

Existence of optimal pairs for optimal control problems with states constrained to Riemannian manifolds

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Abstract

In this paper, we investigate the existence of optimal pairs for optimal control problems with their states constrained pointwise to Riemannian manifolds. For this purpose, by means of the Riemannian geometric tool, we introduce a crucial Cesari-type property, which is an extension of the classical Cesari property (See [3, Definition 3.3, p. 51]) from the setting of Euclidean spaces to that of Riemannian manifolds. Moreover, we show the efficiency of our result by a concrete example.

Keywords: Existence of optimal pairs, optimal controls, Cesari-type property, Riemannian manifolds.

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1. Introduction

Let $n \in \mathbb{N}$ and M be an n dimensional complete Riemannian manifold with a Riemannian metric g . Denote by $T_x M$ the tangent space of M at $x \in M$, by $TM \triangleq \bigcup_{x \in M} T_x M$ the tangent bundle on M , and by $\langle \cdot, \cdot \rangle$ and $|\cdot|$ the inner product and the norm over $T_x M$ related to g , respectively (See [13, Chapter 3] for more material on these notions). Given $T > 0$, a metric space U , maps $f : [0, T] \times M \times U \rightarrow TM$ and $\Gamma : [0, T] \times M \rightarrow 2^U$ (the class of all subsets of U), and subsets $Q \subseteq M$ and $S \subseteq M \times M$, we consider the following control system:

$$\dot{y}(t) = f(t, y(t), u(t)) \quad a.e. \ t \in [0, T], \quad (1.1)$$

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with the control $u(\cdot)$ and state $y(\cdot)$ satisfying the following constrains:

$$\begin{cases} u(t) \in \Gamma(t, y(t)), & a.e. t \in [0, T], \\ y(t) \in Q, & \forall t \in (0, T), \\ (y(0), y(T)) \in S. \end{cases} \quad (1.2)$$

Write $\mathcal{U} \triangleq \{w : (0, T) \rightarrow U \mid w(\cdot) \text{ is measurable}\}$. A pair $(u(\cdot), y(\cdot))$ is said to be feasible, if $u(\cdot) \in \mathcal{U}$, $y(\cdot)$ is absolutely continuous, and both (1.1) and (1.2) are satisfied.

Given maps $f^0 : [0, T] \times M \times U \rightarrow \mathbb{R}$ and $h : M \times M \rightarrow \mathbb{R}$, a pair $(u(\cdot), y(\cdot))$ is said to be admissible, if it is feasible and $f^0(\cdot, y(\cdot), u(\cdot)) \in L^1(0, T)$. Hereafter, we denote by \mathcal{P}_{ad} the set of all admissible pairs, and introduce the cost functional with respect to the control system as follows:

$$J(u(\cdot), y(\cdot)) \triangleq \int_0^T f^0(t, y(t), u(t)) dt + h(y(0), y(T)), \quad \forall (u(\cdot), y(\cdot)) \in \mathcal{P}_{ad}.$$

Our optimal control problem is formulated as follows.

(P) Find $(\bar{u}(\cdot), \bar{y}(\cdot)) \in \mathcal{P}_{ad}$ such that

$$J(\bar{u}(\cdot), \bar{y}(\cdot)) = \inf_{(u(\cdot), y(\cdot)) \in \mathcal{P}_{ad}} J(u(\cdot), y(\cdot)).$$

If the above pair $(\bar{u}(\cdot), \bar{y}(\cdot))$ does exist, then we say that $(\bar{u}(\cdot), \bar{y}(\cdot))$, $\bar{u}(\cdot)$ and $\bar{y}(\cdot)$ are an *optimal pair*, *optimal control* and *optimal trajectory*, respectively. In this paper, we are concerned with the existence of optimal pairs for Problem (P).

The study of existence of optimal pairs for optimal control problems began in 1960s. L. Cesari in [5] studied this sort of problems for finite dimensional systems evolved in Euclidean spaces, in which a key condition, now known as the Cesari property, was introduced to guarantee the existence of optimal pairs. Later, for problems with weaker conditions (compared to the classical Cesari property), there have been a lot of works pursuing further in this direction, see for example [11, 18, 24, 3, 10, 15, 6, 22, 23, 4, 19, 20] and the references cited therein.

