# LOGARITHMIC STABILIZATION OF AN ACOUSTIC SYSTEM WITH A DAMPING TERM OF BRINKMAN TYPE

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Abstract. We study the problem of stabilization for the acoustic system with a spatially distributed damping. Without imposing any hypotheses on the structural properties of the damping term, we identify logarithmic decay of solutions with growing time. Logarithmic decay rate is shown by using a frequency domain method and combines a contradiction argument with the multiplier technique and a new Carleman estimate to carry out a special analysis for the resolvent.

### CONTENTS



# <span id="page-0-0"></span>1. Introduction

We consider the following system of equations:

<span id="page-0-1"></span>(1.1) 
$$
\begin{cases} u_t + \nabla r + b u = 0, \text{ in } \Omega \times \mathbb{R}^+, \\ r_t + \text{div } u = 0, \text{ in } \Omega \times \mathbb{R}^+, \\ u \cdot n = 0, \text{ on } \Gamma \times \mathbb{R}^+, \\ u(0, x) = u^0(x), r(0, x) = r^0(x), x \in \Omega, \end{cases}
$$

where  $\Omega$  is a bounded domain in  $\mathbb{R}^d$ ,  $d \geq 2$ , with a smooth boundary  $\Gamma$ , div =  $\nabla$  is the divergence operator and  $b \in L^{\infty}(\Omega)$ , with  $b \geq 0$  on  $\Omega$  and such that

<span id="page-0-2"></span>(1.2) 
$$
\exists b_{-} > 0 \text{ such that } b \geq b_{-} \text{ on } \omega.
$$

Here  $\omega \neq \emptyset$  stands for the open subset of  $\Omega$  on which the feedback is active. As usual n denotes the unit outward normal vector along Γ.

The system of equations  $(1.1)$  is a linearization of the *acoustic equation* governing the propagation of acoustic waves in a compressible medium, see Lighthill  $[21, 22, 23]$  $[21, 22, 23]$  $[21, 22, 23]$  $[21, 22, 23]$ , where b u represents

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a damping term of Brinkman type. This kind of damping arises also in the process of homog-enization (see Allaire [\[1\]](#page-21-4)), and is frequently used as a suitable *penalization* in fluid mechanics models, see Angot, Bruneau, and Fabrie [\[4\]](#page-21-5). Our main goal is to prove the logarithmic decay of solutions of  $(1.1)$  with growing time.

Let  $L^2(\Omega)$  denote the standard Hilbert space of square integrable functions in  $\Omega$  and its closed subspace  $L_m^2(\Omega) = \{f \in L^2(\Omega) : \int_{\Omega} f(x) dx = 0\}$ . To avoid abuse of notation, we shall write  $\| \cdot \|$  for the  $L^2(\Omega)$ -norm or the  $L^2(\Omega)$ <sup>d</sup>-norm.

Denoting  $H = (L^2(\Omega))^d \times L^2_m(\Omega)$ , we introduce the operator

$$
\mathcal{A} = \begin{pmatrix} 0 & \nabla \\ \text{div} & 0 \end{pmatrix} : \mathcal{D}(\mathcal{A}) = \left\{ (u, r) \in H, (\nabla r, \text{div } u) \in H, u \cdot n_{|\Gamma} = 0 \right\} \subset H \to H,
$$

and

<span id="page-1-1"></span>
$$
\mathcal{B} = \begin{pmatrix} \sqrt{b} \\ 0 \end{pmatrix} \in \mathcal{L}((L^2(\Omega))^d, H), \mathcal{B}^* = \begin{pmatrix} \sqrt{b} & 0 \end{pmatrix} \in \mathcal{L}(H, (L^2(\Omega))^d).
$$

We recall that for  $u \in (L^2(\Omega))^d$  with div  $u \in L^2(\Omega)$ ,  $u \cdot n_{|\Gamma}$  make sens in  $H^{-1/2}(\Gamma)$  (see Girault-Raviart [\[13,](#page-21-6) Chp 1, Theorem 2.5]).

Accordingly, the problem [\(1.1\)](#page-0-1) can be recasted in an abstract form:

(1.3) 
$$
\begin{cases} Z_t(t) + \mathcal{A}Z(t) + \mathcal{B}\mathcal{B}^*Z(t) = 0, t > 0, \\ Z(0) = Z^0, \end{cases}
$$

where  $Z = (u, r)$ , or, equivalently,

(1.4) 
$$
\begin{cases} Z_t(t) = \mathcal{A}_d Z(t), t > 0, \\ Z(0) = Z^0, \end{cases}
$$

with  $\mathcal{A}_d = -\mathcal{A} - \mathcal{B}\mathcal{B}^*$  with  $\mathcal{D}(\mathcal{A}_d) = \mathcal{D}(\mathcal{A})$ .

It can be shown (see [\[2\]](#page-21-7)) that for any initial data  $(u^0, r^0) \in \mathcal{D}(\mathcal{A})$  the problem  $(1.1)$  admits a unique solution

<span id="page-1-2"></span><span id="page-1-0"></span>
$$
(u,r) \in C([0,\infty); \mathcal{D}(\mathcal{A})) \cap C^1([0,\infty); H).
$$

Moreover, the solution  $(u, r)$  satisfies, the energy identity

(1.5) 
$$
E(0) - E(t) = \int_0^t \left\| \sqrt{b} u(s) \right\|_{(L^2(\Omega))^d}^2 ds, \text{ for all } t \ge 0
$$

with

(1.6) 
$$
E(t) = \frac{1}{2} ||(u(t), r(t))||_H^2, \forall t \ge 0,
$$

where we have denoted

$$
\langle (u,r),(v,p) \rangle_H = \int_{\Omega} \left( u(x).v(x) + r(x)p(x) \right) dx, \, \|(u,r)\|_H = \sqrt{\int_{\Omega} \left( |u(x)|^2 + r^2(x) \right) dx}.
$$

Using  $(1.5)$  and a standard density argument, we can extend the solution operator for data  $(u^0, r^0) \in H$ . Consequently, we associate with the problem  $(1.1)$  (or to the abstract Cauchy problems  $(1.3)$  or  $(1.4)$  a semi-group that is globally bounded in H.

As the energy  $E$  is nonincreasing along trajectories, we want to determine the set of initial data  $(u^0, r^0)$  for which

<span id="page-1-3"></span>
$$
(1.7) \t\t\t E(t) \to 0 \text{ as } t \to \infty.
$$

Such a question is of course intimately related to the structural properties of the function  $b$ , notably to the geometry of the set  $\omega$  on which the damping is effective. In fact, when the damping term is globally distributed Ammari, Feireisl and Nicaise [\[2\]](#page-21-7) showed an exponential decay rate of the energy by the means of an observability inequality associated with the conservative problem of [\(1.1\)](#page-0-1). Besides, it is also shown that if the damping coefficient is not uniformly positive definite (i.e inf  $b(x) = 0$ ) then the system [\(1.1\)](#page-0-1) is not exponentially stable. In this paper we consider  $x\in\overline{\Omega}$ a damping which is locally distributed over the domain  $\Omega$  without any geometrical control condition in particular this including the case when the damping coefficient is not uniformly positive defined. So we expect to prove a weaker decay rate then given in [\[2\]](#page-21-7). More precisely, we prove a logarithm decay rate of the energy. Our approach is based in the frequency domain method which consist to prove an exponential loss on the resolvent estimate [\[7,](#page-21-8) [6,](#page-21-9) [10\]](#page-21-10) where the main tool for establishing a such decay is the Carleman estimate.

The theory of Carleman estimates for scalar equations is rather well developed by now. We refer to Hörmander [\[14\]](#page-21-11) and Lebeau and Robbiano [\[17,](#page-21-12) [18,](#page-21-13) [19\]](#page-21-14) for the second-order elliptic and hyperbolic PDE's and to Isakov [\[15\]](#page-21-15) second-order parabolic and Schrödinger operators. However, it turned out that Carleman estimates for systems in more than two variables are difficult to obtain and still somehow very limited: The first results to systems go back to then Carleman's original work [\[9\]](#page-21-16) which is written for a system in two independent variables, and we refer to Calderón [\[8\]](#page-21-17) and Kreiss [\[16\]](#page-21-18) for more relevant systems. Recently, Eller and Toundykov [\[11\]](#page-21-19) have established a Carleman estimate for some first-order elliptic systems. This estimate is extended to elliptic boundary value problems provided the boundary condition satisfies a Lopatinskii-type requirement. In this paper we provide a Carleman estimate for a system of first-order which does not fit into the same framework as that of Eller and Toundykov [\[11\]](#page-21-19). Unlike their approach, our method is based into the Hörmander approach which is essentially based on the sub-ellipticity condition and the Gårding inequality in order to control the non-elliptical regions.

The paper is organized as follows. Section [2](#page-2-0) summarizes some well known facts concerning the acoustic system [\(1.1\)](#page-0-1). In section [3,](#page-4-0) we establish a new Carleman estimate needed for the stabilization problem of the system  $(1.1)$ . In Section [4,](#page-19-0) we prove the logarithmic stability for the system  $(1.1)$ .

## <span id="page-2-0"></span>2. Preliminaries

We start with a simple observation that the problem  $(1.1)$  can be viewed as a bounded (in H) perturbation of the conservative system

(2.8) 
$$
\begin{cases} u_t + \nabla r = 0, \text{ in } \Omega \times \mathbb{R}^+, \\ r_t + \text{div } u = 0, \text{ in } \Omega \times \mathbb{R}^+, \end{cases}
$$

which can be recast as the standard wave equation

<span id="page-2-1"></span>
$$
r_{tt} - \Delta r = 0.
$$

Consequently, the basic existence theory for  $(1.1)$  derives from that of  $(2.8)$ . Hence  $\mathcal{A}_d$  generates a C<sub>0</sub>-semigroup  $(S(t))_{t\geq0}$  in H that is even of contraction because  $\mathcal{A}_d$  is dissipative (see [\(1.5\)](#page-1-0)).

The first main difficulty is that the operator  $\mathcal{A}_d$  possesses a non-trivial (and large) kernel that is left invariant by the evolution. Indeed if  $(u, r)$  belongs to ker  $\mathcal{A}_d$ , then it is solution of the "stationary" problem

(2.9) 
$$
\nabla r + bu = 0, \text{ div } u = 0, \text{ in } \Omega.
$$

Thus multiplying the first identity of [\(2.9\)](#page-2-2) by  $\overline{u}$  and integrating over  $\Omega$  yields

<span id="page-2-2"></span>
$$
\int_{\Omega} (\nabla r \cdot \overline{u} + b|u|^2) \, \mathrm{d}x = 0.
$$

By an integration by parts, using the fact that  $u$  is solenoidal and the boundary condition  $u \cdot n = 0$  on Γ, we get

$$
\int_{\Omega} \nabla r \cdot \overline{u} \, dx = 0,
$$

and therefore we obtain

In other words, we have

and coming back to  $(2.9)$ , we find

Accordingly, we have shown that

$$
\ker \mathcal{A}_d = \{ (u, 0) \in \mathcal{D}(\mathcal{A}) \mid \text{ div } u = 0, \ u|_{\text{supp } b} = 0, \ u \cdot n|_{\Gamma} = 0 \}.
$$

For shortness set  $E = \ker A_d$  and introduce also its orthogonal complement  $H_0$  in H.

It is easy to check that

$$
\langle \mathcal{A}_d(w,s), (u,r) \rangle_H = 0 \text{ for any } (w,s) \in \mathcal{D}(\mathcal{A}), (u,r) \in E;
$$

in particular, the semi-group associated with  $(1.1)$  leaves both E and  $H_0$  invariant. Consequently, the decay property  $(1.7)$  may only hold for initial data emenating from the set  $H_0$ .

The following observation can be shown by a simple density argument:

**Lemma 2.1.** The solution  $(u, r)$  of  $(1.1)$  with initial datum in  $\mathcal{D}(\mathcal{A}_d)$  satisfies

(2.10) 
$$
E'(t) = -\int_{\Omega} b |u|^2 dx \le 0.
$$

Therefore the energy is non-increasing and  $(1.5)$  holds for all initial datum in H.

As already shown in the above, the strong stability result [\(1.7\)](#page-1-3) may hold only if we take the initial data

$$
(u^0, r^0) \in H_0 = \ker[\mathcal{A}_d]^\perp.
$$

There are several ways how to show [\(1.7\)](#page-1-3), here we make use of the following result due to Arendt and Batty [\[5\]](#page-21-20):

<span id="page-3-0"></span>**Theorem 2.1.** Let  $(T(t))_{t\geq0}$  be a bounded  $C_0$ -semigroup on a reflexive Banach space X. Denote by A the generator of  $(T(t))$  and by  $\sigma(A)$  the spectrum of A. If  $\sigma(A) \cap i\mathbb{R}$  is countable and no eigenvalue of A lies on the imaginary axis, then  $\lim_{t\to+\infty} T(t)x = 0$  for all  $x \in X$ .

In view of this theorem we need to identify the spectrum of  $A_d$  lying on the imaginary axis, and we have according to [\[2\]](#page-21-7):

- Suppose that  $|\omega| > 0$ . If  $\lambda$  is a non-zero real number, then  $i\lambda$  is not an eigenvalue of  $\mathcal{A}_d$ .
- Suppose that  $|\omega| > 0$ . If  $\lambda$  is a non-zero real number, then  $i\lambda$  belongs to the resolvent set  $\rho(\mathcal{A}_d)$  of  $\mathcal{A}_d$ .

Now, Theorem [2.1](#page-3-0) leads to

**Corollary 2.1** ([\[2\]](#page-21-7)). Let  $(u, r)$  be the unique semi-group solution of the problem [\(1.1\)](#page-0-1) emanating from the initial data  $(u^0, r^0) \in H$ . Let  $P_E$  be the orthogonal projection onto the space  $E = \text{ker}[\mathcal{A}_d]$ in H, and let

$$
(w, s) = P_E(u^0, r^0).
$$

Then

$$
||(u,r)(t,\cdot)-(w,s)||_H\to 0 \text{ as } t\to\infty
$$

We now state the main result of this article. We begin by a proposition on an estimate of the resolvent.

$$
\int_{\Omega} b|u|^2 \, \mathrm{d}x = 0.
$$

 $u = 0$  on supp b,

 $\nabla r = 0.$ 

<span id="page-4-1"></span>**Proposition 2.1.** There exist  $C > 0$  such that for every  $|\mu| \geq 1$ , and  $(f, g) \in H = (L^2(\Omega))^d \times$  $L^2_m(\Omega)$ , the solution  $(u, r) \in \mathcal{D}(\mathcal{A})$  of  $(\mathcal{A}_d + i\mu)(u, r) = (f, g)$  satisfied

(2.11) 
$$
\|(u,r)\|_H \leq Ce^{C|\mu|} \|(f,g)\|_H,
$$

or equivalently

(2.12) 
$$
\|(\mathcal{A}_d + i\mu)^{-1}\|_{\mathcal{L}(H)} \leq Ce^{C|\mu|}.
$$

We recall the following result.

<span id="page-4-2"></span>**Theorem 2.2.** Let B a generator of a  $C_0$ -semigroup  $(T(t))_{t>0}$  on H, a Hilbert space. We assume

(2.13)  $||T(t)||_{\mathcal{L}(H)}$  is uniformly bounded with respect  $t \geq 0$ ,

(2.14)  $B + i\mu$  is invertible for every  $\mu \in \mathbb{R}$ ,

(2.15) There exists 
$$
C > 0
$$
 such that  $||(B + i\mu)^{-1}||_{\mathcal{L}(H)} \le Ce^{C|\mu|}$ .

Then there exist  $C_1 > 0$  such that for all  $u \in \mathcal{D}(B)$  we have

$$
||T(t)u||_{\mathcal{H}} \leq C_1 \frac{||Bu||_{\mathcal{H}}}{\ln(3+t)}, \,\forall \, t \geq 0.
$$

One has also, for every  $k \ge 1$  there exists  $C_2 > 0$  such that if  $u \in \mathcal{D}(B^k)$ , we have

$$
||T(t)u||_{\mathcal{H}} \leq C_1 \frac{||B^k u||_H}{\ln^k(3+t)}, \,\forall \, t \geq 0.
$$

A weak version of this theorem was first proven by Lebeau [\[17\]](#page-21-12), next Burq [\[6\]](#page-21-9) gives the precise statement. We also refer to Batty and Duyckaerts [\[7\]](#page-21-8) for some generalizations.

On  $H_0 = \text{ker}[\mathcal{A}_d]^{\perp}$ , as seen above  $\mathcal{A}_d + i\mu$  is invertible on  $H_0$ , in fact  $\mathcal{A}_d$  is invertible on  $H_0$  and  $\mathcal{A}_d + i\mu$  is invertible on H for  $\mu \neq 0$ . The semigroup is bounded as the norm on H is non-increasing by [\(1.5\)](#page-1-0). With Proposition [2.1,](#page-4-1) we can apply Theorem [2.2.](#page-4-2) We then obtain.

