_x(x(s^-), r(s), s)g(x(s^-), x((s-\delta_s)^-), r(s), s)dB(s) \\ &+ \int_0^t \int_{0<|z|<c} \Big[V(x(s^-) + h(x(s^-), x((s-\delta_s)^-), r(s), s, z), r(s), s) \\ &\qquad - V(x(s^-), r(s), s) \Big] \widetilde{N}(ds, dz) \\ &+ \int_0^t \int_R \Big[V(x(s^-), r(0) + b(r(s), \iota), s) - V(x(s^-), r(s), s) \Big] \mu^*(ds, d\iota),\end{aligned}$$

where the function b from $S \times \mathbb{R}$ to $\mathbb{R}$ is defined by

$$b(i, l) = \begin{cases} j - i, & \text{if } l \in \Delta_{ij}, \\ 0, & \text{otherwise}, \end{cases}$$

and $\mu^*(ds, d\iota) = \vartheta^*(ds, d\iota) - ds \times m(d\iota)$ is a martingale measure. Here $\vartheta^*(ds, d\iota)$ is a Poisson measure on $\mathbb{R}_+ \times \mathbb{R}$ with intensity $ds \times m(d\iota)$, in which $m(d\iota)$ is the Lebesgue measure on $\mathbb{R}$ and Δ_{ij} is consecutive, left closed, right open intervals of the real line each have length γ_{ij}. Further details can be found in [19, pp. 46–48].

3. Controlled SDDE. In this section, we aim to design a delay feedback control $u(x((t-\tau_t)^-), r(t), t)$ for the controlled SDDE

$$\begin{aligned}dx(t) = &\Big[f(x(t^-), x((t-\delta_t)^-), r(t), t) + u(x((t-\tau_t)^-), r(t), t) \Big] dt \\ &+ g(x(t^-), x((t-\delta_t)^-), r(t), t)dB(t) \\ &+ \int_{0<|z|<c} h(x(t^-), x((t-\delta_t)^-), r(t), t, z)\widetilde{N}(dt, dz)\end{aligned} \tag{3.1}$$

to become stable. Here the control function $u : \mathbb{R}^d \times S \times \mathbb{R}_+ \to \mathbb{R}^d$ is Borel measurable and satisfies the following assumption.

ASSUMPTION 3.1. *There exists a positive constant β such that*

$$|u(x,i,t) - u(y,i,t)| \le \beta |x-y| \tag{3.2}$$

for all $(x,y,i,t) \in \mathbb{R}^d \times \mathbb{R}^d \times S \times \mathbb{R}_+$. Moreover, for the stability purpose, we require that $u(0,i,t) \equiv 0$.

The following theorem shows that the controlled SDDE (3.1) preserves the property of the unique global solution.

THEOREM 3.2. *Let the control time lag τ_t be a Borel measurable function from $\mathbb{R}_+$ to $[0,\bar{\tau}]$, where $\bar{\tau}$ is a positive number. Under Assumptions 2.1, 2.4, 2.5 and 3.1, the controlled SDDE (3.1) with initial data*

$$\{x(t) : -\lambda_0 \le t \le 0\} = \xi \in D^b_{\mathcal{F}_0}([-\lambda_0, 0]; \mathbb{R}^d) \text{ and } r(0) = i_0 \tag{3.3}$$

has a unique global solution $x(t)$ on $[-\lambda_0, \infty)$, and the solution has properties (2.15) and (2.16), where $\lambda_0 = \lambda \vee \bar{\tau}$. Moreover, if we also make $\bar{\alpha}_1 > \bar{\alpha}_2 \bar{\lambda}$ hold, where $\bar{\alpha}_1$ and $\bar{\alpha}_2$ have been given in (2.27), the solution has properties (2.28) and (2.29).

This theorem can be proved in a similar fashion as Lemmas 2.7 and 2.9 were proved. As mentioned in the previous section, we consider the situation in this paper where both f and g satisfy the polynomial growth condition. The following assumption describes this situation.

ASSUMPTION 3.3. *There exist constants $K > 0$, $q_1 > 1$ and $q_i \ge 1$ $(i = 2,3,4)$ such that*

$$\begin{aligned} &|f(x,y,i,t)| \le K(|x| + |y| + |x|^{q_1} + |y|^{q_2}), \\ &|g(x,y,i,t)| \le K(|x| + |y| + |x|^{q_3} + |y|^{q_4}) \end{aligned} \tag{3.4}$$

for all $(x,y,i,t) \in \mathbb{R}^d \times \mathbb{R}^d \times S \times \mathbb{R}_+$. In addition, p and q in Assumption 2.4 also need to meet

$$q > (p + q_1 - 1) \vee (2(q_1 \vee q_2 \vee q_3 \vee q_4)), \tag{3.5}$$

$$p \ge 2(q_1 \vee q_2 \vee q_3 \vee q_4) - q_1 + 1. \tag{3.6}$$

This assumption guarantees, for example, $\mathbb{E}|f(x(t^-), x((t-\delta_t)^-), r(t), t)|^2 < \infty$, and hence the stabilization analysis below can be carried out in L^2. To make the controlled SDDE (3.1) stable, the control function needs to meet more conditions. Our first key condition is:

CONDITION 3.4. *Design the control function $u : \mathbb{R}^d \times S \times \mathbb{R}_+ \to \mathbb{R}^d$ so that we can find real numbers a_i, $\bar{a}_i$, positive numbers $\hat{a}_i$, $\hat{b}_i$, c_i, $\bar{c}_i$ and nonnegative numbers $b_i, \bar{b}_i, d_i, \bar{d}_i$ $(i \in S)$ such that for all $(x,y,i,t) \in \mathbb{R}^d \times \mathbb{R}^d \times S \times \mathbb{R}_+$,*

$$\begin{aligned} &2\Big[x^T[f(x,y,i,t) + u(x,i,t)] + \frac{1}{2}|g(x,y,i,t)|^2\Big] \\ &+ \int_{0<|z|<c} \Big[|x + h(x,y,i,t,z)|^2 - |x|^2 - 2x^T h(x,y,i,t,z)\Big] \vartheta(dz) \\ &\le a_i|x|^2 + b_i|y|^2 - c_i|x|^p + d_i|y|^p, \end{aligned} \tag{3.7}$$

$$x^T[f(x,y,i,t)+u(x,i,t)]+\frac{q_1}{2}|g(x,y,i,t)|^2 \le \bar{a}_i|x|^2+\bar{b}_i|y|^2-\bar{c}_i|x|^p+\bar{d}_i|y|^p, \tag{3.8}$$

and

$$\begin{aligned}&\int_{0<|z|<c}\Big[|x+h(x,y,i,t,z)|^{q_1+1}-|x|^{q_1+1}-(q_1+1)|x|^{q_1-1}x^T h(x,y,i,t,z)\Big]\vartheta(dz)\\ &\le \hat{a}_i|x|^{q_1+1}+\hat{b}_i|y|^{q_1+1},\end{aligned} \tag{3.9}$$