As far as we know, when the states of the control systems are constrained to differentiable manifolds, there are only quite a few existence results on optimal pairs, for which we mention that A. Agrachev, D. Barilari and U. Boscain considered an affine control system (i.e. a system depending linearly on the control) in [1, Theorem 3.43, p. 89].

There are two motivations for us to study the existence of optimal pairs for Problem (P). First, the study of Problem (P) is of practical importance. As we have explained in [8, Section 1], the states of many optimal control problems are actually constrained to manifolds. Next, the study on the existence of optimal pairs for Problem (P) is far from satisfactory. Even though sometimes one may transform Problem (P) to an optimal control

problem on some Euclidean space (i.e., embed M isometrically into a Euclidean space by the Nash embedding theorem (see [17, Theorem 3], then rewrite the control system (1.1) as a controlled ordinary differential equation on that flat space, and replace the hidden state constraint $y(t) \in M$ a.e. $t \in [0, T]$ in (1.1) by a pointwise equality-type constraint $c(y(t)) = 0$ a.e. $t \in [0, T]$ for some function c), and then apply the known existence results in the literature to the resulting system, the transformed problem may not satisfy the conditions required in these existence results. On the other hand, as mentioned before, the results obtained by geometric tools are considerably limited, for which, to the best of our knowledge, only an affine control system has been studied (See [1, Theorem 3.43, p. 89] for example). We refer Remark 2.2 for a more detailed explanation.

The main purpose of this paper is to find some verifiable conditions guaranteeing the existence of optimal pairs for Problem (P). A fundamental difficulty for this aim is caused by the curved state space M . Actually, a usual way to prove the existence of optimal pairs (in flat spaces) can be described as follows: Choose a minimizing sequence (with respect to the cost functional) in the set of admissible pairs first and then prove the limit point (in a suitable sense) of this sequence is an admissible pair. For our present problem, the curved state space M makes this procedure hard to proceed. To overcome this difficulty, we use Riemannian geometry as an analytic tool and prove the existence of optimal pairs under some proper conditions. It is worth mentioning that, the Cesari-type property (A6) (that we shall introduce in Subsection 2.1) is crucial for the existence of optimal pairs, which is an extension of the classical Cesari property (e.g., [3, Definition 3.3, p. 51]) from the setting of Euclidean spaces to that of Riemannian manifolds. When the manifolds degenerate to Euclidean spaces, our condition (A6) is exactly the classical Cesari condition. Our main existence result is stated in Theorem 2.1. Also, we provide a method (in Proposition 2.1) to verify the above mentioned Cesari-type property, and give an example (See Example 2.1) as an application.

This paper has the following novelties: First, although Theorem 2.1 is an extension of [3, Theorem 4.2, p. 58] from the Euclidean case to the case of differentiable manifolds, in the present work we have not assumed that the pointwise state-constraint set Q is compact (See the condition (A5)). Next, Proposition 2.1 looks very similarly to [14, Proposition 4.3, p. 107]. However, the state space discussed in Proposition 2.1 is curved, while the state space in the latter is a Banach space (which is still flat). Thirdly, [1, Theorem 3.43, p.89] exhibits the existence of the minimizing curve connecting two fixed points on a sub-Riemannian manifold. When these points are close enough, the problem is exactly the existence of optimal pairs for an optimal control problem with an affine control system (for which the system is linear with respect to the control variable) on a differentiable manifold and with a quite special cost functional. Comparing to [1, Theorem 3.43, p.89], our results have two differences: 1) We study a more general control system, which is

allowed to be nonlinear with respect to the control variable; 2) Our control set U is not required to be bounded. We also mention that, the optimal control problem discussed in [1, Theorem 3.43, p. 89] fulfills the Cesari-type property (A6) (although such a condition was not introduced there), which will be verified by means of Proposition 2.1. In Remark 2.2, we shall explain that, neither [3, Theorem 4.2, p. 58] nor [1, Theorem 3.43, p. 89] works for the optimal control problem in Example 2.1.

The rest of this paper is organized as follows. In Section 2, the main results of this paper are stated, and two examples are also given. Section 3 is devoted to a proof of our main results.

2. Statement of the main results

2.1. Notations and main assumptions

First, we introduce some notations. For the Riemannian manifold (M, g) , we denote by $\rho(\cdot, \cdot)$, ∇ , and $i(x)$ the distance function, Levi-Civita connection, and the injectivity radius at $x \in M$, respectively. For $x, y \in M$ with $\rho(x, y) < \min\{i(x), i(y)\}$, we denote by L_{xy} the parallel translation of tensors at x from x to y , along the unique shortest geodesic connecting x and y . For more details of these notions, we refer the readers to [9, 7].

