Subdifferentiation of nonconvex sparsity-promoting functionals on Lebesgue spaces

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July 21, 2021

Sparsity-promoting terms are incorporated into the objective functions of optimal control problems in order to ensure that optimal controls vanish on large parts of the underlying domain. Typical candidates for those terms are integral functions on Lebesgue spaces based on the ℓ_p-metric for $p \in [0, 1)$ which are nonconvex as well as non-Lipschitz and, thus, variationally challenging. In this paper, we derive exact formulas for the Fréchet, limiting, and singular subdifferential of these functionals. These generalized derivatives can be used for the derivation of necessary optimality conditions for optimal control problems comprising such sparsity-promoting terms.

Keywords: Integral functionals, Sparsity-promoting functionals, Subdifferentiation, Variational analysis
MSC: 28B05, 49J52, 49J53, 49K27

1 Introduction

For a measurable and bounded set $\Omega \subset \mathbb{R}^d$ as well as real numbers $s \in [1, \infty)$ and $p \in [0, 1)$, we investigate the functional $q_{s,p} \colon L^s(\Omega) \to \mathbb{R}$ given by

$$\forall u \in L^s(\Omega)\colon \quad q_{s,p}(u) := \int_\Omega |u(x)|^p \,\mathrm{d}x \tag{1.1}$$

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for $p \in (0,1)$ and by

$$\forall u \in L^s(\Omega): \quad q_{s,0}(u) := \int_\Omega |u(x)|_0 \,\mathrm{d}x \tag{1.2}$$

for $p = 0$ where we used the mapping $|\cdot|_0 : \mathbb{R} \to \mathbb{R}$ defined as follows:

$$\forall y \in \mathbb{R}: \quad |y|_0 := \begin{cases} 0 & \text{if } y = 0, \\ 1 & \text{otherwise.} \end{cases}$$

We note that $q_{s,0}(u) = \boldsymbol{\lambda}(\{u \neq 0\})$ holds for all $u \in L^s(\Omega)$, i.e., $q_{s,0}$ measures the size of the *support* of its argument. In optimal control, the functional $q_{s,p}$ is popular due to its property to be *sparsity-promoting*, see e.g. Casas and Wachsmuth (2020); Ito and Kunisch (2014); Merino (2019); Natemeyer and Wachsmuth (2020); Wachsmuth (2019), i.e., to enforce control functions to be zero on large parts of their domain. This property of $q_{s,p}$ is induced by the fact that the mappings $y \mapsto |y|^p$, $p \in (0,1)$, and $y \mapsto |y|_0$ possess a uniquely determined global minimizer as well as infinite growth at zero. Let us underline that the case $p = 1$, in which the associated mapping $q_{s,1}$ given as in (1.1) reduces to the (convex) norm of the space $L^1(\Omega)$, is well-studied in the literature, see e.g. Cases et al. (2012); Stadler (2009); Vossen and Maurer (2006); Wachsmuth and Wachsmuth (2011). Clearly, $q_{s,p}$ is not convex for $p \in [0,1)$.

This paper is devoted to the computation of generalized derivatives of the mapping $q_{s,p}$. More precisely, we aim for the derivation of exact formulas for its so-called Fréchet, limiting, and singular subdifferential, see Mordukhovich (2006), which can be used in order to characterize local minimizers of optimal control problems involving $q_{s,p}$ within the objective function. The investigation of calculus rules for subdifferentials of nonconvex integral functions on Lebesgue spaces has been an active topic of research throughout the last decades. Exemplary, we would like to mention Clarke (1983); Giner (2017); Giner and Penot (2018); Mordukhovich and Sagara (2018) where calculus rules were established in situations where the integrand satisfies Lipschitzianity assumptions. We note, however, that these assumptions typically do *not* hold for the integrands of our interest. In Correa et al. (2020), some upper estimates for the subdifferentials of integral functions with potentially non-Lipschitzian integrand have been obtained. Finally, we would like to mention the papers Chieu (2009); Penot (2011) where some emphasis is laid on nonconvex integral functions on $L^1(\Omega)$ without assuming any Lipschitzianity of the integrand. However, as far as we can see, the available results from the literature are of limited practical use for the actual computation of the subdifferentials associated with $q_{s,p}$. That is why we directly compute the subdifferentials of interest from their respective definition. Therefore, we distinguish the cases $p = 0$ and $p \in (0,1)$ where slightly different arguments are necessary in order to proceed.

The remainder of the paper is organized as follows: In Section 2, we comment on the basic notation used in this paper and put some special emphasis on Lebesgue spaces and the underlying tools of variational analysis. Furthermore, we briefly investigate the continuity properties of $q_{s,p}$. Finally, we introduce and study the concept of *slowly decreasing functions* on Lebesgue spaces which will be used to characterize the points where the Fréchet subdifferential of $q_{s,0}$ is nonempty. In Section 3, we first compute the Fréchet subdifferential of $q_{s,0}$ and then turn our attention to the characterization of the limiting and singular subdifferential of this functional. In a similar way, we proceed in Section 4 in order to address the functional $q_{s,p}$ for $p \in (0,1)$. Some concluding remarks close the paper in Section 5.

2 Preliminaries

2.1 Basic notation

For a sequence $\{z_k\}_{k\in\mathbb{N}} \subset Z$ in a real Banach space Z and some point $z \in Z$, we exploit $z_k \to z$ ($z_k \rightharpoonup z$) in order to denote that $\{z_k\}_{k\in\mathbb{N}}$ converges strongly (weakly) to z. Furthermore, for a sequence $\{t_k\}_{k\in\mathbb{N}} \subset \mathbb{R}$ and $\alpha \in \mathbb{R}$, $t_k \searrow \alpha$ means $\{t_k\}_{k\in\mathbb{N}} \subset (\alpha,\infty)$ and $t_k \to \alpha$. Similarly, we interpret $t_k \nearrow \alpha$.

Whenever a function $f\colon Z \to \mathbb{R}$ is Fréchet differentiable at $z \in Z$, its Fréchet derivative will be denoted by $f'(z) \in Z^*$ where Z^* is the (topological) dual of Z.

2.2 Lebesgue spaces

Throughout this paper, we assume that $\Omega \subset \mathbb{R}^d$ is Lebesgue-measurable with positive and finite Lebesgue measure. We equip Ω with the σ-algebra of all Lebesgue-measurable subsets of Ω as well as Lebesgue's measure $\boldsymbol{\lambda}$. For brevity, we will suppress the prefix *Lebesgue* in the course of the manuscript. The characteristic function of a measurable set $A \subset \Omega$, being 1 on A while vanishing on $\Omega \setminus A$, will be denoted by $\chi_A\colon \Omega \to \{0,1\}$. Whenever $u\colon \Omega \to \mathbb{R}$ is measurable, the associated function $\operatorname{sgn} u\colon \Omega \to \{-1,0,1\}$, which assigns to each $x \in \Omega$ the sign of $u(x)$, is measurable as well. For $s \in [1,\infty)$, we use the classical Lebesgue spaces $L^s(\Omega)$ of all (equivalence classes of) real-valued, measurable, s-integrable functions which are equipped with the classical norm

$$\forall u \in L^s(\Omega)\colon \qquad \|u\|_s := \left(\int_\Omega |u(x)|^s \,\mathrm{d}x\right)^{1/s}.$$

For the purpose of completeness, let us recall that $L^\infty(\Omega)$ comprises all (equivalence classes of) real-valued, measurable functions which are essentially bounded, and that this set becomes a Banach space when equipped with the norm

$$\forall u \in L^\infty(\Omega)\colon \qquad \|u\|_\infty := \operatorname*{ess\,sup}_{x\in\Omega} |u(x)|.$$

Fixing $s \in [1,\infty]$, for each function $u \in L^s(\Omega)$, we use

$$\{u \neq 0\} := \{x \in \Omega \,|\, u(x) \neq 0\} \qquad \text{and} \qquad \{u = 0\} := \{x \in \Omega \,|\, u(x) = 0\}$$

for brevity of notation and note that these sets are measurable and well-defined up to sets of measure zero. Similarly, we define the measurable sets $\{u \geq 0\}$, $\{u > 0\}$, $\{u \leq 0\}$, and $\{u < 0\}$ as well as sets with non-zero or bilateral bounds. Throughout the paper,

$r \in (1, \infty]$ given by $\frac{1}{s} + \frac{1}{r} = 1$ is the conjugate coefficient associated with $s \in [1, \infty)$. It is well known that $L^r(\Omega)$ can be identified with the (topological) dual space of $L^s(\Omega)$. Finally, if $\Omega' \subset \Omega$ is measurable, we exploit the notation

$$\forall u \in L^s(\Omega),\, \forall \nu \in (0, s]\colon \quad \|u\|_{\nu,\Omega'} := \left(\int_{\Omega'} |u(x)|^\nu \,\mathrm{d}x \right)^{1/\nu}$$

and note that $\|u\|_{\nu,\Omega'}$ is a finite number for each $u \in L^s(\Omega)$ and each $\nu \in (0, s]$.

We briefly mention that $\Omega \subset \mathbb{R}^d$ can be replaced by an arbitrary measure space (Ω, Σ, μ) such that μ is finite, separable, and non-atomic. Indeed, such a measure space is isomorphic to the Lebesgue measure on some interval, see, e.g., (Bogachev, 2007, Theorem 9.3.4).

2.3 Tools from variational analysis

In this paper, we are concerned with the computation of subdifferentials associated with the functional $q_{s,p}$ with $s \in [1, \infty)$ and $p \in [0, 1)$. Recall that, for a given point $\bar{u} \in L^s(\Omega)$, the so-called Fréchet (or regular) subdifferential of $q_{s,p}$ at $\bar{u}$ is defined by means of

$$\widehat{\partial} q_{s,p}(\bar{u}) := \left\{ \eta \in L^r(\Omega) \,\middle|\, \liminf_{\|h\|_s \searrow 0} \frac{q_{s,p}(\bar{u} + h) - q_{s,p}(\bar{u}) - \int_\Omega \eta(x) h(x) \,\mathrm{d}x}{\|h\|_s} \geq 0 \right\}.$$

Furthermore, in case $s > 1$, the limiting (or Mordukhovich) and the singular subdifferential of $q_{s,p}$ at $\bar{u}$ are defined as stated below:

$$\partial q_{s,p}(\bar{u}) := \left\{ \eta \in L^r(\Omega) \,\middle|\, \begin{array}{l} \exists \{u_k\}_{k\in\mathbb{N}} \subset L^s(\Omega),\, \exists \{\eta_k\}_{k\in\mathbb{N}} \subset L^r(\Omega)\colon \\ \quad u_k \to \bar{u} \text{ in } L^s(\Omega),\, q_{s,p}(u_k) \to q_{s,p}(\bar{u}), \\ \quad \eta_k \rightharpoonup \eta \text{ in } L^r(\Omega), \eta_k \in \widehat{\partial} q_{s,p}(u_k)\, \forall k \in \mathbb{N} \end{array} \right\},$$

$$\partial^\infty q_{s,p}(\bar{u}) := \left\{ \eta \in L^r(\Omega) \,\middle|\, \begin{array}{l} \exists \{u_k\}_{k\in\mathbb{N}} \subset L^s(\Omega),\, \exists \{t_k\}_{k\in\mathbb{N}} \subset (0, \infty), \\ \exists \{\eta_k\}_{k\in\mathbb{N}} \subset L^r(\Omega)\colon \\ \quad u_k \to \bar{u} \text{ in } L^s(\Omega),\, q_{s,p}(u_k) \to q_{s,p}(\bar{u}),\, t_k \searrow 0, \\ \quad t_k\eta_k \rightharpoonup \eta \text{ in } L^r(\Omega),\, \eta_k \in \widehat{\partial} q_{s,p}(u_k)\, \forall k \in \mathbb{N} \end{array} \right\}.$$

Noting that $L^1(\Omega)$ is not a so-called Asplund space, i.e., a Banach space where every convex, continuous functional is *generically* Fréchet differentiable, one cannot simply define the limiting and singular subdifferential of $q_{1,p}$ as a set-limit of the associated Fréchet subdifferential while preserving its variational properties. Instead, the larger so-called ε-subdifferential of $q_{1,p}$ would be needed within the limiting procedure. We would like to point out that working with the limiting variational tools in spaces which do not possess the Asplund property has been shown to be problematic. A detailed discussion can be found in (Mordukhovich, 2006, Section 2.2). Nevertheless, due to (Chieu, 2009, Theorem 3.2), we have $\partial q_{1,p}(\bar{u}) = \widehat{\partial} q_{1,p}(\bar{u})$ for all $\bar{u} \in L^1(\Omega)$ and all $p \in [0, 1)$ (with respect to the correct definition of the limiting subdifferential in non-Asplund spaces), and, thus, our results from Theorems 3.3 and 4.3 yield explicit formulas for the limiting subdifferential of $q_{1,p}$ as well. In variational analysis, the limiting subdifferential has turned out to be a valuable tool for the derivation of necessary optimality conditions for constrained optimization problems, see (Mordukhovich, 2006, Section 5). On the other hand, the singular subdifferential provides a measure of Lipschitzianity of nonsmooth functionals. Applied in the context of this paper, thanks to (Mordukhovich, 2006, Corollary 2.39, Theorem 3.52) and the continuity properties of $q_{s,p}$, see Section 2.4, we have the following result.

