

Convergence of a spatial semi-discretization for a backward semilinear stochastic parabolic equation*

Binjie Li[†] and Xiaoping Xie[‡]

School of Mathematics, Sichuan University, Chengdu 610064,
China

Abstract

This paper studies the convergence of a spatial semi-discretization for a backward semilinear stochastic parabolic equation. The filtration is general, and the spatial semi-discretization uses the standard continuous piecewise linear finite element method. Firstly, higher regularity of the solution to the continuous equation is derived. Secondly, the first-order spatial accuracy is derived for the spatial semi-discretization. Thirdly, an application of the theoretical results to a stochastic linear quadratic control problem is presented.

Keywords. backward semilinear stochastic parabolic equation, Brownian motion, semi-discretization, stochastic linear quadratic control.

AMS subject classifications. 49M25, 65C30, 60H35, 65K10

1 Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a given complete probability space with a normal filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}$ (i.e., \mathbb{F} is right continuous and \mathcal{F}_0 contains all \mathbb{P} -null sets of \mathcal{F}). Assume that $W(\cdot)$ is an \mathbb{F} -adapted one-dimensional real Brownian motion. Let $\mathcal{O} \subset \mathbb{R}^d$ ($d = 1, 2, 3$) be a bounded convex polygonal domain, and let Δ be the realization of the Laplace operator with homogeneous Dirichlet boundary condition in $L^2(\mathcal{O})$. We consider the following backward semilinear stochastic parabolic equation:

$$\begin{cases} dp(t) = -(\Delta p(t) + f(t, p(t), z(t))) dt + z(t) dW(t), & 0 \leq t \leq T, \\ p(T) = p_T, \end{cases} \quad (1)$$

where $p_T \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \dot{H}^1)$ and f satisfies the following conditions:

*This work was supported in part by the National Natural Science Foundation of China (11901410, 12171340) and the Fundamental Research Funds for the Central Universities in China (2020SCU12063).

[†]libinjie@scu.edu.cn

[‡]Corresponding author: xpxie@scu.edu.cn

- $f(\cdot, p, z) \in L^2_{\mathbb{P}}(0, T; H)$ for each $p, z \in H$;
- there exists a positive constant \mathcal{M} such that, \mathbb{P} -almost surely for almost every $t \in [0, T]$,

$$\|f(t, p_1, z_1) - f(t, p_2, z_2)\|_H \leq \mathcal{M}(\|p_1 - p_2\|_H + \|z_1 - z_2\|_H) \quad (2)$$

for all $p_1, p_2, z_1, z_2 \in H$.

The above notations are defined later in Section 2.

Bismut [4, 5] proposed the finite-dimensional linear backward stochastic differential equations (BSDEs for short) and used them to form the stochastic maximum principle for finite-dimensional stochastic optimal control problems. Bensoussan [3] used the infinite-dimensional linear BSDEs to form the maximum stochastic principle of stochastic distributed parameter systems. Later, Pardoux and Peng [44] made a significant breakthrough by establishing the well-posedness of a class of finite-dimensional nonlinear BSDEs, and soon Hu and Peng [28] proposed a highly non-trivial extension to the infinite-dimensional BSDEs. Since then the theory of BSDEs began to develop quickly, mainly motivated by applications to stochastic optimal control, partial differential equations and mathematical finance; see [30, 43, 45, 46, 55] and the references therein. We particularly refer the reader to [21, Chapter 6] and [41] and the references therein for the applications of the infinite-dimensional BSDEs to the stochastic optimal control problems.

The above mentioned works on the BSDEs all require that the filtration is generated by the underlying Wiener process. Motivated by the transposition method for non-homogeneous boundary value problems for partial differential equations (see [36]), Lü and Zhang [37, 38] proposed a new notion of solution, the transposition solution, to BSDEs with general filtration. The transposition solution coincides with the usual strong solution when the filtration is natural. The transposition solution has been successfully used to investigate the stochastic maximal principle for the infinite-dimensional distributed parameter systems; see [39, 40] and the references therein.

By now, the numerical solutions of the finite-dimensional BSDEs have been extensively studied; see [6, 13, 14, 20, 27, 29, 42, 56] and the references therein. In particular, we note that, based on the definition of the transposition solution, Wang and Zhang [51] proposed a numerical method for solving finite-dimensional BSDEs. In view of the fact that the numerical analysis of the infinite-dimensional stochastic differential equations (SDEs for short) differs considerably from that of the finite-dimensional SDEs, generally it is difficult to extend the numerical analysis of finite-dimensional BSDEs to the infinite-dimensional BSDEs directly. We note that there is a huge list of papers in the literature on the numerical analysis of the infinite-dimensional SDEs; see [1, 2, 7, 8, 11, 12, 15, 16, 17, 18, 22, 23, 31, 32, 54, 57] and the references therein. Despite this fact, the numerical analysis of the infinite-dimensional nonlinear BSDEs is expected to be a challenging problem, since the SDEs and the BSDEs are essentially different.

So far, the numerical analysis of the infinite-dimensional BSDEs is very limited. Wang [52] analyzed a semi-discrete Galerkin scheme for a backward semi-linear stochastic parabolic equation. Since this scheme uses the eigenvectors of the Laplace operator, its application appears to be limited. Li and Tang [35]

developed a splitting-up method for solving backward stochastic partial differential equations. To our best knowledge, no convergence rate is available for the spatial semi-discretization with finite element method of backward semilinear stochastic parabolic equations with general filtration.

In this paper, we use the notion of transposition solution introduced by Lü and Zhang [37, 38], and make the following threefold main contributions.

- Firstly, higher regularity of the transposition solution is derived, which is essential for deriving convergence rate of the spatial semi-discretization. We note that [38] gives the basic regularity result

$$(p, z) \in D_{\mathbb{F}}([0, T]; L^2(\Omega; H)) \times L_{\mathbb{F}}^2(0, T; H).$$

Using the regularity estimates of the deterministic backward parabolic equations, we prove that

$$(p, z) \in (L_{\mathbb{F}}^2(0, T; \dot{H}^2) \cap D_{\mathbb{F}}([0, T]; L^2(\Omega; \dot{H}^1))) \times L_{\mathbb{F}}^2(0, T; \dot{H}^1).$$

- Secondly, for a spatial semi-discretization of (1), we derive the error estimate

$$\sup_{0 \leq t \leq T} \|\| (p - p_h)(t) \|\|_H + \|\| p - p_h \|\|_{L^2(0, T; \dot{H}^1)} + \|\| z - z_h \|\|_{L^2(0, T; H)} \leq ch,$$

which is optimal with respect to the regularity of the transposition solution. This spatial semi-discretization adopts the standard continuous piecewise linear finite element method. The case that \mathbb{F} is the natural filtration of $W(\cdot)$ is also covered by our numerical analysis, since in this case the transposition solution coincides with the usual strong solution.

- Thirdly, with the derived higher order regularity of the transposition solution and the convergence estimate, the first-order accuracy is derived for a spatially semi-discrete stochastic linear quadratic control problem, where the filtration is general and the diffusion term of the state equation contains the control variable. Here we note that the stochastic optimal control problems governed by stochastic partial differential equations have been extensively studied in the past four decades; however, these problems have rarely been numerically studied. To our best knowledge, this paper provides the first convergence rate for a spatial semi-discretization of a general stochastic linear quadratic control problem with general filtration.

We believe that the obtained theoretical results in this paper are useful for further numerical analysis of backward semilinear stochastic parabolic equations and stochastic linear quadratic control problems.

The rest of this paper is organized as follows. Section 2 introduces the preliminaries. Section 3 investigates the higher regularity of the transposition solution to (1). In Section 4, we derive the first-order spatial accuracy for a spatial semi-discretization of (1). Finally, using the derived higher regularity result and the convergence estimate, we establish the convergence of a spatially semi-discrete stochastic linear quadratic control problem, and provide some numerical results in Section 5.

2 Preliminaries

Assume that X is a separable Hilbert space with norm $\|\cdot\|_X$. We simply write the Hilbert space $L^2(\Omega, \mathcal{F}_T, \mathbb{P}; X)$ as $L^2(\Omega; X)$, and denote by $\|\cdot\|_X$ its norm. For any $v \in L^2(\Omega; X)$, we use $\mathbb{E}v$ and $\mathbb{E}_t v$ to denote respectively the expectation of v and the conditional expectation of v with respect to \mathcal{F}_t for each $0 \leq t \leq T$. Let $L^2_{\mathbb{F}}(0, T; X)$ be the space of all \mathbb{F} -progressively measurable processes φ such that

$$\|\varphi\|_{L^2(0, T; X)} := \left(\int_0^T \|\varphi(t)\|_X^2 dt \right)^{1/2} < \infty.$$

For any $0 < t < T$, the space $L^2_{\mathbb{F}}(0, t; X)$ is defined analogously to $L^2_{\mathbb{F}}(0, T; X)$. Let $L^2_{\mathbb{F}}(\Omega; C([0, T]; X))$ be the space of all \mathbb{F} -progressively measurable processes φ with continuous paths in X such that

$$\|\varphi\|_{C([0, T]; X)} := \left(\mathbb{E} \sup_{t \in [0, T]} \|\varphi(t)\|_X^2 \right)^{1/2} < \infty.$$

Let $D_{\mathbb{F}}([0, T]; L^2(\Omega; X))$ be the space of all X -valued and \mathbb{F} -adapted processes that are right continuous with left limits in $L^2(\Omega; X)$ with respect to the time variable. This is a Banach space with the norm

$$\|\varphi\|_{D_{\mathbb{F}}([0, T]; L^2(\Omega; X))} := \max_{t \in [0, T]} \|\varphi(t)\|_X \quad \forall \varphi \in D_{\mathbb{F}}([0, T]; L^2(\Omega; X)).$$

Denote $H := L^2(\mathcal{O})$. For each $\gamma \geq 0$, define

$$\dot{H}^\gamma := \{(-\Delta)^{-\gamma/2} v \mid v \in H\}$$

and endow this space with the norm

$$\|v\|_{\dot{H}^\gamma} := \|(-\Delta)^{\gamma/2} v\|_H \quad \forall v \in \dot{H}^\gamma.$$

We use $\dot{H}^{-\gamma}$ to denote the dual space of \dot{H}^γ . The operator Δ can be extended as a bounded linear operator from H to \dot{H}^{-2} by

$$\langle \Delta v, \varphi \rangle_{\dot{H}^2} = \int_{\mathcal{O}} v \Delta \varphi \quad \text{for all } v \in H \text{ and } \varphi \in \dot{H}^2,$$

where $\langle \cdot, \cdot \rangle_{\dot{H}^2}$ denotes the duality pairing between \dot{H}^{-2} and \dot{H}^2 .

