# SETS AVOIDING SIX-TERM ARITHMETIC PROGRESSIONS IN  $\mathbb{Z}_6^n$  are EXPONENTIALLY SMALL

PÉTER PÁL PACH AND RICHÁRD PALINCZA

ABSTRACT. We show that sets avoiding 6-term arithmetic progressions in  $\mathbb{Z}_6^n$  have size at most  $5.709^n$ . It is also pointed out that the "product construction" does not work in this setting, specially, we show that for the extremal sizes in small dimensions we have  $r_6(\mathbb{Z}_6) = 5$ ,  $r_6(\mathbb{Z}_6^2) = 25$  and  $116 \le r_6(\mathbb{Z}_6^3) \le 124$ .

#### 1. INTRODUCTION

There has been great interest in finding progression-free sets in  $\mathbb{Z}_m^n := (\mathbb{Z}/(m\mathbb{Z}))^n$ , especially when  $m = 3$  or 4. Let  $r_k(\mathbb{Z}_m^n)$  denote the maximal size of a set  $A \subset \mathbb{Z}_m^n$  with no k distinct elements in arithmetic progression. Note that for  $m = 3, 4, 5$  the properties "no arithmetic progression of length 3 modulo  $m^{\prime\prime}$  and "no 3 points on any line" are equivalent. The last property is also well known under the name caps.

The following is known [\[3,](#page-6-0) [4,](#page-6-1) [5,](#page-6-2) [2\]](#page-6-3) for the cases  $k = 3, m \in \{3, 4\}$ :

$$
2.21738...n \le r_3(\mathbb{Z}_3^n) \le 2.755...n,
$$
  

$$
3n/\sqrt{n} \ll r_3(\mathbb{Z}_4^n) \le 3.61...n,
$$

and more generally, for primes  $p \geq 3$  and some positive constant  $\delta_p$ 

 $r_3(\mathbb{Z}_p^n) \leq (p - \delta_p)^n$ .

(Note that the lower bound for  $r_3(\mathbb{Z}_3^n)$  holds only for sufficiently large values of n, the upper bounds hold for every *n*.) Indeed the argument yields [\[1\]](#page-6-4) the bound

$$
r_3(\mathbb{Z}_p^n) \le (J(p)p)^n,
$$

where

(1.1) 
$$
J(p) = \frac{1}{p} \min_{0 < t < 1} \frac{1 - t^p}{(1 - t) t^{(p-1)/3}}.
$$

As  $J(p)$  is decreasing and  $J(3) \leq 0.9184$  one can conclude that for every  $m \geq 3$  the following holds (see e.g.  $|1|$  and  $|8|$ ):

(1.2) 
$$
r_3(\mathbb{Z}_m^n) \le (0.9184m)^n
$$

for every  $m \geq 3$ .

Note that the method could be applied for any finite field  $\mathbb{F}_q$  with  $q = p^{\alpha}$ , however, since  $r_3(\mathbb{F}_q^n) = r_3(\mathbb{F}_p^{\alpha n})$  the relevant cases are those when the prime power q is a prime. (The resulting upper bound from the application to  $\mathbb{F}_{p^{\alpha}}$  is worse than the bound coming from the case of  $\mathbb{F}_p$ .)

It is easy to see that the sequence  $(r_3(\mathbb{Z}_m^n))^{1/n}$  converges to some limit  $\alpha_{3,m}$ . The main idea behind this observation is that with the help of the product construction one can bubble up

Date: September 28, 2020.

constructions found in small dimensions. Namely, if A avoids  $3AP$ 's in dimension n, then the *t*-fold direct product  $\underbrace{A \times A \times \cdots \times A}_{t}$ also avoids  $3AP$ 's in dimension  $tn$ .

