EMSO(FO^2) 0-1 law fails for all dense random graphs

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Abstract

In this paper, we disprove EMSO(FO^2) convergence law for the binomial random graph $G(n,p)$ for any constant probability p. More specifically, we prove that there exists an existential monadic second order sentence with 2 first order variables such that, for every $p \in (0,1)$, the probability that it is true on $G(n,p)$ does not converge.

1 Introduction

For undirected graphs, *sentences in the monadic second order logic* (MSO sentences) are constructed using relational symbols $\sim$ (interpreted as adjacency) and $=$, logical connectives $\neg, \to, \leftrightarrow, \vee, \wedge$, first order (FO) variables $x, y, x_1, \ldots$ that express vertices of a graph, MSO variables $X, Y, X_1, \ldots$ that express unary predicates, quantifiers $\forall, \exists$ and parentheses (for formal definitions, see [9]). If, in an MSO sentence ϕ, all the MSO variables are existential and in the beginning, then the sentence is called *existential* monadic second order (EMSO). For example, the EMSO sentence

$$\exists X \quad [\exists x_1 \exists x_2\ X(x_1) \wedge \neg X(x_2)] \wedge \neg[\exists y \exists z\ X(y) \wedge \neg X(z) \wedge y \sim z]$$

expresses the property of being disconnected. Note that this sentence has 1 monadic variable and 4 FO variables but it can be easily rewritten with only 2 FO variables by identifying y with x_1 and z with x_2. In what follows, for a sentence ϕ, we use the usual notation from model theory $G \models \phi$ if ϕ is true for G.

In [8], Kaufmann and Shelah disproved the MSO 0-1 law (*0-1 law for a logic* $\mathcal{L}$ states that every sentence $\varphi \in \mathcal{L}$ is either true on (asymptotically) almost all graphs on the vertex set $[n] := \{1, \ldots, n\}$ as $n \to \infty$, or false on almost all graphs). Moreover, they even disproved a weaker logical law which is called the MSO convergence law (*convergence law for a logic* $\mathcal{L}$ states that, for every sentence $\varphi \in \mathcal{L}$, the fraction of graphs on the vertex set $[n]$ satisfying φ converges as $n \to \infty$). In terms of random graphs, their result can be formulated as follows: there exists an MSO sentence φ such that $\mathsf{P}(G(n,1/2) \models \varphi)$ does not converge as $n \to \infty$.

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Recall that, for $p \in (0,1)$, the binomial random graph $G(n,p)$ is a graph on $[n]$ with each pair of vertices connected by an edge with probability p and independently of other pairs. For more information, we refer readers to the books [1, 3, 7]. In contrast, $G(n,1/2)$ obeys first-order (FO) 0-1 law [4, 5]. In 2001, Le Bars [2] disproved EMSO convergence law for $G(n,1/2)$ and conjectured that, for EMSO sentences with 2 FO variables (or, shortly, EMSO(FO^2) sentences), $G(n,1/2)$ obeys the zero-one law. In 2019, Popova and the second author [11] disproved this conjecture. Notice that all the above results but the last one can be easily generalized to arbitrary constant edge probability p. In [11], it is noticed that the Le Bars conjecture fails for a dense set of $p \in (0,1)$. In this paper, we disprove the Le Bars conjecture for all $p \in (0,1)$. We prove something even stronger: there exists a EMSO(FO^2) sentence φ such that, for every $p \in (0,1)$, $\{\mathbf{P}[G(n,p) \models \varphi]\}_n$ does not converge. Notice that this one sentence disproves the convergence law for all p. Let us define the sentence.

Let $X(k,\ell,m)$ be the number of 6-tuples $(X_1,x_1,X_2,x_2,X_3,x_3)$, consisting of sets $X_1,X_2,X_3 \subset [n]$ and vertices $x_1 \in X_1$, $x_2 \in X_2$, $x_3 \in X_3$, such that

- $|X_1| = k, |X_2| = \ell, |X_3| = m$ and $X_i \cap X_j = \varnothing$ for $i \neq j$,
- each X_i dominates $[n] \setminus (X_1 \sqcup X_2 \sqcup X_3)$, i.e. every vertex from $[n] \setminus (X_1 \sqcup X_2 \sqcup X_3)$ has at least one neighbor in each X_i,
- for any distinct $i,j \in \{1,2,3\}$, there is exactly one edge between X_i and X_j — namely, the edge between x_i and x_j.

Theorem 1. *For any constant $p \in (0,1)$, $\mathsf{P}(\exists k,\ell,m\ X(k,\ell,m) > 0)$ does not converge as $n \to \infty$.*

Clearly, the property $\{\exists k,\ell,m\ X(k,\ell,m) > 0\}$ can be defined in EMSO(FO^2), e.g., by the following sentence:

$$\exists X_1 \exists X_2 \exists X_3 \quad \mathrm{DIS}(X_1,X_2,X_3) \wedge \mathrm{DOM}(X_1,X_2,X_3) \wedge \phi_1(X_1,X_2,X_3) \wedge \phi_2(X_1,X_2,X_3),$$

where the formula

$$\mathrm{DIS}(X_1,X_2,X_3) = \bigwedge_{1 \le i < j \le 3} \Big(\forall x \forall y \quad [X_i(x) \wedge X_j(y)] \Rightarrow [x \neq y] \Big)$$

says that X_1,X_2,X_3 are disjoint; the formula

$$\mathrm{DOM}(X_1,X_2,X_3) = \forall x \quad \Big[\neg(X_1(x) \vee X_2(x) \vee X_3(x)) \Big] \Rightarrow \Big[\bigwedge_{j=1}^{3} (\exists y \ \ X_j(y) \wedge (x \sim y)) \Big]$$

says that each vertex from $[n] \setminus (X_1 \sqcup X_2 \sqcup X_3)$ has a neighbor in each X_i; the formula