Lemma 2.1. *Fix $s \in (1, \infty)$ and $p \in [0, 1)$. Then $q_{s,p}$ is Lipschitz continuous at some point $\bar{u} \in L^s(\Omega)$ if and only if the following conditions hold:*

(a) we have $\partial^\infty q_{s,p}(\bar{u}) = \{0\}$ and

(b) for sequences $\{u_k\}_{k\in\mathbb{N}} \subset L^s(\Omega)$, $\{t_k\}_{k\in\mathbb{N}} \subset (0, \infty)$, and $\{\eta_k\}_{k\in\mathbb{N}} \subset L^r(\Omega)$ satisfying $u_k \to \bar{u}$ in $L^s(\Omega)$, $q_{s,p}(u_k) \to q_{s,p}(\bar{u})$, $t_k \searrow 0$, $t_k\eta_k \rightharpoonup 0$ in $L^r(\Omega)$, and $\eta_k \in \widehat{\partial} q_{s,p}(u_k)$ for each $k \in \mathbb{N}$, we already have $t_k\eta_k \to 0$ in $L^r(\Omega)$.

Let us note that the property from Lemma 2.1 (b) is referred to as *sequential normal epi-compactness* of $q_{s,p}$ at $\bar{u}$ in variational analysis. The latter is related to the so-called *sequential normal compactness* property of sets which has been shown to be problematic in Lebesgue spaces, see (Mehlitz, 2019, Section 4).

2.4 Continuity properties of sparsity-promoting functionals

In this section, we briefly comment on the continuity properties of the sparsity-promoting functional $q_{s,p}$ for $s \in [1, \infty)$ and $p \in [0, 1)$. We split our investigation into two lemmas.

Lemma 2.2. *For each $s \in [1, \infty)$, $q_{s,0}$ is lower semicontinuous.*

Proof. Fix a sequence $\{u_k\}_{k\in\mathbb{N}} \subset L^s(\Omega)$ converging to some $\bar{u} \in L^s(\Omega)$. For subsequent use, we set $\alpha := \liminf_{k\to\infty} q_{s,0}(u_k)$. We pick a subsequence (without relabeling) with $q_{s,0}(u_k) \to \alpha$ and assume without loss of generality (w.l.o.g.) that $\{u_k\}_{k\in\mathbb{N}}$ converges pointwise almost everywhere to $\bar{u}$ along this subsequence. Noting that $y \mapsto |y|_0$ is lower semicontinuous, we find

$$\begin{aligned} \alpha = \lim_{k\to\infty} q_{s,0}(u_k) &= \lim_{k\to\infty} \int_\Omega |u_k(x)|_0 \,\mathrm{d}x \\ &\geq \int_\Omega \left(\liminf_{k\to\infty} |u_k(x)|_0 \right) \mathrm{d}x \geq \int_\Omega |\bar{u}(x)|_0 \,\mathrm{d}x = q_{s,0}(\bar{u}) \end{aligned}$$

from Fatou's lemma, and this shows the claim. □

It is also clear that $q_{s,0}$ is not continuous at points $\bar{u} \in L^s(\Omega)$ with $\boldsymbol{\lambda}(\{\bar{u} = 0\}) > 0$.

Lemma 2.3. *For each $s \in [1, \infty)$ and $p \in (0, 1)$, $q_{s,p}$ is uniformly continuous.*

Proof. Noting that $\varphi\colon \mathbb{R} \to \mathbb{R}$ given by $\varphi(y) := |y|^p$ for all $y \in \mathbb{R}$ is continuous and that the associated Nemytskii operator maps $L^s(\Omega)$ into $L^1(\Omega)$, the latter is continuous due to (Goldberg et al., 1992, Theorem 4). Furthermore, integration of functions is a linear, continuous operation on $L^1(\Omega)$. Thus, $q_{s,p}$ is the composition of two continuous mappings and, thus, continuous.

We note that φ is subadditive, i.e., $|y_1 + y_2|^p \leq |y_1|^p + |y_2|^p$ holds for all $y_1, y_2 \in \mathbb{R}$. Applying this inequality first to $y_1 := z_1 - z_2$ as well as $y_2 := z_2$ and second to $y_1 := z_2 - z_1$ as well as $y_2 := z_1$ for $z_1, z_2 \in \mathbb{R}$ yields the estimate

$$\forall z_1, z_2 \in \mathbb{R}\colon \quad \big||z_1|^p - |z_2|^p\big| \leq |z_1 - z_2|^p.$$

Fix an arbitrary $\varepsilon > 0$. By continuity of $q_{s,p}$ at 0, we find $\delta > 0$ such that $q_{s,p}(u) < \varepsilon$ holds for all $u \in L^s(\Omega)$ such that $\|u\|_s < \delta$. Thus, for any two functions $u, v \in L^s(\Omega)$ satisfying $\|u - v\|_s < \delta$, we find

$$\begin{aligned} |q_{s,p}(u) - q_{s,p}(v)| &= \left| \int_\Omega \big(|u(x)|^p - |v(x)|^p\big)\,\mathrm{d}x \right| \leq \int_\Omega \big||u(x)|^p - |v(x)|^p\big|\,\mathrm{d}x \\ &\leq \int_\Omega |u(x) - v(x)|^p\,\mathrm{d}x = q_{s,p}(u - v) < \varepsilon, \end{aligned}$$

and this yields uniform continuity of $q_{s,p}$. ∎

The inherent nonconvexity of the functional $q_{s,p}$ indicates that it is not weakly lower semicontinuous. Thus, one has to face essential issues regarding the existence of solutions whenever $q_{s,p}$ is used as an additional sparsity-promoting term in the objective function of an optimal control problem where the controls are chosen from a Lebesgue space, see Ito and Kunisch (2014); Wachsmuth (2019) where this is discussed in detail.

2.5 Slowly decreasing functions

Fix $s \in (1, \infty)$. In the course of the paper, it will become clear that the Fréchet subdifferential of $q_{s,0}$ at some point $\bar{u} \in L^s(\Omega)$ such that $\{\bar{u} \neq 0\}$ is of positive measure is likely to be empty if $\bar{u}$ approaches zero on $\{\bar{u} \neq 0\}$ too fast. In this regard, the following definition aims to characterize functions tending to zero slowly enough on their support.

Definition 2.4. *Fix $s \in (1, \infty)$ as well as a function $\bar{u} \in L^s(\Omega)$. We call $\bar{u}$* order s slowly decreasing *(an* s-SD function *for short) whenever for each sequence $\{\Omega_k\}_{k \in \mathbb{N}}$ of measurable subsets of $\{\bar{u} \neq 0\}$, we have*

$$\boldsymbol{\lambda}(\Omega_k) \searrow 0 \qquad \Longrightarrow \qquad \frac{\boldsymbol{\lambda}(\Omega_k)}{\|\bar{u}\|_{s,\Omega_k}} \searrow 0.$$

Note that whenever $\bar{u} \in L^s(\Omega)$ vanishes almost everywhere on Ω, then it is trivially s-SD since there are no measurable subsets Ω_k of $\{\bar{u} \neq 0\}$ with positive measure.

In the subsequently stated lemma, we present a sufficient condition implying that a given function $\bar{u} \in L^s(\Omega)$ is an s-SD function. Its proof follows straight from the definition and is, therefore, omitted.

Lemma 2.5. *Fix $s \in (1, \infty)$ and $\bar{u} \in L^s(\Omega)$. Assume that $\bar{u}$ is bounded away from zero on $\{\bar{u} \neq 0\}$, i.e., that one can find $\varepsilon > 0$ such that $\boldsymbol{\lambda}(\{0 < |\bar{u}| < \varepsilon\}) = 0$ is valid. Then $\bar{u}$ is an s-SD function.*

The following example shows that the condition from Lemma 2.5 is only sufficient but not necessary for the property of $\bar{u} \in L^s(\Omega)$ to be an s-SD function.

Example 2.6. *Consider $\Omega := (0, 1)$ and the function $\bar{u} \in L^s(\Omega)$, $s > 1$, given by $\bar{u}(x) := x^\alpha$ for each $x \in \Omega$ and some $\alpha > 0$. We want to check for which choices of s and α, $\bar{u}$ actually is an s-SD function. Considering sets $\Omega_k \subset \Omega$ of positive measure such that $\boldsymbol{\lambda}(\Omega_k)$ is fixed, the quotient $\boldsymbol{\lambda}(\Omega_k)/\|\bar{u}\|_{s,\Omega_k}$ gets maximal whenever $\bar{u}$ is as small as possible on Ω_k. Thus, by strict monotonicity of $\bar{u}$, it suffices to consider sequences of sets $\{\Omega_k\}_{k\in\mathbb{N}}$ of the form $\Omega_k := (0, t_k)$ where $\{t_k\}_{k\in\mathbb{N}}$ satisfies $t_k \searrow 0$. In this case, we obtain*

$$\frac{\boldsymbol{\lambda}(\Omega_k)}{\|\bar{u}\|_{s,\Omega_k}} = (\alpha s + 1)^{1/s}\, t_k^{1-\alpha-1/s},$$

and this shows that $\bar{u}$ is an s-SD function if and only if $\alpha + 1/s < 1$ holds true. Particularly, $\alpha < 1$ is necessary, and, in this case, $\bar{u}$ tends to 0 quite slowly. Observe that $\alpha < 1 - 1/s$ is equivalent to $|\bar{u}|^{-1} \in L^r(\Omega)$. Recall that $r \in (1, \infty)$ is the conjugate coefficient associated with s.

In the remainder of the section, we aim to find a more tractable characterization of s-SD functions. The above example motivates the subsequently stated lemma.

Lemma 2.7. *Let $s \in (1, \infty)$ be given. Furthermore, fix a function $\bar{u} \in L^s(\Omega)$ with $|\bar{u}|^{-1}\chi_{\{\bar{u}\neq 0\}} \in L^r(\Omega)$. Then $\bar{u}$ is an s-SD function.*

Proof. Pick an arbitrary sequence $\{\Omega_k\}_{k\in\mathbb{N}}$ of measurable subsets of $\{\bar{u} \neq 0\}$ satisfying $\boldsymbol{\lambda}(\Omega_k) \searrow 0$. For each $k \in \mathbb{N}$, we find

$$\boldsymbol{\lambda}(\Omega_k) = \int_{\Omega_k} |\bar{u}(x)|\, |\bar{u}(x)|^{-1}\, \mathrm{d}x \leq \|\bar{u}\|_{s,\Omega_k} \|\, |\bar{u}|^{-1}\, \|_{r,\Omega_k}$$

by applying Hölder's inequality on Ω_k. From $|\bar{u}|^{-1}\chi_{\{\bar{u}\neq 0\}} \in L^r(\Omega)$ we find the convergence $\|\, |\bar{u}|^{-1}\, \|_{r,\Omega_k} \searrow 0$ since $\boldsymbol{\lambda}(\Omega_k) \searrow 0$. Hence, $\bar{u}$ is an s-SD function. $\square$

Note that the requirements of Lemma 2.5 are sufficient for the ones of Lemma 2.7. The next example shows the existence of s-SD functions $\bar{u}$ for which $|\bar{u}|^{-1}\chi_{\{\bar{u}\neq 0\}} \notin L^r(\Omega)$ holds.