As  $\alpha_{3,m} < m$  we may say that 3AP-free sets in  $\mathbb{Z}_m^n$  are exponentially small when  $m \geq 3$ . Prior to this work for longer progressions it has not yet been decided in any of the cases  $4 \leq k \leq m$  whether  $r_k(\mathbb{Z}_m^n)$  is also exponentially small or of order of magnitude  $(m - o(1))^n$ (as  $n \to \infty$ ). In this note we will prove that whenever 6 | m and  $k \in \{4, 5, 6\}$  the quantity  $r_k(\mathbb{Z}_m^n)$  is exponentially small, specially,  $r_6(\mathbb{Z}_6^n) \leq 5.709^n$ . It is tempting to also formulate this statement as  $\lim (r_6(\mathbb{Z}_6^n))^{1/n} \leq 5.709$ , however, somewhat surprisingly, we do not see a proof of the statement that  $r_6(\mathbb{Z}_6^n)^{1/n}$  converges (although we believe it surely does). The convergence is not immediate, because the product construction does not work in general. When  $k = 3$  or m is a prime power, the t-fold direct product  $\underbrace{A \times A \times \cdots \times A}_{t}$ avoids  $k$ -AP's

when A itself is  $k$ -AP-free, however, for general k and m this fails to hold. Let us illustrate this by the case  $k = 6, m = 6$ . In dimension 1 we clearly have  $r_6(\mathbb{Z}_6) = 5$ , and, for instance, the set  $A = \{0, 1, 2, 3, 4\}$  is 6AP-free. By taking  $A \times A = \{0, 1, 2, 3, 4\} \times \{0, 1, 2, 3, 4\}$  we obtain a 25-element subset of  $\mathbb{Z}_6^2$  which contains the following 6AP:

$$
(0,0), (2,3), (4,0), (0,3), (2,0), (4,3).
$$

Although the product construction is not applicable, the value of  $r_6(\mathbb{Z}_6^2)$  still turns out to be  $25 = 5^2$ , however, we will show that  $r_6(\mathbb{Z}_6^3) < 125 = (r_6(\mathbb{Z}_6))^3$ .

Summarizing our results we prove the following bounds:

<span id="page-1-2"></span>**Theorem 1.1.** For sets without arithmetic progression of length 6 we have the following results in small dimensions:

$$
r_6(\mathbb{Z}_6^1) = 5, r_6(\mathbb{Z}_6^2) = 25, 116 \le r_6(\mathbb{Z}_6^3) \le 124.
$$

<span id="page-1-0"></span>**Theorem 1.2.** For sets without arithmetic progression of length 6 we have the following results:

$$
4.434n \le 2n r_3(\mathbb{Z}_3^n) \le r_6(\mathbb{Z}_6^n) \le 5.709n,
$$

assuming that n is sufficiently large.

If 6 | m, then  $\mathbb{Z}_6^n$  is a subgroup of  $\mathbb{Z}_m^n$ , and by using the bound from Theorem [1.2](#page-1-0) in each of the  $(m/6)^n$  cosets the following corollary is obtained:

**Corollary 1.3.** If 6 | m and  $k \in \{4, 5, 6\}$ , then  $r_k(\mathbb{Z}_m^n) \leq (0.948m)^n$ , if n is sufficiently large.

Finally, we provide another upper bound for  $r_6(\mathbb{Z}_6^n)$  in terms of  $r_3(\mathbb{Z}_3^n)$ .

<span id="page-1-1"></span>**Theorem 1.4.** For sets without arithmetic progression of length 6 we have the following result:

$$
r_6(\mathbb{Z}_6^n) \leq 2^{n+1} \sqrt{3^n r_3(\mathbb{Z}_3^n)}.
$$

Note that by using the bound  $r_3(\mathbb{Z}_3^n) \leq 2.756^n$  Theorem [1.4](#page-1-1) implies that  $r_6(\mathbb{Z}_6^n) \leq 5.75^n$ which bound is worse than the one in Theorem [1.2,](#page-1-0) however, if  $r_3(\mathbb{Z}_3^n) \leq 2.69^n$ , then Theorem [1.4](#page-1-1) gives a better estimation than Theorem [1.2.](#page-1-0)

The paper is organized as follows: In Section [2](#page-2-0) we give a reformulation for the problem of finding  $r_k(\mathbb{Z}_6^n)$  with  $k \in \{3, 4, 5, 6\}$  in terms of possible total sizes of systems of subsets of  $\mathbb{Z}_3^n$  satisfying certain properties. In Section [3](#page-3-0) we prove Theorem [1.1,](#page-1-2) Theorem [1.2](#page-1-0) and Theorem [1.4.](#page-1-1) Some concluding remarks are given in Section [4.](#page-6-6)