$$\phi_1(X_1,X_2,X_3) = \bigwedge_{i=1}^{3} \Bigg(\exists x \quad X_i(x) \wedge \Bigg(\forall y \ \ [(y \neq x) \wedge X_i(y)] \Rightarrow \Bigg[\forall x \ \ \Bigg(\bigvee_{j \neq i} X_j(x) \Bigg) \Rightarrow (x \nsim y) \Bigg] \Bigg) \Bigg)$$

says that, for every $i \in \{1,2,3\}$, there is at most one vertex that has neighbors in sets X_j, $j \neq i$; the formula

$$\phi_2(X_1,X_2,X_3) = \bigwedge_{1\leq i<j\leq 3}\Big(\exists x \exists y \quad X_i(x) \wedge X_j(y) \wedge (x \sim y)\Big)$$

says that, for any two distinct X_i, X_j, there exists an edge between them. Clearly, $\phi_1 \wedge \phi_2$ is true if and only if there exist $x_1 \in X_1$, $x_2 \in X_2$, $x_3 \in X_3$, such that, for any distinct $i,j \in \{1,2,3\}$, there is exactly one edge between X_i and X_j — the edge between x_i and x_j.

We prove Theorem 1 in the following way. First, we show that, for some sequence of positive integers $(n_i^{(1)}, i \in \mathbb{N})$, $\sum_{k,\ell,m} \mathsf{E}X(k,\ell,m) \to 0$ (random variables are defined on $G(n_i^{(1)},p)$) as $i \to \infty$. Then, we show that, for another sequence $(n_i^{(2)}, i \in \mathbb{N})$, there exists $k = k(i)$ such that $\mathsf{P}(X(k,k,k) > 0)$ is bounded away from 0 for all large enough i (using second moment methods).

We compute $\mathsf{E}X(k,\ell,m)$ and study its behavior in Section 2. Sections 3, 4 present the sequences $n_i^{(1)}$, $n_i^{(2)}$ respectively and prove that they are as desired.

Remark. It is easy to see, using the union bound, that with asymptotical probability 1 in $G(n,p)$, there are no three sets X_1, X_2, X_3 such that each X_i dominates $[n] \setminus (X_1 \sqcup X_2 \sqcup X_3)$ and there are no edges between distinct X_i and X_j. It means that there exists a sequence $\{n_i\}_i$ such that, with a probability that is bounded away from 0 for large enough i, one can remove at most 3 edges from $G(n_i,p)$ such that the modified graph and $G(n_i,p)$ are EMSO(FO^2)–distinguishable. On the other hand, it is impossible to remove a bounded number of edges from $G(n,p)$ to make it FO–distinguishable from the original graph (with a probability that is bounded away from 0 for large enough n). Indeed, the FO almost sure theory $\mathcal{T}$ of $G(n,p)$ is complete and its set of axioms $\mathcal{E}$ consists of so called extension axioms (see, e.g., [12]). It is straightforward that all axioms from $\mathcal{E}$ hold with asymptotical probability 1 after a deletion of any bounded set of edges from $G(n,p)$. From the completeness and the FO 0-1 law, our observation follows.

2 Expectation

Let $D_n := \{x,y,z \geq 1 : x+y+z \leq n\}$ and consider integers $k,\ell,m \in D_n$. Then, clearly,

$$\mathsf{E}X(k,\ell,m) = \frac{n!}{k!\ell!m!(n-k-\ell-m)!}(k\cdot\ell\cdot m) \times (1-p)^{k\ell+\ell m+km-3}p^3 \times \\ \times \prod_{v\in[n]\setminus(X_1\cup X_2\cup X_3)} \left[(1-(1-p)^k)(1-(1-p)^\ell)(1-(1-p)^m)\right] \leq \quad (1)$$

$$\frac{n^{k+\ell+m}e^{k+\ell+m}}{k^k\ell^\ell m^m}\exp\Big(\ln(k\ell m) + (k\ell+km+\ell m-3)\ln(1-p) + 3\ln p \\ - (n-k-\ell-m)[(1-p)^k+(1-p)^\ell+(1-p)^m]\Big) = e^{f(k,\ell,m)+g(k,\ell,m)}, \quad (2)$$

where f and g are two functions defined on D_n as follows:

$$f(k,\ell,m) = k\ln(n/k) + \ell\ln(n/\ell) + m\ln(n/m) + \ln(k\ell m) + k + \ell + m \\ - n((1-p)^k + (1-p)^\ell + (1-p)^m) + (k\ell + km + \ell m - 3)\ln(1-p) + 3\ln p, \quad (3)$$

$$g(k,\ell,m) = (k+\ell+m)[(1-p)^k + (1-p)^\ell + (1-p)^m]. \quad (4)$$

Let us now compute the partial derivatives:

$$\frac{\partial f}{\partial k} = \ln\frac{n}{k} + (\ell+m)\ln(1-p) + \frac{1}{k} - n(1-p)^k\ln(1-p) + 1,$$
$$\frac{\partial^2 f}{\partial k^2} = -\frac{1}{k} - \frac{1}{k^2} - n(1-p)^k\ln^2(1-p),$$
$$\frac{\partial^2 f}{\partial k\partial\ell} = \frac{\partial^2 f}{\partial\ell\partial m} = \frac{\partial^2 f}{\partial k\partial m} = \ln(1-p).$$

Other derivatives can be obtained by using the symmetry of f. Let us find k^* such that $\frac{\partial f}{\partial k}\big|_{(k^*,k^*,k^*)} = \frac{\partial f}{\partial \ell}\big|_{(k^*,k^*,k^*)} = \frac{\partial f}{\partial m}\big|_{(k^*,k^*,k^*)} = 0$. There is exactly one such k^* since the equation

$$\ln\frac{n}{k} + 2k\ln(1-p) + \frac{1}{k} - n(1-p)^k\ln(1-p) + 1 = 0$$

has the unique solution

$$k^* = \frac{\ln n - \ln\ln n + \ln\ln\frac{1}{1-p}}{\ln\frac{1}{1-p}} + O\left(\frac{\ln\ln n}{\ln n}\right). \quad (5)$$