For any $g \in L^2_{\mathbb{F}}(0, T; \dot{H}^\gamma)$ with $-2 \leq \gamma < \infty$, let $S_0 g$ be the mild solution of the stochastic parabolic equation

$$\begin{cases} dy(t) = \Delta y(t) dt + g(t) dW(t) & \forall t \in [0, T], \\ y(0) = 0. \end{cases} \quad (3)$$

It is standard that (see, e.g., [24, Chapter 3]) for any $0 \leq t \leq T$,

$$(S_0 g)(t) = \int_0^t e^{(t-s)\Delta} g(s) dW(s) \quad \mathbb{P}\text{-a.s.} \quad (4)$$

Moreover, a routine argument with Itô's formula gives

$$\|(S_0 g)(t)\|_{\dot{H}^\gamma}^2 + 2 \|S_0 g\|_{L^2(0, t; \dot{H}^{\gamma+1})}^2 = \|g\|_{L^2(0, t; \dot{H}^\gamma)}^2 \quad \forall 0 < t \leq T. \quad (5)$$

Finally, we introduce the mild solutions to a forward parabolic equation and a backward parabolic equation, respectively. For any $g \in L^2(0, T; H)$, let S_1g and S_2g be the mild solutions of the equations

$$\begin{cases} y'(t) = \Delta y(t) + g(t) & \forall t \in [0, T], \\ y(0) = 0 \end{cases}$$

and

$$\begin{cases} z'(t) = -\Delta z(t) - g(t) & \forall t \in [0, T], \\ z(T) = 0, \end{cases}$$

respectively. We have (see, e.g., [53, Chapter 3])

$$(S_1g)(t) = \int_0^t e^{(t-s)\Delta} g(s) ds \quad \forall t \in [0, T], \quad (6)$$

$$(S_2g)(t) = \int_t^T e^{(s-t)\Delta} g(s) ds \quad \forall t \in [0, T]. \quad (7)$$

It is standard that, for any $v, w \in L^2(0, T; H)$,

$$(S_1v, w)_{L^2(0, T; H)} = (v, S_2w)_{L^2(0, T; H)}, \quad (8)$$

where $(\cdot, \cdot)_{L^2(0, T; H)}$ denotes the inner product of the Hilbert space $L^2(0, T; H)$.

3 Regularity

Following [38], we call

$$(p, z) \in D_{\mathbb{F}}([0, T]; L^2(\Omega; H)) \times L^2_{\mathbb{F}}(0, T; H)$$

a transposition solution to (1) if

$$\begin{aligned} & \int_t^T [p(s), g(s)] + [z(s), \sigma(s)] ds + [p(t), v] \\ &= \int_t^T [f(s, p(s), z(s)), (S_0\sigma + S_1g)(s) + e^{(s-t)\Delta}v] ds \\ & \quad + [(S_0\sigma + S_1g)(T) + e^{(T-t)\Delta}v, p_T] \end{aligned} \quad (9)$$

for all $0 \leq t \leq T$, $(g, \sigma) \in (L^2_{\mathbb{F}}(0, T; H))^2$ and $v \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; H)$, where $[\cdot, \cdot]$ denotes the inner product in $L^2(\Omega; H)$. For the unique existence of the transposition solution to (1), we refer the reader to [38, Theorem 3.1]. Moreover, the proof of [38, Theorem 3.1] contains that, for any $0 \leq t \leq T$,

$$p(t) = \mathbb{E}_t \left(\int_t^T e^{(s-t)\Delta} f(s, p(s), z(s)) ds + e^{(T-t)\Delta} p_T \right) \quad \mathbb{P}\text{-a.s.} \quad (10)$$

In particular,

$$p(T) = p_T \quad \mathbb{P}\text{-a.s.} \quad (11)$$

The main result of this section is the following theorem.

Theorem 3.1. *The transposition solution (p, z) of (1) possesses the following properties:*

(i) $p \in L_{\mathbb{F}}^2(0, T; \dot{H}^2);$

(ii) p admits a modification in $D_{\mathbb{F}}([0, T]; L^2(\Omega; \dot{H}^1));$

(iii) $z \in L_{\mathbb{F}}^2(0, T; \dot{H}^1).$

Remark 3.1. *When \mathbb{F} is the natural filtration of $W(\cdot)$, by the theory in [25, 28] we easily obtain*

$$(p, z) \in (L_{\mathbb{F}}^2(0, T; \dot{H}^2) \cap L_{\mathbb{F}}^2(\Omega; C([0, T]; \dot{H}^1))) \times L_{\mathbb{F}}^2(0, T; \dot{H}^1).$$

The rest of this section is devoted to proving the above theorem. We first introduce three technical lemmas, i.e. Lemmas 3.1, 3.2 and 3.3.

Lemma 3.1. *Assume that $0 \leq a < b \leq T$ and $v \in L^2(\Omega; X)$, with X being a separable Hilbert space. Then there exists a unique $y \in D_{\mathbb{F}}([a, b]; L^2(\Omega; X))$ such that*

$$\mathbb{P}(y(t) = \mathbb{E}_t v) = 1 \quad \forall t \in [a, b], \quad (12)$$

where $D_{\mathbb{F}}([a, b]; L^2(\Omega; X))$ is defined analogously to $D_{\mathbb{F}}([0, T]; L^2(\Omega; X))$.

Lemma 3.2. *Assume that $w \in L^2(\Omega; C([0, T]; X))$, where X is a separable Hilbert space. Then*

$$\lim_{m \rightarrow \infty} \sup_{\substack{0 \leq r < s \leq T \\ s-r \leq 1/m}} \|w(r) - w(s)\|_X = 0. \quad (13)$$

Lemma 3.3. *Assume that $v \in \dot{H}^1$ and $g \in L^2(0, T; H)$. Define*

$$w(t) := e^{(T-t)\Delta} v + \int_t^T e^{(s-t)\Delta} g(s) ds \quad \forall t \in [0, T].$$

Then

$$\|w\|_{C([0, T]; \dot{H}^1)} + \|w\|_{L^2(0, T; \dot{H}^2)} \leq C(\|v\|_{\dot{H}^1} + \|g\|_{L^2(0, T; H)}),$$

where C is a positive constant independent of v, g, T and \mathcal{O} .

The proofs of Lemmas 3.1 and 3.2 are straightforward, and Lemma 3.3 is standard; see, e.g., [10, Theorem 10.11].

Based on the three lemmas above, we are in a position to prove Theorem 3.1 as follows.

Proof of Theorem 3.1. Let us first prove (i). Inserting $t = 0, \sigma = 0$ and $v = 0$ into (9), by (8) we obtain

$$\int_0^T [p(s), g(s)] ds = \int_0^T [\eta(s), g(s)] ds \quad \forall g \in L_{\mathbb{F}}^2(0, T; H),$$

where

$$\eta(s) := \int_s^T e^{(r-s)\Delta} f(r, p(r), z(r)) dr + e^{(T-s)\Delta} p_T \quad \forall s \in [0, T].$$

It follows that

$$p = \mathcal{E}_{\mathbb{F}}\eta \quad \text{in } L^2_{\mathbb{F}}(0, T; H), \quad (14)$$

where $\mathcal{E}_{\mathbb{F}}$ is the $L^2(\Omega; L^2(0, T; H))$ -orthogonal projection onto $L^2_{\mathbb{F}}(0, T; H)$. For each $n > 0$, define

$$\eta_n(t) := \begin{cases} p_T & \text{if } t = T, \\ \frac{n}{T} \int_{\frac{jT}{n}}^{\frac{(j+1)T}{n}} \eta(t) dt & \text{if } t \in [\frac{jT}{n}, \frac{(j+1)T}{n}) \text{ with } 0 \leq j < n. \end{cases} \quad (15)$$

By Lemma 3.3 we have

$$\eta \in L^2(\Omega; L^2(0, T; \dot{H}^2)) \cap L^2(\Omega; C([0, T]; \dot{H}^1)), \quad (16)$$

and a routine density argument yields

$$\lim_{n \rightarrow \infty} \eta_n = \eta \quad \text{in } L^2(\Omega; L^2(0, T; \dot{H}^2)). \quad (17)$$

By Lemma 3.1 we conclude that there exists a unique $p_n \in D_{\mathbb{F}}([0, T]; L^2(\Omega; \dot{H}^2))$ satisfying that

$$\mathbb{P}(p_n(t) = \mathbb{E}_t \eta_n(t)) = 1 \quad \forall t \in [0, T]. \quad (18)$$

Hence, by the inequality

$$\|(p_m - p_n)(t)\|_{\dot{H}^2} = \|\mathbb{E}_t(\eta_m - \eta_n)(t)\|_{\dot{H}^2} \leq \|(\eta_m - \eta_n)(t)\|_{\dot{H}^2}$$

for any $m, n > 0$ and $t \in [0, T]$, we obtain

$$\lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} \|p_m - p_n\|_{L^2(0, T; \dot{H}^2)}^2 \leq \lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} \|\eta_m - \eta_n\|_{L^2(0, T; \dot{H}^2)}^2 dt = 0 \quad (\text{by (17)}).$$

It follows that $\{p_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $L^2_{\mathbb{F}}(0, T; \dot{H}^2)$, and so there exists a unique $\tilde{p} \in L^2_{\mathbb{F}}(0, T; \dot{H}^2)$ such that

$$\lim_{n \rightarrow \infty} p_n = \tilde{p} \quad \text{in } L^2_{\mathbb{F}}(0, T; \dot{H}^2). \quad (19)$$

By (18), it is easy to verify that, for each $n > 0$,

$$\mathcal{E}_{\mathbb{F}}\eta_n = p_n \quad \text{in } L^2_{\mathbb{F}}(0, T; H),$$

so that by (14) and (17) we get

$$\lim_{n \rightarrow \infty} p_n = \lim_{n \rightarrow \infty} \mathcal{E}_{\mathbb{F}}\eta_n = \mathcal{E}_{\mathbb{F}}\eta = p \quad \text{in } L^2_{\mathbb{F}}(0, T; H). \quad (20)$$

In view of (19) and (20), we readily obtain $p \in L^2_{\mathbb{F}}(0, T; \dot{H}^2)$.

Secondly, let us prove (ii). For any $0 < m < n < \infty$, we have

$$\begin{aligned} & \|p_m - p_n\|_{D_{\mathbb{F}}([0, T]; L^2(\Omega; \dot{H}^1))} \\ &= \max_{0 \leq t \leq T} \|\mathbb{E}_t(\eta_m(t) - \eta_n(t))\|_{\dot{H}^1} \quad (\text{by (18)}) \\ &\leq \max_{0 \leq t \leq T} \|\eta_m(t) - \eta_n(t)\|_{\dot{H}^1} \\ &\leq \max_{\substack{0 \leq r < s \leq T \\ s-r \leq 2T/m}} \|\eta(r) - \eta(s)\|_{\dot{H}^1} \quad (\text{by (15)}), \end{aligned}$$

so that Lemma 3.2 implies

$$\lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} \|p_m - p_n\|_{D_{\mathbb{F}}([0, T]; L^2(\Omega; \dot{H}^1))} = 0.$$

It follows that $\{p_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $D_{\mathbb{F}}([0, T]; L^2(\Omega; \dot{H}^1))$. Hence, there exists a unique $\bar{p} \in D_{\mathbb{F}}([0, T]; L^2(\Omega; \dot{H}^1))$ such that

$$\lim_{n \rightarrow \infty} p_n = \bar{p} \quad \text{in } D_{\mathbb{F}}([0, T]; L^2(\Omega; \dot{H}^1)), \quad (21)$$

which, together with (20), yields

$$p = \bar{p} \quad \text{in } L_{\mathbb{F}}^2(0, T; H).$$

It follows that

$$p(t) = \bar{p}(t) \quad \text{in } L^2(\Omega, \mathcal{F}_t, \mathbb{P}; H) \quad \text{a.e. } t \in [0, T].$$

Since $p \in D_{\mathbb{F}}([0, T]; L^2(\Omega; H))$ and $\bar{p} \in D_{\mathbb{F}}([0, T]; L^2(\Omega; \dot{H}^1))$, we then obtain

$$p(t) = \bar{p}(t) \quad \text{in } L^2(\Omega, \mathcal{F}_t, \mathbb{P}; H) \quad \forall t \in [0, T]. \quad (22)$$

Since (15) and (18) imply $p_n(T) = p_T$, \mathbb{P} -a.s., by (21) we get $\bar{p}(T) = p_T$, \mathbb{P} -a.s., and so (11) implies

$$p(T) = \bar{p}(T) \quad \text{in } L^2(\Omega, \mathcal{F}_T, \mathbb{P}; H).$$

By virtue of this equality and (22), we conclude that \bar{p} is exactly the modification of p in $D_{\mathbb{F}}([0, T]; L^2(\Omega; \dot{H}^1))$.