## 2. Subset reformulation

<span id="page-2-0"></span>We may express  $\mathbb{Z}_6^n$  as  $\mathbb{Z}_6^n = F \oplus R$ , where  $F = \{0, 2, 4\} \cong \mathbb{Z}_3^n$  and  $R = \{0, 3\}^n \cong \mathbb{Z}_2^n$ . A sequence  $a_1 = f_1 + r_1, a_2 = f_2 + r_2, \ldots, a_k = f_k + r_k$  (where  $f_i \in F, r_i \in R$ ) forms an arithmetic progression in  $\mathbb{Z}_6^n$  if and only if  $f_1, f_2, \ldots, f_k$  is an arithmetic progression in  $\mathbb{Z}_3^n$ and  $r_1, r_2, \ldots, r_k$  is an arithmetic progression in  $\mathbb{Z}_2^n$ , respectively. Note that if the elements are distinct, then  $k \leq 6$ . If  $k = 3$ , then the progression consists of pairwise different elements if and only if  $f_1, f_2, f_3$  are distinct. Since the sequence  $r_1, r_2, \ldots$  is alternating, for  $k \in \{4, 5, 6\}$ the necessary and sufficient conditions for getting k distinct elements is that  $f_1, f_2, f_3$  are distinct and  $r_1, r_2$  are distinct. Using this decomposition we may reformulate the property that "a subset  $A \subseteq \mathbb{Z}_6^n$  avoids k-term arithmetic progressions" in terms of a property of systems of subsets of  $\mathbb{Z}_3^n$ . Namely, let  $A(r) = \{f \in \mathbb{Z}_3^n : f + r \in A\}$  for  $r \in R$  and let us define properties  $(*)_3,(*)_4,(*)_5,(*)_6$  as follows:

The system of subsets  $A(r)$   $(r \in \mathbb{Z}_2^n)$  satisfies

- property  $(*)_3$ , if  $A(r') \cup A(r'')$  is 3AP-free for every pair  $r', r'' \in \mathbb{Z}_2^n$ ,
- property  $(*)_4$ , if it is not possible to choose two different indices  $r', r'' \in \mathbb{Z}_2^n$  and a 3AP  $a, b, c$  in  $\mathbb{Z}_3^n$  such that  $a, b \in A(r')$  and  $a, c \in A(r'')$ ,
- property  $(*)_5$ , if it is not possible to choose two different indices  $r', r'' \in \mathbb{Z}_2^n$  and a 3AP  $a, b, c$  in  $\mathbb{Z}_3^n$  such that  $a, b, c \in A(r')$  and  $a, b \in A(r'')$ ,
- property  $(*)_6$ , if  $A(r') \cap A(r'')$  is 3AP-free for every pair of distinct indices  $r', r'' \in \mathbb{Z}_2^n$ .

Note that in this reformulation  $\mathbb{Z}_2^n$  serves only as an index set of size  $2^n$ , its structure does not play any role.

Let us summarize in the following statement how the reformulation can be used to study the  $r_k(\mathbb{Z}_6^n)$  values.

**Proposition 2.1.** Let  $k \in \{3, 4, 5, 6\}$ . The maximum total size of a system of subsets  $A(r) \subseteq$  $\mathbb{Z}_3^n$   $(r \in \mathbb{Z}_2^n)$  satisfying property  $(*)_k$  is  $r_k(\mathbb{Z}_6^n)$ .

Proof. The statements immediately follow from the structural description of arithmetic progressions in  $\mathbb{Z}_6^n$ .

Let us mention that the problem of determining the size of the largest 3AP-free subset of  $\mathbb{Z}_6^n$  is equivalent with doing so in case of  $\mathbb{Z}_3^n$ :

<span id="page-2-1"></span>**Proposition 2.2.** For sets without arithmetic progression of length three the following holds:

$$
r_3(\mathbb{Z}_6^n) = 2^n r_3(\mathbb{Z}_3^n).
$$

*Proof.* If  $A_0 \subseteq \mathbb{Z}_3^n$  is 3AP-free, then the system  $A(x) = A_0$   $(x \in \mathbb{Z}_2^n)$  satisfies property  $(*)_3$ , thus  $r_3(\mathbb{Z}_6^n) \geq 2^n r_3(\mathbb{Z}_3^n)$ .