while both

$$\begin{aligned} A_1 &:= -diag(a_1,\cdots,a_N)-\Gamma\\ \text{and } A_2 &:= -diag((q_1+1)\bar{a}_1+\hat{a}_1,\cdots,(q_1+1)\bar{a}_N+\hat{a}_N)-\Gamma,\end{aligned} \tag{3.10}$$

are nonsingular M-matrices; and moreover,

$$\begin{cases} 1>\zeta_1, \zeta_2>\bar{\lambda}\zeta_3,\\ 1>\frac{\zeta_4[q_1-1+2\bar{\lambda}]}{q_1+1},\\ \zeta_5>\frac{\zeta_6[q_1-1+p\bar{\lambda}]}{p+q_1-1},\end{cases} \tag{3.11}$$

where q_1 is the same as in Assumption 3.3,

$$\begin{aligned} &\zeta_1=\max_{i\in S}\theta_i b_i, && \zeta_2=\min_{i\in S}\theta_i c_i,\\ &\zeta_3=\max_{i\in S}\theta_i d_i, && \zeta_4=\max_{i\in S}[(q_1+1)\bar{b}_i+\hat{b}_i]\bar{\theta}_i,\\ &\zeta_5=\min_{i\in S}(q_1+1)\bar{\theta}_i\bar{c}_i, && \zeta_6=\max_{i\in S}(q_1+1)\bar{\theta}_i\bar{d}_i,\end{aligned} \tag{3.12}$$

in which

$$\begin{aligned} (\theta_1,\cdots,\theta_N)^T &= A_1^{-1}(1,\cdots,1)^T,\\ (\bar{\theta}_1,\cdots,\bar{\theta}_N)^T &= A_2^{-1}(1,\cdots,1)^T.\end{aligned} \tag{3.13}$$

It is useful to point out that all θ_i and $\bar{\theta}_i$ defined by (3.13) are positive as both A_1 and A_2 are nonsingular M-matrices (see, e.g., [19, Section 2.6]).

Let us explain that there are lots of such control functions available under Assumptions 2.4 and 2.5. To make the explanation simpler, we assume $\alpha_2>\alpha_3\bar{\lambda}$ in addition to Assumptions 2.4 and 2.5. For example, design the control function $u(x,i,t)=Ax^T$, where A is a symmetric $d\times d$ real-valued negative-definite matrix such that $\lambda_{\max}(A)\le -(k+1)\alpha_1-0.5\tilde{\beta}_1$ with $k>1$, where $\tilde{\beta}_1$ will be determined later. Then

$$x^T u(x,i,t)\le -[(k+1)\alpha_1+0.5\tilde{\beta}_1]|x|^2, \ \forall\,(x,i,t)\in\mathbb{R}^d\times S\times\mathbb{R}_+.$$

Using Assumption 2.4 while noting that $q-1\ge q_1>1$ and $q_1+1>2$, we have

$$x^T[f(x,y,i,t)+u(x,i,t)]+\frac{1}{2}|g(x,y,i,t)|^2 \le -\Big(k\alpha_1+\frac{\tilde{\beta}_1}{2}\Big)|x|^2+\alpha_1|y|^2-\alpha_2|x|^p+\alpha_3|y|^p$$

and

$$\begin{aligned}&x^T[f(x,y,i,t)+u(x,i,t)]+\frac{q_1}{2}|g(x,y,i,t)|^2\\ &\le -\Big(k\alpha_1+\frac{\tilde{\beta}_1}{q_1+1}\Big)|x|^2+\alpha_1|y|^2-\alpha_2|x|^p+\alpha_3|y|^p.\end{aligned}$$

By Assumption 2.5, we can show as property (2.17) was proved that there exist two positive numbers $\tilde{\beta}_1$ and $\tilde{\beta}_2$ such that

$$\int_{0<|z|<c}\Big[|x+h(x,y,i,t,z)|^2-|x|^2-2x^Th(x,y,i,t,z)\Big]\vartheta(dz)\le\tilde{\beta}_1|x|^2+\tilde{\beta}_2|y|^2$$

and

$$\int_{0<|z|<c}\Big[|x+h(x,y,i,t,z)|^{q_1+1}-|x|^{q_1+1}-(q_1+1)|x|^{q_1-1}x^Th(x,y,i,t,z)\Big]\vartheta(dz)$$
$$\le\tilde{\beta}_1|x|^{q_1+1}+\tilde{\beta}_2|y|^{q_1+1}.$$

In other words, we have already verified (3.7) - (3.9). Consequently, we further have

$$\begin{aligned} A_1 &:= 2k\,\mathrm{diag}(\alpha_1,\cdots,\alpha_1)-\Gamma \\ \text{and }\ A_2 &:= (q_1+1)k\,\mathrm{diag}(\alpha_1,\cdots,\alpha_1)-\Gamma, \end{aligned}$$

which are nonsingular M-matrices (see, e.g., [19, Section 2.6]). Moreover, when k is sufficiently large, $\theta_i\approx 1/(2k\alpha_1)$ and $\bar{\theta}_i\approx 1/((q_1+1)k\alpha_1)$ for all $i\in S$. Hence, ζ_1 - ζ_6 defined by (3.12) are

$$\zeta_1\approx\frac{2\alpha_1+\tilde{\beta}_2}{2k\alpha_1},\ \zeta_2=\zeta_5\approx\frac{\alpha_2}{k\alpha_1},\ \zeta_4\approx\frac{(q_1+1)\alpha_1+\tilde{\beta}_2}{(q_1+1)k\alpha_1},\ \zeta_3=\zeta_6\approx\frac{\alpha_3}{k\alpha_1}.$$

It then easy to see (3.11) is satisfied, bearing in mind that $\alpha_2>\alpha_3\bar{\lambda}$. In other words, for a sufficiently large number k, the control function $u(x,i,t)=Ax^T$ meets Condition 3.4 as long as $\lambda_{\max}(A)\le-(k+1)\alpha_1-0.5\tilde{\beta}_1$. Of course, in application, we need to make full use of the special forms of the coefficients f, g and h to design the control function u more wisely.