Example 2.8. *Again we consider $\Omega := (0, 1)$. Let $s \in (1, \infty)$ be arbitrary. For each $k \in \mathbb{N}$, we define $t_k := 2^{-k}$. For some monotonically decreasing sequence $\{\gamma_k\}_{k\in\mathbb{N}} \subset (0, \infty)$ satisfying $\sum_{j=1}^{\infty} \gamma_j^s 2^{-j} < \infty$, we consider the (monotonically increasing) function*

$$\bar{u} := \sum_{j=1}^{\infty} \gamma_j \chi_{(t_{j+1}, t_j]} \in L^s(\Omega).$$

Furthermore, we make use of $\Omega_k := (0, t_k)$ for each $k \in \mathbb{N}$ and note that $\boldsymbol{\lambda}(\Omega_k) = t_k$ is valid. We obtain the estimates

$$\gamma_k^s t_k \geq \|\bar{u}\|_{s,\Omega_k}^s = \sum_{j=k}^{\infty} \gamma_j^s (t_j - t_{j+1}) \geq \gamma_k^s (t_k - t_{k+1}) = 2^{-1}\gamma_k^s t_k,$$

$$\gamma_k^{-1} t_k^{1-1/s} \leq \frac{\boldsymbol{\lambda}(\Omega_k)}{\|\bar{u}\|_{s,\Omega_k}} \leq 2^{1/s}\gamma_k^{-1} t_k^{1-1/s}.$$

This shows that $\bar{u}$ is s-SD only if $\gamma_k^{-r} t_k \to 0$. Actually, this is already sufficient for the s-SD property. Indeed, by monotonicity of $\bar{u}$, is suffices to consider sequences $\{\Omega_k'\}_{k\in\mathbb{N}}$ of type $\Omega_k' := (0, t_k')$ for sequences $\{t_k'\}_{k\in\mathbb{N}} \subset (0,1)$ satisfying $t_k' \searrow 0$. Then, for each $k \in \mathbb{N}$, we find $\ell_k \in \mathbb{N}$ such that $t_{\ell_k+1} \leq t_k' < t_{\ell_k}$ leading to $\ell_k \to \infty$, $\boldsymbol{\lambda}(\Omega_k') < t_{\ell_k} = 2t_{\ell_k+1}$, and

$$\|\bar{u}\|_{s,\Omega_k'}^s = \sum_{j=\ell_k+1}^{\infty} \gamma_j^s (t_j - t_{j+1}) + \gamma_{\ell_k}^s (t_k' - t_{\ell_k+1}) \geq 2^{-1}\gamma_{\ell_k+1}^s t_{\ell_k+1}.$$

This yields

$$\frac{\boldsymbol{\lambda}(\Omega_k')}{\|\bar{u}\|_{s,\Omega_k'}} \leq \frac{2t_{\ell_k+1}}{2^{-1/s}\gamma_{\ell_k+1} t_{\ell_k+1}^{1/s}} = 2^{1+1/s}\gamma_{\ell_k+1}^{-1} t_{\ell_k+1}^{1-1/s} \to 0$$

which shows validity of the s-SD property if $\gamma_k^{-r} t_k \to 0$ is valid.

In particular, choosing $\gamma_k := (k2^{-k})^{1/r}$ for each $k \in \mathbb{N}$, the function $\bar{u}$ is s-SD, but

$$\|\,|\bar{u}|^{-1}\,\|_r^r = \sum_{j=1}^{\infty} \gamma_j^{-r}(t_j - t_{j+1}) = \sum_{j=1}^{\infty} j^{-1} 2^j 2^{-j-1} = \infty,$$

i.e., the sufficient condition from Lemma 2.7 does not hold.

In Example 2.8, the coupling between t_k and γ_k is decisive for the s-SD property. This will be made more precise in Theorem 2.10. For the proof of it, we need an auxiliary lemma, which provides an intermediate value theorem for monotonic functions.

Lemma 2.9. *Let $g\colon [0,\infty) \to [0,\infty)$ be monotonically increasing and not identically 0. Moreover, let $\alpha > 0$ be fixed. Then, for each $C > 0$, there exists a unique $\gamma_C > 0$ such that*

$$\lim_{\gamma \nearrow \gamma_C} g(\gamma) \leq \frac{C}{\gamma_C^\alpha} \leq \lim_{\gamma \searrow \gamma_C} g(\gamma).$$

If $C \searrow 0$ we have $\gamma_C \searrow 0$.

Proof. Observing that $\gamma \mapsto \gamma^\alpha g(\gamma)$ is monotonically increasing on $[0,\infty)$, one can check that

$$\gamma_C := \inf\{\gamma \in (0,\infty) \mid g(\gamma) \geq C\gamma^{-\alpha}\} = \sup\{\gamma \in (0,\infty) \mid g(\gamma) < C\gamma^{-\alpha}\}$$

satisfies the desired inequalities. The uniqueness follows since $\gamma \mapsto C\gamma^{-\alpha}$ is strictly monotonically decreasing. Moreover, if $\varepsilon > 0$ is arbitrary and $C \leq \varepsilon^\alpha g(\varepsilon)$ holds then ε belongs to the set under the infimum and, therefore, $\gamma_C \leq \varepsilon$ follows. $\square$

Theorem 2.10. *Let $s \in (1,\infty)$ be given. Then $\bar{u} \in L^s(\Omega)$ is an s-SD function if and only if*

$$\lim_{\gamma \searrow 0} \boldsymbol{\lambda}(\{0 < |\bar{u}| \le \gamma\})\gamma^{-r} = 0.$$

Proof. "$\Longrightarrow$": For an arbitrary sequence $\{\gamma_k\}_{k\in\mathbb{N}} \subset (0,\infty)$ with $\gamma_k \searrow 0$, we set $\Omega_k := \{0 < |\bar{u}| \le \gamma_k\}$ for each $k \in \mathbb{N}$. Exploiting $\bigcap_{k\in\mathbb{N}} \Omega_k = \varnothing$, one can easily check that $\boldsymbol{\lambda}(\Omega_k) \to 0$ is valid. In case where $\{\boldsymbol{\lambda}(\Omega_k)\}_{k\in\mathbb{N}}$ vanishes along the tail of the sequence, we have nothing to show. Thus, let us assume $\boldsymbol{\lambda}(\Omega_k) \searrow 0$. Then we have

$$0 \le \lim_{k\to\infty} \boldsymbol{\lambda}(\Omega_k)^{1/r}\gamma_k^{-1} = \lim_{k\to\infty} \boldsymbol{\lambda}(\Omega_k)^{1-1/s}\gamma_k^{-1} \le \lim_{k\to\infty} \frac{\boldsymbol{\lambda}(\Omega_k)}{\|\bar{u}\|_{s,\Omega_k}} = 0$$

by definition of an s-SD function.

"$\Longleftarrow$": Let $\{\Omega_k\}_{k\in\mathbb{N}}$ be a sequence of measurable subsets of $\{\bar{u} \ne 0\}$ with $\boldsymbol{\lambda}(\Omega_k) \searrow 0$. For an arbitrary $\gamma > 0$ and $k \in \mathbb{N}$, we have

$$\Omega_k = \{x \in \Omega_k \mid |\bar{u}| \ge \gamma\} \cup \{x \in \Omega_k \mid |\bar{u}| < \gamma\}.$$

Using Chebyshev's inequality, we get

$$\boldsymbol{\lambda}(\Omega_k) \le \frac{\|\bar{u}\|_{s,\Omega_k}^s}{\gamma^s} + \boldsymbol{\lambda}(\{0 < |\bar{u}| < \gamma\}). \tag{2.1}$$

In order to equilibrate the addends on the right-hand side, we apply Lemma 2.9 with $g(t) := \boldsymbol{\lambda}(\{0 < |\bar{u}| < t\})$ and $\alpha := s$ in order to obtain $\gamma_k > 0$ such that

$$\boldsymbol{\lambda}(\{0 < |\bar{u}| < \gamma_k\}) \le \frac{\|\bar{u}\|_{s,\Omega_k}^s}{\gamma_k^s} \le \boldsymbol{\lambda}(\{0 < |\bar{u}| \le \gamma_k\}).$$

Due to $\|\bar{u}\|_{s,\Omega_k} \searrow 0$, Lemma 2.9 guarantees $\gamma_k \searrow 0$. From (2.1), we infer

$$\boldsymbol{\lambda}(\Omega_k) \le 2\|\bar{u}\|_{s,\Omega_k}^s \gamma_k^{-s} \qquad \text{and} \qquad \boldsymbol{\lambda}(\Omega_k) \le 2\boldsymbol{\lambda}(\{0 < |\bar{u}| \le \gamma_k\}).$$

We raise these two inequalities to the powers $r/(r+s)$ and $s/(r+s)$, respectively, and multiply them to obtain

$$\boldsymbol{\lambda}(\Omega_k) \le 2\boldsymbol{\lambda}(\{0 < |\bar{u}| \le \gamma_k\})^{s/(r+s)}\gamma_k^{-rs/(r+s)}\|\bar{u}\|_{s,\Omega_k}^{rs/(r+s)}.$$

Using $rs/(r+s) = 1$ and $\gamma_k \searrow 0$, we get

$$\frac{\boldsymbol{\lambda}(\Omega_k)}{\|\bar{u}\|_{s,\Omega_k}} \le 2\big(\boldsymbol{\lambda}(\{0 < |\bar{u}| \le \gamma_k\})\gamma_k^{-r}\big)^{s/(r+s)} \to 0$$

which completes the proof. □

Note that the condition from Theorem 2.10 is a little bit stronger than $|\bar{u}|^{-1}\chi_{\{\bar{u}\ne 0\}} \in L^{r,\infty}(\Omega)$, where $L^{r,\infty}(\Omega)$ is a weak Lebesgue space (or Lorentz space), which would require that $\boldsymbol{\lambda}(\{0 < |\bar{u}| \le \gamma\})\gamma^{-r}$ is bounded with respect to $\gamma \in (0,\infty)$. Moreover, Chebyshev's inequality can be used to see that $|\bar{u}|^{-1}\chi_{\{\bar{u}\ne 0\}} \in L^r(\Omega)$ implies the condition from Theorem 2.10. Indeed,

$$\boldsymbol{\lambda}(\{0 < |\bar{u}| \le \gamma\})\gamma^{-r} \le \|\,|\bar{u}|^{-1}\|_{r,\{|\bar{u}|^{-1}\ge\gamma^{-1}\}}^r \to 0 \qquad \text{as } \gamma \searrow 0$$

holds if $|\bar{u}|^{-1}\chi_{\{\bar{u}\ne 0\}} \in L^r(\Omega)$ is valid.

3 The case $p = 0$

We start our analysis by investigating the variational properties of the discontinuous functional $q_{s,0}$.

3.1 Fréchet subdifferential

The aim of this subsection is to provide a full characterization of the Fréchet subdifferential associated with the functional $q_{s,0}$ for each $s \in [1, \infty)$. We start our investigations by providing a simple upper bound of the Fréchet subdifferential of $q_{s,0}$.

Lemma 3.1. *For given $s \in [1, \infty)$ and $\bar{u} \in L^s(\Omega)$, we have*

$$\widehat{\partial} q_{s,0}(\bar{u}) \subset \{\eta \in L^r(\Omega) \mid \{\eta \neq 0\} \subset \{\bar{u} = 0\}\}.$$

Proof. Let $\eta \in L^r(\Omega)$ be given such that there exists a measurable set $\Omega' \subset \{\bar{u} \neq 0\}$ of non-zero measure where η is non-vanishing. We assume w.l.o.g. that $|\eta(x)| \geq \rho$ holds for some $\rho > 0$ and almost all $x \in \Omega'$. Define a sequences $\{h_k\}_{k\in\mathbb{N}} \subset L^s(\Omega)$ by means of $h_k := \frac{1}{2k} |\bar{u}| \chi_{\Omega'} \operatorname{sgn} \eta$ for each $k \in \mathbb{N}$. Clearly, we have $\|h_k\|_s \searrow 0$. Furthermore, we find

$$\frac{q_{s,0}(\bar{u} + h_k) - q_{s,0}(\bar{u}) - \int_\Omega \eta(x) h_k(x)\,\mathrm{d}x}{\|h_k\|_s} = \frac{-\frac{1}{2k} \int_{\Omega'} |\eta(x)||\bar{u}(x)|\,\mathrm{d}x}{\frac{1}{2k} \|\bar{u}\|_{s,\Omega'}} \leq -\rho\, \frac{\|\bar{u}\|_{1,\Omega'}}{\|\bar{u}\|_{s,\Omega'}} < 0.$$

Hence, $\eta \notin \widehat{\partial} q_{s,0}(\bar{u})$ and this finishes the proof. $\square$

In the subsequently stated result, we characterize all points in $L^s(\Omega)$ where the Fréchet subdifferential of $q_{s,0}$ is nonempty. Therefore, the concept of slowly decreasing functions discussed in Section 2.5 turns out to be essential.

Lemma 3.2. *Fix $s \in [1, \infty)$ and $\bar{u} \in L^s(\Omega)$. Then $\widehat{\partial} q_{s,0}(\bar{u})$ is nonempty if and only if one of the following conditions is valid:*

(a) $\bar{u} = 0$ holds almost everywhere on Ω,

(b) it holds $s > 1$ and $\bar{u}$ is an s-SD function.