For a.e. $t \in (0, T)$ and each $x \in M$, we define a map $\mathcal{E} : [0, T] \times TM \rightarrow 2^{\mathbb{R} \times TM}$ as follows:

$$\mathcal{E}(t, x) \triangleq \{(x^0, X) \in \mathbb{R} \times T_x M \mid \exists u \in \Gamma(t, x) \text{ s.t. } x^0 \geq f^0(t, x, u) \text{ and } X = f(t, x, u)\},$$

and denote by

$$L_{xy}\mathcal{E}(t, x) \triangleq \{(x^0, L_{xy}X) \mid (x^0, X) \in \mathcal{E}(t, x)\} \quad \text{if } \rho(x, y) < i(y).$$

We introduce the following assumptions that we shall need in the sequel.

(A1) (U, d) is a complete separable metric space;

(A2) The function $f : [0, T] \times M \times U \rightarrow TM$ is measurable in (t, x, u) . For a.e. $t \in (0, T)$, the map $(x, u) \mapsto f(t, x, u)$ is continuous, and there exists a positive constant $K > 1$, $x_0 \in M$ and $\ell(\cdot) \in L^p(0, T)$ with $p > 1$ such that

$$|L_{x_1 x_2} f(s, x_1, u) - f(s, x_2, u)| \leq K \rho(x_1, x_2), \quad (2.1)$$

$$|f(s, x_0, u)| \leq \ell(s) \quad (2.2)$$

holds for a.e. $s \in (0, T)$ and all $u \in U$ and $x_1, x_2 \in M$ with $\rho(x_1, x_2) < \min\{i(x_1), i(x_2)\}$;

(A3) The function $f^0 : [0, T] \times M \times U \rightarrow \mathbb{R}$ is Borel measurable in (t, x, u) and lower semicontinuous in (x, u) , $h : M \times M \rightarrow \mathbb{R}$ is lower semicontinuous, and there exists a positive constant K such that

$$\begin{aligned} \inf\{f^0(t, x, u) \mid (x, u) \in M \times U, \text{a.e. } t \in (0, T)\} &\geq -K, \\ \inf\{h(x_1, x_2) \mid x_1, x_2 \in M\} &\geq -K. \end{aligned}$$

(A4) The map $\Gamma : [0, T] \times M \rightarrow 2^U$ is a set-valued map with closed images. For every $x \in M$, the map $t \mapsto \Gamma(t, x)$ is measurable (See [2, Definition 8.1.1, p. 307]), and for a.e. $t \in [0, T]$, the map $x \mapsto \Gamma(t, x)$ is continuous (See [2, Definitions 1.4.1, 1.4.2 and 1.4.3, pp. 38–40]), i.e., for each $x \in M$ the following two conditions hold:

- (i) For any neighborhood \mathcal{O} of $\Gamma(t, x)$, there exists $\delta > 0$ such that, for any $y \in M$ with $\rho(x, y) \leq \delta$, it holds that $\Gamma(t, y) \subset \mathcal{O}$.
- (ii) For any $v \in \Gamma(t, x)$ and for any sequence $\{x_i\} \subset M$ converging to x , there exists a sequence $\{v_i\} \subset U$ satisfying $v_i \in \Gamma(t, x_i)$ for each $i \in \mathbb{N}$ such that, v_i converges to v as i goes to infinity;

(A5) $Q \subseteq M$ is closed, and $S \subseteq M \times M$ is bounded and closed;

(A6) For a.e. $t \in (0, T)$, the map $\mathcal{E}(t, \cdot) : M \rightarrow 2^{\mathbb{R} \times TM}$ satisfies the Cesari-type property on Q , i.e. the following property

$$\bigcap_{i(x) > \delta > 0} \text{cl co} \left(\bigcup_{\rho(y, x) < \delta} L_{yx} \mathcal{E}(t, y) \right) = \mathcal{E}(t, x) \quad (2.3)$$

holds for all $x \in Q$, where $\text{cl } C$ and $\text{co } C$ denote the closure and convex hulls of the set C , respectively.