Let us now explain why we propose Condition 3.4. If there is no time delay in the controller (i.e., $\tau_t\equiv 0$), the controlled SDDE (3.1) becomes

$$\begin{aligned} dx(t)=&\Big[f(x(t^-),x((t-\delta_t)^-),r(t),t)+u(x(t^-),r(t),t)\Big]dt\\ &+g(x(t^-),x((t-\delta_t)^-),r(t),t)dB(t)\\ &+\int_{0<|z|<c}h(x(t^-),x((t-\delta_t)^-),r(t),t,z)\widetilde{N}(dt,dz). \end{aligned} \tag{3.14}$$

Define a function $U:\mathbb{R}^d\times S\to\mathbb{R}_+$ by

$$U(x,i)=\theta_i|x|^2+\bar{\theta}_i|x|^{q_1+1},\ (x,i)\in\mathbb{R}^d\times S, \tag{3.15}$$

and then, according to Lemma 2.10, the function $\mathcal{L}U:\mathbb{R}^d\times\mathbb{R}^d\times S\times\mathbb{R}_+\to\mathbb{R}$ is given by

$$
\begin{aligned}
\mathcal{L}U(x,y,i,t) = {} & 2\theta_i \Big[x^T[f(x,y,i,t)+u(x,i,t)] + \frac{1}{2}|g(x,y,i,t)|^2\Big] \\
& + (q_1+1)\bar{\theta}_i \Big[|x|^{q_1-1}x^T[f(x,y,i,t)+u(x,i,t)] + \frac{1}{2}|x|^{q_1-1}|g(x,y,i,t)|^2 \\
& + \frac{q_1-1}{2}|x|^{q_1-3}|x^T g(x,y,i,t)|^2\Big] + \sum_{j=1}^{N}\gamma_{ij}(\theta_j|x|^2 + \bar{\theta}_j|x|^{q_1+1}) \\
& + \int_{0<|z|<c} \bar{\theta}_i \Big[|x+h(x,y,i,t,z)|^{q_1+1} - |x|^{q_1+1} - (q_1+1)|x|^{q_1-1}x^T h(x,y,i,t,z)\Big]\vartheta(dz) \\
& + \int_{0<|z|<c} \theta_i \Big[|x+h(x,y,i,t,z)|^2 - |x|^2 - 2x^T h(x,y,i,t,z)\Big]\vartheta(dz). \qquad (3.16)
\end{aligned}
$$

By making use of (3.7)-(3.11) and the Young inequality, (3.16) can be estimated by

$$
\begin{aligned}
\mathcal{L}U(x,y,i,t) \le {} & -|x|^2 + \zeta_1|y|^2 - \zeta_2|x|^p + \zeta_3|y|^p - \left(1 - \frac{\zeta_4(q_1-1)}{q_1+1}\right)|x|^{q_1+1} \\
& + \frac{2\zeta_4}{q_1+1}|y|^{q_1+1} - \left(\zeta_5 - \frac{\zeta_6(q_1-1)}{p+q_1-1}\right)|x|^{p+q_1-1} + \frac{\zeta_6 p}{p+q_1-1}|y|^{p+q_1-1}. \qquad (3.17)
\end{aligned}
$$

Now we propose the second condition to cope with the highly nonlinear nature of the underlying SDDE.

CONDITION 3.5. *Find nine positive numbers* v_j $(1 \le j \le 9)$ *and a function* $W \in C(\mathbb{R}^d;\mathbb{R}_+)$ *such that*

$$
\begin{aligned}
& \mathcal{L}U(x,y,i,t) + v_1(2\theta_i|x| + (q_1+1)\bar{\theta}_i|x|^{q_1})^2 + v_2|f(x,y,i,t)|^2 + v_3|g(x,y,i,t)|^2 \\
& + v_4\int_{0<|z|<c}|h(x,y,i,t,z)|^2\vartheta(dz) \le -v_5|x|^2 + v_6|y|^2 - W(x) + v_7W(y), \qquad (3.18)
\end{aligned}
$$

and

$$
v_8|x|^{p+q_1-1} \le W(x) \le v_9(1+|x|^{p+q_1-1}), \qquad (3.19)
$$

for all $(x,y,i,t,z) \in \mathbb{R}^d \times \mathbb{R}^d \times S \times \mathbb{R}_+ \times \mathbb{R}_0^n$, *where* $v_5 > v_6\bar{\lambda}$ *and* $v_7 \in (0, 1/\bar{\lambda})$.

Let us now explain why it is always possible to meet this rule under Assumptions 2.4, 2.5 and 3.3, and property (2.17). In fact, by (3.4),

$$
\begin{aligned}
& \text{the left-hand-side terms of (3.18)} \\
& \le \mathcal{L}U(x,y,i,t) + 8v_1\theta_i^2|x|^2 + 2v_1(q_1+1)^2\bar{\theta}_i^2|x|^{2q_1} + v_4(|x|^2+|y|^2) \\
& + 4v_2K^2(|x|^2+|y|^2+|x|^{2q_1}+|y|^{2q_2}) + 4v_3K^2(|x|^2+|y|^2+|x|^{2q_3}+|y|^{2q_4}). \qquad (3.20)
\end{aligned}
$$

From (3.6), it is easy to see that $p+q_1-1 \ge 2(q_1 \vee q_2 \vee q_3 \vee q_4)$ and hence

$$
w^{2q_i} \le w^2 + w^{p+q_1-1}, \ \forall\, w \ge 0,\ 1 \le i \le 4.
$$

By using these inequalities and (3.17), we can always choose v_1, v_2, v_3 and v_4 sufficiently small such that

$$
\begin{aligned}
& \text{the left-hand-side terms of (3.18)} \\
& \le -v_5|x|^2 - \bar{\xi}_1|x|^p - \bar{\xi}_3|x|^{q_1+1} - \bar{\xi}_5|x|^{p+q_1-1} \\
& + v_6|y|^2 + \bar{\xi}_2|y|^p + \bar{\xi}_4|y|^{q_1+1} + \bar{\xi}_6|y|^{p+q_1-1}, \qquad (3.21)
\end{aligned}
$$

where v_5, v_6 and $\bar{\xi}_j$ $(1 \le j \le 6)$ are all positive numbers such that $v_5 > v_6\bar{\lambda}$ and $\bar{\xi}_{2k-1} > \bar{\xi}_{2k}\bar{\lambda}$ for $1 \le k \le 3$. Letting

$$W(x) = \bar{\xi}_1|x|^p + \bar{\xi}_3|x|^{q_1+1} + \bar{\xi}_5|x|^{p+q_1-1} \text{ for } x \in \mathbb{R}^d$$

and $v_7 = \max_{1\le k\le 3} \bar{\xi}_{2k}/\bar{\xi}_{2k-1}$, $v_8 = \bar{\xi}_5$ and $v_9 = \bar{\xi}_1 + \bar{\xi}_3 + \bar{\xi}_5$. Therefore, we see that $v_7 \in (0, 1/\bar{\lambda})$,

$$\text{the left-hand-side terms of (3.18)} \le -v_5|x|^2 + v_6|y|^2 - W(x) + v_7W(y),$$

and $v_8|x|^{p+q_1-1} \le W(x) \le v_9(1 + |x|^{p+q_1-1})$.

Hence, we have shown that it is always possible to satisfy Condition 3.5. Of course, in application, we need to make full use of the special forms of the coefficients f, g and h to choose $v_1 - v_9$ more wisely in order to have a larger bound on $\bar{\tau}$ as described in the statements of theorems in the following section.