Thirdly, let us prove (iii). It is standard that there exists an orthonormal basis $\{\phi_k\}_{k=0}^{\infty} \subset \dot{H}^2$ of H satisfying that

$$-\Delta \phi_k = \lambda_k \phi_k,$$

where $\{\lambda_k\}_{k=0}^{\infty}$ is a nondecreasing sequence of strictly positive numbers with limit $+\infty$. For each $n \in \mathbb{N}$, define

$$\begin{aligned} z_n(t) &:= \sum_{k=0}^n (z(t), \phi_k)_H \phi_k, \quad 0 \leq t \leq T, \\ F_n(t) &:= \sum_{k=0}^n (f(t, p(t), z(t)), \phi_k)_H \phi_k, \quad 0 \leq t \leq T, \end{aligned}$$

where $(\cdot, \cdot)_H$ is the inner product of H . For any $0 < m < n < \infty$, define $\delta_{m,n} := z_n - z_m$. Inserting $t = 0$, $g = 0$, $\sigma = -\Delta \delta_{m,n}$ and $v = 0$ into (9) yields

$$\begin{aligned} & \int_0^T [z, -\Delta \delta_{m,n}] \, ds \\ &= \int_0^T [f(s, p(s), z(s)), -(S_0 \Delta \delta_{m,n})(s)] \, ds + [-(S_0 \Delta \delta_{m,n})(T), p_T] \\ &= \int_0^T [(F_n - F_m)(s), -(S_0 \Delta \delta_{m,n})(s)] \, ds + \left[-(S_0 \Delta \delta_{m,n})(T), \sum_{k=m+1}^n (p_T, \phi_k)_H \phi_k \right] \\ &\leq \|F_n - F_m\|_{L^2(0, T; H)} \|S_0 \Delta \delta_{m,n}\|_{L^2(0, T; H)} + \\ &\quad \| (S_0 \Delta \delta_{m,n})(T) \|_{\dot{H}^{-1}} \left(\sum_{k=m+1}^n \lambda_k \| (p_T, \phi_k)_H \|^2_{\mathbb{R}} \right)^{1/2} \\ &\leq \|\delta_{m,n}\|_{L^2(0, T; \dot{H}^1)} \left(\|F_n - F_m\|_{L^2(0, T; H)} + \left(\sum_{k=m+1}^n \lambda_k \| (p_T, \phi_k)_H \|^2_{\mathbb{R}} \right)^{1/2} \right), \end{aligned}$$

where we have used (5) and the equality

$$\|\Delta\delta_{m,n}\|_{L^2(0,T;\dot{H}^{-1})} = \|\delta_{m,n}\|_{L^2(0,T;\dot{H}^1)}.$$

Hence, by the equality

$$\int_0^T [z(t), -\Delta\delta_{m,n}(t)] dt = \|\delta_{m,n}\|_{L^2(0,T;\dot{H}^1)}^2,$$

we get

$$\|\delta_{m,n}\|_{L^2(0,T;\dot{H}^1)} \leq \|F_n - F_m\|_{L^2(0,T;H)} + \left(\sum_{k=m+1}^n \lambda_k \|(p_T, \phi_k)_H\|_{\mathbb{R}}^2 \right)^{1/2}.$$

Since

$$\lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} \|F_n - F_m\|_{L^2(0,T;H)} + \left(\sum_{k=m+1}^n \lambda_k \|(p_T, \phi_k)_H\|_{\mathbb{R}}^2 \right)^{1/2} = 0,$$

we obtain

$$\lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} \|\delta_{m,n}\|_{L^2(0,T;\dot{H}^1)} = 0.$$

This implies that $\{z_n\}_{n=0}^\infty$ is a Cauchy sequence in $L^2_{\mathbb{F}}(0,T;\dot{H}^1)$. Hence, there exists a unique $\tilde{z} \in L^2_{\mathbb{F}}(0,T;\dot{H}^1)$ such that

$$\lim_{n \rightarrow \infty} z_n = \tilde{z} \quad \text{in } L^2_{\mathbb{F}}(0,T;\dot{H}^1).$$

By definition, we also have

$$\lim_{n \rightarrow \infty} z_n = z \quad \text{in } L^2_{\mathbb{F}}(0,T;H).$$

Consequently, we obtain $z \in L^2_{\mathbb{F}}(0,T;\dot{H}^1)$ and thus conclude the proof. \square

4 Spatial semi-discretization

Let \mathcal{K}_h be a conventional conforming, shape regular and quasi-uniform triangulation of \mathcal{O} consisting of d -simplexes, and let h denote the maximum diameter of the elements in \mathcal{K}_h . Define

$$\mathcal{V}_h := \{v_h \in C(\overline{\mathcal{O}}) \mid v_h \text{ is linear on each } K \in \mathcal{K}_h \text{ and } v_h = 0 \text{ on } \partial\mathcal{O}\}.$$

Let Q_h be the $L^2(\mathcal{O})$ -orthogonal projection onto \mathcal{V}_h , and define the discrete Laplace operator $\Delta_h : \mathcal{V}_h \rightarrow \mathcal{V}_h$ by

$$\int_{\mathcal{O}} (\Delta_h v_h) w_h dx = - \int_{\mathcal{O}} \nabla v_h \cdot \nabla w_h dx \quad \text{for all } v_h, w_h \in \mathcal{V}_h.$$

For each $\gamma \in \mathbb{R}$, let \dot{H}_h^γ be the space \mathcal{V}_h endowed with the norm

$$\|v_h\|_{\dot{H}_h^\gamma} := \|(-\Delta_h)^{\gamma/2} v_h\|_H \quad \forall v_h \in \mathcal{V}_h.$$

For any $g \in L^2_{\mathbb{F}}(0, T; H)$, let $S_0^h g$ and $S_1^h g$ be the mild solutions of the equations

$$\begin{cases} dy_h(t) = \Delta_h y_h(t) dt + Q_h g(t) dW(t) & \forall 0 \leq t \leq T, \\ y_h(0) = 0 \end{cases} \quad (23)$$

and

$$\begin{cases} dy_h(t) = (\Delta_h y_h + Q_h g)(t) dt & \forall 0 \leq t \leq T, \\ y_h(0) = 0, \end{cases} \quad (24)$$

respectively. It is standard that, for any $0 \leq t \leq T$,

$$(S_0^h g)(t) = \int_0^t e^{(t-s)\Delta_h} Q_h g(s) dW(s) \quad \mathbb{P}\text{-a.s.}, \quad (25)$$

$$(S_1^h g)(t) = \int_0^t e^{(t-s)\Delta_h} Q_h g(s) ds \quad \mathbb{P}\text{-a.s.} \quad (26)$$

In the rest of this paper, c denotes a generic positive constant independent of h , and its value may differ in different places.

We consider the following spatial semi-discretization of equation (1):

$$\begin{cases} dp_h(t) = -(\Delta_h p_h(t) + Q_h f(t, p_h(t), z_h(t))) dt + z_h(t) dW(t), & 0 \leq t \leq T, \\ p_h(T) = Q_h p_T. \end{cases} \quad (27)$$

Similarly to (1), this equation has a unique transposition solution

$$(p_h, z_h) \in D_{\mathbb{F}}([0, T]; L^2(\Omega; \mathcal{V}_h)) \times L^2_{\mathbb{F}}(0, T; \mathcal{V}_h),$$

which is defined by

$$\begin{aligned} & \int_t^T [p_h(s), g_h(s)] + [z_h(s), \sigma_h(s)] ds + [p_h(t), v_h] \\ &= \int_t^T [f(s, p_h(s), z_h(s)), (S_0^h \sigma_h + S_1^h g_h)(s) + e^{(s-t)\Delta_h} v_h] ds \\ & \quad + [(S_0^h \sigma_h + S_1^h g_h)(T) + e^{(T-t)\Delta_h} v_h, p_T] \end{aligned} \quad (28)$$

for all $0 \leq t \leq T$, $(g_h, \sigma_h) \in (L^2_{\mathbb{F}}(0, T; \mathcal{V}_h))^2$ and $v_h \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathcal{V}_h)$. Similarly to (10), we have, for any $0 \leq t \leq T$,

$$p_h(t) = \mathbb{E}_t \left(\int_t^T e^{(s-t)\Delta_h} Q_h f(s, p_h(s), z_h(s)) ds + e^{(T-t)\Delta_h} Q_h p_T \right) \quad \mathbb{P}\text{-a.s.} \quad (29)$$

The main result of this section is the following error estimate.

Theorem 4.1. *Let (p, z) and (p_h, z_h) be the transposition solutions of (1) and (27), respectively. Then*

$$\sup_{0 \leq t \leq T} \|(p - p_h)(t)\|_H + \|p - p_h\|_{L^2(0, T; \dot{H}^1)} + \|z - z_h\|_{L^2(0, T; H)} \leq ch. \quad (30)$$

4.1 Some auxiliary estimates

Lemma 4.1. For any $v \in \dot{H}^1$, we have

$$\max_{0 \leq t \leq T} \|(e^{t\Delta} - e^{t\Delta_h} Q_h)v\|_H \leq ch \|v\|_{\dot{H}^1}, \quad (31)$$

$$\left(\int_0^T \|(e^{t\Delta} - e^{t\Delta_h} Q_h)v\|_{\dot{H}^1}^2 dt \right)^{1/2} \leq ch \|v\|_{\dot{H}^1}. \quad (32)$$

Lemma 4.2. For any $g \in L^2(0, T; H)$, we have

$$\max_{0 \leq t \leq T} \left\| \int_t^T (e^{(s-t)\Delta} - e^{(s-t)\Delta_h} Q_h)g(s) ds \right\|_H \leq ch \|g\|_{L^2(0, T; H)}, \quad (33)$$

$$\left(\int_0^T \left\| \int_t^T (e^{(s-t)\Delta} - e^{(s-t)\Delta_h} Q_h)g(s) ds \right\|_{\dot{H}^1}^2 dt \right)^{1/2} \leq ch \|g\|_{L^2(0, T; H)}. \quad (34)$$

Lemma 4.3. For any $g_h \in L^2(0, T; \mathcal{V}_h)$, we have

$$\left(\int_0^T \left\| \int_t^T e^{(s-t)\Delta_h} g_h(s) ds \right\|_{\dot{H}_h^1}^2 dt \right)^{1/2} \leq \|g_h\|_{L^2(0, T; \dot{H}_h^{-1})}. \quad (35)$$

For the proof of (31), we refer the reader to [50, Theorem 3.5]. The proofs of (32), (33) and (34) are similar to that of [50, Lemma 3.6]. The inequality (35) can be proved by a routine energy argument.