**Theorem 2.3.** Let  $(u, r)$  be the unique semi-group solution of the problem [\(1.1\)](#page-0-1) emanating from the initial data  $(u^0, r^0) \in \mathcal{D}(\mathcal{A})$ . Let  $P_E$  be the orthogonal projection onto the space  $E = \text{ker}[\mathcal{A}_d]$ in H, and let

 $(w, s) = P_E(u^0, r^0).$ 

$$
Then
$$

$$
||(u,r)(t,\cdot)-(w,s)||_H \leq C \frac{\|\mathcal{A}_d(u^0,r^0)\|_H}{\ln(3+t)}, \,\forall \, t \geq 0,
$$

for some  $C > 0$  independent of  $(u^0, r^0)$ .

Proposition [2.1](#page-4-1) is obtained from Carleman estimates. We need two kinds of such estimates, first an estimate far away the boundary, second an estimate up to the boundary. Both estimates are proven in the next section.

#### <span id="page-4-0"></span>3. Carleman estimates

Let  $\Omega$  be an open bounded subset of  $\mathbb{R}^d$ . Let  $(u, r)$  be a solution of the resolvent problem  $(\mathcal{A}_0 + i\mu)(u, r) = (f, g) \in (L^2(\Omega))^d \times L^2_m(\Omega)$ , that is

<span id="page-4-3"></span>(3.1) 
$$
\begin{cases} -\nabla r + i\mu u = f \text{ in } \Omega, \\ -\text{div } u + i\mu r = g \text{ in } \Omega. \end{cases}
$$

Here we moreover assume that  $(u, r)$  are supported in  $K \subset \Omega$  where K is a compact set. Taking the divergence of the first line and using that div  $u = i\mu r - g$ , we obtain

<span id="page-5-1"></span>(3.2) 
$$
-\Delta r - \mu^2 r = i\mu g + \text{div } f \text{ in } \Omega.
$$

We have to give a Carleman estimate for the solution of this type of equation. This is done in Section [3.2.](#page-6-0) to do that we need some tools on pseudo-differential operators we recall below.

<span id="page-5-0"></span>3.1. Pseudo-differential operators. We start this section with some useful notations. If  $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$  is a multi-index, we introduce the following notation:

$$
\xi^{\alpha} = \xi_1^{\alpha_1} \dots \xi_n^{\alpha_n}, \ \ \partial^{\alpha} = \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}, \ \ D^{\alpha} = D_1^{\alpha_1} \dots D_n^{\alpha_n} \ \text{ and } \ |\alpha| = \alpha_1 + \dots + \alpha_n
$$

where  $D_k = -i \frac{\partial}{\partial x}$  $\frac{\partial}{\partial x_k} = -i\partial_{x_k}$ . We denote by  $\mathscr{C}_c^{\infty}(V)$  the set of functions of class  $\mathscr{C}^{\infty}$  compactly supported in V. For a compact subset K of  $\mathbb{R}^n$ , we note by  $\mathscr{C}_c^{\infty}(K)$  the set of functions in  $\mathscr{C}_c^{\infty}(\mathbb{R}^n)$  supported in K. The space  $L^2(V)$  is equipped with the usual norm denoted by  $||u||_0$ . For  $s \in \mathbb{N}$  we set  $H^s(V) = \{u \in \mathscr{D}'(V); \ \partial^\alpha u \in L^2(V) \ \forall \, |\alpha| \leq s\}$ . The Schwartz space  $\mathscr{S}(\mathbb{R}^n)$  is the set of functions of  $\mathscr{C}^{\infty}$  class with rapid decay rate. Its dual,  $\mathscr{S}'(\mathbb{R}^n)$  is the set of temperate distributions. If  $u \in \mathscr{S}(\mathbb{R}^n)$  its Fourier transform denoted by  $\hat{u}$  is defined by  $\hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-iy.\xi} u(y) dy$  where  $y.\xi = \sum_{i=1}^n$  $i=1$  $y_i\xi_i$  stands for the euclidean inner production in  $\mathbb{R}^n$ . Let f and g be two smooth functions defined in  $V \times \mathbb{R}^n$ , we define the Poisson bracket by  $\{f, g\} = \sum^{n}$  $j=1$  $(\partial_{\xi_j} f . \partial_{x_j} g - \partial_{x_j} f . \partial_{\xi_j} g)$ . And if A and B are two operators we define there commutator by  $[A, B] = A \circ B - B \circ A$ .

<span id="page-5-2"></span>**Definition 3.1.** Let  $a(., ., \tau) \in \mathscr{C}^{\infty}(\mathbb{R}^n \times \mathbb{R}^n)$  where  $\tau \geq 1$  is a large parameter, such that for every muti-index  $\alpha, \beta \in \mathbb{N}^n$  we have

$$
|\partial_x^{\alpha} \partial_{\xi}^{\beta} a(x,\xi,\tau)| \leq C_{\alpha,\beta} \langle \xi,\tau \rangle^{m-|\beta|}, \quad \forall x \in \mathbb{R}^n, \ \forall \xi \in \mathbb{R}^n, \ \forall \tau \geq 1,
$$

where we denoted by  $\langle \xi, \tau \rangle = (|\xi|^2 + \tau^2)^{\frac{1}{2}}$ . In this case we say that a is a symbol of order m and we write  $a \in S_{\tau}^m$ . We call principal symbol of  $a \in S_{\tau}^m$  the equivalence class of a in  $S_{\tau}^m/S_{\tau}^{m-1}$ . We also define  $S_{\tau}^{-\infty} = \bigcap$ r∈R  $S^r_\tau$  and  $S^{+\infty}_\tau = \bigcup$ r∈R  $S^r_\tau.$ 

Definition 3.2. We define the pseudo-differential operator of order m by

$$
a(x, D, \tau)u(x) = Op(a)u(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix.\xi} a(x, \xi, \tau) \hat{u}(\xi) d\xi, \quad \forall u \in \mathscr{S}(\mathbb{R}^n),
$$

where  $a \in S_{\tau}^{m}$ . The set of the pseudo-differential operator of order m is denoted by  $\Psi_{\tau}^{m}$ . If  $A \in \Psi_{\tau}^{m}$ , we denote by  $\sigma_{p}(A)$  his principal symbol.

**Remarks 3.1.** Let  $s \in \mathbb{R}$  for  $u \in \mathcal{S}'(\mathbb{R}^n)$  we set the following norm

 $||u||_{\tau,s} = ||\Lambda^s_\tau u||_0$  with  $\Lambda^s_\tau := Op(\langle \xi, \tau \rangle^s)$ .

Hence we can define the corresponding space

$$
H^s_\tau(\mathbb{R}^n) = \{ u \in \mathscr{S}'(\mathbb{R}^n); \ \|u\|_{\tau,s} < \infty \}.
$$

**Theorem 3.1.** Let  $s \in \mathbb{R}$  and  $a(x,\xi,\tau) \in S_{\tau}^{m}$ , then the operator  $Op(a): H_{\tau}^{s} \longrightarrow H_{\tau}^{s-m}$  maps continuously and uniformly for  $\tau > 1$ .

**Lemma 3.1.** Let  $m \in \mathbb{R}$  and  $a_j \in S^{m-j}_\tau$  with  $j \in \mathbb{N}$ . Then there exist  $a \in S^m_\tau$  such that

$$
\forall N \in \mathbb{N}, \quad a - \sum_{j=0}^{N} a_j \in S_{\tau}^{m-N-1}.
$$

We then write  $a \sim \sum a_j$ . The symbol a is unique up to  $S_{\tau}^{-\infty}$  in the sens that the difference of j two symbols is in  $S_{\tau}^{-M}$  for all  $M \in \mathbb{N}$ . Hence, we identify  $a_0$  with the principal symbol of a.

**Theorem 3.2.** Let  $a \in S_{\tau}^{m}$  and  $b \in S_{\tau}^{m'}$ , then  $Op(a) \circ Op(b) = Op(c)$  with  $c \in S_{\tau}^{m+m'}$  which admits the following asymptotic expansion

$$
c(x,\xi,\tau) \sim \sum_{\alpha} \frac{1}{i^{|\alpha|}\alpha!} \partial_{\xi}^{\alpha} a(x,\xi,\tau) \partial_{x}^{\alpha} b(x,\xi,\tau).
$$

**Theorem 3.3.** Let  $a \in S_{\tau}^{m}$  and  $b \in S_{\tau}^{m'}$ , then  $[Op(a), Op(b)] = Op(c)$  with  $c \in S_{\tau}^{m+m'-1}$  and principal symbol

$$
\sigma(c)(x,\xi,\tau) = \frac{1}{i} \{a,b\}(x,\xi,\tau)
$$

which admits the following asymptotic expansion

$$
c(x,\xi,\tau) \sim \sum_{\alpha} \frac{1}{i^{|\alpha|}\alpha!} \left( \partial_{\xi}^{\alpha} a(x,\xi,\tau) \partial_x^{\alpha} b(x,\xi,\tau) - \partial_{\xi}^{\alpha} b(x,\xi,\tau) \partial_x^{\alpha} a(x,\xi,\tau) \right).
$$

**Theorem 3.4.** Let  $a \in S_{\tau}^m$ , then  $Op(a)^* = Op(b)$  with  $b \in S_{\tau}^m$  which admits the following asymptotic expansion

$$
b(x,\xi,\tau) \sim \sum_{\alpha} \frac{1}{i^{|\alpha|}\alpha!} \partial_{\xi}^{\alpha} \partial_{x}^{\alpha} \bar{a}(x,\xi,\tau).
$$

In particular we have  $\sigma_p(Op(a)^*) = \bar{a}$ .

<span id="page-6-2"></span>**Theorem 3.5** (Gårding inequality). Let K be a compact subset of  $\mathbb{R}^n$  and  $a(x,\xi,\tau) \in S_\tau^m$ , of principal symbol  $a_m$ . We suppose that there exist  $C > 0$  and  $R > 0$  such that

$$
\operatorname{Re} a_m(x,\xi,\tau) \ge C\langle \xi,\tau \rangle^m, \quad \forall \, x \in K, \, \xi \in \mathbb{R}^n, \, \tau \ge 1, \, \langle \xi,\tau \rangle \ge R.
$$

Then for any  $0 < C' < C$  there exists  $\tau_* > 0$  we have

$$
\operatorname{Re}(Op(a)u, u)_{L^2(\mathbb{R}^n)} \ge C' \|u\|_{\tau, \frac{m}{2}}, \quad \forall u \in \mathscr{C}_c^{\infty}(K), \ \tau \ge \tau_*.
$$

# <span id="page-6-0"></span>3.2. Local Carleman estimate away from the boundary. We set the operator

$$
P(x,D) = -\mu^2 - \Delta,
$$

a real values function  $\varphi$  and then we define the conjugate operator by

<span id="page-6-1"></span>
$$
P_{\varphi}(x,D) = e^{\tau \varphi} P(x,D) e^{-\tau \varphi},
$$

where  $\mu$  is a parameter that depends on  $\tau$ , precisely we suppose that

(3.3) 
$$
c_0 \tau \leq |\mu| \leq c'_0 \tau \quad \forall \tau \geq 1,
$$

for some constants  $c'_0 > c_0 > 0$ . Then we have

$$
P_{\varphi}(x, D)w = -\mu^2 w - \Delta w + 2\tau \nabla \varphi \cdot \nabla w - \tau^2 |\nabla \varphi|^2 w + \tau \Delta \varphi w
$$

whose symbol is given by

$$
\sigma(P_{\varphi}) = |\xi|^2 + 2i\tau \nabla \varphi \cdot \xi - \tau^2 |\nabla \varphi|^2 + \tau \Delta \varphi - \mu^2
$$

and with principal symbol  $p_{\varphi}$  given by

$$
p_{\varphi}(x,\xi,\tau) = |\xi + i\tau \nabla \varphi|^2 - \mu^2 = |\xi|^2 + 2i\tau \nabla \varphi \cdot \xi - \tau^2 |\nabla \varphi|^2 - \mu^2.
$$

We define the following self-adjoint operators

$$
Q_2 = \frac{P_{\varphi} + P_{\varphi}^*}{2} \quad \text{and} \quad Q_1 = \frac{P_{\varphi} - P_{\varphi}^*}{2i}
$$

with principal symbols respectively

$$
q_2(x,\xi,\tau) = |\xi|^2 - \tau^2 |\nabla \varphi|^2 - \mu^2 \quad \text{and} \quad q_1(x,\xi,\tau) = 2\tau \nabla \varphi.\xi.
$$
 Noting that  $P_{\varphi} = Q_2 + iQ_1$  and  $p_{\varphi} = q_2 + iq_1$ .

We assume that the weight function  $\varphi \in \mathscr{C}^{\infty}(\mathbb{R}^n, \mathbb{R})$  satisfies the following sub-ellipticity condition in K a compact set of  $\mathbb{R}^d$ , if

<span id="page-7-1"></span> $(3.4)$   $|\nabla \varphi| > 0$  in K

 $\forall (x, \xi, \tau) \in K \times \mathbb{R}^n \times [1, +\infty); p_{\varphi}(x, \xi, \tau) = 0 \Longrightarrow \{q_2, q_1\}(x, \xi, \tau) \ge C \langle \xi, \tau \rangle^3 > 0.$ 

Note that the constant C does not depend on  $\mu$  assuming [\(3.3\)](#page-6-1).

Remark 3.1. Noting that

$$
p_{\varphi}(x,\xi,\tau) = 0 \iff |\xi|^2 = \tau^2 |\nabla \varphi|^2 + \mu^2 \text{ and } \nabla \varphi.\xi = 0.
$$

<span id="page-7-2"></span>**Lemma 3.2.** Let  $\psi \in \mathscr{C}^{\infty}(\mathbb{R}^n, \mathbb{R})$  such that  $|\nabla \psi| > 0$  in K. Then for  $\lambda$  large enough  $\varphi = e^{\lambda \psi}$ satisfies the sub-ellipticity assumption in K.

*Proof.* We can assume that  $\psi \geq 0$ , as we can add a constant to  $\psi$  and  $\varphi$  is multiplied by a constant  $\beta$ . Changing  $\tau$  in  $\tau/\beta$  we can see that sub-ellipticity condition is also satisfied for a different constant C. A straightforward calculation shows that

$$
\{q_2,q_1\}(x,\xi,\tau)=4\tau\left(\sqrt[t+\xi]\varphi''\xi+\tau^2\sqrt[t+\xi]\nabla\varphi\right).
$$

Using the fact that  $\varphi = e^{\lambda \psi}$  then we have

$$
\nabla \varphi = \lambda \nabla \psi \varphi, \quad \varphi_j' = \lambda \varphi \psi_j' \text{ and } \varphi_{jk}'' = \lambda \varphi \psi_{jk}'' + \lambda^2 \varphi \psi_j' \psi_k', \ 1 \le j, k \le n,
$$

therefore we obtain

$$
\{q_2,q_1\} = 4\tau\lambda^3\varphi^3\left(\lambda\tau^2|\nabla\psi|^4 + \tau^{2-t}(\nabla\psi)\psi''\nabla\psi + |\lambda\varphi|^{-2-t}\xi\psi''\xi + \lambda^{-1}|\varphi|^{-2}|\nabla\psi.\xi|^2\right).
$$

Now if  $p_{\varphi} = 0$  then  $|\xi|^2 = \tau^2 |\nabla \varphi|^2 + \mu^2 = \tau^2 \lambda^2 \varphi^2 |\nabla \psi|^2 + \mu^2$ , which gives that

$$
|\lambda \varphi|^{-2} \ {}^t \xi \psi'' \xi \ge -|\psi''| \left( \tau^2 |\nabla \psi|^2 + |\lambda \varphi|^{-2} \mu^2 \right) \ge -C \tau^2 \left( |\nabla \psi|^2 + \lambda^{-2} \right).
$$

Besides, we have

$$
\tau^{2-t}(\nabla\psi)\psi''\nabla\psi \geq -C\tau^{2}|\nabla\psi|^{2}.
$$

Then it follows from these estimates that

$$
\{q_2, q_1\} \ge 4\tau\lambda^3\varphi^3\left(\lambda\tau^2|\nabla\psi|^4 + \tau^2 \ {}^t(\nabla\psi)\psi''\nabla\psi + |\lambda\varphi|^{-2} \ {}^t\xi\psi''\xi\right)
$$
  
 
$$
\ge 4\tau\lambda^3\varphi^3(\lambda\tau^2|\nabla\psi|^4 - C\tau^2|\nabla\psi|^2 - C\tau^2\lambda^{-2}).
$$

Since  $|\nabla \psi| > 0$  in the compact set K then for  $\lambda$  large enough we have  $\{q_2, q_1\} \ge C_\lambda \tau^3 > 0$ . As |ξ| is comparable to  $\tau$  on  $p_{\varphi} = 0$ , we obtain the result.