4. Exponential stabilization. In this section, we will establish several new theorems on the stabilization by the delay feedback control.

THEOREM 4.1. *Let Assumptions 2.1, 2.4, 2.5 and 3.3 hold. Design a control function u satisfying Assumption 3.1 to meet Condition 3.4 and then find nine positive constants v_j $(1 \le j \le 9)$ and a function $W \in C(\mathbb{R}^d; \mathbb{R}_+)$ to meet Condition 3.5. If the upper bound $\bar{\tau}$ of time lag τ_t satisfies*

$$\bar{\tau} < \frac{\sqrt{(v_5 - v_6\bar{\lambda})v_1}}{\sqrt{3}\beta^2} \wedge \frac{\sqrt{v_1v_2}}{\sqrt{2}\beta} \wedge \frac{v_1v_3}{\beta^2} \wedge \frac{v_1v_4}{\beta^2} \wedge \frac{1}{12\beta}, \tag{4.1}$$

then the solution of the controlled SDDE (3.1) with initial value (3.3) has the following property

$$\limsup_{t\to\infty} \frac{1}{t} \log(\mathbb{E}|x(t)|^2) < 0. \tag{4.2}$$

That is, the controlled system (3.1) is exponentially stable in mean square.

Proof. We will use the method of Lyapunov functionals (see, e.g., [19]) to prove the theorem. For this purpose, we define two segments $\widetilde{x}_t := \{x(t+s) : -2\lambda_0 \le s \le 0\}$ and $\widetilde{r}_t := \{r(t+s) : -2\lambda_0 \le s \le 0\}$ for $t \ge 2\lambda_0$, so $\widetilde{x}_t$ and $\widetilde{r}_t$ will be defined for $0 \le t \le 2\lambda_0$. Let $x(s) = \xi(-\lambda_0)$ for $s \in [-2\lambda_0, -\lambda_0)$ and $r(s) = r(0)$ for $s \in [-2\lambda_0, 0)$.

Step 1. The Lyapunov functional used in this proof has the form

$$V(\widetilde{x}_t, \widetilde{r}_t, t) = U(x(t), r(t)) + \frac{\beta^2}{v_1}\psi(t) \tag{4.3}$$

for $t \ge 2\lambda_0$, where U has been defined by (3.15) and

$$\psi(t) = \int_{-\bar{\tau}}^0 \int_{t+s}^t \Big[\bar{\tau}|f_{v^-} + u_{v^-}|^2 + |g_{v^-}|^2 + \int_{0<|z|<c} |h_{v^-}(z)|^2\vartheta(dz)\Big]dvds. \tag{4.4}$$

In this proof, we use $f_{v^-} = f(x(v^-), x((v-\delta_v)^-), r(v), v)$, $u_{v^-} = u(x((v-\tau_v)^-), r(v), v)$, $g_{v^-} = g(x(v^-), x((v-\delta_v)^-), r(v), v)$ and $h_{v^-}(z) = h(x(v^-), x((v-\delta_v)^-), r(v), v, z)$ for $v \ge 0$ to simplify notations.

Let ε is a sufficiently small positive number which will be determined later. Applying Lemma 2.10, we get that

$$e^{\varepsilon t}V(\widetilde{x}_t, \widetilde{r}_t, t) = C + \int_{2\lambda_0}^t e^{\varepsilon s}\Big(\varepsilon V(\widetilde{x}_{s^-}, \widetilde{r}_s, s) + \mathbb{L}V(\widetilde{x}_{s^-}, \widetilde{r}_s, s)\Big)ds + M_t, \tag{4.5}$$

where $\widetilde{x}_{s^-} = \lim_{v\uparrow s}\widetilde{x}_v$, $C = e^{2\varepsilon\lambda_0}V(\widetilde{x}_{2\lambda_0}, \widetilde{r}_{2\lambda_0}, 2\lambda_0)$,

$$\begin{aligned} M_t = &\int_{2\lambda_0}^t e^{\varepsilon s}V_x(\widetilde{x}_{s^-}, \widetilde{r}_s, s)g_s dB(s) \\ &+ \int_{2\lambda_0}^t \int_R e^{\varepsilon s}\Big[V(\widetilde{x}_{s^-}, i_0 + b(\widetilde{r}_s, \iota), s) - V(\widetilde{x}_{s^-}, \widetilde{r}_s, s)\Big]\mu(ds, d\iota) \\ &+ \int_{2\lambda_0}^t \int_{0<|z|<c} e^{\varepsilon s}\Big[V(\widetilde{x}_{s^-} + h_{s^-}(z), \widetilde{r}_s, s) - V(\widetilde{x}_{s^-}, \widetilde{r}_s, s)\Big]\widetilde{N}(ds, dz) \end{aligned}$$

is a real-valued local martingale (see, e.g., [2, 12]), and

$$\begin{aligned} \mathbb{L}V(\widetilde{x}_{s^-}, \widetilde{r}_s, s) = &\frac{\beta^2\bar{\tau}}{v_1}\Big[\bar{\tau}|f_{s^-} + u_{s^-}|^2 + |g_{s^-}|^2 + \int_{0<|z|<c}|h_{s^-}(z)|^2\vartheta(dz)\Big] \\ &+ \mathcal{L}U(x(s^-), x((s-\delta_s)^-), r(s), s) + [2\theta_{r(s)} + (q_1+1)\bar{\theta}_{r(s)}|x(s^-)|^{q_1-1}]x^T(s^-) \\ &\quad\times [u(x((s-\tau_s)^-), r(s), s) - u(x(s^-), r(s), s)] \\ &- \frac{\beta^2}{v_1}\int_{s-\bar{\tau}}^s \Big[\bar{\tau}|f_{v^-} + u_{v^-}|^2 + |g_{v^-}|^2 + \int_{0<|z|<c}|h_{v^-}(z)|^2\vartheta(dz)\Big]dv. \end{aligned} \tag{4.6}$$

By Assumptions 2.4, 2.5, 3.1, 3.3 and Theorem 3.2 as well as Condition 3.4, it is obvious that

$$\mathbb{E}|\mathbb{L}V(\widetilde{x}_{s^-}, \widetilde{r}_s, s)| < \infty, \quad \forall s \geq 2\lambda_0. \tag{4.7}$$

This enables us to proceed without using the technique of stopping times in the next steps.