Lemma 4.4. For any $g \in L^2_{\mathbb{F}}(0, T; H)$, we have

$$\|(S_0 g - S_0^h g)(T)\|_{\dot{H}^{-1}} + \|(S_0 - S_0^h)g\|_{L^2(0, T; H)} \leq ch \|g\|_{L^2(0, T; H)}. \quad (36)$$

Proof. Let $y := S_0 g$ and $y_h := S_0^h g$. By (3) we have

$$\begin{cases} dQ_h y(t) = Q_h \Delta y dt + Q_h g(t) dW(t) & \forall t \in [0, T], \\ Q_h y(0) = 0, \end{cases}$$

and so from (23) we conclude that

$$\begin{cases} de_h(t) = \Delta_h e_h(t) dt + (\Delta_h Q_h y - Q_h \Delta y)(t) dt, & \forall t \in [0, T], \\ e_h(0) = 0, \end{cases}$$

where $e_h := y_h - Q_h y$. It is standard that

$$\begin{aligned} & \|e_h(T)\|_{\dot{H}_h^{-1}} + \|e_h\|_{L^2(0, T; \dot{H}_h^0)} \\ & \leq c \|\Delta_h Q_h y - Q_h \Delta y\|_{L^2(0, T; \dot{H}_h^{-2})} \\ & = c \|\Delta_h(Q_h y - \Delta_h^{-1} Q_h \Delta y)\|_{L^2(0, T; \dot{H}_h^{-2})} \\ & = c \|Q_h y - \Delta_h^{-1} Q_h \Delta y\|_{L^2(0, T; \dot{H}_h^0)}. \end{aligned}$$

Hence,

$$\begin{aligned} & \|(y - y_h)(T)\|_{\dot{H}^{-1}} + \|y - y_h\|_{L^2(0, T; H)} \\ & \leq \|(y - Q_h y)(T)\|_{\dot{H}^{-1}} + c \|y - Q_h y\|_{L^2(0, T; H)} + c \|y - \Delta_h^{-1} Q_h \Delta y\|_{L^2(0, T; H)} \\ & \leq ch \|y(T)\|_H + ch \|y\|_{L^2(0, T; \dot{H}^1)}, \end{aligned}$$

by the following two standard estimates (see, e.g., [9, Theorems 4.4.20 and 5.7.6]):

$$\begin{aligned} \|(I - Q_h)v\|_{\dot{H}^{-1}} &\leq ch\|v\|_H \quad \forall v \in H, \\ \|(I - Q_h)v\|_H + \|v - \Delta_h^{-1}Q_h\Delta v\|_H &\leq ch\|v\|_{\dot{H}^1} \quad \forall v \in \dot{H}^1. \end{aligned}$$

Therefore, the desired estimate (36) follows from

$$\|y(T)\|_H + \|y\|_{L^2(0,T;\dot{H}^1)} \leq c\|g\|_{L^2(0,T;H)},$$

which can be obtained by inserting $\gamma = 0$ and $t = T$ into (5). This completes the proof. \blacksquare

4.2 Proof of Theorem 4.1

To be specific, in this proof c denotes a positive constant depending only on f , p_T , \mathcal{O} , T and the regularity parameters of \mathcal{K}_h . Let

$$e_h^p := p_h - p, \quad e_h^z := z_h - z.$$

By $f(\cdot, 0, 0) \in L^2_{\mathbb{F}}(0, T; H)$, (2) and

$$(p, z) \in D_{\mathbb{F}}([0, T]; L^2(\Omega; H)) \times L^2_{\mathbb{F}}(0, T; H),$$

we have

$$f(\cdot, p(\cdot), z(\cdot)) \in L^2_{\mathbb{F}}(0, T; H). \quad (37)$$

We divide the proof into the following four steps.

Step 1. Let us prove, for any $0 \leq t < T$,

$$\|e_h^z\|_{L^2(t,T;H)} \leq c\left(h + \sqrt{T-t}\left(\|e_h^p\|_{L^2(t,T;H)} + \|e_h^z\|_{L^2(t,T;H)}\right)\right). \quad (38)$$

To this end, let $0 \leq t < T$ be arbitrary but fixed. Define

$$\sigma_h(s) := \begin{cases} 0 & \text{if } 0 \leq s < t, \\ (z_h - Q_h z)(s) & \text{if } t \leq s \leq T. \end{cases} \quad (39)$$

By (25) and (39) we get \mathbb{P} -a.s.

$$(S_0^h \sigma_h)(s) = \begin{cases} 0 & \text{if } 0 \leq s \leq t, \\ \int_t^s e^{(r-t)\Delta_h} (z_h - Q_h z)(r) dW(r) & \text{if } t < s \leq T. \end{cases} \quad (40)$$

It follows that, for any $t < s \leq T$,

$$\begin{aligned} \|(S_0^h \sigma_h)(s)\|_H &= \left\| \int_t^s e^{(r-t)\Delta_h} (z_h - Q_h z)(r) dW(r) \right\|_H \\ &= \left(\int_t^s \left\| e^{(r-t)\Delta_h} (z_h - Q_h z)(r) \right\|_H^2 dr \right)^{1/2} \\ &\leq \left(\int_t^s \|z_h - Q_h z(r)\|_H^2 dr \right)^{1/2} \\ &\leq \|z_h - Q_h z\|_{L^2(t,T;H)}. \end{aligned} \quad (41)$$

Inserting $g = 0$, $\sigma = \sigma_h$ and $v = 0$ into (9) yields

$$\int_t^T [Q_h z(s), \sigma_h(s)] ds = \int_t^T [(S_0 \sigma_h)(s), f(s, p(s), z(s))] ds + [(S_0 \sigma_h)(T), p_T],$$

and inserting $g_h = 0$ and $v_h = 0$ into (28) gives

$$\int_t^T [z_h(s), \sigma_h(s)] ds = \int_t^T [(S_0^h \sigma_h)(s), f(s, p_h(s), z_h(s))] ds + [(S_0^h \sigma_h)(T), p_T].$$

Combining the two equalities above yields

$$\|z_h - Q_h z\|_{L^2(t, T; H)}^2 = \int_t^T [(z_h - Q_h z)(s), \sigma_h(s)] ds = \mathbb{I}_1 + \mathbb{I}_2 + \mathbb{I}_3, \quad (42)$$

where

$$\begin{aligned} \mathbb{I}_1 &:= \int_t^T [(S_0^h \sigma_h)(s), f(s, p_h(s), z_h(s)) - f(s, p(s), z(s))] ds, \\ \mathbb{I}_2 &:= \int_t^T [(S_0^h \sigma_h - S_0 \sigma_h)(s), f(s, p(s), z(s))] ds, \\ \mathbb{I}_3 &:= [(S_0^h \sigma_h - S_0 \sigma_h)(T), p_T]. \end{aligned}$$

For \mathbb{I}_1 we have

$$\begin{aligned} \mathbb{I}_1 &\leq \int_t^T \left\| (S_0^h \sigma_h)(s) \right\|_H \left\| f(s, p_h(s), z_h(s)) - f(s, p(s), z(s)) \right\|_H ds \\ &\leq \|z_h - Q_h z\|_{L^2(t, T; H)} \int_t^T \left\| f(s, p_h(s), z_h(s)) - f(s, p(s), z(s)) \right\|_H ds \quad (\text{by (41)}) \\ &\leq c \|z_h - Q_h z\|_{L^2(t, T; H)} \int_t^T \left\| e_h^p(s) \right\|_H ds + \left\| e_h^z(s) \right\|_H ds \quad (\text{by (2)}) \\ &\leq c\sqrt{T-t} \|z_h - Q_h z\|_{L^2(t, T; H)} \left(\left\| e_h^p \right\|_{L^2(t, T; H)} + \left\| e_h^z \right\|_{L^2(t, T; H)} \right). \end{aligned}$$

For \mathbb{I}_2 we have

$$\begin{aligned} \mathbb{I}_2 &\leq \left\| (S_0^h - S_0) \sigma_h \right\|_{L^2(t, T; H)} \left\| f(\cdot, p(\cdot), z(\cdot)) \right\|_{L^2(t, T; H)} \\ &\leq ch \left\| \sigma_h \right\|_{L^2(t, T; H)} \left\| f(\cdot, p(\cdot), z(\cdot)) \right\|_{L^2(t, T; H)} \quad (\text{by (36)}) \\ &= ch \|z_h - Q_h z\|_{L^2(t, T; H)} \left\| f(\cdot, p(\cdot), z(\cdot)) \right\|_{L^2(t, T; H)} \quad (\text{by (39)}) \\ &\leq ch \|z_h - Q_h z\|_{L^2(t, T; H)} \quad (\text{by (37)}). \end{aligned}$$

For \mathbb{I}_3 we have

$$\begin{aligned} \mathbb{I}_3 &\leq \left\| (S_0^h \sigma_h - S_0 \sigma_h)(T) \right\|_{\dot{H}^{-1}} \|p_T\|_{\dot{H}^1} \\ &\leq ch \left\| \sigma_h \right\|_{L^2(0, T; H)} \|p_T\|_{\dot{H}^1} \quad (\text{by (36)}) \\ &\leq ch \left\| \sigma_h \right\|_{L^2(0, T; H)} \\ &= ch \|z_h - Q_h z\|_{L^2(t, T; H)} \quad (\text{by (39)}). \end{aligned}$$

Combining (42) and the above estimates of \mathbb{I}_1 , \mathbb{I}_2 and \mathbb{I}_3 , we obtain

$$\begin{aligned} &\|z_h - Q_h z\|_{L^2(t, T; H)}^2 \\ &\leq c \left(h + \sqrt{T-t} \left\| e_h^p \right\|_{L^2(t, T; H)} + \sqrt{T-t} \left\| e_h^z \right\|_{L^2(t, T; H)} \right) \|z_h - Q_h z\|_{L^2(t, T; H)}, \end{aligned}$$

which implies

$$\|z_h - Q_h z\|_{L^2(t, T; H)} \leq ch + c\sqrt{T-t} (\|e_h^p\|_{L^2(t, T; H)} + \|e_h^z\|_{L^2(t, T; H)}). \quad (43)$$

By Theorem 3.1 we have $z \in L^2_{\mathbb{F}}(0, T; \dot{H}^1)$, and so we have the standard estimate

$$\|z - Q_h z\|_{L^2(0, T; H)} \leq ch. \quad (44)$$

Hence, the desired estimate (38) follows from (43) and (44).