Let us show that $A = (k^*,k^*,k^*)$ is a point of local maximum of f for all n large enough. Consider the Hessian matrix

$$C = \begin{pmatrix} \frac{\partial^2 f}{\partial k^2}\big|_A & \frac{\partial^2 f}{\partial k\partial\ell}\big|_A & \frac{\partial^2 f}{\partial k\partial m}\big|_A \\ \frac{\partial^2 f}{\partial k\partial\ell}\big|_A & \frac{\partial^2 f}{\partial \ell^2}\big|_A & \frac{\partial^2 f}{\partial \ell\partial m}\big|_A \\ \frac{\partial^2 f}{\partial k\partial m}\big|_A & \frac{\partial^2 f}{\partial \ell\partial m}\big|_A & \frac{\partial^2 f}{\partial m^2}\big|_A \end{pmatrix} = \ln(1-p)\begin{pmatrix} \ln n(1+o(1)) & 1 & 1 \\ 1 & \ln n(1+o(1)) & 1 \\ 1 & 1 & \ln n(1+o(1)) \end{pmatrix}.$$

By Sylvester's criterion [6, Theorem 7.2.5], it is negative-definite for all n large enough: the leading principal minors equal

$$\ln(1-p)\ln n(1+o(1)) < 0,$$
$$\det\left[\ln(1-p)\begin{pmatrix} \ln n(1+o(1)) & 1 \\ 1 & \ln n(1+o(1)) \end{pmatrix}\right] = \ln^2(1-p)\ln^2 n(1+o(1)) > 0,$$
$$\det C = \ln^3(1-p)\ln^3 n(1+o(1)) < 0.$$

Therefore, A is indeed a local maximum point.

We have

$$\begin{aligned}
f(A) &= 3k^*\left(\ln n - \ln k^*\right) - 3n(1-p)^{k^*} + 3(k^*)^2\ln(1-p) + 3k^* + O(\ln\ln n)\\
&= 3k^*\left(\ln n - \ln k^* + k^*\ln(1-p) + 1\right) - \frac{3\ln n}{\ln\frac{1}{1-p}} + O(\ln\ln n)\\
&= 3k^* - \frac{3\ln n}{\ln\frac{1}{1-p}} + O(\ln\ln n) = O(\ln\ln n).
\end{aligned}$$

Notice that k^* is not necessarily an integer. In Section 3, we show that n can be chosen in a way such that $k^* = \lfloor k^*\rfloor + \frac{1}{2} + o(1)$. In this case, the following lemma appears to be useful for bounding from above $\mathsf{E}X(k,\ell,m)$ for all possible k,ℓ,m (in particular, it implies that, for such n, $f(A)$ bounds from above $f(k,\ell,m)$ for all integer $(k,\ell,m)\in D_n$).

Lemma 2. *Uniformly over all $(k,\ell,m)\in D_n$ such that $\min\{|k-k^*|,|\ell-k^*|,|m-k^*|\}\geq\frac{1}{2}+o(1)$,*

$$f(k,\ell,m) \leq -\frac{\ln\frac{1}{1-p}}{2}\ln n(1+o(1))\Big[(k-k^*)^2+(\ell-k^*)^2+(m-k^*)^2\Big]. \tag{6}$$

Proof. Let us set $\Delta_1 = k-k^*, \Delta_2 = \ell-\ell^*, \Delta_3 = m-m^*$. Due to (3),

$$f(k,\ell,m) - f(k^*,k^*,k^*) \quad \leq \quad -\ln\frac{1}{1-p}(\Delta_1\Delta_2+\Delta_1\Delta_3+\Delta_2\Delta_3) \quad +$$

$$\sum_{i=1}^{3}\left(\Delta_i\ln n - \ln\frac{(k^*+\Delta_i)^{k^*+\Delta_i}}{(k^*)^{k^*}} - n(1-p)^{k^*}\left((1-p)^{\Delta_i}-1\right) - 2\Delta_i k^*\ln\frac{1}{1-p} + 2|\Delta_i|\right) \leq$$

$$\sum_{i=1}^{3}\left(\frac{\Delta_i^2\ln\frac{1}{1-p}}{2} - k^*\ln\frac{k^*+\Delta_i}{k^*} - \Delta_i\ln k^* - \frac{\ln n\left((1-p)^{\Delta_i}-1+o(1)\right)}{\ln\frac{1}{1-p}} - \Delta_i\ln n(1+o(1))\right),$$

where the last inequality follows from the inequalities $-\Delta_1\Delta_2-\Delta_1\Delta_3-\Delta_2\Delta_3\leq\frac{1}{2}(\Delta_1^2+\Delta_2^2+\Delta_3^2)$ and $-\Delta_i\ln\left(1+\frac{\Delta_i}{k^*}\right)\leq 0$.