Proof. We start the proof by showing that $0 \in \widehat{\partial} q_{s,0}(\bar{u})$ holds in the presence of each of the given conditions, i.e., we need to show that for all sequences $\{h_k\}_{k\in\mathbb{N}} \subset L^s(\Omega)$ with $\|h_k\|_s \searrow 0$, we have

$$\liminf_{k\to\infty} \frac{q_{s,0}(\bar{u} + h_k) - q_{s,0}(\bar{u})}{\|h_k\|_s} \geq 0.$$

This obviously holds true whenever $q_{s,0}(\bar{u}) = 0$ holds, i.e., if $\{\bar{u} \neq 0\}$ is of measure zero which is the case in (a). Thus, let us assume that (b) holds. For each $k \in \mathbb{N}$, we define $\Omega_k := \{\bar{u} \neq 0\} \cap \{\bar{u} + h_k = 0\}$ and obtain

$$\frac{q_{s,0}(\bar{u} + h_k) - q_{s,0}(\bar{u})}{\|h_k\|_s} \geq \frac{\int_{\{\bar{u}\neq 0\}} \big(|\bar{u}(x) + h_k(x)|_0 - 1\big)\,\mathrm{d}x}{\|h_k\|_s} = -\frac{\boldsymbol{\lambda}(\Omega_k)}{\|h_k\|_s}. \tag{3.1}$$

By $\|h_k\|_s \searrow 0$, we get $\boldsymbol{\lambda}(\Omega_k) \to 0$. In the case where $\boldsymbol{\lambda}(\Omega_k) = 0$ holds along the tail of the sequence, we get $\boldsymbol{\lambda}(\Omega_k)/\|h_k\|_s \to 0$. Otherwise, we may assume w.l.o.g. $\boldsymbol{\lambda}(\Omega_k) > 0$ for all $k \in \mathbb{N}$. Thus, we have $\|h_k\|_{s,\Omega_k} = \|\bar{u}\|_{s,\Omega_k} > 0$ for all $k \in \mathbb{N}$ and, consequently,

$$-\frac{\boldsymbol{\lambda}(\Omega_k)}{\|h_k\|_s} \geq -\frac{\boldsymbol{\lambda}(\Omega_k)}{\|h_k\|_{s,\Omega_k}} = -\frac{\boldsymbol{\lambda}(\Omega_k)}{\|\bar{u}\|_{s,\Omega_k}}.$$

The latter term, however, tends to zero since $\bar{u}$ is an s-SD function. Thus, taking the limit inferior in (3.1) yields $0 \in \widehat{\partial} q_{s,0}(\bar{u})$ in the presence of (b).

In order to show the converse statement, we assume that there exists some $\eta \in \widehat{\partial} q_{s,0}(\bar{u})$. Lemma 3.1 shows $\{\eta \neq 0\} \subset \{\bar{u} = 0\}$. Suppose that $\bar{u}$ is not identically zero almost everywhere on Ω.

For $s = 1$, choose $\rho > 0$ such that $\Omega' := \{0 < |\bar{u}| \leq \rho\}$ is of positive measure. Next, we pick a sequence $\{\Omega'_k\}_{k\in\mathbb{N}}$ of measurable subsets of Ω' which satisfy $\boldsymbol{\lambda}(\Omega'_k) \searrow 0$. For each $k \in \mathbb{N}$, we set $h_k := -\bar{u}\chi_{\Omega'_k}$. By construction, we have $\|h_k\|_1 \searrow 0$. Furthermore, we find

$$\frac{q_{1,0}(\bar{u} + h_k) - q_{1,0}(\bar{u}) - \int_\Omega \eta(x) h_k(x)\,\mathrm{d}x}{\|h_k\|_1} = -\frac{\boldsymbol{\lambda}(\Omega'_k)}{\int_{\Omega'_k} |\bar{u}(x)|\,\mathrm{d}x} \leq -\frac{\boldsymbol{\lambda}(\Omega'_k)}{\rho\,\boldsymbol{\lambda}(\Omega'_k)} = -\frac{1}{\rho},$$

contradicting $\eta \in \widehat{\partial} q_{1,0}(\bar{u})$. Consequently, $s > 1$ holds.

Finally, suppose that $\bar{u}$ is not an s-SD function. Then there is a sequence $\{\Omega_k\}_{k\in\mathbb{N}}$ of measurable subsets of $\{\bar{u} \neq 0\}$ such that $\boldsymbol{\lambda}(\Omega_k) \searrow 0$ while the quotients $\boldsymbol{\lambda}(\Omega_k)/\|\bar{u}\|_{s,\Omega_k}$ do not converge to zero. For simplicity, we assume that there is $\beta > 0$ such that $\boldsymbol{\lambda}(\Omega_k)/\|\bar{u}\|_{s,\Omega_k} \geq \beta$ holds for all $k \in \mathbb{N}$ (otherwise, consider a suitable subsequence). Once more, we make use of the sequence $\{h_k\}_{k\in\mathbb{N}}$ given by $h_k := -\bar{u}\chi_{\Omega_k}$ for each $k \in \mathbb{N}$. As above, we exploit $\{\eta \neq 0\} \subset \{\bar{u} = 0\}$ and $\{h_k \neq 0\} \subset \{\bar{u} \neq 0\}$ in order to find

$$\frac{q_{s,0}(\bar{u} + h_k) - q_{s,0}(\bar{u}) - \int_\Omega \eta(x) h_k(x)\,\mathrm{d}x}{\|h_k\|_s} = -\frac{\boldsymbol{\lambda}(\Omega_k)}{\|\bar{u}\|_{s,\Omega_k}} \leq -\beta,$$

yielding a contradiction to $\eta \in \widehat{\partial} q_{s,0}(\bar{u})$ since $\|h_k\|_s \searrow 0$. $\square$

Now, we are in position to fully characterize the Fréchet subdifferential of $q_{s,0}$. First, we investigate the case $s = 1$ which needs to be treated separately.

Theorem 3.3. *We have*

$$\forall \bar{u} \in L^1(\Omega)\colon \quad \widehat{\partial} q_{1,0}(\bar{u}) = \begin{cases} \{0\} & \text{if } \bar{u} = 0 \text{ a.e. on } \Omega, \\ \varnothing & \text{otherwise.} \end{cases}$$

Proof. Due to Lemma 3.2, we already know that $\widehat{\partial} q_{1,0}(\bar{u})$ is empty for each $\bar{u} \in L^1(\Omega) \setminus \{0\}$. Thus, assume that $\bar{u}$ vanishes almost everywhere on Ω. In the proof of Lemma 3.2, we verified $0 \in \widehat{\partial} q_{1,0}(\bar{u})$. Consequently, we only need to show the converse inclusion. Thus, fix $\eta \in \widehat{\partial} q_{1,0}(\bar{u})$ and assume that η is not identically zero almost everywhere on Ω. Then we find a measurable set $\Omega' \subset \Omega$ of positive measure as well as some $\rho > 0$ such that $|\eta(x)| \geq \rho$ holds for almost all $x \in \Omega'$. Consider a sequence $\{\Omega_k\}_{k\in\mathbb{N}}$ of measurable subsets of Ω' which satisfy $\boldsymbol{\lambda}(\Omega_k) \searrow 0$. For each $k \in \mathbb{N}$, we define $h_k := \frac{2}{\rho}\chi_{\Omega_k} \operatorname{sgn} \eta$. Clearly, $\|h_k\|_1 \searrow 0$ holds. Furthermore, we find

$$\frac{q_{1,0}(h_k) - \int_\Omega \eta(x) h_k(x)\,\mathrm{d}x}{\|h_k\|_1} = \frac{\boldsymbol{\lambda}(\Omega_k) - \frac{2}{\rho}\int_{\Omega_k} |\eta(x)|\,\mathrm{d}x}{\frac{2}{\rho}\boldsymbol{\lambda}(\Omega_k)} \leq -\frac{\boldsymbol{\lambda}(\Omega_k)}{\frac{2}{\rho}\boldsymbol{\lambda}(\Omega_k)} = -\frac{\rho}{2} < 0,$$

contradicting $\eta \in \widehat{\partial} q_{1,0}(\bar{u})$. ☐

Remark 3.4. *Let us consider the unconstrained minimization of the function* $f + q_{1,0}$ *on* $L^1(\Omega)$ *where* $f\colon L^1(\Omega) \to \mathbb{R}$ *is Fréchet differentiable. Exploiting the sum rule from (Mordukhovich, 2006, Proposition 1.107) and Fermat's rule from (Mordukhovich, 2006, Proposition 1.114), a necessary condition for* $\bar{u} \in L^1(\Omega)$ *to be a local minimizer of* $f + q_{1,0}$ *is* $-f'(\bar{u}) \in \widehat{\partial} q_{1,0}(\bar{u})$. *Due to Theorem 3.3, this amounts to* $\bar{u} = 0$ *and* $f'(\bar{u}) = 0$ *almost everywhere on* Ω. *A similar result can be obtained when applying Pontryagin's maximum principle to the problem of interest, see (Ito and Kunisch, 2014, Theorem 2.2) or (Natemeyer and Wachsmuth, 2020, Section 2.1).*

Next, we characterize the Fréchet subdifferential of $q_{s,0}$ for $s \in (1, \infty)$.

Theorem 3.5. *Fix* $s \in (1, \infty)$. *Then we have*

$$\forall \bar{u} \in L^s(\Omega)\colon \quad \widehat{\partial} q_{s,0}(\bar{u}) = \begin{cases} \{\eta \in L^r(\Omega) \mid \{\eta \neq 0\} \subset \{\bar{u} = 0\}\} & \text{if } \bar{u} \text{ is } s\text{-SD}, \\ \varnothing & \text{otherwise.} \end{cases}$$

Proof. Due to $s \in (1, \infty)$, $\widehat{\partial} q_{s,0}(\bar{u})$ is nonempty if and only if $\bar{u} \in L^s(\Omega)$ is an s-SD function, see Lemma 3.2. Thus, fix an s-SD function $\bar{u} \in L^s(\Omega)$. The inclusion "$\subset$" follows from Lemma 3.1. For the reverse inclusion, let $\eta \in L^r(\Omega)$ with $\{\eta \neq 0\} \subset \{\bar{u} = 0\}$ be given. We have to show

$$\liminf_{k\to\infty} \frac{q_{s,0}(\bar{u} + h_k) - q_{s,0}(\bar{u}) - \int_\Omega \eta(x) h_k(x)\,\mathrm{d}x}{\|h_k\|_s} \geq 0$$

for all sequences $\{h_k\}_{k\in\mathbb{N}} \subset L^s(\Omega)$ with $\|h_k\|_s \searrow 0$. For such a sequence, we set

$$\begin{aligned} D_k &:= \frac{q_{s,0}(\bar{u} + h_k) - q_{s,0}(\bar{u}) - \int_\Omega \eta(x) h_k(x)\,\mathrm{d}x}{\|h_k\|_s} \\ &= \frac{\int_{\{\bar{u}=0\}} (|h_k(x)|_0 - \eta(x) h_k(x))\,\mathrm{d}x}{\|h_k\|_s} + \frac{\int_{\{\bar{u}\neq 0\}} (|\bar{u}(x) + h_k(x)|_0 - 1)\,\mathrm{d}x}{\|h_k\|_s} =: D_k^1 + D_k^2. \end{aligned}$$

Let us validate $\liminf_{k\to\infty} D_k^1 \geq 0$. Using Hölder's inequality on $\{\bar{u} = 0\} \cap \{h_k \neq 0\}$ and $\|h_k\|_{s,\{\bar{u}=0\}\cap\{h_k\neq 0\}} = \|h_k\|_{s,\{\bar{u}=0\}}$, we have

$$D_k^1 \geq \frac{\int_{\{\bar{u}=0\}} |h_k(x)|_0\,\mathrm{d}x}{\|h_k\|_s} - \frac{\int_{\{\bar{u}=0\}} |\eta(x) h_k(x)|\,\mathrm{d}x}{\|h_k\|_{s,\{\bar{u}=0\}}}$$

$$\geq \frac{\boldsymbol{\lambda}(\{\bar u = 0\} \cap \{h_k \neq 0\})}{\|h_k\|_s} - \|\eta\|_{r,\{\bar u=0\}\cap\{h_k\neq 0\}}.$$

In case that $\boldsymbol{\lambda}(\{\bar u = 0\} \cap \{h_k \neq 0\}) \not\to 0$, this yields $D_k^1 \to \infty$. On the other hand, if we have $\boldsymbol{\lambda}(\{\bar u = 0\} \cap \{h_k \neq 0\}) \to 0$, we get $\|\eta\|_{r,\{\bar u=0\}\cap\{h_k\neq 0\}} \to 0$. In any case, $\liminf_{k\to\infty} D_k^1 \geq 0$.