Step 2. Let c^* be the particular constant c in the inequality (38). Let $t^* := \max\{0, T - 1/(2c^*)^2\}$, and so by (38) we get

$$\|e_h^z\|_{L^2(t, T; H)} \leq c(h + \|e_h^p\|_{L^2(t, T; H)}) \quad \forall t \in [t^*, T]. \quad (45)$$

By (10) and (29) we have, for any $0 \leq t \leq T$,

$$e_h^p(t) = \mathbb{I}_4 + \mathbb{I}_5 + \mathbb{I}_6 \quad \mathbb{P}\text{-a.s.},$$

where

$$\begin{aligned} \mathbb{I}_4 &:= \mathbb{E}_t \int_t^T e^{(s-t)\Delta_h} Q_h (f(s, p_h(s), z_h(s)) - f(s, p(s), z(s))) \, ds, \\ \mathbb{I}_5 &:= \mathbb{E}_t \int_t^T (e^{(s-t)\Delta_h} Q_h - e^{(s-t)\Delta}) f(s, p(s), z(s)) \, ds, \\ \mathbb{I}_6 &:= \mathbb{E}_t (e^{(T-t)\Delta_h} Q_h - e^{(T-t)\Delta}) p_T. \end{aligned}$$

For \mathbb{I}_4 we have

$$\begin{aligned} \|\mathbb{I}_4\|_H^2 &\leq \left\| \int_t^T e^{(s-t)\Delta_h} Q_h (f(s, p_h(s), z_h(s)) - f(s, p(s), z(s))) \, ds \right\|_H^2 \\ &\leq \left(\int_t^T \left\| e^{(s-t)\Delta_h} Q_h (f(s, p_h(s), z_h(s)) - f(s, p(s), z(s))) \right\|_H \, ds \right)^2 \\ &\leq \left(\int_t^T \|f(s, p_h(s), z_h(s)) - f(s, p(s), z(s))\|_H \, ds \right)^2 \\ &\leq c \int_t^T \|f(s, p_h(s), z_h(s)) - f(s, p(s), z(s))\|_H^2 \, ds \\ &\leq c (\|e_h^p\|_{L^2(t, T; H)}^2 + \|e_h^z\|_{L^2(t, T; H)}^2) \quad (\text{by (2)}). \end{aligned}$$

For \mathbb{I}_5 we have

$$\begin{aligned} \|\mathbb{I}_5\|_H^2 &\leq \left\| \int_t^T (e^{(s-t)\Delta_h} Q_h - e^{(s-t)\Delta}) f(s, p(s), z(s)) \, ds \right\|_H^2 \\ &\leq ch^2 \|f(\cdot, p(\cdot), z(\cdot))\|_{L^2(t, T; H)}^2 \quad (\text{by (33)}) \\ &\leq ch^2 \quad (\text{by (37)}). \end{aligned}$$

For \mathbb{I}_6 we have

$$\begin{aligned} \|\mathbb{I}_6\|_H^2 &\leq \left\| (e^{(T-t)\Delta_h} Q_h - e^{(T-t)\Delta}) p_T \right\|_H^2 \\ &\leq ch^2 \|p_T\|_{\dot{H}^1}^2 \quad (\text{by (31)}) \\ &\leq ch^2. \end{aligned}$$

Combining the above estimates of \mathbb{I}_4 , \mathbb{I}_5 and \mathbb{I}_6 , we obtain

$$\|e_h^p(t)\|_H^2 \leq c(h^2 + \|e_h^p\|_{L^2(t,T;H)}^2 + \|e_h^z\|_{L^2(t,T;H)}^2) \quad \forall t \in [0, T], \quad (46)$$

which, together with (45), implies

$$\|e_h^p(t)\|_H^2 \leq c(h^2 + \|e_h^p\|_{L^2(t,T;H)}^2) \quad \forall t \in [t^*, T].$$

Using the Gronwall's inequality then gives

$$\sup_{t^* \leq t \leq T} \|e_h^p(t)\|_H \leq ch.$$

Hence, by (45) we get

$$\sup_{t^* \leq t \leq T} (\|e_h^p(t)\|_H + \|e_h^z\|_{L^2(t^*,T;H)}) \leq ch. \quad (47)$$

Step 3. Note that t^* depends only on f , p_T , \mathcal{O} , T and the regularity parameters of \mathcal{K}_h . With the estimate $\|e_h^p(t^*)\|_H \leq ch$ (see (47)) and similar arguments in the proof of (45), we get

$$\|e_h^z\|_{L^2(t,t^*;H)} \leq c(h + \|e_h^p\|_{L^2(t,t^*;H)}) \quad \forall t \in [\max\{0, 2t^* - T\}, t^*], \quad (48)$$

which, together with (45), implies

$$\|e_h^z\|_{L^2(t,T;H)} \leq c(h + \|e_h^p\|_{L^2(t,T;H)}) \quad \forall t \in [\max\{0, 2t^* - T\}, T].$$

Combining the above estimate and (46), we obtain

$$\|e_h^p(t)\|_H^2 \leq c(h^2 + \|e_h^p\|_{L^2(t,T;H)}^2) \quad \forall t \in [\max\{0, 2t^* - T\}, T].$$

The Gronwall's inequality then leads to

$$\sup_{\max\{0, 2t^* - T\} \leq t \leq T} \|e_h^p(t)\|_H \leq ch,$$

and so by (48) we obtain

$$\sup_{\max\{0, 2t^* - T\} \leq t \leq T} (\|e_h^p(t)\|_H + \|e_h^z\|_{L^2(\max\{0, 2t^* - T\}, T; H)}) \leq ch.$$

Repeating the above procedure several times finally yields

$$\sup_{0 \leq t \leq T} (\|e_h^p(t)\|_H + \|e_h^z\|_{L^2(0,T;H)}) \leq ch. \quad (49)$$

Step 4. Let us prove

$$\|p - p_h\|_{L^2(0,T;\dot{H}^1)} \leq ch. \quad (50)$$

Set

$$\begin{aligned} F(t) &:= f(t, p(t), z(t)), \quad 0 \leq t \leq T, \\ F_h(t) &:= Q_h f(t, p_h(t), z_h(t)), \quad 0 \leq t \leq T. \end{aligned}$$

By (2) and (49), we have

$$\begin{aligned}
\|F_h - Q_h F\|_{L^2(0,T;H)} &= \left\| Q_h(f(\cdot, p_h(\cdot), z_h(\cdot)) - f(\cdot, p(\cdot), z(\cdot))) \right\|_{L^2(0,T;H)} \\
&\leq \|f(\cdot, p_h(\cdot), z_h(\cdot)) - f(\cdot, p(\cdot), z(\cdot))\|_{L^2(0,T;H)} \\
&\leq c(\|e_h^p\|_{L^2(0,T;H)} + \|e_h^z\|_{L^2(0,T;H)}) \\
&\leq ch.
\end{aligned} \tag{51}$$

By (10) and (29) we obtain, for any $t \in [0, T]$,

$$(p - p_h)(t) = \mathbb{E}_t(\eta_1 + \eta_2 + \eta_3)(t) \quad \mathbb{P}\text{-a.s.},$$

where

$$\begin{aligned}
\eta_1(t) &:= \int_t^T (e^{(s-t)\Delta} - e^{(s-t)\Delta_h} Q_h) F(s) \, ds, \\
\eta_2(t) &:= \int_t^T e^{(s-t)\Delta_h} (Q_h F - F_h)(s) \, ds, \\
\eta_3(t) &:= (e^{(T-t)\Delta} - e^{(T-t)\Delta_h} Q_h) p_T.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|p - p_h\|_{L^2(0,T;\dot{H}^1)} &\leq \|\eta_1 + \eta_2 + \eta_3\|_{L^2(0,T;\dot{H}^1)} \\
&\leq \|\eta_1\|_{L^2(0,T;\dot{H}^1)} + \|\eta_2\|_{L^2(0,T;\dot{H}^1)} + \|\eta_3\|_{L^2(0,T;\dot{H}^1)}. \tag{52}
\end{aligned}$$

For η_1 we have

$$\begin{aligned}
\|\eta_1\|_{L^2(0,T;\dot{H}^1)} &= \left(\mathbb{E} \int_0^T \left\| \int_t^T (e^{(s-t)\Delta} - e^{(s-t)\Delta_h} Q_h) F(s) \, ds \right\|_{\dot{H}^1}^2 dt \right)^{1/2} \\
&\leq ch \left(\mathbb{E} \|F\|_{L^2(0,T;H)}^2 \right)^{1/2} \quad (\text{by (34)}) \\
&\leq ch \quad (\text{by (37)}).
\end{aligned}$$

For η_2 we have

$$\begin{aligned}
\|\eta_2\|_{L^2(0,T;\dot{H}^1)} &= \left(\mathbb{E} \int_0^T \left\| \int_t^T e^{(s-t)\Delta_h} (Q_h F - F_h)(s) \, ds \right\|_{\dot{H}^1}^2 dt \right)^{1/2} \\
&\leq \left(\mathbb{E} \|F_h - Q_h F\|_{L^2(0,T;H)}^2 \right)^{1/2} \quad (\text{by (35)}) \\
&= \|F_h - Q_h F\|_{L^2(0,T;H)} \\
&\leq ch \quad (\text{by (51)}).
\end{aligned}$$

For η_3 we have

$$\begin{aligned}
\|\eta_3\|_{L^2(0,T;\dot{H}^1)} &= \left(\mathbb{E} \int_0^T \| (e^{(T-t)\Delta} - e^{(T-t)\Delta_h} Q_h) p_T \|_{\dot{H}^1}^2 dt \right)^{1/2} \\
&\leq ch \left(\mathbb{E} \|p_T\|_{\dot{H}^1}^2 \right)^{1/2} \quad (\text{by (32)}) \\
&= ch \|p_T\|_{\dot{H}^1} \leq ch.
\end{aligned}$$

Combining (52) and the above estimates of η_1 , η_2 and η_3 yields (50). Finally, summing up (49) and (50) proves (30) and thus concludes the proof of Theorem 4.1.

5 Application to a stochastic linear quadratic control problem

5.1 Continuous problem

We consider the following stochastic linear quadratic control problem:

$$\min_{\substack{u \in L^2_{\mathbb{F}}(0, T; H) \\ y \in L^2_{\mathbb{F}}(0, T; H)}} \frac{1}{2} \| \|y - y_d\| \|_{L^2(0, T; H)}^2 + \frac{\nu}{2} \| \|u\| \|_{L^2(0, T; H)}^2, \quad (53)$$

subject to the state equation

$$\begin{cases} dy(t) = (\Delta y + \alpha_0 y + \alpha_1 u)(t) dt + (\alpha_2 y + \alpha_3 u)(t) dW(t), & 0 \leq t \leq T, \\ y(0) = 0, \end{cases} \quad (54)$$

where $0 < \nu, T < \infty$, $\alpha_0, \alpha_1, \alpha_2, \alpha_3 \in L^\infty(0, T)$, and $y_d \in L^2_{\mathbb{F}}(0, T; H)$. It is standard that (see [38, Theorem 8.1]) problem (53) admits a unique solution (\bar{u}, \bar{y}) , and

$$\bar{u} = -\nu^{-1}(\alpha_1 \bar{p} + \alpha_3 \bar{z}), \quad (55)$$

where (\bar{p}, \bar{z}) is the transposition solution of the backward stochastic parabolic equation

$$\begin{cases} d\bar{p}(t) = -(\Delta \bar{p} + \alpha_0 \bar{p} + \bar{y} - y_d + \alpha_2 \bar{z})(t) dt + \bar{z}(t) dW(t), & 0 \leq t \leq T, \\ \bar{p}(T) = 0. \end{cases} \quad (56)$$

Since Theorem 3.1 implies

$$\alpha_1 \bar{p} + \alpha_3 \bar{z} \in L^2_{\mathbb{F}}(0, T; \dot{H}^1),$$

we then obtain

$$\bar{u} \in L^2_{\mathbb{F}}(0, T; \dot{H}^1). \quad (57)$$

Moreover, we have

$$\bar{y} \in L^2_{\mathbb{F}}(\Omega; C([0, T]; \dot{H}^1)) \cap L^2_{\mathbb{F}}(0, T; \dot{H}^2). \quad (58)$$

Remark 5.1. For the theoretical treatment of the stochastic linear quadratic control problems, we refer the reader to [41] and the references therein.