Notice that $-k^*\ln(1+\Delta_i/k^*)\leq-\Delta_i\ln k^* I(\Delta_i\leq 0)$. Indeed, for positive Δ_i, the inequality is obvious. If $\Delta_i\leq 0$, then it is sufficient to verify the inequality only for boundary values $\Delta_i=0$ and $\Delta_i=1-k^*$ (the function $-k^*\ln(1+x/k^*)+x\ln k^*$ changes its monotonicity only once on $[1-k^*,0]$: first, it decreases and, after $x=k^*/\ln k^*-k^*$, it increases). We get

$$f(k,\ell,m)-f(k^*,k^*,k^*)\leq\sum_{i=1}^{3}\left(\frac{\Delta_i^2\ln\frac{1}{1-p}}{2}-\Delta_i\ln n-\frac{\ln n}{\ln\frac{1}{1-p}}\left((1-p)^{\Delta_i}-1\right)\right)(1+o(1)) =$$

$$\left[\sum_{i=1}^{3}\frac{\Delta_i^2\ln\frac{1}{1-p}}{2}-\frac{\ln n}{\ln\frac{1}{1-p}}\sum_{i=1}^{3}\gamma\left(\Delta_i\ln\frac{1}{1-p}\right)\right](1+o(1))\leq\frac{1+o(1)}{2}\ln\frac{1}{1-p}\sum_{i=1}^{3}\Delta_i^2(1-\ln n),$$

where $\gamma(x)=x+e^{-x}-1\leq x^2/2$ for all $x>0$. Inequality (6) follows.

□

3 A sequence of small probabilities

Let us find a sequence $(n_i^{(1)},\ i \in \mathbb{N})$ such that $\mathsf{P}(\exists k,\ell,m\ X(k,\ell,m) > 0) \to 0$ as $i \to \infty$. For $i \in \mathbb{N}$, set

$$n := n_i^{(1)} = \left\lfloor \left(\frac{1}{1-p}\right)^{i+\frac{1}{2}} i \right\rfloor.$$

Clearly, $k^* = k^*(n) = i + \frac{1}{2} + o(1)$ (k^* is defined in (5)).
Using Lemma 2 and inequality (2), we get that, uniformly over all $k,\ell,m \in D_n$,

$$\mathsf{E}X(k,\ell,m) \leq e^{-\frac{1}{2}\ln\frac{1}{1-p}\ln n(1+o(1))\left[(k-k^*)^2+(\ell-k^*)^2+(m-k^*)^2\right]+g(k,\ell,m)}.$$

Notice that

$$g(k,\ell,m) < 3\Big[|k-k^*|+|\ell-k^*|+|m-k^*|\Big] + 3k^*\Big[(1-p)^k+(1-p)^\ell+(1-p)^m\Big]$$

and

$$3k^*\Big[(1-p)^k+(1-p)^\ell+(1-p)^m\Big] = o(1)\ln n\Big[(k-k^*)^2+(\ell-k^*)^2+(m-k^*)^2\Big].$$

Therefore,

$$\mathsf{E}X(k,\ell,m) \leq e^{-\frac{1}{2}\ln\frac{1}{1-p}\ln n(1+o(1))\left[(k-k^*)^2+(\ell-k^*)^2+(m-k^*)^2\right]}.$$

By the union bound and Markov's inequality,

$$\mathsf{P}\Big(\exists k,\ell,m \in D_n \quad X(k,\ell,m) > 0\Big) \leq \sum_{k,\ell,m\in D_n} \mathsf{E}X(k,\ell,m) \leq \left[\sum_{j=1}^{\infty} e^{-\frac{j}{8}\ln\frac{1}{1-p}\ln n(1+o(1))}\right]^3 = o(1).$$

Therefore, $(n_i^{(1)},\ i \in \mathbb{N})$ is the desired sequence.

4 A sequence of large probabilities

Here, we introduce a sequence $(n_i^{(2)},\ i \in \mathbb{N})$, such that, for some $k = k(n_i^{(2)})$, $\mathsf{P}(X(k,k,k) > 0)$ is bounded away from 0 for all i large enough. For $i \in \mathbb{N}$, define

$$n_i^{(2)} = \left\lfloor \left(\frac{1}{1-p}\right)^{i} i \right\rfloor. \qquad (7)$$

Notice that $k^* = k^*(n_i^{(2)}) = i + o(1)$, where k^* is defined in (5). Setting $n = n_k^{(2)}$ for any

$k \in \mathbb{N}$, we have

$$\begin{aligned}
\mathsf{E}X(k,k,k) &= \frac{n!}{k!k!k!(n-3k)!}k^3(1-p)^{3k^2-3}\cdot p^3\cdot\left[(1-(1-p)^k)\right]^{3(n-3k)} \\
&= \frac{n^n\sqrt{2\pi n}}{k^{3k}\sqrt{(2\pi k)^3}\cdot(n-3k)^{n-3k}\sqrt{2\pi(n-3k)}}\cdot k^3(1-p)^{3k^2-3}p^3\cdot e^{-3n(1-p)^k}(1+o(1)) \\
&= \frac{n^{3k}e^{3k}}{k^{3k}\sqrt{(2\pi)^3}}\left(\frac{p}{1-p}\right)^3 k^{3/2}(1-p)^{3k^2}e^{-3k}(1+o(1)) \\
&= \left(\frac{p}{1-p}\right)^3\frac{k^{3/2}}{\sqrt{(2\pi)^3}}(1+o(1)).
\end{aligned} \tag{8}$$

So, $\mathsf{E}X(k,k,k)\to\infty$ as $k\to\infty$. It remains to prove that $[\mathsf{E}X(k,k,k)]^2/\mathsf{E}X^2(k,k,k)$ is bounded away from 0 and apply the Paley–Zygmund inequality [10] (stated below).

Theorem 3 (Paley–Zygmund inequality)**.** *Let X be a non-negative random variable with $\mathsf{E}X^2<\infty$. Then for any $0\le\lambda<1$,*

$$\mathsf{P}[X>\lambda\mathsf{E}X]\ge(1-\lambda)^2\frac{(\mathsf{E}X)^2}{\mathsf{E}\left[X^2\right]}.$$

4.1 Second moment

Let us call a tuple $(X_1,x_1,X_2,x_2,X_3,x_3)$ a *k-tuple* if sets $X_1,X_2,X_3\subset[n]$ are disjoint, $|X_1|=|X_2|=|X_3|=k$ and $x_1\in X_1$, $x_2\in X_2$, $x_3\in X_3$. Let us call a k-tuple $(X_1,x_1,X_2,x_2,X_3,x_3)$ *special*, if it satisfies the conditions given in Section 1:

- every vertex v from $[n]\setminus(X_1\cup X_2\cup X_3)$ has at least one neighbor in each X_i,
- for any distinct $i,j\in\{1,2,3\}$, there is exactly one edge between X_i and X_j — namely, the edge between x_i and x_j.