It remains to check $\liminf_{k\to\infty} D_k^2 \geq 0$. This, however, can be distilled from the first part of the proof of Lemma 3.2 since $\bar u$ is an s-SD function.

Combining these estimates, we have shown $\liminf_{k\to\infty} D_k \geq 0$ which yields the claim. □

Remark 3.6. *Similar to Remark 3.4, we consider the unconstrained minimization of the function $f + q_{s,0}$ on $L^s(\Omega)$ where $f\colon L^s(\Omega) \to \mathbb{R}$ is Fréchet differentiable and $s \in (1,\infty)$ Then $-f'(\bar u) \in \widehat{\partial} q_{s,0}(\bar u)$ is a necessary condition for $\bar u \in L^s(\Omega)$ to be a local minimizer of $f + q_{s,0}$. Theorem 3.5 now yields that $f'(\bar u) \in L^r(\Omega)$ has to vanish on $\{\bar u \neq 0\}$. Moreover, the implicitly demanded nonemptiness of $\widehat{\partial} q_{s,0}(\bar u)$ requires that either $\bar u$ is equal to zero almost everywhere on Ω or that $\bar u$ tends to zero if at all slowly enough if $\{\bar u \neq 0\}$ is of positive measure since $\bar u$ must be an s-SD function, see Section 2.5. In this regard, the obtained necessary optimality conditions clearly promote* sparse *controls $\bar u$.*

3.2 Limiting subdifferential

We now exploit Theorem 3.5 in order to characterize the limiting and singular subdifferential of $q_{s,0}$ for each $s \in (1,\infty)$. As already pointed out in Section 2.3, the limiting subdifferential of $q_{1,0}$ coincides with its Fréchet subdifferential due to (Chieu, 2009, Theorem 3.2). Anyway, the fact that $L^1(\Omega)$ is not an Asplund space underlines that the case $s = 1$ might be of limited importance here.

Theorem 3.7. *Fix $s \in (1,\infty)$. Then we have*

$$\forall \bar u \in L^s(\Omega)\colon \quad \partial q_{s,0}(\bar u) = \{\eta \in L^r(\Omega) \,|\, \{\eta \neq 0\} \subset \{\bar u = 0\}\}.$$

Proof. Fix $\bar u \in L^s(\Omega)$. In case where $\bar u = 0$ holds almost everywhere on Ω, Theorem 3.5 already gives us $\widehat{\partial} q_{s,0}(\bar u) = L^r(\Omega)$ which implies $\partial q_{s,0}(\bar u) = L^r(\Omega)$. Thus, we assume that $\{\bar u \neq 0\}$ possesses positive measure for the remainder of the proof and verify both inclusions separately.

In order to show the inclusion "$\supset$", we fix $\eta \in L^r(\Omega)$ satisfying $\{\eta \neq 0\} \subset \{\bar u = 0\}$. For each $k \in \mathbb{N}$, we define $\Omega_k := \{|\bar u| \geq 1/k\}$. Clearly, these sets are measurable and provide a nested exhaustion of $\{\bar u \neq 0\}$. Now, set $u_k := \bar u \chi_{\Omega_k}$ for each $k \in \mathbb{N}$ and observe that $\{u_k = 0\} \supset \{\bar u = 0\}$ holds. Invoking Lemma 2.5, u_k is an s-SD function for each $k \in \mathbb{N}$, so that Theorem 3.5 yields $\eta \in \widehat{\partial} q_{s,0}(u_k)$ for each $k \in \mathbb{N}$. Due to

$$\|u_k - \bar u\|_s^s = \int_\Omega |\bar u(x)|^s (1 - \chi_{\Omega_k}(x))\,\mathrm{d}x \leq \frac{\boldsymbol{\lambda}(\Omega)}{k^s} \to 0,$$

we find $u_k \to \bar{u}$ in $L^s(\Omega)$. Exploiting $\Omega_k \subset \{\bar{u} \neq 0\}$ for each $k \in \mathbb{N}$ and lower semicontinuity of $q_{s,0}$, see Lemma 2.2, we find

$$\begin{aligned}\boldsymbol{\lambda}(\{\bar{u} \neq 0\}) = q_{s,0}(\bar{u}) &\leq \liminf_{k\to\infty} q_{s,0}(u_k) \\ &\leq \limsup_{k\to\infty} q_{s,0}(u_k) = \limsup_{k\to\infty} \boldsymbol{\lambda}(\Omega_k) \leq \boldsymbol{\lambda}(\{\bar{u} \neq 0\}),\end{aligned}$$

i.e., $q_{s,0}(u_k) \to q_{s,0}(\bar{u})$. Thus, by definition of the limiting subdifferential, we have shown $\eta \in \partial q_{s,0}(\bar{u})$.

In order to prove "$\subset$", we fix $\eta \in \partial q_{s,0}(\bar{u})$. Thus, we find sequences $\{u_k\}_{k\in\mathbb{N}} \subset L^s(\Omega)$ and $\{\eta_k\}_{k\in\mathbb{N}} \subset L^r(\Omega)$ which satisfy $u_k \to \bar{u}$ in $L^s(\Omega)$, $q_{s,0}(u_k) \to q_{s,0}(\bar{u})$, $\eta_k \rightharpoonup \eta$ in $L^r(\Omega)$, and $\eta_k \in \widehat{\partial} q_{s,0}(u_k)$ for all $k \in \mathbb{N}$. Along a subsequence (without relabeling), we may assume that $\{u_k\}_{k\in\mathbb{N}}$ converges pointwise almost everywhere to $\bar{u}$. Thus, for almost every $x \in \{\bar{u} \neq 0\}$, we have $u_k(x) \to \bar{u}(x) \neq 0$, i.e., $x \in \{u_k \neq 0\}$ and, thus, $x \in \{\eta_k = 0\}$ for sufficiently large $k \in \mathbb{N}$. Thus, almost everywhere on $\{\bar{u} \neq 0\}$, $\{\eta_k\}_{k\in\mathbb{N}}$ converges pointwise to 0. From $\eta_k \rightharpoonup \eta$ in $L^r(\Omega)$, we infer that the weak limit needs to vanish on $\{\bar{u} \neq 0\}$, i.e., $\{\bar{u} \neq 0\} \subset \{\eta = 0\}$. This, however, also means $\{\eta \neq 0\} \subset \{\bar{u} = 0\}$. □

Reprising the above proof while incorporating some nearby minor adjustments, one can show the following result regarding the singular subdifferential of $q_{s,0}$.

Theorem 3.8. *Fix $s \in (1, \infty)$. Then we have*

$$\forall \bar{u} \in L^s(\Omega): \quad \partial^\infty q_{s,0}(\bar{u}) = \{\eta \in L^r(\Omega) \,|\, \{\eta \neq 0\} \subset \{\bar{u} = 0\}\}.$$

As a corollary of Theorems 3.5 and 3.8, we can fully characterize the Lipschitzian properties of $q_{s,0}$.

Corollary 3.9. *For $s \in (1, \infty)$, $q_{s,0}$ is nowhere Lipschitz continuous.*

Proof. Using Lemma 2.1, Theorem 3.8 shows that $q_{s,0}$ cannot be Lipschitz continuous at all points $\bar{u} \in L^s(\Omega)$ which satisfy $\boldsymbol{\lambda}(\{\bar{u} = 0\}) > 0$ since $\partial^\infty q_{s,0}(\bar{u})$ does not reduce to $\{0\}$ in this situation.

Thus, let us consider $\bar{u} \in L^s(\Omega)$ such that $\bar{u} \neq 0$ holds almost everywhere on Ω. In the reminder of this proof, we show that $q_{s,0}$ violates the condition from Lemma 2.1 (b) at $\bar{u}$ which implies that $q_{s,p}$ cannot be Lipschitz at $\bar{u}$. Thus, pick a scalar $\alpha > 0$ such that $\{|\bar{u}| \geq \alpha\}$ possesses positive measure and set $\Omega_k := \{|\bar{u}| \geq \alpha/k\}$ for each $k \in \mathbb{N}$. By construction, $\{\Omega_k\}_{k\in\mathbb{N}}$ is an exhaustion of $\{\bar{u} \neq 0\}$, and each of the sets Ω_k, $k \in \mathbb{N}$, possesses positive measure. Thus, we can pick a sequence $\{\Omega'_k\}_{k\in\mathbb{N}}$ of measurable subsets of Ω such that $\boldsymbol{\lambda}(\Omega'_k) \searrow 0$ and, for each $k \in \mathbb{N}$, $\Omega'_k \subset \Omega_k$. For each $k \in \mathbb{N}$, we define $u_k := \bar{u}\chi_{\Omega_k \setminus \Omega'_k}$ and $\eta_k := \boldsymbol{\lambda}(\Omega'_k)^{-1/r}\chi_{\Omega'_k}$. Similar as in the proof of Theorem 3.7, we can show $u_k \to \bar{u}$ in $L^s(\Omega)$ and $q_{s,0}(u_k) \to q_{s,0}(\bar{u})$. Furthermore, for each $h \in L^s(\Omega)$, we find

$$\left| \int_\Omega \eta_k(x) h(x) \,\mathrm{d}x \right| \leq \|h\|_{s,\Omega'_k} \|\eta_k\|_{r,\Omega'_k} = \|h\|_{s,\Omega'_k}$$

by applying Hölder's inequality on Ω'_k, and due to $\|h\|_{s,\Omega'_k} \to 0$, the above estimate yields $\eta_k \rightharpoonup 0$ in $L^r(\Omega)$. Furthermore, $\|\eta_k\|_r = 1$ for each $k \in \mathbb{N}$ guarantees that this convergence is not strong. Finally, observe that due to Lemma 2.5 and Theorem 3.5, we find $\eta_k \in \frac{1}{k}\widehat{\partial} q_{s,0}(u_k)$ for each $k \in \mathbb{N}$. Thus, Lemma 2.1 shows that $q_{s,0}$ cannot be Lipschitz continuous at $\bar{u}$. $\square$

4 The case $p \in (0, 1)$

Throughout the section, we assume that $p \in (0, 1)$ holds. Here, we study the variational properties of the functional $q_{s,p}$. Basically, although some proofs seem to be a little technical, we proceed in similar way as in Section 3 in order to compute the subdifferentials of interest.

4.1 Fréchet subdifferential

Again, we start to prove validity of a natural upper bound for the Fréchet subdifferential of $q_{s,p}$.

Lemma 4.1. *For given $s \in [1, \infty)$ and $\bar{u} \in L^s(\Omega)$, we have*

$$\widehat{\partial} q_{s,p}(\bar{u}) \subset \{\eta \in L^r(\Omega) \,|\, \eta = p\,|\bar{u}|^{p-2}\,\bar{u} \text{ a.e. on } \{\bar{u} \neq 0\}\}.$$

Proof. Let $\eta \in \widehat{\partial} q_{s,0}(\bar{u})$ be given. For $\varepsilon > 0$, we set $A_\varepsilon := \{|\bar{u}| > \varepsilon\}$. For an arbitrary measurable subset $B \subset A_\varepsilon$ of positive measure, we define a sequence $\{h_k\}_{k\in\mathbb{N}} \subset L^s(\Omega)$ by means of $h_k := k^{-1}\chi_B$ for each $k \in \mathbb{N}$. Clearly, we have $\|h_k\|_s \searrow 0$, so the definition of the Fréchet subdifferential yields

$$\begin{aligned}
0 &\leq \boldsymbol{\lambda}(B)^{1/s} \liminf_{k\to\infty} \frac{q_{s,p}(\bar{u} + h_k) - q_{s,p}(\bar{u}) - \int_\Omega \eta(x)h_k(x)\,\mathrm{d}x}{\|h_k\|_s} \\
&= \liminf_{k\to\infty} \int_B \big(k(|\bar{u}(x) + 1/k|^p - |\bar{u}(x)|^p) - \eta(x)\big)\,\mathrm{d}x = \int_B \big(p\,|\bar{u}(x)|^{p-2}\,\bar{u}(x) - \eta(x)\big)\,\mathrm{d}x.
\end{aligned}$$

Note that we used the dominated convergence theorem with the integrable, dominating function $(p\varepsilon^{p-1} + |\eta|)\chi_B$ for the last equality. Similarly, we can use the sequence $\{\tilde{h}_k\}_{k\in\mathbb{N}} \subset L^s(\Omega)$ given by $\tilde{h}_k := -k^{-1}\chi_B$ for each $k \in \mathbb{N}$ to obtain the reverse inequality. Since $B \subset A_\varepsilon$ was arbitrary, this shows $\eta = p\,|\bar{u}|^{p-2}\,\bar{u}$ almost everywhere on A_ε. Since $\{\bar{u} \neq 0\} = \bigcup_{\varepsilon>0} A_\varepsilon$ holds, the claim has been shown. $\square$

We note that, technically, the above proof also applies to the setting $p = 0$ and, thus, provides another possible validation of Lemma 3.1. However, let us emphasize that the proof we provided for Lemma 3.1 is much simpler and does not exploit deeper results from integration theory like the dominated convergence theorem.