2.2. Main results

Our main existence result on optimal pairs for Problem (P) is stated as follows:

Theorem 2.1. *Suppose that the assumptions (A1)-(A6) hold and $\mathcal{P}_{ad} \neq \emptyset$. Then Problem (P) admits at least one optimal pair.*

Remark 2.1. *Clearly, Theorem 2.1 is an extension of [3, Theorem 4.2, p. 58] from the Euclidean case to the case of manifolds. In particular, the Cesari-type property (A6) is an extension of the condition in [3, (3.5), p. 51] by means of the “parallel translation”, which is an important notion in Riemannian geometry. When the state space M degenerates to a Euclidean space, the condition (A6) is exactly that in [3, (3.5), p. 51]. Moreover, compared to [3, Theorem 4.2, p. 58], we do not require the set Q to be compact. Finally, in the case that the control system (1.1) is linear, the assumption (A2) holds for many cases,*

say when the control set U is bounded (which is reasonable for many concrete problems). In fact, in the existing results for linear control systems (except the problems with special cost functional), similar assumption on the control set is always intrinsically needed, e.g. [3, Theorem 4.2, p. 58] and [6].

As that in the flat spaces, the Cesari-type property (A6) is a kind of convexity condition. The following result provides a method to verify such a property.

Proposition 2.1. *Assume that for a.e. $t \in (0, T)$, one of the following conditions holds:*

(A3') $\Gamma(t, \cdot) : M \rightarrow 2^U$ is upper semicontinuous (See [14, Definition 2.1, p. 89]), the map $f(t, \cdot, \cdot) : M \times U \rightarrow TM$ is uniformly continuous with respect to $u \in U$ and $f^0(t, \cdot, \cdot) : M \times U \rightarrow \mathbb{R}$ is uniformly lower semicontinuous in $u \in U$, i.e., for any $y \in M$ and $\epsilon > 0$, there exists $\sigma \in (0, i(y))$ such that

$$|L_{y'y}f(t, y', u') - f(t, y, u)| < \epsilon \quad \text{and} \quad f^0(t, y', u') > f^0(t, y, u) - \epsilon$$

hold for any $(y', u', u) \in M \times U \times U$ with $\max\{\rho(y, y'), d(u, u')\} < \sigma$.

(A4') $\Gamma(t, x) \equiv \Gamma(t)$ for all $x \in M$, the maps $f(t, \cdot, \cdot) : M \times U \rightarrow TM$ and $f^0(t, \cdot, \cdot) : M \times U \rightarrow \mathbb{R}$ satisfy the following condition: for any $y \in M$ and $\epsilon > 0$, there exists $\sigma \in (0, i(y))$ such that, whenever $(y', u) \in M \times U$ satisfies $\rho(y, y') < \sigma$, one has

$$|L_{y'y}f(t, y', u) - f(t, y, u)| < \epsilon \quad \text{and} \quad f^0(t, y', u) > f^0(t, y, u) - \epsilon.$$

Then, the map $\mathcal{E}(t, \cdot)$ satisfies the Cesari-type property on M , i.e. (2.3) holds, if and only if $\mathcal{E}(t, x)$ is convex and closed for each $x \in M$.

We shall prove Theorem 2.1 and Proposition 2.1 in Section 3. The following example is an application of Theorem 2.1 and Proposition 2.1.

Example 2.1. For $x = (x_1, x_2, x_3)^\top \in \mathbb{R}^3$, set

$$f_1(x) = (0, e^{-x_1^2} x_1 \sin x_3, x_3)^\top, \quad f_2(x) = (e^{-x_2^2} x_2 \sin x_3, 0, x_3)^\top.$$

Consider the following optimal control problem: Minimize

$$J(u(\cdot), v(\cdot)) = \int_0^T \left(u^2(t) \left(e^{-2x_1(t)^2} x_1(t)^2 \frac{(\sin x_3(t))^2}{x_3(t)^2} + 1 \right) + v(t) \sqrt{e^{-2x_2(t)^2} x_2(t)^2 \frac{(\sin x_3(t))^2}{x_3(t)^2} + 1} \right) dt$$

over $(u(\cdot), v(\cdot)) \in \mathcal{U} \triangleq \{(u, v) : [0, T] \rightarrow [0, \pi] \times [0, 1] \mid (u, v)(\cdot) \text{ is measurable}\}$, where $x(\cdot) = (x_1(\cdot), x_2(\cdot), x_3(\cdot))^\top : [0, T] \rightarrow \mathbb{R}^3$ is the solution to the following control system:

$$\begin{cases} \dot{x}(t) = f_1(x(t)) \sin u(t) + f_2(x(t))v(t), & \text{a.e. } t \in [0, T], \\ x_3(t) > 0, & \forall t \in [0, T], \\ x(0) = (0, 0, 1)^\top, & e^T \leq x_3(T) \leq 2e^T. \end{cases} \quad (2.4)$$

Then, as we shall show below, Problem (P) for the above optimal control system admits at least one optimal pair.