On the other hand, if  $\Sigma$  $\sum_{x \in \mathbb{Z}_2^n} |A(x)| > 2^n r_3(\mathbb{Z}_3^n)$ , then for some x we have  $|A(x)| > r_3(\mathbb{Z}_3^n)$ , thus  $A(x)$  contains a 3AP, and  $(*)_3$  fails to hold. Hence,  $r_3(\mathbb{Z}_6^n) = 2^n r_3(\mathbb{Z}_3^n)$  $\Box$ 

In fact the argument only used that 6 has residue 2 modulo 4, and in general it yields the following statement:

**Proposition 2.3.** If  $m = 4M + 2$  for some integer M, then

$$
r_3(\mathbb{Z}_m^n) = 2^n r_3(\mathbb{Z}_{m/2}^n).
$$

While studying  $r_3(\mathbb{Z}_m^n)$  there are some technical differences between the cases when m is odd and when  $m$  is divisible by 4, but the case when  $m$  is an even number not divisible by 4 simply reduces to the odd case. We shall mention that for certain composite values of  $m$  there has been some improvements on the trivial corollaries of the prime case, like  $r_3(\mathbb{Z}_9^n) \leq 3^n r_3(\mathbb{Z}_3^n)$ . Namely, the method was adapted to odd prime powers [\[1,](#page-6-4) [7,](#page-6-7) [9\]](#page-6-8) and also to the technically more difficult even case for  $m = 2^3 = 8$ . [\[8\]](#page-6-5)

## 3. Proofs

<span id="page-3-0"></span>*Proof of Theorem [1.1.](#page-1-2)* Dimension 1. Clearly,  $r_6(\mathbb{Z}_6^1) = 5$ . Any 5-element subset of  $\mathbb{Z}_6^1$  is trivially 6AP-free.

**Dimension [2](#page-2-0).** Now, we show that  $r_6(\mathbb{Z}_6^2) = 25$ . Using the reformulation from Section 2 we are interested in the maximal possible total size of a system of four subsets of  $\mathbb{Z}_3^2$  satisfying property  $(*)_6$ . That is, we would like to determine the maximum of  $\sum_{n=1}^{4}$  $\sum_{i=1}^{\infty}$  |  $A_i$ |, where  $A_i \subseteq$  $\mathbb{Z}_3^2$  ( $1 \leq i \leq 4$ ) such that no 3AP is contained in at least two of the subsets  $A_i$ . The total number of 3AP's in  $\mathbb{Z}_3^2$  is  $\frac{9\cdot 8}{6} = 12$ , thus the four subsets  $A_1$ ,  $A_2$ ,  $A_3$ ,  $A_4$  can contain at most twelve 3AP's in total. It is easy to determine the smallest possible number of 3AP's that must be contained in a subset of a given size (by hand or by a computer search). Let us summarize the results in the table below:



Let  $x_i$  denote the number of *i*-element subsets among  $A_1, A_2, A_3, A_4$  (where  $0 \le i \le 9$ ). Since each 3AP can appear in at most one set  $A_i$ , the optimal value for  $\sum^4$  $\sum_{i=1}^{\infty} |A_i|$  can not be more than the solution of the following integer program:

> $\max x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 + 6x_6 + 7x_7 + 8x_8 + 9x_9$ subject to  $x_0 + x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 = 4$  $x_5 + 2x_6 + 5x_7 + 8x_8 + 12x_9 \leq 12$  $x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9:$  nonnegative integers

(The first constraint ensures that four subsets are chosen, and the second constraint holds, since the total number of 3AP's contained in the four subsets can not be more than the total number of 3AP's in  $\mathbb{Z}_3^2$ .)

By solving the above integer program we obtain that the optimal value is 25 which is attained at  $x_6 = 3, x_7 = 1$  (everything else is 0). That is, to achieve 25, one of the subsets must have size 7, and the three other subsets must have size 6.