Similar to Lemma 3.2, we aim to characterize all points in $L^s(\Omega)$ where the associated Fréchet subdifferential of $q_{s,p}$ is nonempty.

Lemma 4.2. *Fix $s \in [1, \infty)$ and $\bar{u} \in L^s(\Omega)$. Then $\widehat{\partial} q_{s,p}(\bar{u})$ is nonempty if and only if one of the following conditions is valid:*

(a) $\bar{u} = 0$ holds almost everywhere on Ω,

(b) it holds $s > 1$ and $|\bar{u}|^{p-1} \chi_{\{\bar{u} \neq 0\}} \in L^r(\Omega)$ is valid.

Proof. In the first part of this proof, we show that, in the presence of (a) or (b), $\bar{\eta} \colon \Omega \to \mathbb{R}$ given by $\bar{\eta} := p\,|\bar{u}|^{p-2}\,\bar{u}\chi_{\{\bar{u}\neq 0\}}$ belongs to $\widehat{\partial} q_{s,p}(\bar{u})$. This is clearly obvious in case where $\bar{u}$ vanishes almost everywhere on Ω, i.e., when (a) holds, so let us focus on the situation given in (b). First, we observe that $\bar{\eta}$ defined above is a function from $L^r(\Omega)$ due to the requirements in (b). Next, we show that for each sequence $\{h_k\}_{k\in\mathbb{N}} \subset L^s(\Omega)$ satisfying $\|h_k\|_s \searrow 0$, we have

$$\liminf_{k\to\infty} \frac{\int_{\{\bar{u}\neq 0\}} \big(|\bar{u}(x) + h_k(x)|^p - |\bar{u}(x)|^p - p|\bar{u}(x)|^{p-2}\bar{u}(x)h_k(x)\big)\,\mathrm{d}x}{\|h_k\|_s} \geq 0. \tag{4.1}$$

One can easily check that by definition of the Fréchet subdifferential and $\bar{\eta}$, this is sufficient for $\bar{\eta} \in \widehat{\partial} q_{s,p}(\bar{u})$.

It will be beneficial to write $h_k = c_k\bar{u} + h_k\chi_{\{\bar{u}=0\}}$ for each $k \in \mathbb{N}$ where the measurable function $c_k \colon \Omega \to \mathbb{R}$ is given by $c_k := h_k\bar{u}^{-1}\chi_{\{\bar{u}\neq 0\}}$. Furthermore, we will make use of the set $\Omega_k := \{h_k \neq 0\} \cap \{\bar{u} \neq 0\}$ for each $k \in \mathbb{N}$. With the aid of these definitions, we can rewrite the quotient of interest by means of

$$\frac{\int_{\Omega_k} \big(|1 + c_k(x)|^p - 1 - pc_k(x)\big)|\bar{u}(x)|^p\,\mathrm{d}x}{\Big(\int_{\Omega_k} |c_k(x)|^s|\bar{u}(x)|^s\,\mathrm{d}x + \int_{\{\bar{u}=0\}} |h_k(x)|^s\,\mathrm{d}x\Big)^{1/s}}. \tag{4.2}$$

Next, for each $k \in \mathbb{N}$, we decompose Ω_k into the four disjoint subsets

$$\Omega_k^1 := \{c_k < -1/p\}, \qquad \Omega_k^2 := \{-1/p \leq c_k \leq -1/2\},$$
$$\Omega_k^3 := \{-1/2 < c_k < 1/2\}, \qquad \Omega_k^4 := \{c_k \geq 1/2\}.$$

This allows us to rewrite the quotient in (4.2) as $Q_k^1 + Q_k^2 + Q_k^3 + Q_k^4$ with

$$Q_k^i := \frac{\int_{\Omega_k^i} \big(|1 + c_k(x)|^p - 1 - pc_k(x)\big)|\bar{u}(x)|^p\,\mathrm{d}x}{\Big(\int_{\Omega_k} |c_k(x)|^s|\bar{u}(x)|^s\,\mathrm{d}x + \int_{\{\bar{u}=0\}} |h_k(x)|^s\,\mathrm{d}x\Big)^{1/s}}$$

for each $i = 1, 2, 3, 4$. By construction, Q_k^1 is nonnegative which yields $\liminf_{k\to\infty} Q_k^1 \geq 0$. Furthermore, in case where Ω_k^2 is of positive measure, we find

$$Q_k^2 \geq (p-2)\frac{\|\bar{u}\|_{p,\Omega_k^2}^p}{\|\bar{u}\|_{s,\Omega_k^2}} \geq (p-2)\frac{\|\bar{u}\|_{s,\Omega_k^2}\|\,|\bar{u}|^{p-1}\,\|_{r,\Omega_k^2}}{\|\bar{u}\|_{s,\Omega_k^2}} = (p-2)\|\,|\bar{u}|^{p-1}\,\|_{r,\Omega_k^2}$$

from $|\bar{u}|^{p-1}\chi_{\{\bar{u}\neq 0\}} \in L^r(\Omega)$ and Hölder's inequality on Ω_k^2. Since we have $\boldsymbol{\lambda}(\Omega_k^2) \searrow 0$ from $\|h_k\|_s \searrow 0$, $\|\,|\bar{u}|^{p-1}\,\|_{r,\Omega_k^2} \searrow 0$ holds which is why $\liminf_{k\to\infty} Q_k^2 \geq 0$ follows. Next, let us investigate the setting where Ω_k^3 is of positive measure. A second-order Taylor expansion of the mapping $y \mapsto (1+y)^p - 1$ at the origin yields $(1+y)^p - 1 - py \geq -y^2$ for all $y \in (-1/2, 1/2)$. Thus, we obtain

$$Q_k^3 \geq -\frac{\int_{\Omega_k^3} |c_k(x)|^2 |\bar{u}(x)|^p \, \mathrm{d}x}{\|c_k \bar{u}\|_{s,\Omega_k^3}} \geq -\frac{\|c_k \bar{u}\|_{s,\Omega_k^3} \|c_k \, |\bar{u}|^{p-1} \, \|_{r,\Omega_k^3}}{\|c_k \bar{u}\|_{s,\Omega_k^3}} = -\|c_k \, |\bar{u}|^{p-1} \, \|_{r,\Omega_k^3}$$

where we used Hölder's inequality on Ω_k^3 and $|\bar{u}|^{p-1} \chi_{\{\bar{u} \neq 0\}} \in L^r(\Omega)$ which, by boundedness of c_k on Ω_k^3, guarantees $c_k \, |\bar{u}|^{p-1} \chi_{\Omega_k^3} \in L^r(\Omega)$. Observing that $\bar{u}$ does not vanish on Ω_k^3, that $h_k = c_k \bar{u}$ holds on Ω_k^3, and that $\|h_k\|_s \searrow 0$ is valid, we obtain the pointwise convergence of $\{c_k\}_{k \in \mathbb{N}}$ to 0 almost everywhere on Ω_k^3. Thus, $c_k(x)|\bar{u}(x)|^{p-1} \to 0$ holds for almost every $x \in \Omega_k^3$. Noting that $\{c_k \, |\bar{u}|^{p-1} \chi_{\Omega_k^3}\}_{k \in \mathbb{N}}$ is dominated by $|\bar{u}|^{p-1} \chi_{\Omega_k^3} \in L^r(\Omega)$, we find $\|c_k \, |\bar{u}|^{p-1} \, \|_{r,\Omega_k^3} \to 0$ from Lebesgue's dominated convergence theorem, i.e., $\liminf_{k \to \infty} Q_k^3 \geq 0$. Finally, we address the situation where Ω_k^4 is of positive measure. Recalling that $h_k = c_k \bar{u}$, $\bar{u} \neq 0$, and $c_k \geq 1/2$ hold on Ω_k^4, $\|h_k\|_s \searrow 0$ implies $\boldsymbol{\lambda}(\Omega_k^4) \searrow 0$. Exploiting $|\bar{u}|^{p-1} \chi_{\Omega_k^4} \in L^r(\Omega)$, we find

$$Q_k^4 \geq -p \frac{\int_{\Omega_k^4} c_k(x) |\bar{u}(x)|^p \, \mathrm{d}x}{\|c_k \bar{u}\|_{s,\Omega_k^4}} \geq -p \frac{\|c_k \bar{u}\|_{s,\Omega_k^4} \| \, |\bar{u}|^{p-1} \, \|_{r,\Omega_k^4}}{\|c_k \bar{u}\|_{s,\Omega_k^4}} = -p \| \, |\bar{u}|^{p-1} \, \|_{r,\Omega_k^4}$$

by applying Hölder's inequality on Ω_k^4. Due to $\boldsymbol{\lambda}(\Omega_k^4) \searrow 0$, we have $\| \, |\bar{u}|^{p-1} \, \|_{r,\Omega_k^4} \searrow 0$ which yields $\liminf_{k \to \infty} Q_k^4 \geq 0$. Combining all these estimates, (4.1) has been shown, i.e., $\bar{\eta} \in \widehat{\partial} q_{s,p}(\bar{u})$ is valid.

Let us show the converse statement. Therefore, we assume that there is some $\eta \in \widehat{\partial} q_{s,p}(\bar{u})$. Due to Lemma 4.1, we know that $\eta = p \, |\bar{u}|^{p-2} \, \bar{u}$ holds almost everywhere on $\{\bar{u} \neq 0\}$. Thus, from $\eta \in L^r(\Omega)$, the condition $|\bar{u}|^{p-1} \chi_{\{\bar{u} \neq 0\}} \in L^r(\Omega)$ follows. We assume that $\bar{u}$ is not identically zero almost everywhere on Ω.

Suppose that $s = 1$ holds. We fix a set $\Omega' \subset \{\bar{u} \neq 0\}$ of positive measure and some $\rho > 0$ such that $|\bar{u}(x)| \geq \rho$ holds almost everywhere on Ω'. Set $A_\ell := \{x \in \Omega' \,|\, 0 < |\bar{u}(x)| \leq \ell\}$ for each $\ell \in \mathbb{N}$. Then we have $\bigcup_{\ell \in \mathbb{N}} A_\ell = \Omega'$, and due to $\boldsymbol{\lambda}(\Omega') > 0$, there exists some $\ell_0 \in \mathbb{N}$ such that A_{ℓ_0} is of positive measure. Let us now fix a sequence $\{\Omega'_k\}_{k \in \mathbb{N}}$ of measurable subsets of A_{ℓ_0} which satisfy $\boldsymbol{\lambda}(\Omega'_k) \searrow 0$. For brevity of notation, set $m_k := \boldsymbol{\lambda}(\Omega'_k)$ for each $k \in \mathbb{N}$ and define $h_k := m_k^{-1/2} \bar{u} \chi_{\Omega'_k}$. By construction, we have

$$\|h_k\|_1 = m_k^{-1/2} \int_{\Omega'_k} |\bar{u}(x)| \, \mathrm{d}x \leq \ell_0 \, m_k^{1/2} \searrow 0.$$

Furthermore, we find

$$\begin{aligned}
&\frac{q_{1,p}(\bar u + h_k) - q_{1,p}(\bar u) - \int_\Omega \eta(x) h_k(x)\,\mathrm{d}x}{\|h_k\|_1} \\
&\qquad = \frac{\int_{\Omega_k'} \big((1+m_k^{-1/2})^p |\bar u(x)|^p - |\bar u(x)|^p - p m_k^{-1/2} |\bar u(x)|^p\big)\,\mathrm{d}x}{m_k^{-1/2}\int_{\Omega_k'} |\bar u(x)|\,\mathrm{d}x} \\
&\qquad = \frac{(1+m_k^{-1/2})^p - 1 - p m_k^{-1/2}}{m_k^{-1/2}} \, \frac{\|\bar u\|_{p,\Omega_k'}^p}{\|\bar u\|_{1,\Omega_k'}}.
\end{aligned} \tag{4.3}$$

Due to $p \in (0,1)$, it holds

$$\frac{(1+m_k^{-1/2})^p - 1 - p m_k^{-1/2}}{m_k^{-1/2}} = \Big(m_k^{1/(2p)} + m_k^{1/(2p)-1/2}\Big)^p - m_k^{1/2} - p \to -p.$$