<span id="page-7-0"></span>**Lemma 3.3.** Let  $f$  and  $g$  be two real continuous functions defined in  $K$  such that  $f$  is positive on a compact subset  $K$  of  $\mathbb{R}^d$  and verifies that

$$
\forall y \in K, \qquad f(y) = 0 \Longrightarrow g(y) \ge L > 0.
$$

We set  $h_{\kappa} = \kappa f + g$ , then for  $\kappa$  sufficiently large then  $h_{\kappa} \geq C$  for some constant  $C > 0$ .

*Proof.* Let  $y_0 \in K$  to prove the result we distinguish two cases.

**Case 1: We assume**  $f(y_0) = 0$ . Then according to the assumption made in this lemma we have  $h_{\kappa}(y_0) = g(y_0) \geq L$ . Then there exists a neighborhood of  $y_0$ ,  $V_{y_0}$  such that for  $y \in V_{y_0}$  and every  $\kappa > 0$ ,  $h_{\kappa}(y) \ge g(y) \ge L/2$ . Let  $\kappa_{y_0} = 1$ .

**Case 2: We assume**  $f(y_0) > 0$ . Since f and g are continuous, there exist  $V_{y_0}$  a neighborhood of  $y_0$  and  $C_1, C_2 > 0$  such that  $f(y) \geq C_1$  and  $|g(y)| \leq C_2$  for all  $y \in V_{y_0}$ . Then for all  $\kappa \ge (L + C_2)/C_1$ ,  $h_{\kappa}(y) \ge L$ . Let  $\kappa_{y_0} = (L + C_2)/C_1$ .

We cover K, by compactness argument, by a finite number of such neighborhoods  $V_{y_1}, \ldots, V_{y_p}$ with associated  $\kappa_j = \kappa_{y_j}$ . Taking  $\kappa = \max_{1 \le j \le p} {\{\kappa_j\}}$ , we have  $h_{\kappa}(y) \ge L/2$  on each  $V_{y_j}$ , then on K. This completes the proof.

<span id="page-8-1"></span>**Lemma 3.4.** Let V an open bounded subset of  $\mathbb{R}^d$  and  $\kappa > 0$ . We suppose that  $\varphi$  verifies the sub-ellipticity assumption on K and we set  $\rho_{\kappa} = \kappa (q_2^2 + q_1^2) + \tau \{q_2, q_1\}$ . Then for  $\kappa$  large enough there exists  $C > 0$  such that for all  $(x, \xi) \in K \times \mathbb{R}^d$  and  $\tau \geq 1$  we have  $\rho_{\kappa}(x, \xi) \geq C \langle \xi, \tau \rangle^4$ .

*Proof.* First we assume  $|\xi|$  large with respect  $\tau$ , that is  $|\xi| \geq \beta \tau$  for  $\beta$  sufficiently large, to be fixed below. We have

<span id="page-8-0"></span>
$$
\rho_{\kappa}(x,\xi) = \kappa (q_2^2 + q_1^2) + \tau \{q_2, q_1\}
$$
\n
$$
= \kappa \left( |\xi|^2 - (\tau^2 |\nabla \varphi|^2 + \mu^2) \right)^2 + 4\kappa \tau^2 (\nabla \varphi \cdot \xi)^2 + 4\tau^{2-t} \xi \varphi'' \xi + \tau^{4-t} (\nabla \varphi) \varphi'' \nabla \varphi
$$
\n(3.5)\n
$$
= \kappa \langle \xi, \tau \rangle^4 \left( 1 - \frac{(\tau^2 (1 + |\nabla \varphi|^2) + \mu^2)}{\langle \xi, \tau \rangle^2} \right)^2 + 4\kappa \tau^2 (\nabla \varphi \cdot \xi)^2 + 4\tau^{2-t} \xi \varphi'' \xi + \tau^{4-t} (\nabla \varphi) \varphi'' \nabla \varphi
$$
\n
$$
\geq C' \langle \xi, \tau \rangle^4 - C\tau^2 |\xi|^2 - C\tau^4,
$$

if  $\beta$  is sufficiently large such that  $\frac{(\tau^2(1+|\nabla\varphi|^2)+\mu^2)}{(\zeta+2\pi)^2}$  $\frac{(\sqrt{y_1} + \mu)}{(\xi, \tau)^2} \leq 1/2$  and for some constants  $C', C > 0$ . If  $\beta$  is sufficiently large we obtain  $C\tau^2|\xi|^2 + C\tau^4 \leq C' \langle \xi, \tau \rangle^4/2$ , from  $(3.5)$  we obtain  $\rho_{\kappa}(x,\xi) \geq$  $C''\langle \xi, \tau \rangle^4$ , for  $C'' > 0$ . This fixes  $\beta$ .

Second we assume  $|\xi| \leq \beta \tau$ . As  $\rho_{\kappa}$  is homogeneous of degree 4 in  $(\xi, \tau, \mu)$ , we can prove the estimate on  $K' = \{(x, \xi, \tau, \mu) \in K \times \mathbb{R}^d \times [0, +\infty) \times \mathbb{R}, |\xi|^2 + \tau^2 + \mu^2 = 1, |\xi| \leq \beta\tau, c_0\tau \leq |\mu| \leq \tau\}$  $c'_0 \tau$  taking into account of [\(3.3\)](#page-6-1). As K' is a compact set, we can apply Lemma [3.3](#page-7-0) by taking  $f = q_2^2 + q_1^2$  and  $g = \tau\{q_2, q_1\}$ . This completes the proof.

<span id="page-8-7"></span>**Theorem 3.6.** Let  $\Omega$  be an open bounded set of  $\mathbb{R}^n$  and  $\varphi$  be a function that satisfies the sub-ellipticity assumption in K. Then there exist  $\tau_* > 0$  and  $C > 0$  such that

<span id="page-8-4"></span>(3.6) 
$$
\tau^3 \|e^{\tau \varphi} r\|_0^2 + \tau \|e^{\tau \varphi} \nabla r\|_0^2 + \tau^{-1} \sum_{\alpha=2} \|e^{\tau \varphi} D^{\alpha} r\|_0^2 \leq C \|e^{\tau \varphi} P r\|_0^2.
$$

for all  $r \in \mathscr{C}_c^{\infty}(K)$ ,  $\tau \geq \tau_*$  and  $\mu$  satisfying [\(3.3\)](#page-6-1).

This theorem is classical, and we can find in Hörmander [\[14\]](#page-21-11). Here we give a proof to be complete.

*Proof.* Let's take  $w = e^{\tau \varphi} r$  then  $Pr = f$  can be written as follow  $P_{\varphi} w = g = f e^{\tau \varphi}$ . Since  $g = Q_2w + iQ_1w$  and  $Q_1$  and  $Q_2$  are symmetric then we have

(3.7) 
$$
||g||_0^2 = ||Q_2 w||_0^2 + ||Q_1 w||_0^2 + i(Q_1 w, Q_2 w) - i(Q_2 w, Q_1 w) = (Q_2 Q_2 w, w) + (Q_1 Q_1 w, w) + i(Q_2 Q_1 w, w) - i(Q_1 Q_2 w, w) = ((Q_1^2 + Q_2^2 + i[Q_2, Q_1]) w, w).
$$

<span id="page-8-2"></span>We fix  $\kappa$  large enough such that the statement of Lemma [3.4](#page-8-1) is fulfilled, then for  $\tau$  sufficiently large satisfying  $\kappa \tau^{-1}$  < 1 then from [\(3.7\)](#page-8-2) we obtain

(3.8) 
$$
\tau^{-1}\left( \left( \kappa(Q_1^2 + Q_2^2) + i\tau[Q_2, Q_1] \right) w, w \right) \le ||g||_0^2.
$$

Since the principal symbol of  $\kappa(Q_1^2+Q_2^2)+i\tau[Q_2,Q_1]$  is given by  $\rho_\kappa(x,\xi,\tau)=\kappa(q_2^2+q_1^2)+\tau\{q_2,q_1\}$ then from Lemma [3.4](#page-8-1) we have  $\rho_{\kappa}(x,\xi,\tau) \geq C \langle \xi,\tau \rangle^4$  therefore by Gårding inequality (Theorem [3.5\)](#page-6-2) it follows that

<span id="page-8-6"></span><span id="page-8-5"></span><span id="page-8-3"></span>Re 
$$
((\kappa(Q_1^2+Q_2^2)+i\tau[Q_2,Q_1])w,w) \ge ||w||_{\tau,2}^2.
$$

Combining this inequality with [\(3.8\)](#page-8-3) we find

(3.9) 
$$
\tau^{-1} \|w\|_{\tau,2}^2 \le \|g\|_0^2.
$$

which reads

(3.10) 
$$
\tau^3 \|w\|_0^2 + \tau \|\nabla w\|_0^2 + \tau^{-1} \sum_{|\alpha|=2} \|D^{\alpha}w\|_0^2 \le C \|e^{\tau \varphi} f\|_0^2.
$$

Since we have

<span id="page-9-0"></span>
$$
e^{\tau\varphi}D_jr = (D_j + i\tau\partial_j\varphi)w, \qquad e^{\tau\varphi}D_jD_kr = (D_j + i\tau\partial_j\varphi)(D_k + i\tau\partial_k\varphi)w,
$$

then we have

(3.11) 
$$
\tau \|e^{\tau \varphi} \nabla r\|_0^2 \leq C \left(\tau^3 \|w\|_0^2 + \tau \|\nabla w\|_0^2\right)
$$

and

<span id="page-9-1"></span>
$$
(3.12) \t\t \tau^{-1} \sum_{|\alpha|=2} \|e^{\tau \varphi} D^{\alpha} r\|_0^2 \leq C \left( \tau^3 \|w\|_0^2 + \tau \|\nabla w\|_0^2 + \tau^{-1} \sum_{|\alpha|=2} \|D^{\alpha} w\|_0^2 \right).
$$

Thus, estimate  $(3.6)$  follows by replacing  $(3.11)$  and  $(3.12)$  into  $(3.10)$ . This concludes the proof.  $\Box$ 

<span id="page-9-2"></span>**Theorem 3.7.** Let  $\Omega$  be an open bounded set of  $\mathbb{R}^d$ , let  $K \in \Omega$  and  $\varphi$  be a function that satisfies the sub-ellipticity assumption in K. Then there exist  $\tau_* > 0$  and  $C > 0$  such that

$$
\tau^{3} \|e^{\tau \varphi} r\|_{0}^{2} + \tau \|e^{\tau \varphi} \nabla r\|_{0}^{2} \leq C\tau^{2} \left( \|e^{\tau \varphi} g\|_{0}^{2} + \|e^{\tau \varphi} f\|_{0}^{2} \right)
$$

for all  $r \in \mathcal{C}_c^{\infty}(K)$  which satisfies  $(3.2)$ ,  $\tau \geq \tau_*$  and  $\mu$  satisfying  $(3.3)$ .

From this theorem we can deduce this corollary.

<span id="page-9-4"></span>**Corollary 3.1.** Let  $\Omega$  be an open bounded set of  $\mathbb{R}^d$ , let  $K \in \Omega$  and  $\varphi$  be a function that satisfies the sub-ellipticity assumption in K. Then there exist  $\tau_* > 0$  and  $C > 0$  such that

$$
\tau^{3} \|e^{\tau \varphi} u\|_{0}^{2} + \tau^{3} \|e^{\tau \varphi} r\|_{0}^{2} + \tau \|e^{\tau \varphi} \nabla r\|_{0}^{2} \leq C\tau^{2} \left( \|e^{\tau \varphi} g\|_{0}^{2} + \|e^{\tau \varphi} f\|_{0}^{2} \right)
$$

for all  $r, u \in \mathcal{C}_c^{\infty}(K)$  which satisfies  $(3.1), \tau \geq \tau_*$  $(3.1), \tau \geq \tau_*$  and  $\mu$  satisfying  $(3.3)$ .

*Proof.* As r satisfies [\(3.2\)](#page-5-1), with Theorem [3.7,](#page-9-2) we only have to estimate  $\tau^3 ||e^{\tau \varphi}u||_0^2$ . From the first equation of [\(3.1\)](#page-4-3) we have  $\tau^3 \|e^{\tau \varphi} u\|_0^2 \lesssim \tau^3 \|e^{\tau \varphi} \mu^{-1} (f + \nabla r) \|_0^2$ , which gives the result using  $(3.3)$ .

Proof of Theorem [3.7.](#page-9-2) We set  $P_{\varphi} = e^{\tau \varphi} P e^{-\tau \varphi}$ ,  $w = e^{\tau \varphi} r$ ,  $F = -e^{\tau \varphi} f$  and  $G = i\mu e^{\tau \varphi} g +$  $\tau e^{\tau \varphi} \nabla \varphi \cdot f$ . Then from  $(3.2)$  we have

$$
P_{\varphi}w = G + \operatorname{div}(F).
$$

Let  $K_1$  be such that  $K \subseteq K_1 \subseteq \Omega$  and let  $\chi \in \mathscr{C}_c^\infty(K_1)$  be such that  $\chi = 1$  on K. Setting  $v = \chi \Lambda_{\tau}^{-1} w$  with  $\Lambda_{\tau} = (\tau^2 - \Delta)^{1/2}$  and we write

$$
P_{\varphi}v = \chi \Lambda_{\tau}^{-1} P_{\varphi} w + [P_{\varphi}, \chi \Lambda_{\tau}^{-1}] w = \chi \Lambda_{\tau}^{-1} (G + \text{div}(F)) + [P_{\varphi}, \chi \Lambda_{\tau}^{-1}] w,
$$

.

then we find

(3.13) 
$$
||P_{\varphi}v||_0 \leq C \left( \tau^{-1} ||G||_0 + ||F||_0 + ||w||_0 \right)
$$

Applying Estimate  $(3.9)$  in the proof of Theorem [3.6](#page-8-7) to v then we obtain

<span id="page-9-3"></span> $\tau^{-\frac{1}{2}} \|v\|_{\tau,2} \leq C \|P_{\varphi}v\|_0$ 

We have  $v = \Lambda_{\tau}^{-1} w + [\chi, \Lambda_{\tau}^{-1}] w$  then  $||w||_{\tau,1} \lesssim ||v||_{\tau,2} + ||w||_0$ . That together with  $(3.13)$ 

$$
\tau^{-\frac{1}{2}} \|w\|_{\tau,1} \leq C \left( \tau^{-1} \|G\|_{0} + \|F\|_{0} + \|w\|_{0} \right).
$$

Multiplying by  $\tau$  which is chosen sufficiently large then we obtain

$$
\tau^{\frac{1}{2}} \|w\|_{\tau,1} \leq C \left( \|G\|_{0} + \tau \|F\|_{0} \right) \leq C \tau \left( \|e^{\tau \varphi} g\|_{0} + \|e^{\tau \varphi} f\|_{0} \right).
$$

As  $||w||_{\tau,1}$  is equivalent to  $\tau ||e^{\tau \varphi} v||_0 + ||e^{\tau \varphi} \nabla v||_0$  arguing as in the proof of  $(3.11)$  we obtain the result of the theorem. <span id="page-10-0"></span>3.3. Local Carleman estimate at the boundary. In this section because the boundary, we use a tangential pseudo-differential calculus. This calculus is completely analogous to the one presented in Section [3.1](#page-5-0) except that a function  $a(x', x_d, \xi')$  is a symbol in  $(x', \xi')$  in the sense of Definition [3.1](#page-5-2) where  $x_d$  is a parameter and the estimates given in Definition 3.1 are uniform with respect  $x_d$ . To avoid confusion we denote by  $S^m_{\tau,\tau}$  the class of tangential symbol of order m,  $Op_{\mathsf{T}}(a)$  the operator associated with the symbol  $a \in S^m_{\mathsf{T},\tau}$ . The class of operators associated with symbols in  $S^m_{\mathsf{T},\tau}$  is denoted by  $\Psi^m_{\mathsf{T},\tau}$ . We refer to [\[20\]](#page-21-21) for details on these symbols and operators. We consider functions in a half space  $\mathbb{R}^{d-1} \times (0, +\infty) = \mathbb{R}^d_+$ , and we denote by  $\|.\|_+ = \|.\|_{L^2(\mathbb{R}^d_+)}$  the  $L^2$ -norm and  $(.,.)_+ = (.,.)_{L^2(\mathbb{R}^d_+)}$  the associated inner product. At the boundary  $x_d = 0$  we denote the  $L^2$ -norm by  $|g|^2 = \int_{\mathbb{R}^{d-1}} |g(x')|^2 dx'$  and the inner product associated by  $(.,.)_{\partial} = (.,.)_{L^2(\mathbb{R}^{d-1})}$ . A set  $W = \omega \times \Gamma$  in  $\mathbb{R}^d \times \mathbb{R}^{d-1} \times \mathbb{R}^+$  is called a conic open set, if there exist  $\omega$  an open set in  $\mathbb{R}^d$ , and  $\Gamma$  an open set in  $\mathbb{R}^{d-1} \times \mathbb{R}^+$  such that for all  $(\xi', \tau) \in \Gamma$  and  $\lambda > 0$  then  $(\lambda \xi', \lambda \tau) \in \Gamma$ . For  $s \in \mathbb{R}$  we denote by  $\Lambda^s_{\tau, \tau}$  the tangential operator defined by  $\Lambda_{\tau,\tau}^s = \text{Op}_{\tau}(\langle \xi', \tau \rangle^s)$ .