By symmetry, we may assume that  $A_1 = \mathbb{Z}_3^2 \setminus \{u, v\}$ , where u and v are two different elements. Let  $w = -u - v$  be the third point on the line uv. Let  $\alpha$  denote the direction of the line uv. Note that in  $\mathbb{Z}_3^2$  there are four possible directions, let us denote the other three directions by  $\beta$ ,  $\gamma$  and  $\delta$ .

Note that  $A_2, A_3, A_4$  must have size 6 and each of them must contain exactly two 3AP's. In  $\mathbb{Z}_3^2$  there are two types of 6-element sets: the complement of a 6-element set is either an affine line or not. To contain only two 3AP's the sets  $A_2$ ,  $A_3$ ,  $A_4$  must all be the complements of affine lines, in other words, each of them is a union of two parallel lines. Moreover, these

SETS AVOIDING SIX-TERM ARITHMETIC PROGRESSIONS IN  $\mathbb{Z}_6^n$  are exponentially small 5

lines must not be parallel with the line  $uv$ , otherwise at least one of them would be contained in two subsets (in  $A_1$  and here).

Also, none of these lines can go through  $w$ , as this would result in a 3AP contained both in  $A_1$  and here.

Finally, a line from  $A_i$  and a line from  $A_j$  (where  $2 \leq i < j \leq 4$ ) must not be parallel with each other because of similar reasons. That is, we may assume that  $A_2$ ,  $A_3$ ,  $A_4$  are the unions of two-two lines of directions  $\beta, \gamma, \delta$ , respectively.

Therefore,  $A_2, A_3, A_4$  can be characterized as follows:  $A_2, A_3, A_4$  are all the unions of two parallel lines, where the directions of the lines are  $\beta, \gamma, \delta$  resp., furthermore each line goes through u or v. (Thus  $\{A_2, A_3, A_4\}$  is uniquely determined.)

The obtained system  $\{A_1, A_2, A_3, A_4\}$  satisfies the conditions, since:

- $A_1$  contains two 3AP's with direction  $\alpha$  and three more 3AP's that contain w.
- None of the 3AP's contained in  $A_2$ ,  $A_3$ ,  $A_4$  have direction  $\alpha$  and none of them contains  $w$ .
- The two-two lines contained in  $A_2$ ,  $A_3$ ,  $A_4$  have directions  $\beta$ ,  $\gamma$ ,  $\delta$ , respectively.

Hence, we proved that the largest  $6AP$ -free set in  $\mathbb{Z}_6^2$  has size 25 (and it is unique in the above described sense).

Dimension 3. Analogously to the previous case, with a quick computer check we find that the minimum number of 3AP's that must be contained in subsets of  $\mathbb{Z}_3^3$  of given sizes are the numbers below. (Let  $m_j$  denote the minimum number of 3AP's that must be contained in a set of size  $j.$ )





Let  $x_i$  denote the number of *i*-element subsets among  $A_1 - A_8$  (where  $0 \le i \le 27$ ).

Since each 3AP can appear in at most one set  $A_i$ , the optimal value for  $\sum^8$  $\sum_{i=1}^{\infty} |A_i|$  can not be more than the solution of the following integer program:

$$
\max \sum_{i=1}^{27} ix_i
$$
  
subject to  

$$
\sum_{i=0}^{27} x_i = 8
$$

$$
\sum_{i=0}^{27} m_i x_i \le 117
$$

$$
x_0, x_1, \dots, x_{27}
$$
: nonnegative integers

With the help of an IP solver we obtained that the optimum is 124 yielding the bound

$$
r_6(\mathbb{Z}_6^3) \le 124.
$$

Turning to the lower bound, with computer help we found the following construction where the total size of the eight subsets is 116:



Hence,

$$
116 \le r_6(\mathbb{Z}_6^3).
$$

 $\Box$ 

*Proof of Theorem [1.2.](#page-1-0)* The lower bound follows from Proposition [2.2](#page-2-1) and Edel's [\[3\]](#page-6-0) lower bound for  $r_3(\mathbb{Z}_3^n)$ .

For proving the upper bound it suffices to show that  $\Sigma$  $\sum_{i\in I} |A_i| \leq 5.709^n$  if the system of subsets (of  $\mathbb{Z}_3^n$ )  $\{A_i : i \in I\}$  satisfies property  $(*)_6$  and  $|I| = 2^n$ .