Setting $\eta_1 = \min_{i\in S}\theta_i$, $\eta_2 = \max_{i\in S}\theta_i$ and $\eta_3 = \max_{i\in S}\bar{\theta}_i$, and taking the expectation on both sides of (4.5), we get

$$\begin{aligned} \eta_1 e^{\varepsilon t}\mathbb{E}|x(t)|^2 \leq &C_1 + \frac{\varepsilon\beta^2}{v_1}\phi_1(t) + \int_{2\lambda_0}^t e^{\varepsilon s}\mathbb{E}\mathbb{L}V(\widetilde{x}_{s^-}, \widetilde{r}_s, s)ds \\ &+ \int_{2\lambda_0}^t \varepsilon e^{\varepsilon s}\Big(\eta_2\mathbb{E}|x(s^-)|^2 + \eta_3\mathbb{E}|x(s^-)|^{q_1+1}\Big)ds, \end{aligned} \tag{4.8}$$

where $C_1 = e^{2\varepsilon\lambda_0}\mathbb{E}V(\widetilde{x}_{2\lambda_0}, \widetilde{r}_{2\lambda_0}, 2\lambda_0)$ and

$$\phi_1(t) = \mathbb{E}\int_{2\lambda_0}^t e^{\varepsilon s}\Big(\int_{-\bar{\tau}}^0\int_{s+u}^s \Big[\bar{\tau}|f_{v^-} + u_{v^-}|^2 + |g_{v^-}|^2 + \int_{0<|z|<c}|h_{v^-}(z)|^2\vartheta(dz)\Big]dvdu\Big)ds.$$

Step 2. Let us estimate $\mathbb{L}V(\widetilde{x}_{s^-}, \widetilde{r}_s, s)$. Firstly, it follows from Assumption 3.1 that

$$\begin{aligned} &[2\theta_{r(s)} + (q_1+1)\bar{\theta}_{r(s)}|x(s^-)|^{q_1-1}]x^T(s^-)[u(x((s-\tau_s)^-), r(s), s) - u(x(s^-), r(s), s)] \\ &\leq v_1[2\theta_{r(s)}|x(s^-)| + (q_1+1)\bar{\theta}_{r(s)}|x(s^-)|^{q_1}]^2 + \frac{\beta^2}{4v_1}|x(s^-) - x((s-\tau_s)^-)|^2. \end{aligned} \tag{4.9}$$

Next we observe from (4.1) that

$$\frac{2\beta^2\bar{\tau}^2}{v_1} \leq v_2, \quad \frac{\beta^2\bar{\tau}}{v_1} \leq v_3, \quad \frac{\beta^2\bar{\tau}}{v_1} \leq v_4. \tag{4.10}$$

It then follows from (4.6) along with Condition 3.5 and Assumption 3.1 that

$$\begin{aligned}\mathbb{L}V(\widetilde{x}_{s^-},\widetilde{r}_s,s) \le & -v_5|x(s^-)|^2+v_6|x((s-\delta_s)^-)|^2-W(x(s^-))+v_7W(x((s-\delta_s)^-))\\ &+\frac{2\beta^4\bar{\tau}^2}{v_1}|x((s-\tau_s)^-)|^2+\frac{\beta^2}{4v_1}|x(s^-)-x((s-\tau_s)^-)|^2\\ &-\frac{\beta^2}{v_1}\int_{s-\bar{\tau}}^{s}\Big[\bar{\tau}|f_{v^-}+u_{v^-}|^2+|g_{v^-}|^2+\int_{0<|z|<c}|h_{v^-}(z)|^2\vartheta(dz)\Big]dv.\end{aligned} \tag{4.11}$$

Noting that $\beta\bar{\tau}\le 1/12$, we have

$$\frac{2\beta^4\bar{\tau}^2}{v_1}|x((s-\tau_s)^-)|^2 \le \frac{3\beta^4\bar{\tau}^2}{v_1}|x(s^-)|^2+\frac{\beta^2}{24v_1}|x(s^-)-x((s-\tau_s)^-)|^2. \tag{4.12}$$

Finally, taking the expectation on both sides of (4.11), and then combing with (4.12), we get

$$\begin{aligned}\mathbb{E}\mathbb{L}V(\widetilde{x}_{s^-},\widetilde{r}_s,s) \le & -\Big(v_5-\frac{3\beta^4\bar{\tau}^2}{v_1}\Big)\mathbb{E}|x(s^-)|^2+v_6\mathbb{E}|x((s-\delta_s)^-)|^2-\mathbb{E}W(x(s^-))\\ &+v_7\mathbb{E}W(x((s-\delta_s)^-))+\frac{7\beta^2}{24v_1}\mathbb{E}|x(s^-)-x((s-\tau_s)^-)|^2\\ &-\frac{\beta^2}{v_1}\mathbb{E}\int_{s-\bar{\tau}}^{s}\Big[\bar{\tau}|f_{v^-}+u_{v^-}|^2+|g_{v^-}|^2+\int_{0<|z|<c}|h_{v^-}(z)|^2\vartheta(dz)\Big]dv.\end{aligned} \tag{4.13}$$

Step 3. It is obvious to see that

$$\begin{aligned}\mathbb{E}|x(s^-)|^{q_1+1} &\le \mathbb{E}|x(s^-)|^2+\mathbb{E}|x(s^-)|^{p+q_1-1}\\ &\le \mathbb{E}|x(s^-)|^2+v_8^{-1}\mathbb{E}W(x(s^-)).\end{aligned} \tag{4.14}$$

By Lemma 2.2, we have

$$\int_{2\lambda_0}^{t} e^{\varepsilon s}\mathbb{E}|x((s-\delta_s)^-)|^2ds \le \bar{\lambda}e^{\varepsilon\lambda}\int_{-\lambda}^{t} e^{\varepsilon s}\mathbb{E}|x(s^-)|^2ds, \tag{4.15}$$

$$\int_{2\lambda_0}^{t} e^{\varepsilon s}\mathbb{E}W(x((s-\delta_s)^-))ds \le \bar{\lambda}e^{\varepsilon\lambda}\int_{-\lambda}^{t} e^{\varepsilon s}\mathbb{E}W(x(s^-))ds. \tag{4.16}$$

Substituting (4.13)-(4.16) into (4.8) we obtain

$$\begin{aligned}\eta_1e^{\varepsilon t}\mathbb{E}|x(t)|^2 \le & C_2+\frac{\varepsilon\beta^2}{v_1}\phi_1(t)-\frac{\beta^2}{v_1}\phi_2(t)+\frac{7\beta^2}{24v_1}\int_{2\lambda_0}^{t} e^{\varepsilon s}\mathbb{E}|x(s^-)-x((s-\tau_s)^-)|^2ds\\ &-\Big(1-v_7\bar{\lambda}e^{\varepsilon\lambda}-\frac{\varepsilon\eta_3}{v_8}\Big)\int_{2\lambda_0}^{t} e^{\varepsilon s}\mathbb{E}W(x(s^-))ds\\ &-\Big(v_5-v_6\bar{\lambda}e^{\varepsilon\lambda}-\frac{3\beta^4\bar{\tau}^2}{v_1}-\varepsilon\eta_2-\varepsilon\eta_3\Big)\int_{2\lambda_0}^{t} e^{\varepsilon s}\mathbb{E}|x(s^-)|^2ds\end{aligned} \tag{4.17}$$

for $t\ge 2\lambda_0$, where $C_2=C_1+\bar{\lambda}e^{\varepsilon\lambda}\int_{-\lambda}^{2\lambda_0}e^{\varepsilon s}\Big[v_6\mathbb{E}|x(s^-)|^2+v_7\mathbb{E}W(x(s^-))\Big]ds$, and

$$\phi_2(t)=\mathbb{E}\int_{2\lambda_0}^{t} e^{\varepsilon s}\Big(\int_{s-\bar{\tau}}^{s}\Big[\bar{\tau}|f_{v^-}+u_{v^-}|^2+|g_{v^-}|^2+\int_{0<|z|<c}|h_{v^-}(z)|^2\vartheta(dz)\Big]dv\Big)ds.$$