We recall the following microlocal Gårding inequality obtained, for instance, by applying sharp Gårding inequality.

<span id="page-10-3"></span>**Theorem 3.8** (Microlocal Gårding inequality). Let K be a compact set of  $\mathbb{R}^d$  and let W be a conic open set of  $\mathbb{R}^d \times \mathbb{R}^{d-1} \times \mathbb{R}^+$  contained in  $K \times \mathbb{R}^{d-1} \times \mathbb{R}^+$ . Let also  $\chi \in S^0_{\mathsf{T},\tau}$  be homogeneous of order 0 (for  $\langle \xi', \tau \rangle \ge 1$ ) and be such that  $\text{supp}(\chi) \subset W$ .

Let  $a(x,\xi',\tau) \in S^m_{\tau,\tau}$ , with principal part  $a_m$  homogeneous of order m. If there exist  $C_0 > 0$ and  $R > 0$  such that

Re 
$$
a_m(x, \xi', \tau) \ge C_0 \langle \xi', \tau \rangle^m
$$
,  $(x, \xi', \tau) \in W$ ,  $\tau \in [1, +\infty)$ ,  $\langle \xi', \tau \rangle \ge R$ ,

then for any  $0 < C_1 < C_0$ ,  $N \in \mathbb{N}$ , there exist  $C_N$  and  $\tau_* \geq 1$  such that

$$
\mathrm{Re}\left(\mathrm{Op}_{\mathsf{T}}(a)\mathrm{Op}_{\mathsf{T}}(\chi)u,\mathrm{Op}_{\mathsf{T}}(\chi)u\right)_{+}\geq C_{1}\|\Lambda_{\mathsf{T},\tau}^{m/2}\mathrm{Op}_{\mathsf{T}}(\chi)u\|_{+}^{2}-C_{N}\|\Lambda_{\mathsf{T},\tau}^{-N}u\|_{+}^{2},
$$

for  $u \in \mathscr{S}(\mathbb{R}^d)$  and  $\tau \geq \tau_*$ .

As we want to change the variables in order to have a flat boundary which is convenient to do the computations, we use the language and usual tools of Riemannian geometry. In this framework the gradient and divergence operators keep forms we can follow after a change of variables. Our purpose is to use these tools locally and we do not use manifold tools as charts, atlas and etc. To fix the notation, let V be an open set in  $\mathbb{R}^d$ . Let  $g(x) = (g_{ij}(x))_{1 \le i,j \le d}$  be a positive symmetric matrix called the metric, we denote by  $g^{-1}(x) = (g^{ij}(x))_{1 \le i,j \le d}$  the inverse of  $g(x)$ . For a smooth function r, we denote by  $(\nabla_g r(x))^i = \sum_{1 \leq j \leq d} g^{ij}(x) \partial_{x_j} r(x)$  the gradient of r. We have  $\nabla_g r(x) \in T_x V$ , this means that  $\nabla_g r$  is a tangent vector field.

For  $u(x) = (u^1(x), \ldots, u^d(x))$  a smooth tangent vector field, we define the divergence operator by

<span id="page-10-2"></span><span id="page-10-1"></span>
$$
\operatorname{div}_g u(x) = \big(\operatorname{det} g(x)\big)^{-1/2} \sum_{1 \le j \le d} \partial_{x_j} \Big(\big(\operatorname{det} g(x)\big)^{1/2} u^j(x)\Big).
$$

For a smooth function  $r$  and a smooth tangent vector field  $u$ , we have

(3.14) 
$$
\operatorname{div}_g(ru) = r \operatorname{div}_g u + g(\nabla_g r, u), \text{ where } g(\nabla_g r, u) = \sum_{1 \le i, j \le d} g_{ij} (\nabla_g r)^i u^j.
$$

For two smooth functions  $r_1$  and  $r_2$  we have

(3.15) 
$$
\nabla_g(r_1r_2) = r_1 \nabla_g r_2 + r_2 \nabla_g r_1.
$$

It is well-known that there exist coordinates (called normal geodesic coordinates) such that the boundary is defined locally by  $x_d = 0$ , the open set  $\Omega \cap V$  is defined by  $x_d > 0$ , the metric g is such that  $g_{id} = g_{di} = 0$  for  $i = 1, ..., d-1$  and,  $g_{dd} = 1$ . We denote by  $\tilde{g} = (g_{ij})_{1 \le i, j \le d-1}$  the metric g on  $x_d$  fixed.

We can define on each manifold  $x_d = const$  the gradient and divergence operators associated with  $\tilde{g}$  and for r a smooth function and  $\tilde{u} = (u^1, \dots, u^{d-1})$  a smooth vector field on  $x_d = const$ , we have

$$
(\nabla_{\tilde{g}}r)^i = \sum_{1 \leq j \leq d-1} g^{ij} \partial_{x_j} r \text{ for } i = 1,\ldots,d-1, \quad \text{div}_{\tilde{g}} \tilde{u} = (\det \tilde{g})^{-1/2} \sum_{1 \leq j \leq d-1} \partial_{x_j} \left( (\det \tilde{g})^{1/2} u^j \right).
$$

In such coordinates, we have det  $g = \det \tilde{g}$ . The gradient and divergence operators take the following form.

<span id="page-11-4"></span>(3.16) 
$$
\nabla_g r = (\nabla_{\tilde{g}} r, \partial_{x_d} r), \text{ div}_g u = \text{div}_{\tilde{g}} \tilde{u} + \partial_{x_d} u^d + hu^d, \text{ where } h = (\det \tilde{g})^{-1/2} \partial_{x_d} (\det \tilde{g})^{1/2}.
$$

We recall that the equation of the resolvent problem  $(A_0 + i\mu)(u, r) = (f, g)$  locally takes the form

<span id="page-11-0"></span>(3.17) 
$$
\begin{cases}\n-\nabla_g r + i\mu u = f \text{ in } x_d > 0, \\
-\text{div}_g u + i\mu r = g \text{ in } x_d > 0, \\
u^d = 0 \text{ on } x_d = 0.\n\end{cases}
$$

We have the following theorem

<span id="page-11-3"></span>**Theorem 3.9.** Let  $x_0 \in \mathbb{R}^{d-1} \times \{0\}$ , we assume there exist a neighborhood of  $x_0$  where  $\varphi$ satisfies [\(3.4\)](#page-7-1) the sub-ellipticity condition and  $\partial_{x_d} \varphi(x_0) > 0$ . Then there exist  $V_0$  be an open set of  $\mathbb{R}^d$  such that  $x_0 \in V_0$ ,  $C > 0$ , and  $\tau_* > 0$  such that

$$
\tau^{1/2} |e^{\tau \varphi} r_{|x_d=0}| + \tau^{1/2} ||e^{\tau \varphi} u||_+ + \tau^{1/2} ||e^{\tau \varphi} r||_+ + \tau^{-1/2} ||e^{\tau \varphi} \nabla_g r||_+ \leq C (||e^{\tau \varphi} f||_+ + ||e^{\tau \varphi} g||_+),
$$
  
for  $u, r \in \mathscr{C}^{\infty}(\mathbb{R}^d)$  supported on  $V_0$ , satisfying (3.17), for every  $\tau \geq \tau_*$  and  $\mu$  satisfying (3.3).

Let  $v = e^{\tau \varphi} u$  and  $w = e^{\tau \varphi} r$ . We have from [\(3.14\)](#page-10-1) and [\(3.15\)](#page-10-2)  $\nabla_{\!g}r = e^{-\tau\varphi}(\nabla_{\!g}w - \tau w\nabla_{\!g}\varphi),$  $\operatorname{div}_g u = e^{-\tau \varphi} \big( \operatorname{div}_g v - \tau g(\nabla_g \varphi, v) \big).$ 

Then System [\(3.17\)](#page-11-0) takes the form

<span id="page-11-1"></span>(3.18) 
$$
\begin{cases}\n-\nabla_g w + \tau w \nabla_g \varphi + i \mu v = F \text{ in } x_d > 0, \\
-\text{div}_g v + \tau g (\nabla_g \varphi, v) + i \mu w = G \text{ in } x_d > 0, \\
v^d = 0 \text{ on } x_d = 0,\n\end{cases}
$$

where  $F = e^{\tau \varphi} f$  and  $G = e^{\tau \varphi} g$ .

In the following, we denote by  $\tilde{F} = (F^1, \ldots, F^{d-1})$  and by  $\tilde{v} = (v^1, \ldots, v^{d-1})$ , then we have  $F = (\tilde{F}, F^d)$  and  $v = (\tilde{v}, v^d)$ . Multiplying [\(3.18\)](#page-11-1) by *i*, we have

<span id="page-11-2"></span>(3.19) 
$$
\begin{cases}\n-i\nabla_{\tilde{g}} w + i\tau w \nabla_{\tilde{g}} \varphi - \mu \tilde{v} = i \tilde{F} \text{ in } x_d > 0, \\
-i \partial_{x_d} w + i\tau w \partial_{x_d} \varphi - \mu v^d = iF^d \text{ in } x_d > 0, \\
-i \operatorname{div}_g v + i\tau \tilde{g} (\nabla_{\tilde{g}} \varphi, \tilde{v}) + i\tau v^d \partial_{x_d} \varphi - \mu w = iG \text{ in } x_d > 0, \\
v^d = 0 \text{ on } x_d = 0.\n\end{cases}
$$

For this system we prove the following Carleman estimate.

<span id="page-11-5"></span>**Proposition 3.1.** Let  $x_0 \in \mathbb{R}^{d-1} \times \{0\}$ , we assume there exist a neighborhood of  $x_0$  where  $\varphi$ satisfies [\(3.4\)](#page-7-1) the sub-ellipticity condition and  $\partial_{x_d} \varphi(x_0) > 0$ . For  $s \in \mathbb{R}$ , there exist  $V_0$  be an open set such that  $x_0 \in V_0$ ,  $C > 0$ , and  $\tau_* > 0$  such that

$$
|\Lambda_{\text{T},\tau}^{s+1/2} w_{|x_d=0}| + \tau^{1/2} \|\Lambda_{\text{T},\tau}^s v\|_+ + \tau^{-1/2} \|\Lambda_{\text{T},\tau}^{s+1} w\|_+ \leq C ( \|\Lambda_{\text{T},\tau}^s F\|_+ + \|\Lambda_{\text{T},\tau}^s G\|_+),
$$

for  $v, w \in \mathscr{S}(\mathbb{R}^d)$  satisfying [\(3.19\)](#page-11-2), supported in  $V_0$ , for every  $\tau \geq \tau_*$  and  $\mu$  satisfying [\(3.3\)](#page-6-1).

From this proposition we deduce Theorem [3.9](#page-11-3) taking  $s = 0$ . Indeed we have  $\tau^{1/2}|w_{x_d=0}| \lesssim$  $|\Lambda_{\tau,\tau}^{1/2} w_{x_d=0}|$  and from  $(3.17)$  we have

$$
\tau^{-1/2} \|e^{\tau \varphi} \nabla_{\!g} r\|_{+} \lesssim \|e^{\tau \varphi} (\nabla_{\!g} r - i\mu u)\|_{+} + |\mu|\tau^{-1/2} \|e^{\tau \varphi} u\|_{+} \lesssim \|e^{\tau \varphi} f\|_{+} + \tau^{1/2} \|e^{\tau \varphi} u\|_{+},
$$

from [\(3.3\)](#page-6-1).

We begin by reducing the system in a 2  $\times$  2 system. We denote by  $\zeta' \in S^1_{\tau,\tau}$  the tangential symbol of the operator  $-i\nabla_{\tilde{g}} + i\tau \nabla_{\tilde{g}} \varphi$ . We have

$$
\zeta_i = \sum_{1 \leq j \leq d-1} g^{ij} (\xi_j + i\tau \partial_{x_j} \varphi) \text{ for } i = 1, \dots, d-1.
$$

Let  $\text{Op}_{\tau}(\delta) := -i \, \text{div}_{\tilde{g}} + i \tau \tilde{g}(\nabla_{\tilde{g}} \varphi, \cdot)$  where  $\delta \in S^1_{\tau, \tau}$ . The principal symbol of the operator  $\delta$  is  $(\xi_1 + i\tau \partial_{x_1} \varphi, \ldots, \xi_{d-1} + i\tau \partial_{x_{d-1}} \varphi)$  modulo symbol in  $S^0_{\tau,\tau}$ . The first equation of  $(3.19)$  reads  $\text{Op}_{\tau}(\zeta')w - \mu\tilde{v} = i\tilde{F}$ . Applying in both side of this equation the operator  $\text{Op}_{\tau}(\delta)$ , we obtain

(3.20) 
$$
\mathrm{Op}_{\tau}(\delta)\tilde{v} = -i\mu^{-1}\mathrm{Op}_{\tau}(\delta)\tilde{F} + \mu^{-1}\mathrm{Op}_{\tau}(\delta)\mathrm{Op}_{\tau}(\zeta')w.
$$

From [\(3.16\)](#page-11-4) we have

$$
-i \operatorname{div}_g v + i\tau \tilde{g}(\nabla_{\tilde{g}} \varphi, \tilde{v}) = \operatorname{Op}_{\tau}(\delta) \tilde{v} - i \partial_{x_d} v^d - i h v^d
$$
  
=  $D_{x_d} v^d - i\mu^{-1} \operatorname{Op}_{\tau}(\delta) \tilde{F} + \mu^{-1} \operatorname{Op}_{\tau}(\delta) \operatorname{Op}_{\tau}(\zeta') w - i h v^d.$ 

From this equation and second and third of  $(3.19)$  we obtain two equations on w and  $v<sup>d</sup>$ , that is (3.21)

<span id="page-12-0"></span>
$$
\begin{cases}\nD_{x_d} w + i\tau w \partial_{x_d} \varphi - \mu v^d = iF^d \text{ in } x_d > 0, \\
D_{x_d} v^d + \mu^{-1} \text{Op}_{\tau}(\delta) \text{Op}_{\tau}(\zeta') w - \mu w + i\tau v^d \partial_{x_d} \varphi - i h v^d = iG + i\mu^{-1} \text{Op}_{\tau}(\delta) \tilde{F} \text{ in } x_d > 0, \\
v^d = 0 \text{ on } x_d = 0.\n\end{cases}
$$

Let  $U = (w, v^d)$ , the system  $(3.21)$  has the form

$$
D_{x_d}U + BU = H, \text{ where } H = (iF^d, iG + \mu^{-1}\text{Op}_{\mathsf{T}}(\delta)\tilde{F}),
$$

and B is a tangential matrix operators with principal symbol

$$
b = \begin{pmatrix} i\tau \partial_{x_d} \varphi & -\mu \\ \mu^{-1} q(x, \xi') - \mu & i\tau \partial_{x_d} \varphi \end{pmatrix},
$$

modulo  $\mu^{-1}S^1_{\tau,\tau}$ , where  $q(x,\xi') = \sum$  $1\leq i,j\leq d-1$  $g^{ij}(x)(\xi_i + i\tau \partial_{x_i}\varphi(x))(\xi_j + i\tau \partial_{x_j}\varphi(x))$ . The char-

acteristic polynomial of b is given by  $P(\lambda) = (\lambda - i\tau \partial_{x_d} \varphi)^2 + q - \mu^2$ . Let  $\alpha \in \mathbb{C}$  such that  $\alpha^2 = q - \mu^2$  with Re  $\alpha \ge 0$ . The definition of  $\alpha$  is ambiguous when  $q - \mu^2 \le 0$  but in this case if  $q - \mu^2 < 0$  the root are simple and the analysis below is independent of the choice of root. In particular the roots are smooth, or if  $q - \mu^2 = 0$  the root is double and below, we give a specific analysis in this case. The root of  $P(\lambda)$  are  $i\tau ∂_{x} \varphi \pm i\alpha$  and the analysis in what follows depends on the location of roots in complex plane. We have the following result, denoting  $s = t^2$  where  $t, s \in \mathbb{C}$  we have for  $r_0 > 0$ ,

<span id="page-12-1"></span>(3.22) 
$$
|\text{Re } t| \leq r_0 \iff 4r_0^2 \text{Re } s - 4r_0^4 + (\text{Im } s)^2 \leq 0.
$$

Indeed, let  $t = x+iy$ , we have Re  $s = x^2 - y^2$  and Im  $s = 2xy$ , we obtain  $4r_0^2$  Re  $s-4r_0^4 + (\text{Im } s)^2 =$  $4(r_0^2 + y^2)(x^2 - r_0^2)$  which gives the result.