Remark 5.2. The regularity result (58) is standard. It can be easily proved by the standard Galerkin method and the theory of the finite-dimensional linear SDEs (see [45, Chapter 3]).

Remark 5.3. We note that Zhou and Li [58] established the convergence of a full discretization for a Neumann boundary control problem governed by a stochastic parabolic equation with additive boundary noise and general filtration. When \mathbb{F} is the natural filtration of the Brownian motion, we refer the reader to [19], [33] and [48] for some related numerical analysis.

5.2 Spatially semi-discrete problem

The spatial semi-discretization of problem (53) reads as follows:

$$\min_{\substack{u_h \in L^2_{\mathbb{F}}(0,T;\mathcal{V}_h) \\ y_h \in L^2_{\mathbb{F}}(0,T;\mathcal{V}_h)}} \frac{1}{2} \|y_h - y_d\|_{L^2(0,T;H)}^2 + \frac{\nu}{2} \|u_h\|_{L^2(0,T;H)}^2, \quad (59)$$

subject to the state equation

$$\begin{cases} dy_h(t) = (\Delta_h y_h + \alpha_0 y_h + \alpha_1 u_h)(t) dt + (\alpha_2 y_h + \alpha_3 u_h)(t) dW(t), & 0 \leq t \leq T, \\ y_h(0) = 0. \end{cases} \quad (60)$$

Similarly to problem (53), problem (59) admits a unique solution (\bar{u}_h, \bar{y}_h) , and

$$\bar{u}_h = -\nu^{-1}(\alpha_1 \bar{p}_h + \alpha_3 \bar{z}_h), \quad (61)$$

where (\bar{p}_h, \bar{z}_h) is the transposition solution of the spatially semi-discrete backward stochastic parabolic equation

$$\begin{cases} d\bar{p}_h(t) = -(\Delta_h \bar{p}_h + \alpha_0 \bar{p}_h + \bar{y}_h - Q_h y_d + \alpha_2 \bar{z}_h)(t) dt + \bar{z}_h(t) dW(t), & 0 \leq t \leq T, \\ \bar{p}_h(T) = 0. \end{cases} \quad (62)$$

The main result of this section is the following error estimate.

Theorem 5.1. *Let (\bar{u}, \bar{y}) and (\bar{u}_h, \bar{y}_h) be the solutions of problems (53) and (59), respectively. Then*

$$\|\bar{u} - \bar{u}_h\|_{L^2(0,T;H)} + \|\bar{y} - \bar{y}_h\|_{L^2(0,T;H)} \leq ch. \quad (63)$$

To prove this theorem, we first introduce three lemmas.

Lemma 5.1. *Let y_h be the solution of the stochastic equation*

$$\begin{cases} dy_h(t) = (\Delta_h y_h + \alpha_0 y_h + g_h)(t) dt + \alpha_2(t) y_h(t) dW(t), & 0 \leq t \leq T, \\ y_h(0) = 0, \end{cases} \quad (64)$$

where $g_h \in L^2_{\mathbb{F}}(0,T;\mathcal{V}_h)$. Then

$$\|y_h\|_{L^2(0,T;H)} \leq c \|g_h\|_{L^2(0,T;\dot{H}_h^{-2})}. \quad (65)$$

Proof. Letting $w_h := (-\Delta_h)^{-1/2} y_h$, by (64) we have

$$\begin{cases} dw_h(t) = (\Delta_h w_h + \alpha_0 w_h + (-\Delta_h)^{-1/2} g_h)(t) dt + \alpha_2(t) w_h(t) dW(t), & 0 \leq t \leq T, \\ w_h(0) = 0. \end{cases}$$

A routine argument with the Itô's formula then yields, for any $0 \leq t \leq T$,

$$\begin{aligned} & \|w_h(t)\|_H^2 + 2 \|w_h\|_{L^2(0,t;\dot{H}_h^1)}^2 \\ &= 2 \int_0^t [w_h(s), \alpha_0(s) w_h(s) + (-\Delta_h)^{-1/2} g_h(s)] ds + \|\alpha_2 w_h\|_{L^2(0,t;H)}^2 \\ &\leq c \|w_h\|_{L^2(0,t;H)}^2 + 2 \|w_h\|_{L^2(0,t;\dot{H}_h^1)} \left\| (-\Delta_h)^{-1/2} g_h \right\|_{L^2(0,t;\dot{H}_h^{-1})} \\ &\leq c \|w_h\|_{L^2(0,t;H)}^2 + \|w_h\|_{L^2(0,t;\dot{H}_h^1)}^2 + \left\| (-\Delta_h)^{-1/2} g_h \right\|_{L^2(0,t;\dot{H}_h^{-1})}^2 \\ &= c \|w_h\|_{L^2(0,t;H)}^2 + \|w_h\|_{L^2(0,t;\dot{H}_h^1)}^2 + \|g_h\|_{L^2(0,t;\dot{H}_h^{-2})}^2. \end{aligned} \quad (66)$$

It follows that

$$\|w_h(t)\|_H^2 \leq c \|w_h\|_{L^2(0,t;H)}^2 + \|g_h\|_{L^2(0,t;\dot{H}_h^{-2})}^2 \quad \forall 0 \leq t \leq T,$$

and so using the Gronwall's inequality yields

$$\sup_{0 \leq t \leq T} \|w_h(t)\|_H \leq c \|g_h\|_{L^2(0,T;\dot{H}_h^{-2})}. \quad (67)$$

In addition, inserting $t = T$ into (66) gives

$$\|w_h\|_{L^2(0,T;\dot{H}_h^1)}^2 \leq c \|w_h\|_{L^2(0,T;H)}^2 + \|g_h\|_{L^2(0,T;\dot{H}_h^{-2})}^2. \quad (68)$$

Finally, combining (67), (68) and the fact

$$\|y_h\|_{L^2(0,T;H)} = \|w_h\|_{L^2(0,T;\dot{H}_h^1)},$$

we readily obtain (65). ■

Lemma 5.2. *Assume that (\bar{u}, \bar{y}) is the solution of problem (53). Let y_h be the mild solution of the stochastic equation*

$$\begin{cases} dy_h(t) = (\Delta_h y_h + \alpha_0 y_h + \alpha_1 Q_h \bar{u})(t) dt + (\alpha_2 y_h + \alpha_3 Q_h \bar{u})(t) dW(t), & 0 \leq t \leq T, \\ y_h(0) = 0. \end{cases} \quad (69)$$

Then

$$\|\bar{y} - y_h\|_{L^2(0,T;H)} \leq ch^2. \quad (70)$$

Proof. By definition we have

$$\begin{cases} d\bar{y}(t) = (\Delta \bar{y} + \alpha_0 \bar{y} + \alpha_1 \bar{u})(t) dt + (\alpha_2 \bar{y} + \alpha_3 \bar{u})(t) dW(t), & 0 \leq t \leq T, \\ \bar{y}(0) = 0, \end{cases}$$

so that

$$\begin{cases} dQ_h \bar{y}(t) = Q_h (\Delta \bar{y} + \alpha_0 \bar{y} + \alpha_1 \bar{u})(t) dt + Q_h (\alpha_2 \bar{y} + \alpha_3 \bar{u})(t) dW(t), & 0 \leq t \leq T, \\ Q_h \bar{y}(0) = 0. \end{cases}$$

Hence, by (69) we get

$$\begin{cases} de_h(t) = (\Delta_h e_h + \alpha_0 e_h + \Delta_h Q_h \bar{y} - Q_h \Delta \bar{y})(t) dt + \alpha_2(t) e_h(t) dW(t), & 0 \leq t \leq T, \\ e_h(0) = 0, \end{cases}$$

where $e_h := y_h - Q_h \bar{y}$. By Lemma 5.1 we then obtain

$$\begin{aligned} \|e_h\|_{L^2(0,T;H)} &\leq c \|\Delta_h Q_h \bar{y} - Q_h \Delta \bar{y}\|_{L^2(0,T;\dot{H}_h^{-2})} \\ &= c \|\Delta_h (Q_h \bar{y} - \Delta_h^{-1} Q_h \Delta \bar{y})\|_{L^2(0,T;\dot{H}_h^{-2})} \\ &= c \|Q_h \bar{y} - \Delta_h^{-1} Q_h \Delta \bar{y}\|_{L^2(0,T;H)}. \end{aligned}$$

It follows that

$$\begin{aligned} &\|\bar{y} - y_h\|_{L^2(0,T;H)} \\ &\leq \|\bar{y} - Q_h \bar{y}\|_{L^2(0,T;H)} + \|e_h\|_{L^2(0,T;H)} \\ &\leq \|\bar{y} - Q_h \bar{y}\|_{L^2(0,T;H)} + c \|Q_h \bar{y} - \Delta_h^{-1} Q_h \Delta \bar{y}\|_{L^2(0,T;H)} \\ &\leq c \|\bar{y} - Q_h \bar{y}\|_{L^2(0,T;H)} + c \|\bar{y} - \Delta_h^{-1} Q_h \Delta \bar{y}\|_{L^2(0,T;H)}. \end{aligned}$$

Hence, the desired estimate (70) follows from (58) and the standard estimate

$$\|v - Q_h v\|_H + \|v - \Delta_h^{-1} Q_h \Delta v\|_H \leq ch^2 \|v\|_{\dot{H}^2} \quad \forall v \in \dot{H}^2.$$

This completes the proof. \blacksquare

Lemma 5.3. *Assume that $g_h, v_h \in L^2_{\mathbb{F}}(0, T; \mathcal{V}_h)$. Let (p_h, z_h) be the transposition solution of the equation*

$$\begin{cases} dp_h(t) = -(\Delta_h p_h + \alpha_0 p_h + g_h + \alpha_2 z_h)(t) dt + z_h(t) dW(t), & 0 \leq t \leq T, \\ p_h(T) = 0, \end{cases}$$

and let y_h be the mild solution of the equation

$$\begin{cases} dy_h(t) = (\Delta_h y_h + \alpha_0 y_h + \alpha_1 v_h)(t) dt + (\alpha_2 y_h + \alpha_3 v_h)(t) dW(t), & 0 \leq t \leq T, \\ y_h(0) = 0. \end{cases}$$

Then

$$\int_0^T [(\alpha_1 p_h + \alpha_3 z_h)(t), v_h(t)] dt = \int_0^T [g_h(t), y_h(t)] dt. \quad (71)$$

Proof. Note that

$$y_h = S_0^h(\alpha_2 y_h + \alpha_3 v_h) + S_1^h(\alpha_0 y_h + \alpha_1 v_h).$$

By definition (see (28)), we then obtain

$$\begin{aligned} & \int_0^T [p_h(t), (\alpha_0 y_h + \alpha_1 v_h)(t)] + [z_h(t), (\alpha_2 y_h + \alpha_3 v_h)(t)] dt \\ &= \int_0^T [(\alpha_0 p_h + g_h + \alpha_2 z_h)(t), y_h(t)] dt, \end{aligned}$$

which implies the desired equality (71). This completes the proof. \blacksquare

Finally, we are in a position to show the proof of Theorem 5.1 as follows.