Indeed, we consider the set

$$H^3 = \{(x_1, x_2, x_3)^\top \in \mathbb{R}^3 \mid x_3 > 0\}$$

as a manifold, with only one coordinate chart $(H^3, I|_{H^3})$, where $I : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is the identity map and $I|_{H^3}$ is the restriction of I to the subset H^3 . We also introduce on H^3 the metric g :

$$g_{ij}(x_1, x_2, x_3) = \delta_{ij}/x_3^2, \quad i, j = 1, 2, 3,$$

where $\delta_{ii} = 1$ for $i = 1, 2, 3$, and $\delta_{ij} = 0$ if $i \neq j$. The Riemannian manifold (H^3, g) is called the hyperbolic space of dimension 3. It was proved in [9, p. 162] that (H^3, g) is a simply connected and complete Riemannian manifold.

It is obvious that, for each $x \in H^3$, $f_1(x), f_2(x) \in T_x H^3$, and their norms in the metric g are as follows:

$$|f_i(x)|_g = \left(\frac{e^{-2x_i^2} x_i^2 (\sin x_3)^2 + x_3^2}{x_3^2} \right)^{1/2}, \quad i = 1, 2,$$

where $|\cdot|_g$ denotes the norm of a tensor with respect to the metric g . Thus, we can view the control system (2.4) as a system on manifold H^3 :

$$\begin{cases} \dot{x}(t) = f_1(x(t)) \sin u(t) + f_2(x(t))v(t), & \text{a.e. } t \in [0, T], \\ x(t) \in H^3, & \forall t \in [0, T], \\ x(0) = (0, 0, 1)^\top, & e^T \leq x_3(T) \leq 2e^T, \end{cases}$$

with the control $(u(\cdot), v(\cdot)) \in \mathcal{U}$. The corresponding cost functional can be rewritten as follows:

$$J(u(\cdot), v(\cdot)) = \int_0^T (u^2(t)|f_1(x(t))|_g^2 + v(t)|f_2(x(t))|_g) dt.$$

In what follows, we will verify that this optimal control problem fulfills the assumptions required in Theorem 2.1.

It is easy to check that (A1), (A3), (A4), (A5), $\mathcal{P}_{ad} \neq \emptyset$ and (2.2) are satisfied. It suffices to verify (2.1) and (A6).

First, we shall use [7, Lemma 4.1] to prove that (2.1) holds. The local expression for the corresponding Levi-Civita connection ∇ is given by the Christoffel symbols Γ_{ij}^k , i.e. $\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum_{k=1}^3 \Gamma_{ij}^k \frac{\partial}{\partial x_k}$ for $i, j = 1, 2, 3$. By some direct computations we obtain the following (See [9, p. 161]):

$$\Gamma_{ij}^k(x) = \frac{1}{x_3} (-\delta_{jk}\delta_{i3} - \delta_{ki}\delta_{j3} + \delta_{ij}\delta_{k3}), \quad i, j, k = 1, 2, 3, x \in H^3. \quad (2.5)$$

Assume $\nabla f_1(x) = \sum_{i,j=1}^3 A_{ij}(x) \frac{\partial}{\partial x_i} \otimes dx_j$ for $x \in H^3$. Then, by the definitions of co-variant derivative of tensors (See [21, p.57]) and Levi-Civita connection (See [21, Theorem 2.2.2, p.53]), we have

$$\begin{aligned} A_{ij}(x) &= \nabla f_1 \left(dx_i, \frac{\partial}{\partial x_j} \right) \\ &= \nabla_{\frac{\partial}{\partial x_j}} \left(e^{-x_1^2} x_1 \sin x_3 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3} \right) (dx_i) \\ &= \frac{\partial}{\partial x_j} \left(e^{-x_1^2} x_1 \sin x_3 \right) \delta_{2i} + e^{-x_1^2} x_1 \sin x_3 \Gamma_{j2}^i(x) + \delta_{j3} \delta_{3i} + x_3 \Gamma_{j3}^i(x). \end{aligned}$$