<span id="page-12-2"></span>From [\(3.22\)](#page-12-1), we obtain that  $|\text{Re }\alpha| \leq \tau |\partial_{x_d} \varphi|$  is equivalent to

(3.23) 
$$
4\tau^2 (\partial_{x_d} \varphi)^2 (\text{Re } q - \mu^2) - 4\tau^4 (\partial_{x_d} \varphi)^4 + (\text{Im } q)^2 \leq 0,
$$

where, from the definition of  $q$ , we have

$$
\begin{cases}\n\operatorname{Re} q(x,\xi') = \sum_{1 \leq i,j \leq d-1} g^{ij}(x) \big(\xi_i \xi_j - \tau^2 \partial_{x_i} \varphi(x) \partial_{x_j} \varphi(x)\big), \\
\operatorname{Im} q(x,\xi') = \tau \sum_{1 \leq i,j \leq d-1} g^{ij}(x) \xi_j \partial_{x_i} \varphi(x).\n\end{cases}
$$

We prove a microlocal Carleman estimate.

<span id="page-13-0"></span>**Lemma 3.5.** Let  $x_0 \in \mathbb{R}^{d-1} \times \{0\}$ , we assume there exist a neighborhood of  $x_0$  where  $\varphi$  satisfies [\(3.4\)](#page-7-1) the sub-ellipticity condition and  $\partial_{x_d} \varphi(x_0) > 0$ . Let  $(\xi'_0, \tau_0) \in \mathbb{R}^{d-1} \times \mathbb{R}^+$  be such that  $|\xi'_0|^2 + \tau_0^2 = 1$ . There exist W be an open conic set of  $(x_0, \xi'_0, \tau_0)$ ,  $\chi_1 \in S^0_{\tau, \tau}$  be an homogenous symbol of order  $0$  for  $\langle \xi', \tau \rangle \ge 1$  supported in W and  $\chi_1 = 1$  in a conic neighborhood of  $(x_0, \xi'_0, \tau_0)$ . For  $s \in \mathbb{R}$ , there exist  $C > 0$ , and  $\tau_* > 0$  such that

$$
\begin{aligned} |\Lambda_{\text{T},\tau}^{s+1/2} \text{Op}_{\text{T}}(\chi_1) w_{|x_d=0}| &+ \tau^{1/2} \|\Lambda_{\text{T},\tau}^s \text{Op}_{\text{T}}(\chi_1) v\|_+ + \tau^{-1/2} \|\Lambda_{\text{T},\tau}^{s+1} \text{Op}_{\text{T}}(\chi_1) w\|_+ \\ &\leq C \big( \|\Lambda_{\text{T},\tau}^s F\|_+ + \|\Lambda_{\text{T},\tau}^s G\|_+ + \|\Lambda_{\text{T},\tau}^s w\|_+ + \|\Lambda_{\text{T},\tau}^s v^d\|_+ \big), \end{aligned}
$$

for  $v, w \in \mathscr{S}(\mathbb{R}^d)$  satisfying [\(3.19\)](#page-11-2), for every  $\tau \geq \tau_*$  and  $\mu$  satisfying [\(3.3\)](#page-6-1).

This lemma implies Proposition [3.1](#page-11-5) as we can cover  $\langle \xi', \tau \rangle = 1$  by a finite number of open sets given by the statement of Lemma [3.5.](#page-13-0)

For the proof of Lemma [3.5,](#page-13-0) we distinguish two cases,  $\alpha \neq 0$  and  $\alpha = 0$ .

Assume that  $\alpha(x_0, \xi'_0, \tau_0) \neq 0$ . By continuity and homogeneity in  $(\xi', \tau)$ ,  $\alpha \neq 0$  in a conic neighborhood W of  $(x_0, \xi'_0, \tau_0)$ . Let  $\chi_0 \in S^0_{\tau, \tau}$  be an homogenous symbol of order 0 for  $\langle \xi', \tau \rangle \ge 1$ such that  $\chi_0 = 1$  in a conic neighborhood of  $(x_0, \xi'_0, \tau_0)$ , supported in W and  $\chi_1$  supported on  $\chi_0 = 1$ . Writing

$$
b = \begin{pmatrix} i\tau \partial_{x_d} \varphi & -\mu \\ \mu^{-1} \alpha^2 & i\tau \partial_{x_d} \varphi \end{pmatrix},
$$

the left eigenvector associated with  $i\tau \partial_{x_d} \varphi + i\alpha$  (resp.  $i\tau \partial_{x_d} \varphi - i\alpha$ ) is  $(-i\alpha \mu)(\text{resp. } (i\alpha \mu))$ .

Let  $\tilde{\alpha} = \chi_0 \alpha$ , as  $\alpha$  is a smooth homogenous function of order 1 in W,  $\tilde{\alpha} \in S^1_{\tau, \tau}$ . Recall the notation  $\Lambda_{\tau,\tau}^s = \text{Op}_{\tau}(\langle \xi', \tau \rangle^s)$ , according with the above algebraic computations and with the left vector found, we define

<span id="page-13-4"></span>(3.24) 
$$
\begin{cases} z_1 = -i\Lambda_{\tau,\tau}^{-1} \text{Op}_{\tau}(\tilde{\alpha}) \text{Op}_{\tau}(\chi_1) w + \mu \Lambda_{\tau,\tau}^{-1} \text{Op}_{\tau}(\chi_1) v^d \\ z_2 = i\Lambda_{\tau,\tau}^{-1} \text{Op}_{\tau}(\tilde{\alpha}) \text{Op}_{\tau}(\chi_1) w + \mu \Lambda_{\tau,\tau}^{-1} \text{Op}_{\tau}(\chi_1) v^d. \end{cases}
$$

As  $v^d = 0$  on  $x_d = 0$  we obtain  $z_1 + z_2 = 0$  on  $x_d = 0$ . Applying  $\pm i\Lambda_{\tau,\tau}^{-1} \text{Op}_{\tau}(\tilde{\alpha}) \text{Op}_{\tau}(\chi_1)$  to the first equation [\(3.21\)](#page-12-0),  $\mu \Lambda_{\tau,\tau}^{-1}$  Op<sub>T</sub>( $\chi_1$ ) to the second equation and summing up, we obtain

<span id="page-13-3"></span>(3.25) 
$$
D_{x_d}z_j + \text{Op}_{\mathsf{T}}(i\tau \partial_{x_d}\varphi + (-1)^j i\tilde{\alpha})z_j = H_j \text{ where}
$$

$$
\|\Lambda_{\mathsf{T},\tau}^s H_j\|_+\lesssim \|\Lambda_{\mathsf{T},\tau}^s F^d\|_+ + \|\Lambda_{\mathsf{T},\tau}^s G\|_+ + \|\Lambda_{\mathsf{T},\tau}^s \tilde{F}\|_+ + \|\Lambda_{\mathsf{T},\tau}^s w\|_+ + \|\Lambda_{\mathsf{T},\tau}^s v^d\|_+.
$$

We compute

<span id="page-13-1"></span>
$$
(3.26) \quad 2\operatorname{Re}(H_j, i\Lambda_{T,\tau}^{2s+1}z_j)_+ = 2\operatorname{Re}(D_{x_d}z_j + \operatorname{Op}_{\mathsf{T}}(i\tau\partial_{x_d}\varphi + (-1)^j i\tilde{\alpha})z_j, i\Lambda_{T,\tau}^{2s+1}z_j)_+ = |\Lambda_{T,\tau}^{s+1/2}(z_j)_{|x_d=0}|^2 + 2\operatorname{Re}(\Lambda_{T,\tau}^{2s+1}\operatorname{Op}_{\mathsf{T}}(\tau\partial_{x_d}\varphi + (-1)^j \tilde{\alpha})z_j, z_j)_+,
$$

using that  $(3.27)$ 

<span id="page-13-5"></span><span id="page-13-2"></span>
$$
2\operatorname{Re}(D_{x_d}h, i\Lambda_{\rm T,\tau}^{2m}h)_+=|\Lambda_{\rm T,\tau}^{m}h_{|x_d=0}|^2
$$

for  $h \in \mathscr{S}(\mathbb{R}^d)$ .

If  $j = 2$ , we have  $\tau \partial_{x_d} \varphi + \text{Re}\,\alpha \gtrsim \tau + |\xi'|$  in W. Let  $\chi_2 \in S^0_{\tau,\tau}$  supported in  $\chi_0 = 1$  and  $\chi_2 = 1$  on the support of  $\chi_1$ . From symbolic calculus we have

,

(3.28) kΛ s <sup>T</sup>,τ z<sup>2</sup> − Op<sup>T</sup> (χ2)z<sup>2</sup> k<sup>+</sup> . kΛ −N <sup>T</sup>,τ wk<sup>+</sup> + kΛ −N <sup>T</sup>,τ v d k+.

Then the tangential Gårding inequality of Theorem [3.8](#page-10-3) applies and we have

$$
2 \operatorname{Re}(\Lambda_{\tau,\tau}^{2s+1} \operatorname{Op}_{\tau}(\tau \partial_{x_d} \varphi + \tilde{\alpha}) z_2, z_2)_+ \geq C_1 \|\Lambda_{\tau,\tau}^{s+1} z_2\|_+^2 - C_N \big( \|\Lambda_{\tau,\tau}^{-N} w\|_+^2 + \|\Lambda_{\tau,\tau}^{-N} v^d\|_+^2 \big).
$$

From [\(3.26\)](#page-13-1), we then deduce

<span id="page-14-0"></span>
$$
(3.29) \quad 2\operatorname{Re}(H_2, i\Lambda_{T,\tau}^{2s+1}z_2)_+ \geq
$$
  

$$
C_1\big(|\Lambda_{T,\tau}^{s+1/2}(z_2)_{|x_d=0}|^2 + \|\Lambda_{T,\tau}^{s+1}z_2\|_+^2\big) - C_N\big(\|\Lambda_{T,\tau}^{-N}w\|_+^2 + \|\Lambda_{T,\tau}^{-N}v^d\|_+^2\big),
$$

for  $C_1 > 0$ , for every  $N > 0$  and  $C_N > 0$ , uniformly with respect to  $\tau$  chosen sufficiently large. This implies

<span id="page-14-1"></span>
$$
(3.30) \t\t\t |\Lambda_{\tau,\tau}^{s+1/2}(z_2)_{|x_d=0}| + \|\Lambda_{\tau,\tau}^{s+1}z_2\|_{+} \lesssim \|\Lambda_{\tau,\tau}^s H_2\|_{+} + \|\Lambda_{\tau,\tau}^{-N}w\|_{+} + \|\Lambda_{\tau,\tau}^{-N}v^d\|_{+}.
$$

<span id="page-14-4"></span>**Lemma 3.6.** Assume that  $\alpha \neq 0$  in W.

If  $\text{Re}\,\alpha - \partial_{x_d}\varphi \neq 0$  on W, we have

<span id="page-14-2"></span>
$$
(3.31) \t\t\t || \Lambda_{\tau,\tau}^{s+1} z_1 ||_{+} \leq C \big( || \Lambda_{\tau,\tau}^{s} H_1 ||_{+} + |\Lambda_{\tau,\tau}^{s+1/2} (z_1)_{|x_d=0}| + || \Lambda_{\tau,\tau}^{s} w ||_{+} + || \Lambda_{\tau,\tau}^{s} v^{d} ||_{+} \big),
$$

for some  $C > 0$ .

If  $\text{Re}\,\alpha - \tau \partial_{x_d}\varphi = 0$  at  $(x_0, \xi'_0, \tau_0)$ , we have

<span id="page-14-3"></span>
$$
(3.32) \qquad \|\Lambda_{\mathsf{T},\tau}^{s+1/2} z_1\|_{+} \le C \big( \|\Lambda_{\mathsf{T},\tau}^{s} H_1\|_{+} + |\Lambda_{\mathsf{T},\tau}^{s+1/2}(z_1)_{|x_d=0}| + \|\Lambda_{\mathsf{T},\tau}^{s} w\|_{+} + \|\Lambda_{\mathsf{T},\tau}^{s} v^d\|_{+} \big),
$$
for some  $C > 0$ 

for some  $C > 0$ .

*Proof.* We have to distinguish three cases, that is  $|\text{Re }\alpha| \leq \tau |\partial_{x_d} \varphi|$  at  $(x_0, \xi'_0, \tau_0)$ .

• If  $| \text{Re } \alpha | < \tau | \partial_{x_d} \varphi |$ , from [\(3.23\)](#page-12-2) this is equivalent to

$$
4\tau^2 (\partial_{x_d} \varphi)^2 (\text{Re } q - \mu^2) - 4\tau^4 (\partial_{x_d} \varphi)^4 + (\text{Im } q)^2 < 0.
$$

We have  $\tau \partial_{x_d} \varphi - \text{Re}\,\alpha \gtrsim \tau + |\xi'|$  in W. Then we have the same computations as in [\(3.29\)](#page-14-0) and  $(3.30)$ , and we have

$$
|\Lambda_{\text{T},\tau}^{s+1/2}(z_1)_{|x_d=0}|+||\Lambda_{\text{T},\tau}^{s+1}z_1||_{+}\lesssim ||\Lambda_{\text{T},\tau}^sH_1||_{+}+||\Lambda_{\text{T},\tau}^{-N}w||_{+}+||\Lambda_{\text{T},\tau}^{-N}v^d||_{+}
$$

which is a better estimate than  $(3.31)$ .

• If  $| \text{Re } \alpha | > \tau | \partial_{x_d} \varphi |$ , from [\(3.22\)](#page-12-1) this is equivalent to

$$
4\tau^{2}(\partial_{x_{d}}\varphi)^{2}(\text{Re }q-\mu^{2})-4\tau^{4}(\partial_{x_{d}}\varphi)^{4}+(\text{Im }q)^{2}>0.
$$

Observe that this case contains the case where  $\tau_0 = 0$  as  $|\xi'_0| = 1$ , and in W we have  $q(x,\xi',\tau) \geq c|\xi'|^2$  and  $|\xi'| \gg \tau$ .

As  $-\tau \partial_{x_d} \varphi + \text{Re}\,\alpha \gtrsim \tau + |\xi'|$  in W, from [\(3.26\)](#page-13-1), we can introduce a cutoff as in [\(3.28\)](#page-13-2) to apply the tangential Gårding inequality of Theorem [3.8,](#page-10-3) we deduce

$$
-2\operatorname{Re}(H_1,i\Lambda_{\tau,\tau}^{2s+1}z_1)_+ + |\Lambda_{\tau,\tau}^{s+1/2}(z_1)|_{x_d=0}|^2 \gtrsim \|\Lambda_{\tau,\tau}^{s+1}z_1\|_+^2 - C_N(\|\Lambda_{\tau,\tau}^{-N}w\|_+^2 + \|\Lambda_{\tau,\tau}^{-N}v^d\|_+^2),
$$

and then

$$
\|\Lambda_{\tau,\tau}^{s+1}z_1\|_{+} \lesssim \|\Lambda_{\tau,\tau}^s H_1\|_{+} + |\Lambda_{\tau,\tau}^{s+1/2}(z_1)_{|x_d=0}| + \|\Lambda_{\tau,\tau}^{-N}w\|_{+} + \|\Lambda_{\tau,\tau}^{-N}v^d\|_{+}
$$

which implies  $(3.31)$ .