Let

$$\mathcal{X}=(X_1,x_1,X_2,x_2,X_3,x_3)\qquad\text{and}\qquad\mathcal{Y}=(Y_1,y_1,Y_2,y_2,Y_3,y_3), \tag{9}$$

be two k-tuples. Everywhere below, we denote

$$r:=|(X_1\sqcup X_2\sqcup X_3)\cap(Y_1\sqcup Y_2\sqcup Y_3)|,$$

$$r_i:=|Y_i\cap(X_1\sqcup X_2\sqcup X_3)|,\ \ r_{j+3}:=|X_j\cap(Y_1\sqcup Y_2\sqcup Y_3)|,\ \ r_{i,j}:=|Y_i\cap X_j|.$$

Let Γ be the set of all k-tuples. For $\mathcal{X}\in\Gamma$, let $\xi_{\mathcal{X}}$ be the Bernoulli random variable that equals 1 if and only if $\mathcal{X}$ is special. Then $X(k,k,k)=\sum_{\mathcal{X}\in\Gamma}\xi_{\mathcal{X}}$. From this,

$$\mathsf{E}X^2(k,k,k)=\sum_{\mathcal{X},\mathcal{Y}\in\Gamma}\xi_{\mathcal{X}}\xi_{\mathcal{Y}}.$$

We compute this value in the usual way by dividing the summation into parts with respect to the value of r:

$$\mathsf{E}X^2(k,k,k)=S_1+S_2+S_3, \tag{10}$$

- $S_1 = \sum_{\mathcal{X},\mathcal{Y}\in\Gamma:\ r\in(r_0,3k-r_0)} \xi_{\mathcal{X}}\xi_{\mathcal{Y}}$,
- $S_2 = \sum_{\mathcal{X},\mathcal{Y}\in\Gamma:\ r\leq r_0} \xi_{\mathcal{X}}\xi_{\mathcal{Y}}$,
- $S_3 = \sum_{\mathcal{X},\mathcal{Y}\in\Gamma:\ r\geq 3k-r_0} \xi_{\mathcal{X}}\xi_{\mathcal{Y}}$,

where $r_0 = \left\lceil \frac{16}{\ln[1/(1-p)]} \right\rceil$.

In Section 4.3, we give upper bounds on S_1 and S_2. An upper bound on S_3 is given in Section 4.4. In Section 4.5, we apply the Paley–Zygmund inequality and finish the proof. Auxiliary lemmas that are used for bounds on S_i are given in Section 4.2.

4.2 Auxiliary lemmas

For a k-tuple $\mathcal{X} = (X_1, x_1, X_2, x_2, X_3, x_3)$, let $\mathcal{N}_{\mathcal{X}}$ be the event saying that there are no edges between X_1, X_2, X_3 except for those between x_1, x_2, x_3.

Lemma 4. *Let $\mathcal{X}, \mathcal{Y}$ be k-tuples. Then*

$$\mathsf{P}(\mathcal{N}_{\mathcal{Y}}|\mathcal{N}_{\mathcal{X}}) \leq (1-p)^{3k^2-\frac{r^2}{3}-3}.$$

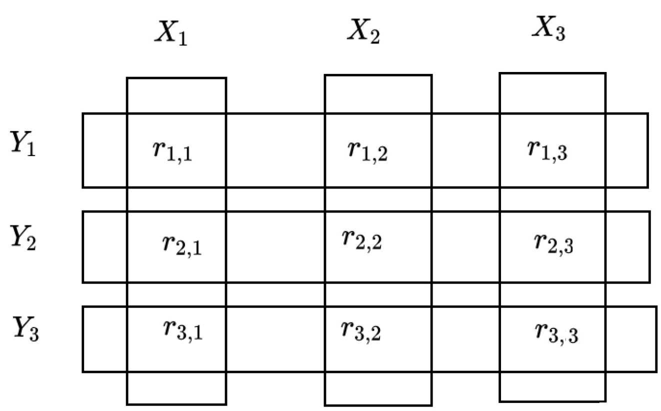

Figure 1: two k-tuples X and Y with given intersections $r_{i,j}$, $i, j \in \{1, 2, 3\}$.

Proof. The number of pairs $(u, v) \in [n^2]$ such that $u \in Y_i, v \in Y_j$ for some $i \neq j$ but u and v do not belong to $X_{\tilde{i}}, X_{\tilde{j}}$ for any distinct $\tilde{i}, \tilde{j} \in \{1, 2, 3\}$ equals

$$3k^2 - \frac{1}{2}\sum_{i,j} r_{i,j} \sum_{\tilde{i}\neq i, \tilde{j}\neq j} r_{\tilde{i},\tilde{j}} \geq 3k^2 - (r_1r_2 + r_2r_3 + r_3r_1) \geq 3k^2 - \frac{r^2}{3},$$

where we used that

$$\frac{r^2}{9} = \left(\frac{r_1 + r_2 + r_3}{3}\right)^2 \geq \frac{r_1r_2 + r_2r_3 + r_3r_1}{3}. \tag{11}$$

Finally, it remains to notice that conditional probability $\mathsf{P}(\mathcal{N}_{\mathcal{Y}}|\mathcal{N}_{\mathcal{X}})$ does not exceed $(1-p)^{3k^2-\frac{r^2}{3}-3}$ since we should exclude no more than 3 pairs of vertices (u, v) that coincide with a pair of vertices from y_1, y_2, y_3. □