We will use a supersaturation extension [\[6\]](#page-6-9) of the cap set result. This says that any subset of  $\mathbb{Z}_3^n$  of density  $\alpha$  has three-term arithmetic progression density at least  $\alpha^C$ , where  $C \approx 13.901$  $C \approx 13.901$  $C \approx 13.901$  is an explicit constant <sup>1</sup>. (Note that this includes counting trivial three-term arithmetic progressions.)

Let  $\beta = 3/2^{1/C} \approx 2.854$ , then we have  $\beta^C = \frac{3^C}{2}$  $\frac{1}{2}$ . The total size of subsets having size at most  $\beta^n$  is at most  $2^n\beta^n$ . Now, we consider the subsets with size larger than  $\beta^n$ . Let  $m_i$ denote the number of those subsets whose size lies in  $(2^i\beta^n, 2^{i+1}\beta^n]$ . Since each 3AP can occur in at most one set, we obtain that

$$
m_i(2^i\beta^n/3^n)^C \le 1
$$

yielding that  $m_i \leq (3/\beta)^{Cn} 2^{-iC}$ . Therefore, the total size of subsets of size larger than  $\beta^n$  is at most

$$
\sum_{i=0}^{\infty} m_i 2^{i+1} \beta^n \le \sum_{i=0}^{\infty} (3/\beta)^{Cn} 2^{-iC} 2^{i+1} \beta^n = (2\beta)^n \sum_{i=0}^{\infty} 2^{1-(C-1)i} \le 2.001(2\beta)^n.
$$

Hence, by adding up the obtained upper bounds for sets of size at most  $\beta^n$  and larger than  $\beta^n$  it is obtained that  $\sum |A_i| \leq 3.001(2\beta)^n$ .

 $\Box$ 

*Proof of Theorem [1.4.](#page-1-1)* It suffices to prove that  $S := \sum$  $\sum_{i\in I} |A_i| \leq 2^{n+1}\sqrt{3^n r_3(\mathbb{Z}_3^n)}$  if the system of subsets (of  $\mathbb{Z}_3^n$ )  $\{A_i : i \in I\}$  satisfies property  $(*)_6$  and  $|I| = 2^n$ .

Let us enumerate the elements of  $\mathbb{Z}_3^n$  by the positive integers from [3<sup>n</sup>]. For  $i \in I$  let  $v_i$ be the characteristic vector of  $A_i$ , that is, the jth entry of  $v_i$  is 1 if the element (from  $\mathbb{Z}_3^n$ )

<span id="page-5-0"></span><sup>1</sup> Namely, 
$$
C = 1 + \frac{\log 3}{\log(3/\alpha)}
$$
, where  $\alpha = 3J(3) = 2.755...$ 

SETS AVOIDING SIX-TERM ARITHMETIC PROGRESSIONS IN  $\mathbb{Z}_6^n$  are exponentially small 7 labeled by j is contained in  $A_i$  and 0 otherwise. Let  $w := \sum$ i∈I  $v_i$ , denote the entries of w by  $w_1, \ldots, w_{3^n}$ . Note that  $w_1 + \cdots + w_{3^n} = \sum_{i \in I}$  $\sum_{i\in I} |A_i| = S.$ 

<span id="page-6-10"></span>By the Cauchy inequality

(3.1) 
$$
w^{2} = w_{1}^{2} + \cdots + w_{3^{n}}^{2} \geq \frac{(w_{1} + \cdots + w_{3^{n}})^{2}}{3^{n}} = \frac{S^{2}}{3^{n}}.
$$

Since  $A_i \cap A_j$  is 3AP-free for any two different indices  $i, j \in I$  we have  $v_i v_j \le r_3(\mathbb{Z}_3^n)$ . Therefore,

(3.2) 
$$
w^{2} = \sum_{i \in I} v_{i}^{2} + \sum_{i,j \in I, i \neq j} v_{i} v_{j} \leq S + 2^{2n} r_{3} (\mathbb{Z}_{3}^{n}).
$$