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• If  $\text{Re }\alpha = \tau \partial_{x_d} \varphi$  at  $(x_0, \xi'_0, \tau_0)$ , from  $(3.22)$  and as  $\text{Re }\alpha$  and  $\partial_{x_d} \varphi$  are positive, this is equivalent to

 $, ,$ 

$$
4\tau^2(\partial_{x_d}\varphi)^2(\text{Re }q-\mu^2) - 4\tau^4(\partial_{x_d}\varphi)^4 + (\text{Im }q)^2 = 0, \text{ at } (x_0, \xi'_0, \tau_0).
$$

We use Carleman technics to obtain an estimate. Before doing that we must translate sub-ellipticity assumption [\(3.4\)](#page-7-1) on  $p<sub>\varphi</sub>$  on analogous condition on  $\alpha$ . First observe that

$$
p_{\varphi}(x,\xi,\tau) = (\xi_d + i\tau \partial_{x_d}\varphi)^2 + \alpha^2 = (\xi_d + i\tau \partial_{x_d}\varphi + i\alpha)(\xi_d + i\tau \partial_{x_d}\varphi - i\alpha).
$$

As  $i\tau_0 \partial_{x_d} \varphi(x_0) - i\alpha(x_0, \xi'_0, \tau_0) \in \mathbb{R}$ ,  $p_{\varphi} = 0$  is equivalent to  $\xi_d + i\tau \partial_{x_d} \varphi - i\alpha = 0$ . Noting that  $i\tau\partial_{x_d}\varphi + i\alpha \notin \mathbb{R}$  thus  $\xi_d + i\tau\partial_{x_d}\varphi + i\alpha \neq 0$  in W. Second, for a smooth function  $q = q_r + iq_i$ where  $q_r, q_i$  are real valued, we have  $\{q, \bar{q}\} = 2i\{q_i, q_r\}$ . Thus on  $p_{\varphi} = 0$  we have

$$
\{p_{\varphi}, \overline{p_{\varphi}}\} = |\xi_d + i\tau \partial_{x_d}\varphi + i\alpha|^2 \{\xi_d + i\tau \partial_{x_d}\varphi - i\alpha, \xi_d - i\tau \partial_{x_d}\varphi + i\bar{\alpha}\}\
$$
  
=  $2i|\xi_d + i\tau \partial_{x_d}\varphi + i\alpha|^2 \{\tau \partial_{x_d}\varphi - \text{Re }\alpha, \xi_d + \text{Im }\alpha\}.$ 

Thus sub-ellipticity condition reads in  $W$ , there exists  $C > 0$  such that

<span id="page-15-0"></span>(3.33) 
$$
\xi_d + i\tau \partial_{x_d} \varphi - i\alpha = 0 \Longrightarrow \{\xi_d + \text{Im}\,\alpha, \tau \partial_{x_d} \varphi - \text{Re}\,\alpha\} \ge C \langle \xi', \tau \rangle.
$$

At  $(x_0, \xi'_0, \tau_0)$ , observe that we can choose  $\xi_d$  such that  $\xi_d + \text{Im}\,\alpha = 0$  and as  $\tau_0 \partial_{x_d} \varphi - \text{Re}\,\alpha = 0$ , the condition  $(3.33)$  means, by continuity and homogeneity, there exists  $C > 0$  such that

(3.34) 
$$
\{\xi_d + \text{Im}\,\alpha, \tau \partial_{x_d}\varphi - \text{Re}\,\alpha\} \ge C\langle \xi', \tau \rangle \text{ in } W,
$$

eventually shrinking W.

Let

<span id="page-15-2"></span>
$$
A = \frac{1}{2} \big( \text{Op}_{\tau} (i\tau \partial_{x_d} \varphi - i\tilde{\alpha}) + \text{Op}_{\tau} (i\tau \partial_{x_d} \varphi - i\tilde{\alpha})^* \big),
$$
  

$$
B = \frac{1}{2i} \big( \text{Op}_{\tau} (i\tau \partial_{x_d} \varphi - i\tilde{\alpha}) - \text{Op}_{\tau} (i\tau \partial_{x_d} \varphi - i\tilde{\alpha})^* \big).
$$

We have  $A = A^*$ ,  $B = B^*$ ,  $Op_\mathsf{T}(i\tau \partial_{x_d} \varphi - i\tilde{\alpha}) = A + iB$ , and principal symbol of A is Im  $\tilde{\alpha}$  and principal symbol of B is  $\tau \partial_{x_d} \varphi - \text{Re}\,\tilde{\alpha}$ .

Now from [\(3.25\)](#page-13-3) we compute for  $z = \text{Op}_{\text{T}}(\chi_0) \Lambda_{\text{T},\tau}^s z_1$ 

<span id="page-15-3"></span>(3.35) 
$$
\| (D_{x_d} + Op_T(i\tau \partial_{x_d} \varphi - i\tilde{\alpha}))z \|_+^2 = \| (D_{x_d} + A)z \|_+^2 + \| Bz \|_+^2 + 2 \operatorname{Re} \left( (D_{x_d} + A)z, iBz \right)_+.
$$

We have

$$
2\operatorname{Re}(D_{x_d}z, iBz)_{+} = ([D_{x_d}, iB]z, z)_{+} + (Bz_{|x_d=0}, z_{|x_d=0})\partial.
$$

As the principal symbol of B is  $\tau \partial_{x_d} \varphi - \text{Re } \tilde{\alpha}$ , we obtain

<span id="page-15-1"></span>(3.36) 
$$
2 \operatorname{Re} (D_{x_d} z, iBz)_+ \geq \operatorname{Re} (i[D_{x_d}, Op_T(\tau \partial_{x_d} \varphi - \operatorname{Re} \tilde{\alpha})]z, z)_+ + \operatorname{Re} (Op_T(\tau \partial_{x_d} \varphi - \operatorname{Re} \tilde{\alpha})z_{|x_d=0}, z_{|x_d=0})\partial - C ||z||_+^2 - C |z_{|x_d=0}|^2,
$$

for some constant  $C > 0$ . We also have

$$
2 \operatorname{Re} (Az, iBz)_{+} = (i[A, B]z, z)_{+} \ge \operatorname{Re} (i[\operatorname{Op}_{\tau}(\operatorname{Im} \tilde{\alpha}), \operatorname{Op}_{\tau}(\tau \partial_{x_d} \varphi - \operatorname{Re} \tilde{\alpha})]z)_{+} - C \|z\|_{+}^{2}.
$$
  
Then from this estimate and (3.36), we obtain

<span id="page-15-4"></span>
$$
(3.37) \quad 2\operatorname{Re}\left(\left(D_{x_d}+A\right)z, iBz\right)_+ \geq \operatorname{Re}\left(i[D_{x_d}+\operatorname{Op}_{\mathsf{T}}(\operatorname{Im}\tilde{\alpha}),\operatorname{Op}_{\mathsf{T}}(\tau\partial_{x_d}\varphi-\operatorname{Re}\tilde{\alpha})]z,z\right)_+\\+ \operatorname{Re}(\operatorname{Op}_{\mathsf{T}}(\tau\partial_{x_d}\varphi-\operatorname{Re}\tilde{\alpha})z_{|x_d=0}, z_{|x_d=0})\partial-C\|z\|_+^2-C|z_{|x_d=0}|^2.
$$

The principal symbol of  $i[D_{x_d} + \text{Op}_{\tau}(\text{Im }\tilde{\alpha}), \text{Op}_{\tau}(\tau \partial_{x_d} \varphi - \text{Re }\tilde{\alpha})]$  is  $\{\xi_d + \text{Im }\alpha, \tau \partial_{x_d} \varphi - \text{Re }\alpha\},\$ then from [\(3.34\)](#page-15-2) and microlocal Gårding inequality of Theorem [3.8,](#page-10-3) we have

<span id="page-15-5"></span>(3.38) Re  $(i[D_{x_d} + \text{Op}_{\tau}(\text{Im }\tilde{\alpha}), \text{Op}_{\tau}(\tau \partial_{x_d} \varphi - \text{Re }\tilde{\alpha})]z, z)_+ \geq C_1 || \Lambda_{\tau, \tau}^{1/2} z ||_+^2 - C_N ||\Lambda_{\tau, \tau}^{-N} z_1||_+^2.$ We have  $1/2$ 

$$
|(\text{Op}_{\mathsf{T}}(\tau \partial_{x_d} \varphi - \text{Re}\,\tilde{\alpha})z_{|x_d=0}, z_{|x_d=0})_{\partial}| \lesssim |\Lambda_{\mathsf{T},\tau}^{1/2} z_{|x_d=0}|^2,
$$

then from  $(3.35)$ ,  $(3.37)$  and  $(3.38)$  we obtain

<span id="page-16-0"></span>
$$
(3.39) \t\t\t || \Lambda_{\tau,\tau}^{1/2} z ||_{+} \lesssim || (D_{x_d} + \mathrm{Op}_{\tau} (i\tau \partial_{x_d} \varphi - i\tilde{\alpha})) z ||_{+} + |\Lambda_{\tau,\tau}^{1/2} z_{|x_d=0}| + ||\Lambda_{\tau,\tau}^{-N} z_1||_{+},
$$

as we can absorb the remainder term  $||z||_+^2$  by the left hand side. Recalling the definition of  $z_1$ given by formula  $(3.24)$ , the symbolic calculus yields

$$
\|\Lambda_{\tau,\tau}^{1/2} \mathcal{O} p_{\tau}(\chi_0)\Lambda_{\tau,\tau}^s z_1 - \Lambda_{\tau,\tau}^{s+1/2} z_1\|_{+} \lesssim \|\Lambda_{\tau,\tau}^{s-1/2} w\|_{+} + \|\Lambda_{\tau,\tau}^{s-1/2} v^d\|_{+}.
$$

From  $z = \text{Op}_{\mathsf{T}}(\chi_0) \Lambda_{\mathsf{T},\tau}^s z_1$ , we deduce

$$
(3.40) \t\t\t\t\|\Lambda_{\tau,\tau}^{s+1/2}z_1\|_{+}\lesssim \|\Lambda_{\tau,\tau}^{1/2}z\|_{+}+\|\Lambda_{\tau,\tau}^{s-1/2}w\|_{+}+\|\Lambda_{\tau,\tau}^{s-1/2}v^d\|_{+}.
$$

Symbolic calculus also gives

(3.41) 
$$
\left\| \left(D_{x_d} + \mathrm{Op}_T(i\tau \partial_{x_d}\varphi - i\tilde{\alpha})\right)z \right\|_+ \lesssim \|\Lambda_{T,\tau}^s H_1\|_+ + \|\Lambda_{T,\tau}^s w\|_+ + \|\Lambda_{T,\tau}^s v^d\|_+,
$$

and

<span id="page-16-1"></span>(3.42) 
$$
|\Lambda_{\tau,\tau}^{1/2} z_{|x_d=0}| \lesssim |\Lambda_{\tau,\tau}^{s+1/2}(z_1)_{|x_d=0}|.
$$

Then from  $(3.39)$ – $(3.42)$  we obtain

$$
(3.43) \qquad \|\Lambda_{\mathsf{T},\tau}^{s+1/2}z_1\|_{+} \lesssim \|\Lambda_{\mathsf{T},\tau}^s H_1\|_{+} + |\Lambda_{\mathsf{T},\tau}^{s+1/2}(z_1)|_{x_d=0}| + \|\Lambda_{\mathsf{T},\tau}^s w\|_{+} + \|\Lambda_{\mathsf{T},\tau}^s v^d\|_{+},
$$

which is  $(3.32)$ . This achieves the proof of Lemma [3.6](#page-14-4) as we have treated the three cases.  $\square$ 

We can prove Lemma [3.5](#page-13-0) in the case  $\alpha \neq 0$ .

If Re $\alpha - \partial_{x_d} \varphi \neq 0$  on W, from [\(3.30\)](#page-14-1), Lemma [3.6,](#page-14-4) and as  $z_1 + z_2 = 0$  on  $x_d = 0$ , we deduce

$$
\begin{aligned}\n|\Lambda_{\text{T},\tau}^{s+1/2}(z_2)_{|x_d=0}|+ \|\Lambda_{\text{T},\tau}^{s+1}z_2\|_{+} &+ \|\Lambda_{\text{T},\tau}^{s+1}z_1\|_{+} \\
&\lesssim \|\Lambda_{\text{T},\tau}^{s}H_2\|_{+} + \|\Lambda_{\text{T},\tau}^{s}H_1\|_{+} + \|\Lambda_{\text{T},\tau}^{s}w\|_{+} + \|\Lambda_{\text{T},\tau}^{s}v^d\|_{+}.\n\end{aligned}
$$

From [\(3.25\)](#page-13-3) we deduce

<span id="page-16-2"></span>
$$
(3.44) \quad |\Lambda_{\mathsf{T},\tau}^{s+1/2}(z_2)_{|x_d=0}| + |\Lambda_{\mathsf{T},\tau}^{s+1} z_2|_{+} + |\Lambda_{\mathsf{T},\tau}^{s+1} z_1|_{+} \lesssim \|\Lambda_{\mathsf{T},\tau}^{s}\mathbf{G}\|_{+} + \|\Lambda_{\mathsf{T},\tau}^{s}\mathbf{F}^d\|_{+} + \|\Lambda_{\mathsf{T},\tau}^{s}\tilde{\mathbf{F}}\|_{+} + \|\Lambda_{\mathsf{T},\tau}^{s}\mathbf{W}\|_{+}.
$$

We have from [\(3.24\)](#page-13-4),  $z_1 + z_2 = 2\mu\Lambda_{\tau,\tau}^{-1}Op_{\tau}(\chi_1)v^d$  and from [\(3.3\)](#page-6-1) we deduce

$$
(3.45) \t\t\t\t\t\t\tau\|\Lambda^s_{\tau,\tau} \mathrm{Op}_{\tau}(\chi_1)v^d\|_{+} \lesssim |\mu| \|\Lambda^s_{\tau,\tau} \mathrm{Op}_{\tau}(\chi_1)v^d\|_{+} \lesssim \|\Lambda^{s+1}_{\tau,\tau}z_1\|_{+} + \|\Lambda^{s+1}_{\tau,\tau}z_2\|_{+}.
$$

We have  $\text{Op}_{\text{T}}(\tilde{\alpha})^*\Lambda_{\text{T},\tau}^s\Lambda_{\text{T},\tau}^s\text{Op}_{\text{T}}(\tilde{\alpha}) = \text{Op}_{\text{T}}(\langle \xi',\tau \rangle^{2s}\tilde{\alpha}^2)$  modulo an operator of order  $2s+1$ . As  $\tilde{\alpha}$ is not 0 on the support of  $\chi_1$ , the tangential Gårding inequality of Theorem [3.8](#page-10-3) yields

$$
\|\Lambda_{\mathsf{T},\tau}^s \mathcal{O} \mathbf{p}_{\mathsf{T}}(\tilde{\alpha}) \mathcal{O} \mathbf{p}_{\mathsf{T}}(\chi_1) w\|_{+} + \|\Lambda_{\mathsf{T},\tau}^{-N} w\|_{+} \gtrsim \|\Lambda_{\mathsf{T},\tau}^s \mathcal{O} \mathbf{p}_{\mathsf{T}}(\chi_1) w\|_{+},
$$

for every  $N > 0$ . From this and as  $z_2 - z_1 = 2i\Lambda_{\tau,\tau}^{-1} \text{Op}_{\tau}(\tilde{\alpha}) \text{Op}_{\tau}(\chi_1) w$  from [\(3.24\)](#page-13-4), we deduce

$$
(3.46) \t\t\t ||\Lambda_{\tau,\tau}^{s+1} \text{Op}_{\tau}(\chi_1)w||_{+} \lesssim ||\Lambda_{\tau,\tau}^{s+1} z_1||_{+} + ||\Lambda_{\tau,\tau}^{s+1} z_2||_{+} + ||\Lambda_{\tau,\tau}^{-N} w||_{+}.
$$

From the first equation of  $(3.19)$  and from  $(3.3)$  we have

$$
\tau \|\Lambda_{\tau,\tau}^s \mathcal{O} \mathbf{p}_{\tau}(\chi_1)\tilde{v}\|_{+} \lesssim \|\Lambda_{\tau,\tau}^{s+1} \mathcal{O} \mathbf{p}_{\tau}(\chi_1)w\|_{+} + \|\Lambda_{\tau,\tau}^s \tilde{F}\|_{+} + \|\Lambda_{\tau,\tau}^s w\|_{+} \lesssim \|\Lambda_{\tau,\tau}^{s+1} z_1\|_{+} + \|\Lambda_{\tau,\tau}^{s+1} z_2\|_{+} + \|\Lambda_{\tau,\tau}^s \tilde{F}\|_{+} + \|\Lambda_{\tau,\tau}^s w\|_{+}.
$$

From [\(3.24\)](#page-13-4),  $(z_2)_{|x_d=0} = i\Lambda_{\tau,\tau}^{-1} \text{Op}_{\tau}(\tilde{\alpha}) \text{Op}_{\tau}(\chi_1) w_{|x_d=0}$ , arguing as from above and using the Gårding estimate of Theorem [3.5,](#page-6-2) we have

<span id="page-16-3"></span>(3.48) 
$$
|\Lambda_{\tau,\tau}^{s+1/2} \text{Op}_{\tau}(\chi_1) w_{|x_d=0}| \lesssim |\Lambda_{\tau,\tau}^{s+1/2}(z_2)_{|x_d=0}|.
$$