Proof of Theorem 5.1. Firstly, we present some preliminary results. Let y_h be the mild solution of (69), let (\bar{p}, \bar{z}) and (\bar{p}_h, \bar{z}_h) be the transposition solutions of (56) and (62), respectively, and let (p_h, z_h) be the transposition solution of the spatially semi-discrete backward stochastic parabolic equation

$$\begin{cases} dp_h(t) = -(\Delta_h p_h + \alpha_0 p_h + Q_h(\bar{y} - y_d) + \alpha_2 z_h)(t) dt + z_h(t) dW(t), & 0 \leq t \leq T, \\ p_h(T) = 0. \end{cases} \quad (72)$$

By Lemma 5.3, it is easy to verify that

$$\int_0^T [\alpha_1(p_h - \bar{p}_h) + \alpha_3(z_h - \bar{z}_h), \bar{u}_h - \bar{u}] dt = \int_0^T [\bar{y} - \bar{y}_h, \bar{y}_h - y_h] dt. \quad (73)$$

By Theorem 4.1 we have

$$\sup_{0 \leq t \leq T} \|\bar{p} - p_h(t)\|_H + \|\bar{p} - p_h\|_{L^2(0, T; \dot{H}^1)} + \|\bar{z} - z_h\|_{L^2(0, T; H)} \leq ch. \quad (74)$$

Secondly, we use the standard argument in the numerical analysis of optimization with PDE constraints (see [26, Theorem 3.4]) to prove (63). By (55) we obtain

$$\nu \int_0^T [\bar{u}, \bar{u} - \bar{u}_h] dt = \int_0^T [\alpha_1 \bar{p} + \alpha_3 \bar{z}, \bar{u}_h - \bar{u}] dt, \quad (75)$$

and by (61) we get

$$-\nu \int_0^T [\bar{u}_h, \bar{u} - \bar{u}_h] dt = \int_0^T [\alpha_1 \bar{p}_h + \alpha_3 \bar{z}_h, \bar{u} - \bar{u}_h] dt. \quad (76)$$

Summing up the above two equalities yields

$$\begin{aligned} \nu \|\bar{u} - \bar{u}_h\|_{L^2(0,T;H)}^2 &= \int_0^T [\alpha_1(\bar{p} - \bar{p}_h) + \alpha_3(\bar{z} - \bar{z}_h), \bar{u}_h - \bar{u}] dt \\ &= \mathbb{I}_1 + \mathbb{I}_2, \end{aligned}$$

where

$$\begin{aligned} \mathbb{I}_1 &:= \int_0^T [\alpha_1(\bar{p} - p_h) + \alpha_3(\bar{z} - z_h), \bar{u}_h - \bar{u}] dt, \\ \mathbb{I}_2 &:= \int_0^T [\alpha_1(p_h - \bar{p}_h) + \alpha_3(z_h - \bar{z}_h), \bar{u}_h - \bar{u}] dt. \end{aligned}$$

For \mathbb{I}_1 we have

$$\begin{aligned} \mathbb{I}_1 &\leq c(\|\bar{p} - p_h\|_{L^2(0,T;H)} + \|\bar{z} - z_h\|_{L^2(0,T;H)}) \|\bar{u} - \bar{u}_h\|_{L^2(0,T;H)} \\ &\leq ch \|\bar{u} - \bar{u}_h\|_{L^2(0,T;H)} \quad (\text{by (74)}). \end{aligned}$$

For \mathbb{I}_2 we have

$$\begin{aligned} \mathbb{I}_2 &= \int_0^T [\bar{y} - \bar{y}_h, \bar{y}_h - y_h] dt \quad (\text{by (73)}) \\ &= -\|\bar{y} - \bar{y}_h\|_{L^2(0,T;H)}^2 + \int_0^T [\bar{y} - \bar{y}_h, \bar{y} - y_h] dt \\ &\leq -\frac{1}{2} \|\bar{y} - \bar{y}_h\|_{L^2(0,T;H)}^2 + \frac{1}{2} \|\bar{y} - y_h\|_{L^2(0,T;H)}^2 \\ &\leq -\frac{1}{2} \|\bar{y} - \bar{y}_h\|_{L^2(0,T;H)}^2 + ch^4 \quad (\text{by (70)}). \end{aligned}$$

Combining the above estimates of \mathbb{I}_1 and \mathbb{I}_2 yields

$$\begin{aligned} \nu \|\bar{u} - \bar{u}_h\|_{L^2(0,T;H)}^2 &+ \frac{1}{2} \|\bar{y} - \bar{y}_h\|_{L^2(0,T;H)}^2 \\ &\leq ch \|\bar{u} - \bar{u}_h\|_{L^2(0,T;H)} + ch^4 \\ &\leq ch^2 + \frac{\nu}{2} \|\bar{u} - \bar{u}_h\|_{L^2(0,T;H)}^2, \end{aligned}$$

which implies (63). This completes the proof. \square

5.3 Numerical results

Let $J \geq 2$ be a positive integer. Set $t_j := j\tau$ for each $0 \leq j \leq J$, where $\tau := T/J$. Define

$$L_{\mathbb{F},\tau}^2(0, T; \mathcal{V}_h) := \{U \in L_{\mathbb{F}}^2(0, T; \mathcal{V}_h) \mid U \text{ is constant on } [t_j, t_{j+1}), \forall 0 \leq j < J\}.$$

For each $0 \leq j < J$, define $\delta W_j := W(t_{j+1}) - W(t_j)$, and let $\text{Net}_j : \mathbb{R}^d \times \mathbb{R}^j \rightarrow \mathbb{R}$ be a neural network. Define the control $U \in L_{\mathbb{F},\tau}^2(0, T; \mathcal{V}_h)$ as follows:

$$U(0, x) := \text{Net}_0(x) \quad \text{for each interior node } x \text{ of } \mathcal{K}_h;$$

for any $1 \leq j < J$,

$$U(t_j, x) := \text{Net}_j(x, \delta W_0, \dots, \delta W_{j-1}) \quad \text{for each interior node } x \text{ of } \mathcal{K}_h.$$

The numerical optimal control \bar{U} is obtained by training these neural networks Net_j with the loss function

$$\mathcal{L}_{h,\tau}(U) := \frac{1}{2} \sum_{j=0}^{J-1} \| \| Y - y_d(t_j) \| \|_{L^2(t_j, t_{j+1}; H)}^2 + \frac{\nu}{2} \| \| U \| \|_{L^2(0, T; H)}^2,$$

where the numerical state $Y \in L_{\mathbb{F},\tau}^2(0, T; \mathcal{V}_h)$ is calculated as follows:

$$\begin{cases} Y(t_{j+1}) - Y(t_j) = \tau \Delta_h Y(t_{j+1}) + \tau(\alpha_0 Y + \alpha_1 U)(t_j) \\ \quad + (\alpha_2 Y + \alpha_3 U)(t_j) \delta W_j, & 0 \leq j < J, \\ Y(0) = 0. \end{cases}$$

Our numerical experiment adopts the following settings: $\mathcal{O} = (0, 1)$, $T = 2 \times 10^{-1}$, $\nu = 1 \times 10^{-2}$, $\alpha_0 = \alpha_1 = \alpha_2 = 1$, $\alpha_3 = 1 \times 10^{-1}$; each Net_j , $0 \leq j < J$, is a fully connected feedforward neural network with four hidden layers, where each hidden layer possesses 200 neurons with the ReLU activation function; in the computation of each numerical optimal control, these neural networks Net_j are trained by the Adam optimization algorithm with 5000 iterations, where each iteration uses 128 paths. The numerical experiment is performed by means of PyTorch with single precision.

In Tables 1 and 2, the reference control U^* is the numerical control with $h = 1/64$, \bar{Y} and Y^* are the numerical states of the controls \bar{U} and U^* , respectively, and $\| \cdot \|$ denotes the norm $\| \cdot \|_{L^2(0, T; H)}$. In addition, the norm $\| \cdot \|_{L^2(0, T; H)}$ is calculated by 2.56×10^6 paths. The numerical results in Tables 1 and 2 demonstrate that

$$\| \bar{U} - U^* \|_{L^2(0, T; H)} + \| \bar{Y} - Y^* \|_{L^2(0, T; H)}$$

is close to $O(h)$, which agrees well with Theorem 5.1.

h	$J = 50$				$J = 80$			
	$\ \bar{U} - U^* \ $	Order	$\ \bar{Y} - Y^* \ $	Order	$\ \bar{U} - U^* \ $	Order	$\ \bar{Y} - Y^* \ $	Order
1/4	3.12×10^{-1}	-	9.14×10^{-3}	-	3.10×10^{-1}	-	9.37×10^{-3}	-
1/8	1.43×10^{-1}	1.12	4.80×10^{-3}	0.93	1.45×10^{-1}	1.10	4.80×10^{-3}	0.97
1/16	6.82×10^{-2}	1.07	1.95×10^{-3}	1.30	7.36×10^{-2}	0.98	2.29×10^{-3}	1.07
1/32	3.87×10^{-2}	0.82	9.37×10^{-4}	1.06	4.24×10^{-2}	0.80	1.07×10^{-3}	1.10

Table 1: Numerical results with $y_d(t, x) = x^{-0.49}$, $(t, x) \in [0, T] \times \mathcal{O}$.

h	$J = 50$				$J = 80$			
	$\ \bar{U} - U^*\ $	Order	$\ \bar{Y} - Y^*\ $	Order	$\ \bar{U} - U^*\ $	Order	$\ \bar{Y} - Y^*\ $	Order
1/4	3.69×10^{-1}	–	1.14×10^{-2}	–	3.63×10^{-1}	–	1.13×10^{-2}	–
1/8	1.66×10^{-1}	1.16	5.97×10^{-3}	0.93	1.73×10^{-1}	1.07	5.92×10^{-3}	0.94
1/16	8.38×10^{-2}	0.98	2.97×10^{-3}	1.01	9.35×10^{-2}	0.89	2.93×10^{-3}	1.01
1/32	4.53×10^{-2}	0.89	1.29×10^{-3}	1.20	5.19×10^{-2}	0.85	1.41×10^{-3}	1.06

Table 2: Numerical results with $y_d(t, x) = (1 + W(t)^2)x^{-0.49}$, $(t, x) \in [0, T] \times \mathcal{O}$.