Noting that the first integration in (4.17) is the same as $\int_{2\lambda_0}^t \mathbb{E}|x(s)-x(s-\tau_s)|^2 ds$, we hence estimate from the SDDE (3.1) that

$$\mathbb{E}|x(s)-x(s-\tau_s)|^2 \le 3\mathbb{E}\int_{s-\bar\tau}^{s}\Big[\bar\tau|f_{v^-}+u_{v^-}|^2+|g_{v^-}|^2+\int_{0<|z|<c}|h_{v^-}(z)|^2\vartheta(dz)\Big]dv. \tag{4.18}$$

Consequently

$$\begin{aligned}\eta_1 e^{\varepsilon t}\mathbb{E}|x(t)|^2 \le & C_2+\frac{\varepsilon\beta^2}{v_1}\phi_1(t)-\frac{\beta^2}{8v_1}\phi_2(t)-\Big(1-v_7\bar\lambda e^{\varepsilon\lambda}-\frac{\varepsilon\eta_3}{v_8}\Big)\int_{2\lambda_0}^t e^{\varepsilon s}\mathbb{E}W(x(s^-))ds\\ & -\Big(v_5-v_6\bar\lambda e^{\varepsilon\lambda}-\frac{3\beta^4\bar\tau^2}{v_1}-\varepsilon\eta_2-\varepsilon\eta_3\Big)\int_{2\lambda_0}^t e^{\varepsilon s}\mathbb{E}|x(s^-)|^2ds.\end{aligned} \tag{4.19}$$

In addition, it is easy to see that $\phi_1(t)\le\bar\tau\phi_2(t)$. As $v_7\bar\lambda<1$ while using condition (4.1), we can choose a sufficiently small $\varepsilon\in(0,1/(8\lambda_0))$ such that

$$v_5-v_6\bar\lambda e^{\varepsilon\lambda}-\frac{3\beta^4\bar\tau^2}{v_1}-\varepsilon\eta_2-\varepsilon\eta_3\ge 0,$$

and

$$1-v_7\bar\lambda e^{\varepsilon\lambda}-\frac{\varepsilon\eta_3}{v_8}\ge 0.$$

Then it follows from (4.19) that

$$\mathbb{E}|x(t)|^2\le\frac{C_2}{\eta_1}e^{-\varepsilon t},\quad \forall t\ge 2\lambda_0, \tag{4.20}$$

which is the required assertion (4.2). The proof is hence complete. ∎

THEOREM 4.2. *Let all the conditions of Theorem 4.1 hold and assume $\bar\alpha_1>\bar\alpha_2\bar\lambda$, where $\bar\alpha_1$ and $\bar\alpha_2$ have been given in (2.27). Then the solution of the controlled system (3.1) with the initial data (3.3) has the property*

$$\limsup_{t\to\infty}\frac{1}{t}\log(\mathbb{E}|x(t)|^{\overline{q}})<0,\quad \forall\overline{q}\in[2,q). \tag{4.21}$$

That is, the controlled system (3.1) is exponentially stable in $L^{\overline{q}}$.

Proof. From (2.28) in Lemma 2.9, we obtain

$$C_3:=\sup_{0\le t<\infty}\mathbb{E}|x(t)|^q<\infty. \tag{4.22}$$

Fix any $\overline{q}\in(2,q)$. For a constant $\rho\in(0,1)$, the Hölder inequality shows

$$\mathbb{E}|x(t)|^{\overline{q}}=\mathbb{E}(|x(t)|^{2\rho}|x(t)|^{\overline{q}-2\rho})\le(\mathbb{E}|x(t)|^2)^{\rho}(\mathbb{E}|x(t)|^{(\overline{q}-2\rho)/(1-\rho)})^{1-\rho}.$$

Letting $\rho=(q-\overline{q})/(q-2)$, it is easy to show that

$$\begin{aligned}\mathbb{E}|x(t)|^{\overline{q}} &\le (\mathbb{E}|x(t)|^2)^{(q-\overline{q})/(q-2)}(\mathbb{E}|x(t)|^q)^{(\overline{q}-2)/(q-2)}\\ &\le C_3^{(\overline{q}-2)/(q-2)}(\mathbb{E}|x(t)|^2)^{(q-\overline{q})/(q-2)}.\end{aligned} \tag{4.23}$$

From (4.20), we get that

$$\mathbb{E}|x(t)|^{\overline{q}}\le C_4 e^{-\varepsilon\rho t} \tag{4.24}$$

for all $t\ge 2\lambda_0$, where $C_4=C_3^{(\overline{q}-2)/(q-2)}C_2^{(q-\overline{q})/(q-2)}$. According to (4.24), the required assertion (4.21) holds. The proof is complete. ∎

THEOREM 4.3. *If all the conditions of Theorem 4.2 hold, the solution of the controlled system (3.1) with the initial data (3.3) has the property*

$$\limsup_{t\to\infty} \frac{1}{t}\log(|x(t)|) < 0 \quad a.s. \tag{4.25}$$

That is, controlled system (3.1) is almost surely exponentially stable.

Proof. Define $t_k = kh_0$, $k = 3, 4, \cdots$. By using Itô's isometry, Hölder inequality and Doob martingale inequality (see, e.g., [2, 6, 19]) , we have

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{t_k\le t\le t_{k+1}} |x(t)|^2\Big) \le 4\mathbb{E}|x(t_k)|^2 \\
&+ 4\lambda_0\mathbb{E}\int_{t_k}^{t_{k+1}} \Big(|f(x(t^-), x((t-\delta_t)^-), r(t), t) + u(x((t-\tau_t)^-), r(t), t)|^2\Big)dt \\
&+ 16\mathbb{E}\int_{t_k}^{t_{k+1}}\int_{0<|z|<c} |h(x(t^-), x((t-\delta_t)^-), r(t), t, z)|^2\vartheta(dz)dt \\
&+ 16\mathbb{E}\int_{t_k}^{t_{k+1}} |g(x(t^-), x((t-\delta_t)^-), r(t), t)|^2 dt.
\end{aligned} \tag{4.26}$$