From  $(3.44)$ – $(3.48)$  we obtain Lemma [3.5.](#page-13-0)

If  $\text{Re }\alpha - \tau \partial_{x_d} \varphi = 0$  at  $(x_0, \xi'_0, \tau_0)$ , adding  $(3.30)$  to  $\varepsilon(3.32)$  $\varepsilon(3.32)$  for  $\varepsilon > 0$ , we deduce

$$
\begin{split} |\Lambda_{\text{T},\tau}^{s+1/2}(z_2)_{|x_d=0}|+ \|\Lambda_{\text{T},\tau}^{s+1/2}z_2\|_{+} & + \varepsilon \|\Lambda_{\text{T},\tau}^{s+1/2}z_1\|_{+} \\ & \lesssim \|\Lambda_{\text{T},\tau}^{s}H_2\|_{+} + \varepsilon \|\Lambda_{\text{T},\tau}^{s}H_1\|_{+} + \|\Lambda_{\text{T},\tau}^{s}w\|_{+} + \|\Lambda_{\text{T},\tau}^{s}v^d\|_{+} + \varepsilon |\Lambda_{\text{T},\tau}^{s+1/2}(z_1)_{|x_d=0}|. \end{split}
$$

From [\(3.25\)](#page-13-3) and as  $z_1 + z_2 = 0$  on  $x_d = 0$ , we deduce for  $\varepsilon$  small enough tat

<span id="page-17-1"></span>
$$
(3.49) \quad |\Lambda_{\tau,\tau}^{s+1/2}(z_2)_{|x_d=0}| + ||\Lambda_{\tau,\tau}^{s+1/2} z_2||_+ + ||\Lambda_{\tau,\tau}^{s+1/2} z_1||_+
$$
  

$$
\lesssim ||\Lambda_{\tau,\tau}^s G||_+ + ||\Lambda_{\tau,\tau}^s F^d||_+ + ||\Lambda_{\tau,\tau}^s \tilde{F}||_+ + ||\Lambda_{\tau,\tau}^s v^d||_+ + ||\Lambda_{\tau,\tau}^s w||_+.
$$

We have from [\(3.24\)](#page-13-4),  $z_1 + z_2 = 2\mu \Lambda_{\tau,\tau}^{-1} \text{Op}_{\tau}(\chi_1) v^d$ . Let  $\chi_2 \in S^0_{\tau,\tau}$  supported in  $\chi_0 = 1$  and  $\chi_2 = 1$ on the support of  $\chi_1$ . From symbolic calculus we have

$$
\|({\rm Op}_{T}(\chi_{2})\Lambda^{s+3/2}_{T,\tau}\mu^{-1})(\mu\Lambda^{-1}_{T,\tau}{\rm Op}_{T}(\chi_{1})v^{d})-\Lambda^{s+1/2}_{T,\tau}{\rm Op}_{T}(\chi_{1})v^{d}\|_{+}\lesssim \|\Lambda^{-N}_{T,\tau}v^{d}\|_{+}.
$$

As  $Op_{T}(\chi_2)\Lambda_{T,\tau}^{s+3/2}\mu^{-1}$  is an operator of order  $s+1/2$  as  $|\mu|$  and  $|\xi'|$  are comparable on the support of  $\chi_2$ , we deduce (3.50)

$$
\tau^{1/2} \|\Lambda_{\rm T,\tau}^s {\rm Op}_{{\rm T}}(\chi_1)v^d\|_{+} \lesssim \|\Lambda_{\rm T,\tau}^{s+1/2} {\rm Op}_{{\rm T}}(\chi_1)v^d\|_{+} \lesssim \|\Lambda_{\rm T,\tau}^{s+1/2}z_1\|_{+} + \|\Lambda_{\rm T,\tau}^{s+1/2}z_2\|_{+} + \|\Lambda_{\rm T,\tau}^{-N}v^d\|_{+}.
$$

We have  $\tau^{-1} \text{Op}_{\tau}(\tilde{\alpha})^* \Lambda^s_{\tau,\tau} \Lambda^s_{\tau,\tau} \text{Op}_{\tau}(\tilde{\alpha}) = \tau^{-1} \text{Op}_{\tau}(\langle \xi', \tau \rangle^{2s} \tilde{\alpha}^2)$  modulo an operator of order 2s. As  $\tilde{\alpha}$  is not 0 on the support of  $\chi_1$ , the tangential Gårding inequality of Theorem [3.8](#page-10-3) yields

$$
\tau^{-1/2}\|\Lambda_{\mathsf{T},\tau}^s\mathrm{Op}_\mathsf{T}(\tilde\alpha)\mathrm{Op}_\mathsf{T}(\chi_1)w\|_+ + \|\Lambda_{\mathsf{T},\tau}^{-N}w\|_+\gtrsim \tau^{-1/2}\|\Lambda_{\mathsf{T},\tau}^{s+1}\mathrm{Op}_\mathsf{T}(\chi_1)w\|_+,
$$

for every  $N > 0$ . From this and as  $z_2 - z_1 = 2i\Lambda_{\text{T},\tau}^{-1} \text{Op}_{\text{T}}(\tilde{\alpha}) \text{Op}_{\text{T}}(\chi_1) w$  from [\(3.24\)](#page-13-4), we deduce, using symbolic calculus and  $\chi_2 \chi_1 = \chi_1$ ,

<span id="page-17-0"></span>
$$
(3.51) \quad \tau^{-1/2} \|\Lambda_{\tau,\tau}^{s+1} \text{Op}_{\tau}(\chi_1)w\|_{+} \lesssim \tau^{-1/2} \|\Lambda_{\tau,\tau}^{s+1} \text{Op}_{\tau}(\chi_2) \Lambda_{\tau,\tau}^{-1} \text{Op}_{\tau}(\tilde{\alpha}) \text{Op}_{\tau}(\chi_1)w\|_{+} + \|\Lambda_{\tau,\tau}^{-N} w\|_{+} \lesssim \|\Lambda_{\tau,\tau}^{s+1/2} z_1\|_{+} + \|\Lambda_{\tau,\tau}^{s+1/2} z_2\|_{+} + \|\Lambda_{\tau,\tau}^{-N} w\|_{+},
$$

as  $\tau^{-1/2} \Lambda_{\tau,\tau}^{s+1}$  Op<sub>T</sub> $(\chi_2)$  is an operator of order  $s+1/2$ .

From the first equation of  $(3.19)$  and  $(3.3)$ , we have

$$
(3.52) \t\tau^{1/2} \|\Lambda_{\tau,\tau}^s \text{Op}_{\tau}(\chi_1)\tilde{v}\|_{+} \lesssim \|\Lambda_{\tau,\tau}^{s+1}\mu^{-1}\tau^{1/2} \text{Op}_{\tau}(\chi_1)w\|_{+} + \|\Lambda_{\tau,\tau}^s \tilde{F}\|_{+} + \|\Lambda_{\tau,\tau}^s w\|_{+} \lesssim \|\Lambda_{\tau,\tau}^{s+1/2} z_1\|_{+} + \|\Lambda_{\tau,\tau}^{s+1/2} z_2\|_{+} + \|\Lambda_{\tau,\tau}^s w\|_{+} + \|\Lambda_{\tau,\tau}^s \tilde{F}\|_{+},
$$

from [\(3.51\)](#page-17-0).

From  $(3.24)$ ,  $(z_2)_{|x_d=0} = i\Lambda_{\tau,\tau}^{-1} \text{Op}_{\tau}(\tilde{\alpha}) \text{Op}_{\tau}(\chi_1) w_{|x_d=0}$ , arguing as from above and using the Gårding estimate of Theorem [3.5,](#page-6-2) we have

<span id="page-17-2"></span>
$$
(3.53) \qquad |\Lambda_{\tau,\tau}^{s+1/2} \text{Op}_{\tau}(\chi_1) w_{|x_d=0}| \lesssim |\Lambda_{\tau,\tau}^{s+1/2}(z_2)_{|x_d=0}|.
$$

From  $(3.49)$ – $(3.53)$  we obtain Lemma [3.5.](#page-13-0)

Now we consider the case  $q - \mu^2 = 0$ . Let  $\varepsilon > 0$ , we can shrink W such that  $|q - \mu^2| \leq \varepsilon \langle \xi', \tau \rangle^2$ in W. Note that  $|\mu| \sim \tau \sim |\xi'|$  on W. Let  $\chi_1$  be the cutoff defined previously supported on W and  $\chi_0$  supported on W and  $\chi_0 = 1$  on the support of  $\chi_1$ . By symbolic calculus we have

<span id="page-17-3"></span>(3.54) 
$$
Op_{T}(\chi_{1})Op_{T}(\delta)Op_{T}(\zeta') = Op_{T}(\delta)Op_{T}(\zeta')Op_{T}(\chi_{1}) + [Op_{T}(\chi_{1}), Op_{T}(\delta)Op_{T}(\zeta')]= Op_{T}(q)Op_{T}(\chi_{1}) + Op_{T}(r_{1})Op_{T}(\chi_{1})+ Op_{T}(\chi_{0})[Op_{T}(\chi_{1}), Op_{T}(\delta)Op_{T}(\zeta')] + Op_{T}(r_{-N}),
$$

where  $r_1 \in S^1_{\tau,\tau}$  and  $r_{-N} \in S^{-N}_{\tau,\tau}$ . Observe that  $\mu^{-1}\chi_j \in S^{-1}_{\tau,\tau}$  for  $j = 1, 2$ .

From  $(3.21)$ ,  $(3.54)$ , and by symbolic calculus we have

$$
\begin{cases}\nD_{x_d} \text{Op}_{\tau}(\chi_1) w + i\tau (\partial_{x_d} \varphi) \text{Op}_{\tau}(\chi_1) w - \mu \text{Op}_{\tau}(\chi_1) v^d = H_1 & \text{in } x_d > 0, \\
D_{x_d} \text{Op}_{\tau}(\chi_1) v^d + \mu^{-1} \text{Op}_{\tau}(q - \mu^2) \text{Op}_{\tau}(\chi_1) w + i\tau (\partial_{x_d} \varphi) \text{Op}_{\tau}(\chi_1) v^d = H_2 & \text{in } x_d > 0, \\
v^d = 0 & \text{on } x_d = 0,\n\end{cases}
$$

where

<span id="page-18-1"></span>(3.55) 
$$
\|\Lambda_{\tau,\tau}^s H_j\|_+\leq C_{\varepsilon}\big(\|\Lambda_{\tau,\tau}^s F^d\|_+ + \|\Lambda_{\tau,\tau}^s G\|_+ + \|\Lambda_{\tau,\tau}^s w\|_+ + \|\Lambda_{\tau,\tau}^s v^d\|_+ + \|\Lambda_{\tau,\tau}^s \tilde{F}\|_+\big),
$$
 for  $j=1,2$  with  $C_{\varepsilon}$  depends on  $\varepsilon$ . We compute

<span id="page-18-0"></span>(3.56) 
$$
2 \operatorname{Re}(H_1, i\Lambda_{\mathsf{T},\tau}^{2s+1} \operatorname{Op}_{\mathsf{T}}(\chi_1) w)_{+}
$$
  
\n
$$
= 2 \operatorname{Re}(D_{x_d} \operatorname{Op}_{\mathsf{T}}(\chi_1) w + i\tau (\partial_{x_d} \varphi) \operatorname{Op}_{\mathsf{T}}(\chi_1) w - \mu \operatorname{Op}_{\mathsf{T}}(\chi_1) v^d, i\Lambda_{\mathsf{T},\tau}^{2s+1} \operatorname{Op}_{\mathsf{T}}(\chi_1) w)_{+}.
$$
  
\nBy microbial Gårding inequality of Theorem 3.8 we have, using  $\tau \sim |\xi'|$  on W  
\n(3.57) 
$$
2 \operatorname{Re}(i\tau (\partial_{x_d} \varphi) \operatorname{Op}_{\mathsf{T}}(\chi_1) w, i\Lambda_{\mathsf{T},\tau}^{2s+1} \operatorname{Op}_{\mathsf{T}}(\chi_1) w)_{+} \geq C_0 \| \Lambda_{\mathsf{T},\tau}^{s+1} \operatorname{Op}_{\mathsf{T}}(\chi_1) w \|_{+}^2 - C_N \| \Lambda_{\mathsf{T},\tau}^{-N} w \|_{-}^2
$$

for  $C_0 > 0$ , for all  $N > 0$  and  $C_N > 0$ .

From this,  $(3.27)$  and  $(3.56)$  we obtain

$$
(3.58) \quad 2 \operatorname{Re}(H_1, i\Lambda_{\tau,\tau}^{2s+1} \operatorname{Op}_{\tau}(\chi_1) w)_{+}
$$
  
\n
$$
\geq |\Lambda_{\tau,\tau}^{s+1/2} \operatorname{Op}_{\tau}(\chi_1) w_{|x_d=0}|^2 + C_1 ||\Lambda_{\tau,\tau}^{s+1} \operatorname{Op}_{\tau}(\chi_1) w||_+^2 - \mu^2 C_2 ||\Lambda_{\tau,\tau}^s \operatorname{Op}_{\tau}(\chi_1) v^d||_+^2 - C_N ||\Lambda_{\tau,\tau}^{-N} w||^2,
$$
  
\nfor  $C_1, C_2 > 0$ , for all  $N > 0$  and  $C_N > 0$ .

We then obtain

<span id="page-18-4"></span>
$$
(3.59) \quad |\Lambda_{\text{T},\tau}^{s+1/2} \text{Op}_{\text{T}}(\chi_1) w_{|x_d=0}|^2 + ||\Lambda_{\text{T},\tau}^{s+1} \text{Op}_{\text{T}}(\chi_1) w_{||_+}^2 \le \mu^2 C_3 ||\Lambda_{\text{T},\tau}^s \text{Op}_{\text{T}}(\chi_1) v^d_{||_+^2} + C_{\varepsilon} (||\Lambda_{\text{T},\tau}^s F^d||_+ + ||\Lambda_{\text{T},\tau}^s G||_+ + ||\Lambda_{\text{T},\tau}^s w_{||_+} + ||\Lambda_{\text{T},\tau}^s v^d_{||_+} + ||\Lambda_{\text{T},\tau}^s \tilde{F}||_+),
$$

for  $C_3 > 0$ , for all  $N > 0$  and  $C_N, C_{\varepsilon} > 0$ .

Now we compute

<span id="page-18-2"></span>
$$
(3.60) \quad 2 \operatorname{Re}(H_2, i\mu \Lambda_{T,\tau}^{2s} \operatorname{Op}_{T}(\chi_1)v^d)_{+} = 2 \operatorname{Re}(D_{x_d} \operatorname{Op}_{T}(\chi_1)v^d, i\mu \Lambda_{T,\tau}^{2s} \operatorname{Op}_{T}(\chi_1)v^d)_{+} + 2 \operatorname{Re}(\mu^{-1} \operatorname{Op}_{T}(q) \operatorname{Op}_{T}(\chi_1)w - \mu \operatorname{Op}_{T}(\chi_1)w + i\tau (\partial_{x_d} \varphi) \operatorname{Op}_{T}(\chi_1)v^d, i\mu \Lambda_{T,\tau}^{2s} \operatorname{Op}_{T}(\chi_1)v^d)_{+}.
$$

From [\(3.27\)](#page-13-5) we have  $2 \text{Re}(D_{x_d} \text{Op}_{T}(\chi_1)v^d, i\Lambda_{T,\tau}^{2s} \text{Op}_{T}(\chi_1)v^d)_{+} = 0$  as  $v^d = 0$  on  $x_d = 0$ .

As  $C\varepsilon^2 \langle \xi', \tau \rangle^2 - \mu^{-2} (q - \mu^2)^2 \ge \varepsilon^2 \langle \xi', \tau \rangle^2$ , on W with  $C > 0$ , using that  $\tau \sim |\mu| \sim |\xi'|$ , we have by microlocal Gårding inequality of Theorem [3.8](#page-10-3)

$$
2 \operatorname{Re}(\mu^{-1} \operatorname{Op}_{T}(q - \mu^{2}) \operatorname{Op}_{T}(\chi_{1})w, \Lambda_{T,\tau}^{2s} \mu^{-1} \operatorname{Op}_{T}(q - \mu^{2}) \operatorname{Op}_{T}(\chi_{1})w)_{+} \leq C_{4} \varepsilon^{2} \|\Lambda_{T,\tau}^{s+1} \operatorname{Op}_{T}(\chi_{1})w\|_{+}^{2} + C_{N,\varepsilon} \|\Lambda_{T,\tau}^{-N} w\|_{+}^{2}.
$$

Then we have

$$
(3.61) \quad 2\operatorname{Re}(\mu^{-1}\operatorname{Op}_{\mathsf{T}}(q)\operatorname{Op}_{\mathsf{T}}(\chi_1)w - \mu \operatorname{Op}_{\mathsf{T}}(\chi_1)w, i\mu \Lambda_{\mathsf{T},\tau}^{2s} \operatorname{Op}_{\mathsf{T}}(\chi_1)v^d)_{+}
$$
  

$$
\leq \varepsilon |\mu| C_5 \|\Lambda_{\mathsf{T},\tau}^{s} \operatorname{Op}_{\mathsf{T}}(\chi_1)v^d\|_{+} \left( \|\Lambda_{\mathsf{T},\tau}^{s+1} \operatorname{Op}_{\mathsf{T}}(\chi_1)w\|_{+} + C_{N,\varepsilon} \|\Lambda_{\mathsf{T},\tau}^{-N}w\|_{+} \right),
$$

for  $C_5 > 0$ .