Lemma 5. *Let $\mathcal{X},\mathcal{Y}$ be k-tuples and there exists $s \in \{1,2,3\}$ such that $r_s > k - r_0$ and $r_{s,j} < k - 6r_0$ for all $j \in \{1,2,3\}$. If $r_0 < \frac{1}{30}k$, then*

$$\mathsf{P}(\mathcal{N}_{\mathcal{Y}}|\mathcal{N}_{\mathcal{X}}) \le (1-p)^{3k^2 - \frac{r^2}{3} - 3 + 4kr_0}.$$

Proof. Repeating previous arguments, it is sufficient to prove that $\frac{1}{2}\sum_{i,j} r_{i,j} \sum_{\tilde{i}\neq i, \tilde{j}\neq j} r_{\tilde{i},\tilde{j}} \le \frac{r^2}{3} - 4kr_0$. Without loss of generality, let us assume that $r_{s,1} \ge r_{s,2} \ge r_{s,3}$.

Applying (11) for r_4, r_5, r_6, we get

$$\frac{1}{2}\sum_{i,j} r_{i,j} \sum_{p\neq i, q\neq j} r_{p,q} = (r_4r_5 + r_4r_6 + r_5r_6) - \sum_{i=1}^{3}(r_{i,1}r_{i,2} + r_{i,2}r_{i,3} + r_{i,1}r_{i,3}) \le$$
$$(r_4r_5 + r_4r_6 + r_5r_6) - (r_{s,1}r_{s,2} + r_{s,1}r_{s,3} + r_{s,2}r_{s,3}) \le$$
$$\frac{r^2}{3} - (r_{s,1}r_{s,2} + r_{s,1}r_{s,3} + r_{s,2}r_{s,3}) \le \frac{r^2}{3} - r_{s,1}(r_s - r_{s,1}) < \frac{r^2}{3} - r_{s,1}(k - r_0 - r_{s,1}) \le$$
$$\frac{r^2}{3} - (k-6r_0)(k - r_0 - (k-6r_0)) = \frac{r^2}{3} - 5r_0(k-6r_0) \le \frac{r^2}{3} - 4r_0k \quad (12)$$

since the function $f(x) = x(k - r_0 - x)$ is concave on $[\frac{k-r_0}{3}, k - 6r_0]$, and so, it achieves the minimum value at one of the ends of the segment. Clearly, the value at the right end is smaller. □

Lemma 6. *Let $r_0 > 0$ be a fixed number. Let $\mathcal{X},\mathcal{Y}$ be k-tuples (9) and $r \le r_0$. Then the probability that each X_j, $j \in \{1,2,3\}$, and each Y_i, $i \in \{1,2,3\}$, are dominating sets in $[n] \setminus ((X_1 \sqcup X_2 \sqcup X_3) \cup (Y_1 \sqcup Y_2 \sqcup Y_3))$ does not exceed $(1 - 6(1-p)^k)^n(1+o(1))$.*

Proof. Fix a vertex $v \in [n] \setminus ((X_1 \sqcup X_2 \sqcup X_3) \cup (Y_1 \sqcup Y_2 \sqcup Y_3))$. Set $X := X_1 \sqcup X_2 \sqcup X_3$, $Y := Y_1 \sqcup Y_2 \sqcup Y_3$. Then $\{v$ has neighbors in each X_j and each $Y_i\} \subset \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$, where

- $\mathcal{A} = \Big\{\forall i \in \{1,2,3\} \quad v \text{ has neighbors both in } X_i \setminus Y \text{ and in } Y_i \setminus X\Big\}$,
- $\mathcal{B} = \Big\{\exists i \in \{1,2,3\} \quad v \text{ has a neighbor in } X_i \cap Y \text{ and does not have a neighbor in } X_i \setminus Y\Big\}$,
- $\mathcal{C} = \Big\{\exists i \in \{1,2,3\} \quad v \text{ has a neighbor in } Y_i \cap X \text{ and does not have a neighbor in } Y_i \setminus X\Big\}$.

Clearly,

$$\mathsf{P}(\mathcal{A}) = \prod_{i=1}^{6}(1 - (1-p)^{k-r_i}).$$

$\mathsf{P}(v \text{ has a neighbor in } Y_i \cap X \text{ and does not have a neighbor in } Y_i \setminus X) = (1-p)^{k-r_i}(1-(1-p)^{r_i})$,

$\mathsf{P}(v \text{ has a neighbor in } X_j \cap Y \text{ and does not have a neighbor in } X_j \setminus Y) = (1-p)^{k-r_{j+3}}(1-(1-p)^{r_{j+3}})$.

Therefore,

$$\mathsf{P}(\mathcal{A}\cup\mathcal{B}\cup\mathcal{C}) \leq \prod_{i=1}^{6}(1-(1-p)^{k-r_i}) + \sum_{i=1}^{6}(1-p)^{k-r_i}(1-(1-p)^{r_i}) =$$

$$1-\sum_{i=1}^{6}(1-p)^{k-r_i} + O\left((1-p)^{2k}\right) + \sum_{i=1}^{6}((1-p)^{k-r_i}-(1-p)^k) = 1-6(1-p)^k + O\left((1-p)^{2k}\right).$$

So, multiplying over $v\in n\setminus((X_1\sqcup X_2\sqcup X_3)\cup(Y_1\sqcup Y_2\sqcup Y_3))$ and recalling that $n=\lfloor k/(1-p)^k\rfloor$, we get the desired bound. □

4.3 Small and medium intersections

In this section, we estimate S_1 and S_2.