6 Conclusion

For the backward semilinear stochastic parabolic equation with general filtration, we have derived the higher regularity of the solution to the continuous problem, and obtained the first-order accuracy of the spatial semi-discretization with the linear finite element method. The derived theoretical results have been applied to a general stochastic linear quadratic control problem, and the first-order spatial accuracy has been derived for a spatially semi-discrete stochastic linear quadratic control problem.

References

- [1] A. BARTH AND A. LANG, *Milstein approximation for advection-diffusion equations driven by multiplicative noncontinuous martingale noises*, Appl. Math. Optim., 66 (2012), pp. 387–413.
- [2] M. BECCARI, M. HUTZENTHALER, A. JENTZEN, R. KURNIAWAN, F. LINDNER, AND D. SALIMOVA, *Strong and weak divergence of exponential and linear-implicit Euler approximations for stochastic partial differential equations with superlinearly growing nonlinearities*, arXiv:1903.06066 (2019).
- [3] A. BENSOUSSAN, *Stochastic maximum principle for distributed parameter systems*, J. Franklin Institute, 315 (1983), pp. 387–406.
- [4] J.-M. BISMUT, *Conjugate convex functions in optimal stochastic control*, J. Math. Anal. Appl., 44 (1973), pp. 384–404.
- [5] J.-M. BISMUT, *An introductory approach to duality in optimal stochastic control*, SIAM Rev., 20 (1978), pp. 62–78.
- [6] B. BOUCHARD AND N. TOUZI, *Discrete-time approximation and Monte-Carlo simulation of backward stochastic differential equations*, Stoch. Process. Appl., 111 (2004), pp. 175–206.
- [7] C.-E. BREHIER, J. CUI AND J. HONG, *Strong convergence rates of semidiscrete splitting approximations for the stochastic Allen-Cahn equation*, IMA J. Numer. Anal., 39 (2019), pp. 2096–2134.
- [8] C.-E. BREHIER AND X. WANG, *On parareal algorithms for semilinear parabolic stochastic PDEs*, SIAM J. Numer. Anal., 58 (2020), pp. 254–278.

- [9] S. C. Brenner and R. Scott. *The mathematical theory of finite element methods*. Springer, New York, 2008.
- [10] H. BREZIS, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer, New York, 2010.
- [11] Y. CAO, J. HONG, AND Z. LIU, *Approximating stochastic evolution equations with additive white and rough noises*, SIAM J. Numer. Anal., 55 (2017), pp. 1958–1981.
- [12] E. CARELLI AND A. PROHL, *Rates of convergence for discretizations of the stochastic incompressible Navier-Stokes equations*, SIAM J. Numer. Anal., 50 (2012), pp. 2467–2496.
- [13] J.-F. CHASSAGNEUX, *Linear multistep schemes for BSDEs*, SIAM J. Numer. Anal., 52 (2014), pp. 2815–2836.
- [14] J.-F. CHASSAGNEUX AND A. RICHOU, *Numerical stability analysis of the Euler scheme for BSDEs*, SIAM J. Numer. Anal., 53 (2015), pp. 1172–1193.
- [15] J. CUI AND J. HONG, *Strong and weak convergence rates of a spatial approximation for stochastic partial differential equation with one-sided Lipschitz coefficient*, SIAM J. Numer. Anal., 57 (2019), pp. 1815–1841.
- [16] J. CUI, J. HONG AND L. SUN, *Strong convergence of full discretization for stochastic Cahn-Hilliard equation driven by additive noise*, SIAM J. Numer. Anal., 59 (2021), pp. 2866–2899.
- [17] J. CUI, J. HONG AND L. SUN, *Weak convergence and invariant measure of a full discretization for parabolic SPDEs with non-globally Lipschitz coefficients.*, Stochastic Process. Appl., 134 (2021), pp. 55–93.
- [18] Q. DU AND T. ZHANG, *Numerical approximation of some linear stochastic partial differential equations driven by special additive noises*, SIAM J. Numer. Anal., 40 (2002), pp. 1421–1445.
- [19] T. DUNST AND A. PROHL, *The forward-backward stochastic heat equation: numerical analysis and simulation*, SIAM J. Sci. Comput., 38 (2016), pp. A2725–A2755.
- [20] W. E, M. HUTZENTHALER, A. JENTZEN, AND T. KRUSE, *On multilevel Picard numerical approximations for high-dimensional nonlinear parabolic partial differential equations and high-dimensional nonlinear backward stochastic differential equations*, J. Sci. Comput., 79 (2019), pp. 1534–1571.
- [21] G. FABBRI, F. GOZZI, AND A. SWIECH, *Stochastic optimal control in infinite dimension*, Springer, 2017.
- [22] X. FENG, Y. LI, AND Y. ZHANG, *Finite element methods for the stochastic Allen-Cahn equation with gradient-type multiplicative noise*, SIAM J. Numer. Anal., 55 (2017), pp. 194–216.

- [23] X. FENG, A. PROHL, AND L. VO, *Optimally convergent mixed finite element methods for the stochastic stokes equations*, IMA J. Numer. Anal., 00 (2021), pp. 1–31.
- [24] L. GAWARECKI AND V. MANDREKAR, *Stochastic differential equations in infinite dimensions*, Springer, Berlin, 2011.
- [25] G. GUATTERI AND G. TESSITORE, *On the backward stochastic Riccati equation in infinite dimensions*, SIAM J. Control Optim., 44 (2005), pp. 159–194.
- [26] M. HINZE, R. PINNAU, M. ULBRICH, AND S. ULBRICH, *Optimization with PDE Constraints*, Springer, Netherlands, 2009.
- [27] Y. HU, D. NUALART, AND X. SONG, *Malliavin calculus for backward stochastic differential equations and application to numerical solutions*, Ann. Appl. Probab., 21 (2011), pp. 2379–2423.
- [28] Y. HU AND S. PENG, *Adapted solution of a backward semilinear stochastic evolution equation*, Stoch. Anal. Appl., 9 (1991), pp. 445–459.
- [29] P. IMKELLER AND G. D. REIS, *Path regularity and explicit convergence rate for BSDE with truncated quadratic growth*, Stochastic Process. Appl., 120 (2010), pp. 348–379.
- [30] N. E. KAROUI, S. PENG, AND M. C. QUENEZ, *Backward stochastic differential equations in finance*, Math. Financ., 7 (1997), pp. 1–71.
- [31] R. KRUSE, *Optimal error estimates of Galerkin finite element methods for stochastic partial differential equations with multiplicative noise*, IMA J. Numer. Anal., 34 (2014), pp. 217–251.
- [32] R. KRUSE, *Strong and weak approximation of semilinear stochastic evolution equations*, Springer, Cham, 2014.
- [33] B. LI AND Q. ZHOU, *Discretization of a distributed optimal control problem with a stochastic parabolic equation driven by multiplicative noise*, J. Sci. Comput., 87 (2021).
- [34] B. LI, Q. ZHOU, AND X. XIE, *Numerical analysis of a distributed control problem with a stochastic parabolic equation*, arXiv:2101.07624v2 (2021).
- [35] Y. LI AND S. TANG, *Approximation of backward stochastic partial differential equations by a splitting-up method*, J. Math. Anal. Appl., 493 (2021), pp. 124518.
- [36] J. LIONS AND E. MAGENES, *Non-Homogeneous Boundary Value Problems and Applications, Vol. 1*, Springer, Berlin, 1972.
- [37] Q. LÜ AND X. ZHANG, *Well-posedness of backward stochastic differential equations with general filtration*, J. Differ. Equ., 254 (2013), pp. 3200–3227.
- [38] Q. LÜ AND X. ZHANG, *General Pontryagin-Type Stochastic Maximum Principle and Backward Stochastic Evolution Equations in Infinite Dimensions*, Springer, Cham, 2014.

- [39] Q. LÜ AND X. ZHANG, *Optimal feedback for stochastic linear quadratic control and backward stochastic Riccati equations in infinite dimensions*, arXiv:1901.00978v2, (2019).
- [40] Q. LÜ AND X. ZHANG, *A concise introduction to control theory for stochastic partial differential equations*, arXiv:2101.10681 (2021).
- [41] Q. LÜ AND X. ZHANG, *Mathematical control theory for stochastic partial differential equations*, Springer, Cham, 2021.
- [42] J. MA, P. PROTTER, AND J. YONG, *Solving forward-backward stochastic differential equations explicitly—a four step scheme*, Probab. Theory Related Fields, 98 (1994), pp. 339–359.
- [43] J. MA AND J. YONG, *Forward-Backward Stochastic Differential Equations and Their Applications*, Springer, Berlin, 1999.
- [44] E. PARDOUX AND S. PENG, *Adapted solution of a backward stochastic differential equation*, Syst. Control Lett., 14 (1990), pp. 55–61.
- [45] E. PARDOUX AND A. RASCANU, *Stochastic differential equations, backward SDEs, partial differential equations*, Springer, Cham, 2014.
- [46] S. PENG, *Backward stochastic differential equations and applications to optimal control*, Appl. Math. Optim., 27 (1993), pp. 125–144.
- [47] A. PROHL AND Y. WANG, *Strong error estimates for a space-time discretization of the linear quadratic control problem with the stochastic heat equation with linear noise*, arXiv:2012.04418v1 (2020).
- [48] A. PROHL AND Y. WANG, *Strong rates of convergence for space-time discretization of the backward stochastic heat equation, and of a linear-quadratic control problem for the stochastic heat equation*, arXiv:2012.10117v1 (2020).
- [49] D. REVUZ AND M. YOR, *Continuous martingales and Brownian motion*, Springer, Berlin, 3 ed., 1999.
- [50] V. THOMÉE, *Galerkin Finite Element Methods for Parabolic Problems*, Springer, Berlin, 2006.
- [51] P. WANG AND X. ZHANG, *Numerical solutions of backward stochastic differential equations: a finite transposition method*, C. R. Math. Acad. Sci. Paris, 349 (2011), pp. 901–903.
- [52] Y. WANG, *A semidiscrete Galerkin scheme for backward stochastic parabolic differential equations*, Math. Control Relat. Fields, 6 (2016), pp. 489–515.
- [53] A. YAGI, *Abstract parabolic evolution equations and their applications*, Springer, Berlin, 2010.
- [54] Y. YAN, *Galerkin finite element methods for stochastic parabolic partial differential equations*, SIAM J. Numer. Anal., 43 (2005), pp. 1363–1384.

- [55] J. YONG AND X. Y. ZHOU, *Stochastic Controls, Hamiltonian Systems and HJB Equations, Applications of Mathematics*, Springer, New York, 1999.
- [56] J. ZHANG, *A numerical scheme for BSDEs*, Ann. Appl. Probab., 14 (2004), pp. 459–488.
- [57] Z. ZHANG AND G. E. KARNIADAKIS, *Numerical methods for stochastic partial differential equations with white noise*, Springer, Cham, 2017.
- [58] Q. ZHOU AND B. LI, *Numerical analysis of a Neumann boundary control problem with a stochastic parabolic equation*, arXiv:2104.09443 (2021).