From microlocal Gårding inequality of Theorem [3.8](#page-10-3) and as  $\partial_{x_d} \varphi(x_0) > 0$ , we have (3.62)

<span id="page-18-3"></span> $2 \text{Re}(i\tau(\partial_{x_d}\varphi) \text{Op}_T(\chi_1)v^d, i\mu\Lambda_{T,\tau}^{2s} \text{Op}_T(\chi_1)v^d)_{+} \geq \mu^2 C_6 \|\Lambda_{T,\tau}^s \text{Op}_T(\chi_1)v^d\|_{+}^2 - C_N \|\Lambda_{T,\tau}^{-N}v^d\|_{+}^2,$ where  $C_6 > 0$  is independent of  $\varepsilon$ , for all  $N > 0$ ,  $C_N > 0$ .

From  $(3.55)$  and  $(3.60)$ – $(3.62)$  we obtain

$$
\mu^{2} \|\Lambda_{\mathsf{T},\tau}^{s} \mathcal{O} \mathbf{p}_{\mathsf{T}}(\chi_{1}) v^{d}\|_{+}^{2} \leq |\mu| \varepsilon C_{2} \|\Lambda_{\mathsf{T},\tau}^{s} \mathcal{O} \mathbf{p}_{\mathsf{T}}(\chi_{1}) v^{d}\|_{+} \left(\|\Lambda_{\mathsf{T},\tau}^{s+1} \mathcal{O} \mathbf{p}_{\mathsf{T}}(\chi_{1}) w\|_{+} + C_{N,\varepsilon} \|\Lambda_{\mathsf{T},\tau}^{-N} w\|_{+}\right) \n+ C_{\varepsilon} \left(\|\Lambda_{\mathsf{T},\tau}^{s} F^{d}\|_{+} + \|\Lambda_{\mathsf{T},\tau}^{s} G\|_{+} + \|\Lambda_{\mathsf{T},\tau}^{s} w\|_{+} + \|\Lambda_{\mathsf{T},\tau}^{s} v^{d}\|_{+} + \|\Lambda_{\mathsf{T},\tau}^{s} \tilde{F}\|_{+}\right)^{2}.
$$

We deduce

<span id="page-19-1"></span>
$$
\mu^{2} \|\Lambda_{\tau,\tau}^{s} \text{Op}_{\tau}(\chi_{1})v^{d}\|_{+}^{2} \leq \varepsilon^{2} C_{7} \big( \|\Lambda_{\tau,\tau}^{s+1} \text{Op}_{\tau}(\chi_{1})w\|_{+}^{2} \big) + C_{\varepsilon} \big( \|\Lambda_{\tau,\tau}^{s} F^{d}\|_{+} + \|\Lambda_{\tau,\tau}^{s} G\|_{+} + \|\Lambda_{\tau,\tau}^{s} w\|_{+} + \|\Lambda_{\tau,\tau}^{s} v^{d}\|_{+} + \|\Lambda_{\tau,\tau}^{s} \tilde{F}\|_{+} \big)^{2}.
$$

By the linear combination  $(3.63)+\varepsilon(3.59)$  $(3.63)+\varepsilon(3.59)$  and fixing  $\varepsilon$  sufficiently small, from  $(3.3)$  and  $\tau$  sufficiently large, we deduce

<span id="page-19-2"></span>
$$
(3.64) \quad \tau \|\Lambda_{\tau,\tau}^s \text{Op}_{\tau}(\chi_1)v^d\|_{+} + |\Lambda_{\tau,\tau}^{s+1/2} \text{Op}_{\tau}(\chi_1)w_{|x_d=0}| + \|\Lambda_{\tau,\tau}^{s+1} \text{Op}_{\tau}(\chi_1)w\|_{+} \n\lesssim \|\Lambda_{\tau,\tau}^s F^d\|_{+} + \|\Lambda_{\tau,\tau}^s G\|_{+} + \|\Lambda_{\tau,\tau}^s w\|_{+} + \|\Lambda_{\tau,\tau}^s v^d\|_{+} + \|\Lambda_{\tau,\tau}^s \tilde{F}\|_{+}.
$$

From the first equation of  $(3.19)$  and from  $(3.3)$  we have

<span id="page-19-3"></span>
$$
(3.65) \quad \tau \|\Lambda_{\mathsf{T},\tau}^{s} \mathbf{O} \mathbf{p}_{\mathsf{T}}(\chi_1) \tilde{v} \|_{+} \lesssim \|\Lambda_{\mathsf{T},\tau}^{s+1} \mathbf{O} \mathbf{p}_{\mathsf{T}}(\chi_1) w \|_{+} + \|\Lambda_{\mathsf{T},\tau}^{s} \tilde{F} \|_{+} + \|\Lambda_{\mathsf{T},\tau}^{s} w \|_{+} \lesssim \|\Lambda_{\mathsf{T},\tau}^{s} F^d \|_{+} + \|\Lambda_{\mathsf{T},\tau}^{s} G \|_{+} + \|\Lambda_{\mathsf{T},\tau}^{s} \tilde{F} \|_{+} + \|\Lambda_{\mathsf{T},\tau}^{s} w \|_{+} + \|\Lambda_{\mathsf{T},\tau}^{s} v^d \|_{+},
$$

<span id="page-19-0"></span>from [\(3.64\)](#page-19-2). From (3.64) and [\(3.65\)](#page-19-3) we obtain Lemma [3.5](#page-13-0) in the case  $\alpha = 0$ .

## 4. Logarithmic stability

The exponential estimate of Proposition [2.1](#page-4-1) is the consequence of the two following results. First a global Carleman estimate with an observability term and second an estimate of the observability term coming from the dissipation.

Let  $\omega_0$  and  $\omega_1$  be open sets such that  $\omega_1 \in \omega_0 \in \omega$ , and, from [\(1.2\)](#page-0-2), we have  $b(x) \geq b_0 > 0$ for  $x \in \omega$ . In what follows we denote by  $\|.\|_0 := \|.\|_{L^2(\Omega)}$ .

<span id="page-19-4"></span>**Theorem 4.1.** Let  $\Omega$  be an open bounded set of  $\mathbb{R}^d$  with smooth boundary. Let  $\varphi \in \mathscr{C}(\mathbb{R}^d)$  be a function that satisfies the sub-ellipticity assumption in  $\overline{\Omega} \setminus \omega_1$ . Then there exist  $\tau_* > 0$  and  $C > 0$  such that

$$
\tau^{3/2} \|e^{\tau \varphi} r\|_0 + \tau^{3/2} \|e^{\tau \varphi} u\|_0 \leq C \left( \tau \|e^{\tau \varphi} g\|_0 + \tau \|e^{\tau \varphi} f\|_0 + \tau^{3/2} \|e^{\tau \varphi} r\|_{L^2(\omega_0)} + \tau^{3/2} \|e^{\tau \varphi} u\|_{L^2(\omega_0)} \right),
$$

for all  $u, r \in \mathscr{C}_c^\infty(\overline{\Omega})$  which satisfies  $(3.1), u \cdot n|_{\Gamma} = 0, \tau \ge \tau_*,$  $(3.1), u \cdot n|_{\Gamma} = 0, \tau \ge \tau_*,$  and  $\mu$  satisfying  $(3.3)$ .

**Remarks 4.1.** It is classical that there exist  $\psi$  such that  $\partial_n \psi(x) < 0$  for  $x \in \partial \Omega$  and  $\nabla \psi \neq 0$  $for x \in \overline{\Omega} \setminus \omega_1$  (see Fursikov-Imanuvilov [\[12\]](#page-21-22)). From Lemma [3.2,](#page-7-2)  $\varphi = e^{\lambda \psi}$  satisfies sub-ellipticity condition in  $\overline{\Omega} \setminus \omega_1$  for  $\lambda$  sufficiently large. In what follows we fix such a function  $\varphi$ .

<span id="page-19-5"></span>**Proposition 4.1.** Let  $(u, r) \in \mathcal{D}(\mathcal{A})$  solution of  $(\mathcal{A}_d + i\mu)(u, r) = (f, g) \in H$ . Then we have

<span id="page-19-6"></span>(4.1) 
$$
|\mu|||\sqrt{b}u||_0^2 \leq C||(u,r)||_H||(f,g)||_H
$$

$$
|\mu|||r||_{L^2(\omega_0)}^2 \leq C||(u,r)||_H||(f,g)||_H
$$

for some constant  $C > 0$ .

From these two results we are able to prove Proposition [2.1.](#page-4-1)

*Proof of Proposition [2.1.](#page-4-1)* Noting that the resolvent problem  $(A - i\mu)(u, r) = (f, g)$  is written as follow

$$
\begin{cases} \nabla r + i\mu u = f - bu & \text{in } \Omega\\ \n\text{div}(u) + i\mu r = g & \text{in } \Omega\\ \nu.n = 0 & \text{on } \Gamma. \n\end{cases}
$$

This allows us to apply Theorem [4.1.](#page-19-4) So let  $C_2 = \max_{x \in \overline{\Omega}} \varphi(x)$  and  $C_1 = \min_{x \in \overline{\Omega}} \varphi(x)$  we deduce from the Carleman estimate of Theorem [4.1](#page-19-4) that

$$
(4.2) \t\t ||r||_0 + ||u||_0 \lesssim e^{(C_2 - C_1)\tau} (||g||_0 + ||f + bu||_0 + ||r||_{L^2(\omega_0)} + ||u||_{L^2(\omega_0)}).
$$

Taking  $\tau = |\mu|/c_0$  accordingly with [\(3.3\)](#page-6-1), by the estimates of Proposition [4.1](#page-19-5) and as  $||bu||_0 \lesssim$  $\parallel$  $\sqrt{b}u\|_0$ , we have

(4.3) 
$$
\|(u,r)\|_H \lesssim Ce^{K|\mu|} \left( \|(f,g)\|_H + \|(u,r)\|_H^{1/2} \|(f,g)\|_H^{1/2} \right),
$$

which yields  $||(u, r)||_H \lesssim e^{K'|\mu|}||(f, g)||_H$ . This is the sought result.

*Proof of Proposition [4.1.](#page-19-5)* From equation, we have  $-\nabla r + i\mu u - bu = f$  taking the inner product with u, we obtain  $-(\nabla r, u) + i\mu \|u\|^2 - (bu, u) = (f, u)$ . Integrating by parts, we have  $-(\nabla r, u) =$  $(r, \text{div } u)$  as  $u \cdot n = 0$  on  $\partial \Omega$ . Using the second equation  $-\text{div } u + i\mu r = g$ , we have  $-(\nabla r, u) =$  $(r, i\mu r - q)$ . We thus obtain

$$
-i\mu||r||^2 - (r,g) + i\mu||u||^2 - (bu, u) = (f, u).
$$

Taking the real part of this equation we have  $|\mu|||\sqrt{b}u||^2 \leq |(f, u)| + |(r, g)|$ . This implies the first estimate of  $(4.1)$ .

Let  $\chi \in \mathscr{C}_c^{\infty}(\mathbb{R}^d)$  such that  $\chi(x) = 1$  for  $x \in \omega_0$  and  $\chi$  supported in  $\omega$ . Taking the inner product between  $-\text{div } u + i\mu r = g$  and  $\chi^2 r$ , we obtain  $(-\text{div } u, \chi^2 r) + i\mu ||\chi r||^2 = (g, \chi^2 r)$ . Integrating by parts we have  $(-\text{div } u, \chi^2 r) = (u, \chi^2 \nabla r) + (u, 2\chi r \nabla \chi)$  and by equation  $-\nabla r +$  $i\mu u - bu = f$  we have

$$
-i\mu|| \chi u||^2 - (u, \chi^2 bu) - (u, \chi^2 f) + (u, 2\chi r \nabla \chi) + i\mu||\chi r||^2 = (g, \chi^2 r).
$$

Taking account that  $b \geq b_-$  in  $\omega$ , thus on the support of  $\chi$ , we have

$$
|\mu| \| \chi r \|^2 \lesssim \| (u, r) \|_H \| (f, g) \|_H + \| u \nabla \chi \| \| \chi r \| + |\mu| \| \chi u \|^2 + \| \sqrt{b} u \|^2.
$$

We can estimate  $||u\nabla \chi||$  and  $|| \chi u||$  by  $||\sqrt{b}u||$  and by the first estimate of Proposition [4.1](#page-19-5) we obtain the second estimate.

*Proof of Theorem [4.1.](#page-19-4)* Let  $x_0 \in \overline{\Omega} \setminus \omega_1$ , from Corollary [3.1](#page-9-4) if  $x_0 \in \Omega$  or from Theorem [3.9](#page-11-3) if  $x_0 \in \partial\Omega$  we obtain, in both cases, an open neighborhood (in  $\mathbb{R}^d$ ) of  $x_0$ , V such that

(4.4) 
$$
\tau^{3/2} \|e^{\tau \varphi} r\|_0 + \tau^{3/2} \|e^{\tau \varphi} u\|_0 \leq C \left( \tau \|e^{\tau \varphi} g\|_0 + \tau \|e^{\tau \varphi} f\|_0 \right),
$$

for  $u, r \in \mathscr{C}_c^{\infty}(V)$ . By compactness of  $\overline{\Omega} \setminus \omega_1$  we can find a finite recovering  $(V_j)_{j \in J}$  of  $\overline{\Omega} \setminus \omega_1$ . Let  $(\chi_j)_{j\in J}$  be a partition of unity subordinated to  $(V_j)_{j\in J}$  such that  $\sum_{j\in J} \chi_j(x) = 1$  for  $x \in \overline{\Omega} \setminus \omega_1$ . Let  $u_j = \chi_j u$  and  $r_j = \chi_j r$  where  $(u, r)$  solution to  $(3.1), u \cdot n|_{\Gamma} = 0$ . We have

<span id="page-20-0"></span>
$$
-\nabla r_j + i\mu u_j = \chi_j f - r\nabla \chi_j
$$
  
- div  $u_j + i\mu r_j = \chi_j g - u \cdot \nabla \chi_j$ 

We can apply the Carleman estimate  $(4.4)$  in each  $V_j$  and we obtain

$$
\tau^{3/2} \|e^{\tau \varphi} r_j\|_0 + \tau^{3/2} \|e^{\tau \varphi} u_j\|_0 \lesssim \tau \|e^{\tau \varphi} (\chi_j g - u \cdot \nabla \chi_j) \|_0 + \tau \|e^{\tau \varphi} (\chi_j f - r \nabla \chi_j) \|_0
$$
  

$$
\lesssim \tau \|e^{\tau \varphi} g\|_0 + \tau \|e^{\tau \varphi} u\|_0 + \tau \|e^{\tau \varphi} f\|_0 + \tau \|e^{\tau \varphi} r\|_0.
$$

We have

$$
\tau^{3/2} \|e^{\tau \varphi} r\|_{0} + \tau^{3/2} \|e^{\tau \varphi} u\|_{0}
$$
  
\n
$$
\lesssim \tau^{3/2} \sum_{j \in J} ( \|e^{\tau \varphi} r_{j}\|_{0} + \|e^{\tau \varphi} u_{j}\|_{0}) + \tau^{3/2} \|e^{\tau \varphi} r\|_{L^{2}(\omega_{0})} + \tau^{3/2} \|e^{\tau \varphi} u\|_{L^{2}(\omega_{0})}
$$
  
\n
$$
\lesssim \tau \|e^{\tau \varphi} g\|_{0} + \tau \|e^{\tau \varphi} u\|_{0} + \tau \|e^{\tau \varphi} f\|_{0} + \tau \|e^{\tau \varphi} r\|_{0} + \tau^{3/2} \|e^{\tau \varphi} r\|_{L^{2}(\omega_{0})} + \tau^{3/2} \|e^{\tau \varphi} u\|_{L^{2}(\omega_{0})}.
$$

This gives the sought result as we can absorb the term  $\tau ||e^{\tau \varphi}u||_0 + \tau ||e^{\tau \varphi}r||_0$  with the left hand side.  $\Box$ 

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