By Lemma 4 and Lemma 6,

$$S_1 \leq \sum_{r=r_0}^{3k-r_0}\binom{n}{3k}3^{3k}k^3\binom{n}{3k-r}\binom{3k}{r}3^{3k}k^3(1-p)^{3k^2-3}(1-p)^{3k^2-\frac{r^2}{3}-3}, \tag{13}$$

$$S_2 \leq \sum_{r\leq r_0}\frac{n!}{k!k!k!(n-3k)!}(1-p)^{3k^2-3}k^3\sum_{s_1,s_2,s_3}\frac{n^{s_1}n^{s_2}n^{s_3}}{s_1!s_2!s_3!}(1-p)^{3k^2-\frac{r^2}{3}-3}k^3(1-6(1-p)^k)^n(1+o(1)), \tag{14}$$

where the second summation is over all non-negative integers s_1,s_2,s_3 such that $s_1+s_2+s_3=3k-r$ and, for each $i\in\{1,2,3\}$, $|s_i-k|\leq r_0$.

From (13), we get that, for k large enough,

$$S_1 \leq \sum_{r=r_0}^{3k-r_0}\left(\frac{en}{3k}\right)^{3k}\left(\frac{en}{3k-r}\right)^{3k-r}2^{3k}3^{6k}k^6(1-p)^{6k^2-6-\frac{r^2}{3}} \leq$$

$$\sum_{r=r_0}^{3k-r_0}\left(\frac{n}{k}\right)^{6k-r}\frac{\left(3^6 2^3\right)^k k^6 e^{6k-r}}{\left(1-\frac{r}{3k}\right)^{3k-r}}(1-p)^{6k^2-6-\frac{1}{3}r^2} \leq \sum_{r=r_0}^{3k-r_0}(1-p)^{rk-6-\frac{r^2}{3}}e^{15k},$$

where the last inequality is obtained from

- the definition of n (we get $n\leq k(1-p)^{-k}$),
- the inequality $\ln(1-x)>-x-x^2$ that is true for all $x\in(0,1)$ (it is applied here with $x=\frac{r}{3k}$),
- and the inequality $k^6<C^k$ that is true for any $C>1$ and large enough k (it is applied here with $C=\frac{e^9}{3^6 2^3}>1$).

Finally, we get that

$$S_1 < \sum_{r=r_0}^{3k-r_0} \left((1-p)^{r-\frac{6}{k}-\frac{r^2}{3k}} \cdot e^{15}\right)^k = o(1) \tag{15}$$

since, for $r \in [r_0, 3k-r_0]$, we have $r - \frac{6}{k} - \frac{r^2}{3k} \geq r_0(1 - \frac{6}{kr_0} - \frac{r_0}{3k}) = r_0(1-o(1))$ and due to the choice of $r_0 > \frac{16}{\ln[1/(1-p)]}$.

It remains to bound S_2. For k large enough, we get

$$S_2 \leq \frac{(n/k)^{3k}}{k^{3/2}\left(1-\frac{3k}{n}\right)^{n-3k}}(1-p)^{6k^2-\frac{r_0^2}{3}-6}k^6(1-6(1-p)^k)^n \times \sum_{r\leq r_0}\sum_{s_1,s_2,s_3}\frac{n^{s_1}n^{s_2}n^{s_3}}{s_1!s_2!s_3!} \leq$$
$$c_1 e^{3k}k^{9/2}(1-p)^{3k^2}(1-6(1-p)^k)^n \sum_{r\leq r_0}\sum_{s_1,s_2,s_3}\frac{n^{3k-r}}{s_1!s_2!s_3!}$$

for some positive constant c_1, where the last inequality follows from the definition of n and the inequality $\ln(1-x) \geq -\frac{x}{1-x}$ applied to $x = \frac{3k}{n}$.

Notice that, for s_1, s_2, s_3 such that $s_1+s_2+s_3 = 3k-r$ and $|s_i - k| \leq r_0$, we get that

$$s_1!s_2!s_3! \geq \sqrt{s_1s_2s_3}\,s_1^{s_1}s_2^{s_2}s_3^{s_3}e^{-s_1-s_2-s_3} \geq \sqrt{(k-r_0)^3}\left(k-\frac{r}{3}\right)^{3k-r}e^{r-3k}$$

since the minimum value of $s_1^{s_1}s_2^{s_2}s_3^{s_3}$ is achieved when $s_1 = s_2 = s_3$. Therefore, we get

$$\begin{aligned} S_2 &\leq c_2 e^{3k}k^3(1-p)^{3k^2}\left(1-6(1-p)^k\right)^n \sum_{r\leq r_0}\left(\frac{en}{k-r/3}\right)^{3k-r} \\ &\leq r_0 c_2 e^{3k}k^3(1-p)^{3k^2}e^{-6n(1-p)^k}\left(\frac{en}{k}\right)^{3k} \leq c_3k^3 \end{aligned} \tag{16}$$

for some positive constants c_2 and c_3.

4.4 Large intersections

In this section we produce an upper bound for S_3. We let $S_3 = S_3^1 + S_3^2$, where S_3^1 is the summation over all $\mathcal{X}, \mathcal{Y}$ such that $r > 3k - r_0$ and for each Y_i there exists X_j which almost coincides with Y_i (see Figure 2):

$$\forall i \in \{1,2,3\}\ \exists j \in \{1,2,3\}: \quad r_{i,j} \geq k - 6r_0. \tag{17}$$

Notice that given $\mathcal{X}$ and r_4, r_5, r_6, the number of ways to choose $Y_i \cap (X_1 \sqcup X_2 \sqcup X_3)$, $i \in \{1,2,3\}$, is bounded from above by

$$\binom{k}{k-r_4}\binom{k}{k-r_5}\binom{k}{k-r_6}3^{3k-r} \leq (3k)^{3k-r}.$$

Also, given $\mathcal{X}$ and Y_1, Y_2, Y_3 such that (17) holds, the number of choices of y_1, y_2, y_3 such that $(Y_1, y_1, Y_2, y_2, Y_3, y_3)$ has a chance to be special is $O(k)$ since, for every possible choice