By comparing [\(3.1\)](#page-6-10) and [\(3.2\)](#page-6-11) we obtain that  $S^2 - 3^nS - 2^{2n}3^n r_3(\mathbb{Z}_3^n) \leq 0$  which yields

<span id="page-6-11"></span>
$$
S \le \frac{3^n + \sqrt{3^{2n} + 2^{2n+2}3^n r_3(\mathbb{Z}_3^n)}}{2} < 2^{n+1} \sqrt{3^n r_3(\mathbb{Z}_3^n)}.
$$

#### 4. Concluding remarks

<span id="page-6-6"></span>In this paper we prove that  $r_6(\mathbb{Z}_6^n) \leq 5.709^n$ , which implies that  $r_k(\mathbb{Z}_m^n)$  is exponentially smaller than  $m^n$  when 6 | m and  $k \in \{4, 5, 6\}$ . Previously this was known only for the cases  $3 = k \le m$ , and according to our knowledge there is no pair of k, m with  $3 \le k \le m$  such that  $r_k(\mathbb{Z}_m^n) = (m - o(1))^n$  is known to be true.

## 5. Acknowledgements

Both authors were supported by the Lendület program of the Hungarian Academy of Sciences (MTA). PPP was also supported by the National Research, Development and Innovation Office NKFIH (Grant Nr. K124171, K129335 and BME NC TKP2020). RP was also supported by the BME-Artificial Intelligence FIKP grant of EMMI (BME FIKP-MI/SC).

#### **REFERENCES**

- <span id="page-6-4"></span>[1] J. Blasiak, T. Church, H. Cohn, J. Grochow, E. Naslund, W. Sawin, C. Umans, On cap sets and the group-theoretic approach to matrix multiplication. Discrete Anal. 2017, Paper No. 3, 27 pp.
- <span id="page-6-3"></span>[2] E. Croot, V.F. Lev, P.P. Pach, Progression-free sets in  $\mathbb{Z}_4^n$  are exponentially small, Ann. of Math. (2) 185 (2017), no. 1, 331–337.
- <span id="page-6-1"></span><span id="page-6-0"></span>[3] Y. Edel, Extensions of generalized product caps, Des. Codes Cryptography 31 (2004), 5 – 14.
- [4] J. S. Ellenberg, D. Gijswijt, On large subsets of  $\mathbb{F}_q^n$  with no three-ter arithmetic progression. Ann. of Math. (2) 185 (2017), no. 1, 339–343.
- <span id="page-6-9"></span><span id="page-6-2"></span>[5] C. Elsholtz, P. P. Pach, Des. Codes Cryptography (2020), https://doi.org/10.1007/s10623-020-00769-0
- [6] J. Fox and L. M. Lovász, A tight bound for Greens arithmetic triangle removal lemma in vector spaces, Advances 321 (2017) 287–297.
- <span id="page-6-7"></span><span id="page-6-5"></span>[7] F. Petrov, Combinatorial results implied by many zero divisors in a group ring, arXiv: 1606.03256
- [8] F. Petrov, C. Pohoata, Improved Bounds for Progression-Free Sets in  $C_8^n$ , Israel J. Math. 236 (2020), no. 1, 345–363.
- <span id="page-6-8"></span>[9] D. Speyer, https://sbseminar.wordpress.com/2016/07/08/bounds-for-sum-free-sets-inprime-powercyclic-groups-three-ways/

# $\rm PÉTER \,\, P\acute{A}L \,\, PACH \,\, AND \,\, RICH \acute{A}RD \,\, PALINCZA \,\,$

E-mail address: ppp@cs.bme.hu

MTA-BME LENDÜLET ARITHMETIC COMBINATORICS RESEARCH GROUP, DEPARTMENT OF COMputer Science and Information Theory, Budapest University of Technology and Economics, 1117 BUDAPEST, MAGYAR TUDÓSOK KÖRÚTJA 2., HUNGARY

E-mail address: pricsi@cs.bme.hu

MTA-BME LENDÜLET ARITHMETIC COMBINATORICS RESEARCH GROUP, DEPARTMENT OF COMputer Science and Information Theory, Budapest University of Technology and Economics, 1117 BUDAPEST, MAGYAR TUDÓSOK KÖRÚTJA 2., HUNGARY