It follows from Assumptions 2.5, 3.1 and 3.3 that

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{t_k\le t\le t_{k+1}} |x(t)|^2\Big) \le 4\mathbb{E}|x(t_k)|^2 \\
&+C_5\mathbb{E}\int_{t_k}^{t_{k+1}}\Big(|x(t^-)|^2+|x((t-\delta_t)^-)|^2+|x(t^-)|^{\overline{q}}+|x((t-\delta_t)^-)|^{\overline{q}}+|x((t-\tau_t)^-)|^2\Big)dt,
\end{aligned}$$

where $\overline{q} = 2(q_1 \vee q_2 \vee q_3 \vee q_4)$ and C_5 is a positive number. Noting that $\overline{q} \in [2, q)$ by Assumption 3.3, we can apply (4.20) and (4.24) to obtain

$$\mathbb{E}\Big(\sup_{t_k\le t\le t_{k+1}} |x(t)|^2\Big) \le C_6 e^{-\varepsilon\rho k h_0},$$

where C_6 is another positive number. Consequently

$$\sum_{k=3}^{\infty} P\Big(\sup_{t_k\le t\le t_{k+1}} |x(t)| > e^{-0.25\varepsilon\rho k h_0}\Big) \le \sum_{k=3}^{\infty} C_6 e^{-0.5\varepsilon\rho k h_0} < \infty.$$

According to Borel-Cantelli lemma (see, e.g., [19]), it shows that for almost all $\omega \in \Omega$, there exists a positive integer $k_0 = k_0(\omega)$ such that

$$\sup_{t_k\le t\le t_{k+1}} |x(t)| \le e^{-0.25\varepsilon\rho k h_0}, \quad k \ge k_0.$$

So we have

$$\frac{1}{t}\log(|x(t)|) \le -\frac{0.25\varepsilon\rho k}{k+1}, t \in [t_k, t_{k+1}], \quad k \ge k_0.$$

This implies

$$\limsup_{t\to\infty} \frac{1}{t}\log(|x(t)|) \le -0.25\varepsilon\rho < 0 \quad a.s,$$

which is the required assertion (4.25). The proof is hence complete. ∎

5. Numerical simulation. In this section, we will discuss an example to illustrate our theoretical results.

EXAMPLE 5.1. To simplify the calculation, we consider the scalar highly nonlinear SDDE with Lévy noise and 2-state Markov switching of the form

$$\begin{aligned} dx(t) =& f(x(t^-), x((t-\delta_t)^-), r(t), t)dt + g(x(t^-), x((t-\delta_t)^-), r(t), t)dB(t) \\ &+ \int_{0<|z|<c} h(x(t^-), x((t-\delta_t)^-), r(t), t, z)\widetilde{N}(dt, dz) \end{aligned} \tag{5.1}$$

on $t \geq 0$ but we will omit mentioning the initial data. Here the coefficients f, g and h are defined by

$$\begin{aligned} f(x,y,1) &= x(1-3x^2+y^2), \qquad & g(x,y,1) &= |x|^{3/2} + 0.5y, \\ f(x,y,2) &= x(1-2x^2-y^2), \qquad & g(x,y,2) &= 0.5|x|^{3/2} - 0.5y, \\ h(x,y,z,1) &= 0.5yz - 0.5xz, \qquad & h(x,y,z,2) &= 0.25yz - 0.5xz \end{aligned}$$

for all $x, y \in \mathbb{R}$ and $z \in \mathbb{R}_0$, where $\mathbb{R}_0 = \mathbb{R} - \{0\}$, $c = 5$, $B(t)$ is a scalar Brownian motion, $r(t)$ is a Markov chain on the state space $S = \{1, 2\}$ with its generator $\Gamma = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}$, and the time delay $\delta_t = 0.1|\sin(t)| + 0.1$.

The Lévy measure ϑ satisfies $\vartheta(dz) = a\phi(dz) = 0.5 \times e^{-2|z|}dz$, where $a = 0.5$ denotes the jump rate and $\phi(\cdot)$ is the jump distribution, and its probability density function satisfies $e^{-2|z|}$, so (2.1) can be met. In addition, it should be pointed out that SDDEs driven by Lévy noise have many applications in financial markets (see, e.g., [8, 22]).

We can verify that Assumption 2.1 holds when $\lambda_1 = 0.1$, $\lambda = 0.2$ and $\bar{\lambda} = 1.1111$. It is also easy to show that Assumption 2.4 holds for $p = 4$, $\alpha_1 = [1 + 0.25(q-1)^2] \vee (q-1)$, $\alpha_2 = 1.25$, $\alpha_3 = 0.5$ and for any $q > 6$. Next Assumption 2.5 can be met with $L = 0.5$ and $\alpha = 1$. According to Lemma 2.7, the SDDE (5.1) has a unique global solution $x(t)$ which has properties (2.15) and (2.16). In order to make (2.27) hold, it is sufficient if $\bar{\lambda}\alpha_3 < 1$, so we know that the solution $x(t)$ has properties (2.28) and (2.29). Assumption 3.3 can be satisfied with $q_1 = q_2 = 3$, $q_3 = 1.5$ and $q_4 = 1$. In the remaining part of this example, we will fix $q = 7$.

To stabilize the SDDE (5.1), we use the delay feedback control to form the controlled system

$$\begin{aligned} dx(t) =& \Big[f(x(t^-), x((t-\delta_t)^-), r(t), t) + u(x((t-\tau_t)^-), r(t), t)\Big]dt \\ &+ g(x(t^-), x((t-\delta_t)^-), r(t), t)dB(t) \\ &+ \int_{0<|z|<c} h(x(t^-), x((t-\delta_t)^-), r(t), t, z)\widetilde{N}(dt, dz), \end{aligned} \tag{5.2}$$

where

$$u(x, 1, t) = -5x; \quad u(x, 2, t) = -4x. \tag{5.3}$$

It is easy to see that Assumption 3.1 holds for $\beta = 5$. By Theorem 3.2, the controlled system (5.2) has the unique solution $x(t)$ which has properties (2.28) and (2.29). Next, we will check Condition 3.4. For $(x, y, i, t, z) \in \mathbb{R} \times \mathbb{R} \times S \times \mathbb{R}_+ \times \mathbb{R}_0$, we have

$$\begin{aligned}
&2\Big[x^T[f(x,y,i,t)+u(x,i,t)]+\frac{1}{2}|g(x,y,i,t)|^2\Big]\\
&+\int_{0<|z|<c}\Big[|x+h(x,y,i,t,z)|^2-|x|^2-2x^Th(x,y,i,t,z)\Big]\vartheta(dz)\\
&\le \begin{cases} -6.8754x^2+0.6246y^2-4x^4+y^4, & i=1,\\ -5.6565x^2+0.5467y^2-2.75x^4+y^4, & i=2,\end{cases}
\end{aligned}$$

$$\begin{aligned}
&x^T[f(x,y,i,t)+u(x,i,t)]+\frac{q_1}{2}|g(x,y,i,t)|^2\\
&\le \begin{cases} -2.5x^2+0.75y^2-x^4+0.5y^4, & i=1,\\ -2.625x^2+0.75y^2-1.125x^4+0.5y^4, & i=2,\end{cases}
\end{aligned}$$

and

$$\begin{aligned}
&\int_{0<|z|<c}\Big[|x+h(x,y,i,t,z)|^4-|x|^4-4|x|^2x^Th(x,y,i,t,z)\Big]\vartheta(dz)\\
&\le \begin{cases} 1.4854x^4+0.7378y^4, & i=1,\\ 0.8547x^4+0.2169y^4, & i=2.\end{cases}
\end{aligned}$$