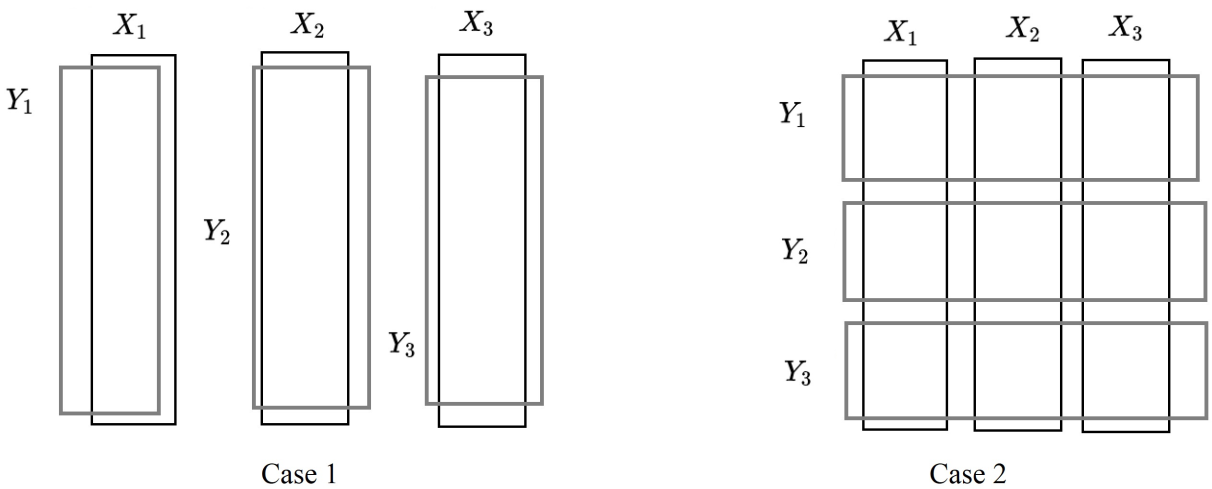

Figure 2: two types of intersections between Y_i and X_j. Case 1 is presented in simplified form (in general, each Y_i may have a non-empty intersection with each X_j).

of (y_1, y_2, y_3), there exists $j \in \{1, 2, 3\}$ such that, for every $i \in \{1, 2, 3\}$, y_i belongs to either $\{x_1, x_2, x_3\}$, or $(Y_1 \sqcup Y_2 \sqcup Y_3) \setminus (X_1 \sqcup X_2 \sqcup X_3)$ (this set has a bounded size), or X_j. Indeed, it is not possible that two y-vertices belong to different X-sets and do not belong to $\{x_1, x_2, x_3\}$ because there are no edges between X_1, X_2, X_3 other than those between x_1, x_2, x_3.

Finally, given $\mathcal{X}$ and $\mathcal{Y}$,

$$\mathsf{P}(\mathcal{N}_{\mathcal{Y}} | \mathcal{N}_{\mathcal{X}}) \leq (1-p)^{-3+\sum_{i\in\{1,2,3\}}(k-r_i)(r-r_i)}.$$

Since $r_1 + r_2 + r_3 = r$, we get

$$\sum_i (k-r_i)(r-r_i) = 3kr - k\sum_i r_i - r^2 + \sum_i r_i^2 = 2kr - r^2 + \sum_i r_i \geq 2kr - r^2 + r^2/3 = 2r(3k-r)/3.$$

Combining all the above bounds, we get that there exists $C > 0$ such that

$$S_3^1 \leq \frac{n!}{k!k!k!(n-3k)!} p^3 (1-p)^{3k^2-3} k^3 (1-(1-p)^k)^{3(n-6k)} \times \\ Ckn^{3k-r}(3k)^{3k-r}(1-p)^{2r(3k-r)/3}. \quad (18)$$

The product in the first line of (18) asymptotically equals to $\mathsf{E}X(k,k,k) = O(k^{3/2})$ due to (8). Moreover,

$$\left(3nk(1-p)^{2r/3}\right)^{3k-r} \leq \left(3\left(\frac{1}{1-p}\right)^k k^2 (1-p)^{2k-2r_0/3}\right)^{3k-r} = O(1).$$

Therefore, $S_3^1 = O(k^{5/2})$.

It remains to bounds S_3^2. Applying Lemma 5 and inequalities $3k-r\leq r_0$, $r_0\geq 16/\ln\frac{1}{1-p}$, we get

$$S_3^2 \leq \left[\frac{n!}{k!k!k!(n-3k)!}(1-p)^{3k^2-3}k^3(1-(1-p)^k)^{3(n-6k)}\right] n^{3k-r}3^{3k}k^3(1-p)^{4kr_0-3} =$$
$$O\left(k^{4.5}3^{3k}\left(\frac{k}{(1-p)^k}\right)^{3k-r}(1-p)^{4kr_0}\right) = O\left(k^{4.5}3^{3k}k^{r_0}(1-p)^{3kr_0}\right) = o(1).$$

Finally,

$$S_3 = S_3^1 + S_3^2 = O(k^{5/2}). \tag{19}$$

4.5 Final steps

Set $X = X(k,k,k)$. Due to (8), $\mathsf{E}X \sim \frac{p^3}{\left(\sqrt{2\pi}(1-p)\right)^3}k^{3/2}$. On the other hand, combining (10) with bounds (15), (16), (19), we get that there exists $c>0$ such that $\mathsf{E}X^2 = S_1+S_2+S_3 < ck^3$.

By Paley-Zygmund inequality (Theorem 3), for k large enough,

$$\mathbb{P}(X>0) \geq \frac{(\mathbb{E}X)^2}{\mathbb{E}X^2} > \frac{p^6}{2\pi(1-p)^6 c}.$$

Therefore, $(n_i^{(2)}, i\in\mathbb{N})$ is the desired sequence.

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