On the other hand, we have

$$\frac{\|\bar u\|_{p,\Omega_k'}^p}{\|\bar u\|_{1,\Omega_k'}} \geq \frac{\rho^p}{\ell_0}$$

by choice of $\Omega_k' \subset A_{\ell_0}$. Hence, for sufficiently large $k \in \mathbb{N}$, (4.3) yields

$$\frac{q_{1,p}(\bar u + h_k) - q_{1,p}(\bar u) - \int_\Omega \eta(x) h_k(x)\,\mathrm{d}x}{\|h_k\|_1} \leq -\frac{p}{2}\frac{\rho^p}{\ell_0} < 0,$$

but this contradicts $\eta \in \widehat{\partial} q_{1,p}(\bar u)$. $\square$

For $s > 1$, fix a function $\bar u \in L^s(\Omega)$ such that $\{\bar u \neq 0\}$ possesses positive measure and some sequence $\{\Omega_k\}_{k\in\mathbb{N}}$ of measurable subsets of $\{\bar u \neq 0\}$ possessing positive measure. Supposing that $|\bar u|^{p-1}\chi_{\{\bar u\neq 0\}} \in L^r(\Omega)$ is valid, Hölder's inequality on Ω_k yields

$$\|\bar u\|_{p,\Omega_k}^p \leq \|\bar u\|_{s,\Omega_k} \|\, |\bar u|^{p-1} \,\|_{r,\Omega_k}$$

for each $k \in \mathbb{N}$. Thus, we find

$$\boldsymbol{\lambda}(\Omega_k) \searrow 0 \qquad \Longrightarrow \qquad \frac{\|\bar u\|_{p,\Omega_k}^p}{\|\bar u\|_{s,\Omega_k}} \searrow 0, \tag{4.4}$$

which can be interpreted as a reasonable adaptation of the s-SD property from Definition 2.4 to the setting $p \in (0,1)$. Note that (4.4) can be used in the proof of Lemma 4.2 in order to show $\liminf_{k\to\infty} Q_k^2 \geq 0$. However, as demonstrated above, (4.4) is implied by $|\bar u|^{p-1}\chi_{\{\bar u\neq 0\}} \in L^r(\Omega)$ which, either way, needs to be postulated in order to show the assertion of Lemma 4.2. We can interpret L^r-regularity of $|\bar u|^{p-1}\chi_{\{\bar u\neq 0\}}$ again as a condition which ensures that whenever $\bar u$ approaches zero on $\{\bar u \neq 0\}$, then this has to happen slowly enough. Recall that in case $p = 0$, see Lemma 3.2, $|\bar u|^{-1}\chi_{\{\bar u\neq 0\}} \in L^r(\Omega)$

is only sufficient but not necessary for the nonemptiness of $\widehat{\partial} q_{s,0}(\bar u)$, see Lemma 2.7 and Example 2.8 as well.

Now, we are in position to fully characterize the Fréchet subdifferential of $q_{s,p}$. Again, we distinguish the cases $s = 1$ and $s \in (1, \infty)$.

Theorem 4.3. *We have*

$$\forall \bar u \in L^1(\Omega)\colon \quad \widehat{\partial} q_{1,p}(\bar u) = \begin{cases} \{0\} & \text{if } \bar u = 0 \text{ a.e. on } \Omega, \\ \varnothing & \text{otherwise.} \end{cases}$$

Proof. Due to Lemma 4.2, we know that $\widehat{\partial} q_{1,p}(\bar u)$ is empty for each $\bar u \in L^1(\Omega) \setminus \{0\}$. Thus, let us assume that $\bar u = 0$ holds almost everywhere on Ω. It is obvious by definition of the Fréchet subdifferential that $0 \in \widehat{\partial} q_{1,p}(\bar u)$ is valid. In order to show the converse inclusion, fix $\eta \in \widehat{\partial} q_{1,p}(\bar u)$ and assume that η does not vanish almost everywhere on Ω. Then we find a measurable set $\Omega' \subset \Omega$ of positive measure as well as some $\rho > 0$ such that $|\eta(x)| \ge \rho$ holds for almost every $x \in \Omega'$. We fix a sequence $\{\Omega_k\}_{k\in\mathbb{N}}$ of measurable subsets of Ω' such that $\boldsymbol{\lambda}(\Omega_k) \searrow 0$ is valid. Furthermore, we choose some constant $\alpha > \rho^{1/(p-1)}$. Due to $p \in (0,1)$, this yields $\alpha^{p-1} < \rho$. Now, we set $h_k := \alpha \chi_{\Omega_k} \operatorname{sgn} \eta$ for each $k \in \mathbb{N}$ and observe that $\|h_k\|_1 \searrow 0$ is valid. Additionally, we find

$$\begin{aligned} \frac{q_{1,p}(h_k) - \int_\Omega \eta(x) h_k(x)\,\mathrm{d}x}{\|h_k\|_1} &= \frac{\alpha^p \boldsymbol{\lambda}(\Omega_k) - \alpha \int_{\Omega_k} |\eta(x)|\,\mathrm{d}x}{\alpha \boldsymbol{\lambda}(\Omega_k)} \\ &\le \frac{\alpha^{p-1}\boldsymbol{\lambda}(\Omega_k) - \rho \boldsymbol{\lambda}(\Omega_k)}{\boldsymbol{\lambda}(\Omega_k)} = \alpha^{p-1} - \rho < 0, \end{aligned}$$

contradicting our assumption $\eta \in \widehat{\partial} q_{1,p}(\bar u)$. □

Theorem 4.4. *Fix $s \in (1,\infty)$. Then we have*

$$\forall \bar u \in L^s(\Omega)\colon \quad \widehat{\partial} q_{s,p}(\bar u) = \{\eta \in L^r(\Omega) \,|\, \eta = p\,|\bar u|^{p-2}\,\bar u \text{ a.e. on } \{\bar u \neq 0\}\}.$$

In particular, the set on the right-hand side is empty if $|\bar u|^{p-1}\,\chi_{\{\bar u \neq 0\}} \notin L^r(\Omega)$.

Proof. The inclusion "$\subset$" follows from Lemma 4.1. For the reverse inclusion, let $\eta \in L^r(\Omega)$ with $\eta = p\,|\bar u|^{p-2}\,\bar u$ almost everywhere on $\{\bar u \neq 0\}$ be given. We have to show

$$\liminf_{k\to\infty} \frac{q_{s,p}(\bar u + h_k) - q_{s,p}(\bar u) - \int_\Omega \eta(x) h_k(x)\,\mathrm{d}x}{\|h_k\|_s} \ge 0$$

for all sequences $\{h_k\}_{k\in\mathbb{N}} \subset L^s(\Omega)$ with $\|h_k\|_s \searrow 0$. For such a sequence, we set

$$\begin{aligned} D_k &:= \frac{q_{s,p}(\bar u + h_k) - q_{s,p}(\bar u) - \int_\Omega \eta(x) h_k(x)\,\mathrm{d}x}{\|h_k\|_s} \\ &= \frac{\int_{\{\bar u = 0\}} \big(|h_k(x)|^p - \eta(x) h_k(x)\big)\,\mathrm{d}x}{\|h_k\|_s} \end{aligned}$$

$$
\begin{aligned}
&+\frac{\int_{\{\bar u\neq 0\}}\big(|\bar u(x)+h_k(x)|^p-|\bar u(x)|^p-p\,|\bar u(x)|^{p-2}\,\bar u(x)h_k(x)\big)\,\mathrm{d}x}{\|h_k\|_s}\\
&=: D_k^1+D_k^2.
\end{aligned}
$$

First, let us validate $\liminf_{k\to\infty} D_k^1 \geq 0$. In case where η equals zero almost everywhere on $\{\bar u = 0\}$, this is obvious. Otherwise, for some arbitrarily chosen $\varepsilon > 0$, choose $t > 0$ large enough such that $\|\eta\chi_{\{|\eta|>t\}}\|_{r,\{\bar u=0\}} \leq \varepsilon$. Next, for each $k \in \mathbb{N}$, we define $h_k^1, h_k^2 \in L^s(\Omega)$ by means of $h_k^1 := h_k\chi_{\{|h_k|\leq t^{1/(p-1)}\}}$ and $h_k^2 := h_k\chi_{\{|h_k|>t^{1/(p-1)}\}}$. By construction, we find

$$
D_k^1 = \frac{\int_{\{\bar u=0\}}\big(|h_k^1(x)|^p-\eta(x)h_k^1(x)\big)\,\mathrm{d}x}{\|h_k\|_s}+\frac{\int_{\{\bar u=0\}}\big(|h_k^2(x)|^p-\eta(x)h_k^2(x)\big)\,\mathrm{d}x}{\|h_k\|_s}. \tag{4.5}
$$

Observing that for all $x \in \{\bar u = 0\} \cap \{|h_k| \leq t^{1/(p-1)}\} \cap \{|\eta| \leq t\}$, we have the estimate $|h_k(x)|^{p-1} \geq t \geq |\eta(x)|$, i.e., $|h_k(x)|^p \geq |\eta(x)h_k(x)| \geq \eta(x)h_k(x)$, it holds

$$
\begin{aligned}
\frac{\int_{\{\bar u=0\}}\big(|h_k^1(x)|^p-\eta(x)h_k^1(x)\big)\,\mathrm{d}x}{\|h_k\|_s} &\geq -\frac{\int_{\{\bar u=0\}}|\eta(x)\chi_{\{|\eta|>t\}}(x)h_k^1(x)|\,\mathrm{d}x}{\|h_k^1\|_{s,\{\bar u=0\}}}\\
&\geq -\|\eta\chi_{\{|\eta|>t\}}\|_{r,\{\bar u=0\}} \geq -\varepsilon
\end{aligned}
$$

where Hölder's inequality on $\{\bar u = 0\}$ was used to obtain the last but one estimate. On the other hand, we find

$$
\frac{\int_{\{\bar u=0\}}\big(|h_k^2(x)|^p-\eta(x)h_k^2(x)\big)\,\mathrm{d}x}{\|h_k\|_s} \geq -\frac{\int_{\{\bar u=0\}}|\eta(x)h_k^2(x)|\,\mathrm{d}x}{\|h_k^2\|_{s,\{\bar u=0\}}} \geq -\|\eta\chi_{\{|h_k|>t^{1/(p-1)}\}}\|_{r,\{\bar u=0\}}
$$

again from Hölder's inequality on $\{\bar u = 0\}$. As a consequence, (4.5) yields the estimate $D_k^1 \geq -\varepsilon - \|\eta\chi_{\{|h_k|>t^{1/(p-1)}\}}\|_{r,\{\bar u=0\}}$ for all $k \in \mathbb{N}$. Since we have $\|h_k\|_s \searrow 0$, the convergence $\boldsymbol{\lambda}(\{|h_k| > t^{1/(p-1)}\}) \to 0$ holds which guarantees $\|\eta\chi_{\{|h_k|>t^{1/(p-1)}\}}\|_{r,\{\bar u=0\}} \to 0$. Observing that $\varepsilon > 0$ is independent of k and can be made arbitrarily small, we have shown $\liminf_{k\to\infty} D_k^1 \geq 0$.

Noting that we can distill $\liminf_{k\to\infty} D_k^2 \geq 0$ from the proof of Lemma 4.2, this already yields $\liminf_{k\to\infty} D_k \geq 0$ and the statement of the theorem has been shown. □

In the subsequent remark, which parallels Remarks 3.4 and 3.6, we comment on necessary optimality conditions for unconstrained optimization problems involving the functional $q_{s,p}$ as a sparsity-promoting term.

Remark 4.5. *Fix $s \in [1,\infty)$, a Fréchet differentiable function $f\colon L^s(\Omega) \to \mathbb{R}$, and consider the unconstrained minimization of $f + q_{s,p}$. A necessary condition for some $\bar u \in L^s(\Omega)$ to be a local minimizer of $f + q_{s,p}$ is given by $-f'(\bar u) \in \widehat{\partial} q_{s,p}(\bar u)$.*

- *(a) In case $s = 1$, Theorem 4.3 shows that this amounts to $\bar u = 0$ and $f'(\bar u) = 0$ almost everywhere on Ω. A similar result can be obtained applying Pontryagin's maximum principle, see (Ito and Kunisch, 2014, Theorem 2.2) or (Natemeyer and Wachsmuth, 2020, Section 2.1).*

(b) *In case $s > 1$, Theorem 4.4 implies that $f'(\bar u) \in L^r(\Omega)$ has to equal $p\,|\bar u|^{p-2}\,\bar u$ almost everywhere on $\{\bar u \neq 0\}$. This implicitly demands $|\bar u|^{p-1}\,\chi_{\{\bar u \neq 0\}} \in L^r(\Omega)$ which promotes sparse controls since $\bar u$ has to approach zero from $\{\bar u \neq 0\}$ if at all slowly enough.*

4.2 Limiting subdifferential

Let us now turn our attention to the limiting subdifferential constructions in the Aspund space setting $s \in (1,\infty)$. Thanks to Theorem 4.4, we can adapt most of the proof strategies directly from Section 3.2.