So (3.7)-(3.9) hold with

$$\begin{aligned}
&a_1=-6.8754, && b_1=0.6246, && c_1=4, && d_1=1,\\
&a_2=-5.6565, && b_2=0.5467, && c_2=2.75, && d_2=1,\\
&\bar a_1=-2.5, && \bar b_1=0.75, && \bar c_1=1, && \bar d_1=0.5,\\
&\bar a_2=-2.625, && \bar b_2=0.75, && \bar c_2=1.125, && \bar d_2=0.5,\\
&\hat a_1=1.4854, && \hat b_1=0.7378, && \hat a_2=0.8547, && \hat b_2=0.2169,
\end{aligned}$$

and

$$A_1=\begin{pmatrix} 8.8754 & -2\\ -2 & 7.6565\end{pmatrix},\quad A_2=\begin{pmatrix} 10.5146 & -2\\ -2 & 11.6453\end{pmatrix},$$

which are both M-matrices. According to (3.13), we get

$$\theta_1=0.1510,\quad \theta_2=0.1701,\quad \bar\theta_1=0.1152,\quad \bar\theta_2=0.1057.$$

Consequently,

$$\begin{aligned}
&\zeta_1=0.0943,\quad \zeta_2=0.4678,\quad \zeta_3=0.1701,\\
&\zeta_4=0.4306,\quad \zeta_5=0.4608,\quad \zeta_6=0.2304,
\end{aligned}$$

which meet (3.11). That is, control function $u(x, i)$ satisfies Condition 3.4. Furthermore, it is clear that

$$U(x,i)=\begin{cases} 0.1510x^2+0.1152x^4, & i=1,\\ 0.1701x^2+0.1057x^4, & i=2.\end{cases}$$

By (3.17), we have

$$\mathcal{L}U(x,y,i,t)\le -x^2+0.0943y^2-1.2525x^4+0.3854y^4-0.384x^6+0.1536y^6.$$

At the same time, we get

$$\begin{aligned}(2\theta_i|x| + (q_1+1)\bar{\theta}_i|x|^{q_1})^2 &\le 0.1157x^2 + 0.2877x^4 + 0.2123x^6,\\ |f(x,y,i,t)|^2 &\le x^2 - 4x^4 + y^4 + 9.3333x^6 + 2y^6,\\ |g(x,y,i,t)|^2 &\le 0.5x^2 + 0.5y^2 + 2x^4,\\ \int_{0<|z|<c} |h(x,y,i,t,z)|^2\vartheta(dz) &\le 0.1246x^2 + 0.1246y^2.\end{aligned}$$

Choosing $v_1 = 0.4$, $v_2 = 0.01$, $v_3 = 0.27$ and $v_4 = 0.27$, we then obtain

$$\begin{aligned}&\mathcal{L}U(x,y,i,t) + v_1(2\theta_i|x| + (q_1+1)\bar{\theta}_i|x|^{q_1})^2 + v_2|f(x,y,i,t)|^2 + v_3|g(x,y,i,t)|^2\\ &+ v_4\int_{0<|z|<c} |h(x,y,i,t,z)|^2\vartheta(dz)\\ &\le -0.7751x^2 + 0.2629y^2 - 0.6374x^4 + 0.3954y^4 - 0.2057x^6 + 0.1736y^6\\ &\le -0.7751x^2 + 0.2629y^2 - W(x) + 0.8439W(y),\end{aligned}$$

where $W(x) = 0.6374x^4 + 0.2057x^6$, $v_5 = 0.7751$, $v_6 = 0.2629$, $v_7 = 0.8439$, $v_8 = 0.2057$ and $v_9 = 0.8431$. By (4.1), we know that the controlled system (5.2) is exponentially stable in $L^{\bar{q}}$ for any $\bar{q} \in [2,7)$ with $\bar{\tau} < 0.0043$, and it is also almost surely exponentially stable.

The computer simulation will be given by using the Euler-Maruyama method (see, e.g., [11]) with step size 10^{-3}, and the conditions for numerical simulation are $\tau_t = 0.004/(1+e^{-t})$, initial value $x(t) = 1 + \sin(t)$, $t \in [-0.2, 0]$ and $r(0) = 1$.

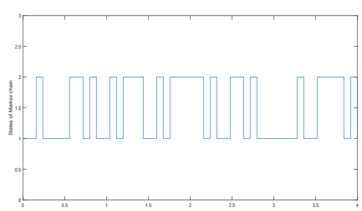

FIG. 1. *Markov chain.*

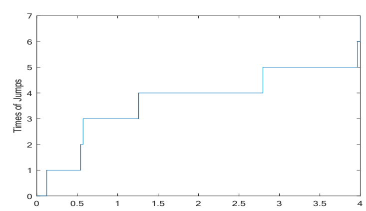

FIG. 2. *Time evolution of the number of jumps.*

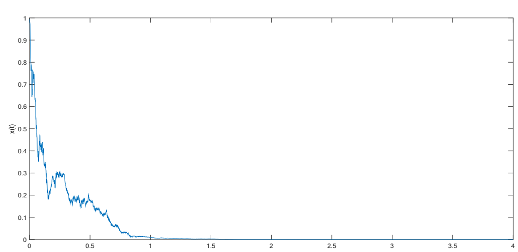

FIG. 3. *The state trajectory of the solution.*

Figs 1 and 2 show the sample paths of 2-state Markov switching and time evolution of the number of jumps respectively. Fig 3 shows the state trajectory of the solution of the controlled SDDE (5.2).

6. Conclusions. In this paper, we have not only showed the existence and uniqueness of the global solution to the highly nonlinear SDDE with Lévy noise and Markov switching, but also the finiteness and boundedness of the moments of the solution. The time delay in the given unstable SDDE is a variable of time which may not have to be differentiable. Moreover, we have studied the qth moment exponential stability and almost surely exponential stability by a delay feedback control. A useful feature is that the time lag in the feedback control can be of time-varying as long as it has a sufficiently small upper bound. The main techniques used in this paper are the theory of M-matrices and the method of Lyapunov functionals. An example with some computer simulations has been presented to illustrate our theory.

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