Theorem 4.6. *Fix $s \in (1,\infty)$. Then we have*

$$\forall \bar u \in L^s(\Omega)\colon \quad \partial q_{s,p}(\bar u) = \widehat{\partial} q_{s,p}(\bar u) = \{\eta \in L^r(\Omega) \,|\, \eta = p\,|\bar u|^{p-2}\,\bar u \text{ a.e. on } \{\bar u \neq 0\}\}.$$

Proof. The second "=" follows from Theorem 4.4 and it remain to verify the first equality. The inclusion "$\supset$" follows from the definition of the limiting subdifferential.

The proof of the converse inclusion "$\subset$" can be directly transferred from the one of Theorem 3.7 exploiting the different pointwise characterization of the Fréchet subdifferential from Theorem 4.4. □

Next, we characterize the singular subdifferential of $q_{s,p}$.

Theorem 4.7. *Fix $s \in (1,\infty)$. Then we have*

$$\forall \bar u \in L^s(\Omega)\colon \quad \partial^\infty q_{s,p}(\bar u) = \{\eta \in L^r(\Omega) \,|\, \{\eta \neq 0\} \subset \{\bar u = 0\}\}.$$

Proof. Fix $\bar u \in L^s(\Omega)$. Observe that in case where $\bar u$ vanishes almost everywhere on Ω, we have $\widehat{\partial} q_{s,p}(\bar u) = L^r(\Omega)$ from Theorem 4.4 which, by definition of the singular subdifferential, already yields $\partial^\infty q_{s,p}(\bar u) = L^r(\Omega)$. Thus, we may assume throughout the remainder of the proof that $\{\bar u \neq 0\}$ possesses positive measure.

In order to prove the inclusion "$\supset$", we fix $\eta \in L^r(\Omega)$ which satisfies $\{\eta \neq 0\} \subset \{\bar u = 0\}$. We set $\Omega_k := \{|\bar u| \geq 1/k\}$ as well as $u_k := \bar u \chi_{\Omega_k}$ for each $k \in \mathbb{N}$ leading to $u_k \to \bar u$ in $L^s(\Omega)$ and $q_{s,p}(u_k) \to q_{s,p}(\bar u)$, see Lemma 2.3. For each $k \in \mathbb{N}$, let us define a measurable function $\eta_k\colon \Omega \to \mathbb{R}$ by means of

$$\forall x \in \Omega\colon \quad \eta_k(x) := \begin{cases} p|\bar u(x)|^{p-2}\bar u(x) & \text{if } x \in \Omega_k, \\ k\,\eta(x) & \text{otherwise.} \end{cases}$$

For each $k \in \mathbb{N}$, we find

$$\begin{aligned} \|\eta_k\|_r^r &= \int_{\{|\bar u| \geq 1/k\}} p^r |\bar u(x)|^{(p-1)r}\,\mathrm{d}x + \int_{\{|\bar u| < 1/k\}} k^r |\eta(x)|^r\,\mathrm{d}x \\ &\leq p^r k^{(1-p)r} \boldsymbol{\lambda}(\{\bar u \neq 0\}) + k^r \|\eta\|_r^r < \infty, \end{aligned}$$

i.e., $\{\eta_k\}_{k\in\mathbb{N}} \subset L^r(\Omega)$. Furthermore, $\eta_k \in \widehat{\partial} q_{s,p}(u_k)$ follows from Theorem 4.4 since $\{u_k \neq 0\} = \Omega_k$ is valid for each $k \in \mathbb{N}$. Noting that η vanishes on Ω_k, we obtain

$$\|\tfrac{1}{k}\eta_k - \eta\|_r^r = \frac{p^r}{k^r}\int_{\Omega_k} |\bar{u}(x)|^{(p-1)r}\,\mathrm{d}x \le \frac{p^r}{k^r}\int_{\Omega_k} k^{(1-p)r}\,\mathrm{d}x \le \frac{p^r\,\boldsymbol{\lambda}(\{\bar{u}\neq 0\})}{k^{pr}}$$

for each $k \in \mathbb{N}$, and this shows $\frac{1}{k}\eta_k \to \eta$ in $L^r(\Omega)$. Particularly, we find $\eta \in \partial^\infty q_{s,p}(\bar{u})$ by definition of the singular subdifferential.

In order to prove the inclusion "$\subset$", let us fix $\eta \in \partial^\infty q_{s,p}(\bar{u})$. By definition of the singular subdifferential, we find sequences $\{u_k\}_{k\in\mathbb{N}} \subset L^s(\Omega)$, $\{\eta_k\}_{k\in\mathbb{N}} \subset L^r(\Omega)$, and $\{t_k\}_{k\in\mathbb{N}} \subset (0,\infty)$ such that $u_k \to \bar{u}$ in $L^s(\Omega)$, $t_k \searrow 0$, and $t_k\eta_k \rightharpoonup \eta$ in $L^r(\Omega)$ hold while $\eta_k \in \widehat{\partial} q_{s,p}(u_k)$ is valid for each $k \in \mathbb{N}$. Along a subsequence (without relabeling), $\{u_k\}_{k\in\mathbb{N}}$ converges pointwise almost everywhere to $\bar{u}$. Thus, for almost all $x \in \{\bar{u} \neq 0\}$, we find $u_k(x) \to \bar{u}(x) \neq 0$, i.e., $x \in \{u_k \neq 0\}$ and, due to Theorem 4.4, $\eta_k(x) = p|u_k(x)|^{p-2}u_k(x)$ for sufficiently large $k \in \mathbb{N}$. Thus, for almost every $x \in \{\bar{u} \neq 0\}$, we have $t_k\,\eta_k(x) \to 0$. Thus, the weak convergence $t_k\eta_k \rightharpoonup \eta$ in $L^r(\Omega)$ ensures that η needs to vanish almost everywhere on $\{\bar{u} \neq 0\}$, i.e., $\{\eta \neq 0\} \subset \{\bar{u} = 0\}$. □

We would like to focus the reader's attention to the fact that the limiting subdifferential $\partial q_{s,p}(\bar{u})$ might be empty for some $\bar{u} \in L^s(\Omega)$ where $\{\bar{u} \neq 0\}$ possesses positive measure while $|\bar{u}|^{p-1}\,\chi_{\{\bar{u}\neq 0\}}$ lacks of L^r-regularity. In contrast, the singular subdifferential $\partial^\infty q_{s,p}(\bar{u})$ has been shown to be never empty.

We close the section by showing that $q_{s,p}$ is nowhere Lipschitz continuous.

Corollary 4.8. *For $s \in (1,\infty)$, $q_{s,p}$ is nowhere Lipschitz continuous.*

Proof. For large parts, the proof parallels the one of Corollary 3.9. Again, the situation is easy whenever $\bar{u} \in L^s(\Omega)$ satisfies $\boldsymbol{\lambda}(\{\bar{u} = 0\}) > 0$ due to Lemma 2.1 and Theorem 4.7. Thus, we assume that $\{\bar{u} = 0\}$ is of measure zero and show that $q_{s,p}$ fails to satisfy the condition from Lemma 2.1 (b) at $\bar{u}$. Therefore, we first choose $\alpha > 0$ such that $\{|\bar{u}| \ge \alpha\}$ is of positive measure, define $\Omega_k := \{|\bar{u}| \ge \alpha/k\}$ for each $k \in \mathbb{N}$, and pick a subset $\Omega_k' \subset \Omega_k$ of positive measure for each $k \in \mathbb{N}$ such that $\boldsymbol{\lambda}(\Omega_k') \searrow 0$ holds. For each $k \in \mathbb{N}$, we define $u_k := \bar{u}\chi_{\Omega_k\setminus\Omega_k'}$. We find $u_k \to \bar{u}$ in $L^s(\Omega)$, and Lemma 2.3 guarantees $q_{s,p}(u_k) \to q_{s,p}(\bar{u})$. For each $k \in \mathbb{N}$, we define $\eta_k \in L^r(\Omega)$ by means of

$$\forall x \in \Omega\colon \quad \eta_k(x) := \begin{cases} \frac{p}{k}|\bar{u}(x)|^{p-2}\bar{u}(x) & \text{if } x \in \Omega_k \setminus \Omega_k', \\ \boldsymbol{\lambda}(\Omega_k')^{-1/r} & \text{if } x \in \Omega_k', \\ 0 & \text{otherwise.} \end{cases}$$

By construction, we have $\|\eta_k\|_r \ge 1$ for each $k \in \mathbb{N}$. On the other hand, for each $h \in L^s(\Omega)$ and $k \in \mathbb{N}$, we find

$$\begin{aligned} \left|\int_\Omega \eta_k(x)h(x)\,\mathrm{d}x\right| &\le \frac{p}{k}\int_{\Omega_k\setminus\Omega_k'} |\bar{u}(x)|^{(p-2)}\bar{u}(x)h(x)\,\mathrm{d}x + \|h\|_{s,\Omega_k'} \\ &\le \frac{p}{\alpha^{1-p}k^p}\boldsymbol{\lambda}(\{\bar{u}\neq 0\})^{1/r}\|h\|_{s,\{\bar{u}\neq 0\}} + \|h\|_{s,\Omega_k'} \end{aligned}$$

from Hölder's inequality on $\Omega_k \setminus \Omega'_k$ and Ω'_k, respectively, and this yields $\eta_k \rightharpoonup 0$ in $L^r(\Omega)$. Due to $\eta_k \in \frac{1}{k}\widehat{\partial} q_{s,p}(u_k)$ for each $k \in \mathbb{N}$, see Theorem 4.4, this shows that $q_{s,p}$ cannot be Lipschitz continuous at $\bar{u}$, see Lemma 2.1. $\square$

5 Concluding remarks

In this paper, we derived exact formulas for the Fréchet, limiting, and singular subdifferential of the functional $q_{s,p}$ defined in (1.1) and (1.2). As Remarks 3.4, 3.6 and 4.5 underline, the formulas for the Fréchet subdifferential can be used in order to derive necessary optimality conditions for the unconstrained minimization of functions $f + q_{s,p}$ where $f\colon L^s(\Omega) \to \mathbb{R}$ is Fréchet differentiable. Let us now assume that $f + q_{s,p}$ has to be minimized with respect to some constraint set $U_{\mathrm{ad}} \subset L^s(\Omega)$. Then Fermat's rule yields validity of $0 \in \widehat{\partial}(f + q_{s,q} + \delta_{U_{\mathrm{ad}}})(\bar{u})$ for each associated local minimizer $\bar{u} \in L^s(\Omega)$ of the problem where the so-called indicator function $\delta_{U_{\mathrm{ad}}}\colon L^s(\Omega) \to \mathbb{R} \cup \{\infty\}$ of U_{ad} equals 0 on U_{ad} and is set to ∞, otherwise. Note that the Fréchet subdifferential does not obey a sum rule as soon as not all but one addends are smooth. In the present situation, the simultaneous non-Lipschitzness of $q_{s,p}$ and $\delta_{U_{\mathrm{ad}}}$ does not even allow to apply the *fuzzy* sum rule of Fréchet subdifferential calculus, see (Mordukhovich, 2006, Theorem 2.33), and take the limit afterwards. Thus, one may try to evaluate the slightly weaker necessary optimality condition $0 \in \partial(f + q_{s,q} + \delta_{U_{\mathrm{ad}}})(\bar{u})$ since the sum rule for the limiting subdifferential applies to non-Lipschitz functions as well, see (Mordukhovich, 2006, Theorem 3.36). Unluckily, this will not be a straight task since both of the functionals $q_{s,p}$ and $\delta_{U_{\mathrm{ad}}}$ are not so-called sequentially normally epi-compact on their respective domains, see comments at the end of Section 2.3 and the proofs of Corollaries 3.9 and 4.8. Nevertheless, for particular choices of U_{ad} like box-constrained sets, there might be a chance to show validity of the sum rule by inherent problem structure and, thus, obtain necessary optimality conditions in terms of the limiting subdifferential.

Acknowledgments

This work is supported by the DFG Grant *Bilevel Optimal Control: Theory, Algorithms, and Applications* (Grant No. WA 3636/4-2) within the Priority Program SPP 1962 (Non-smooth and Complementarity-based Distributed Parameter Systems: Simulation and Hierarchical Optimization).

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