Before estimating the norm of $|\nabla f_1|_g$, we are going to calculate the norms of a vector and covector at $x \in H^3$. Taking any $X = \sum_{i=1}^3 X_i \frac{\partial}{\partial x_i} \in T_x H^3$, we have $|X|_g = \frac{1}{x_3} |X|_{\mathbb{R}^3}$, where $|\cdot|_{\mathbb{R}^3}$ denotes the Euclidean norm of X . Take any covector $\eta = \sum_{i=1}^3 \eta_i dx_i$. According to the definition for the norm of a tensor (See [7, (2.5)]), we have

$$\begin{aligned} |\eta|_g &= \sup \{ \eta(Y) \mid Y = \sum_{i=1}^3 Y_i \frac{\partial}{\partial x_i} \in T_x H^3, |Y|_g \leq 1 \} \\ &= \sup \left\{ \sum_{i=1}^3 \eta_i Y_i \mid \frac{1}{x_3} |(Y_1, Y_2, Y_3)^\top|_{\mathbb{R}^3} \leq 1 \right\} \\ &= x_3 |(\eta_1, \eta_2, \eta_3)^\top|_{\mathbb{R}^3}. \end{aligned}$$

By means of [7, (2.5)] again, we also have

$$\begin{aligned} |\nabla f_1|_g &= \sup \{ \nabla f_1(\eta, Y) \mid |\eta|_g \leq 1, |Y|_g \leq 1 \} \\ &= \sup \left\{ \sum_{i,j=1}^3 A_{ij}(x) \eta_i X_j \mid |(\eta_1, \eta_2, \eta_3)^\top|_{\mathbb{R}^3} \leq 1/x_3, |(X_1, X_2, X_3)^\top|_{\mathbb{R}^3} \leq x_3 \right\} \\ &\leq C_1, \end{aligned}$$

for a big constant C_1 , where we have used (2.5) and the expressions for $A_{ij}(x)$. Similarly, we can also show $|\nabla f_2(x)|_g \leq C_1$ for any $x \in H^3$. Thus, (2.1) follows from [7, Lemma 4.1].

Next, we will show that (A6) holds by means of Proposition 2.1. It is obvious that the assumptions (A3') and (A4') hold. We claim that, for a.e. $t \in (0, T)$ the set

$$\mathcal{E}(t, x) = \bigcup_{\substack{(u,v) \\ \in [0,\pi] \times [0,1]}} \left\{ \begin{pmatrix} z^0 \\ \sin u f_1(x) + v f_2(x) \end{pmatrix} \mid \begin{array}{l} (u, v) \in [0, \pi] \times [0, 1], \\ z^0 \geq u^2 |f_1(x)|_g^2 + v |f_2(x)|_g \end{array} \right\}$$

is convex and closed for each $x \in H^3$.

It is obvious that $\mathcal{E}(t, x)$ is closed. We only have to show that it is convex. To this end, take any

$$\begin{pmatrix} z_0 \\ \sin u f_1(x) + v f_2(x) \end{pmatrix}, \begin{pmatrix} \xi_0 \\ \sin \hat{u} f_1(x) + \hat{v} f_2(x) \end{pmatrix} \in \mathcal{E}(t, x)$$

and $\lambda \in [0, 1]$, where $(u, v), (\hat{u}, \hat{v}) \in [0, \pi] \times [0, 1]$. We set $u_\lambda = \lambda u + (1 - \lambda)\hat{u}$ and $v_\lambda = \lambda v + (1 - \lambda)\hat{v}$. Since $\sin x$ with $x \in [0, \pi]$ is concave, we have

$$\lambda \sin u + (1 - \lambda) \sin \hat{u} \leq \sin u_\lambda.$$

On the other hand, there exists $\hat{u}_\lambda \in [\min\{u, \hat{u}\}, \max\{u, \hat{u}\}]$ such that

$$\lambda \sin u + (1 - \lambda) \sin \hat{u} = \sin \hat{u}_\lambda.$$

Let $\hat{u}_{\lambda,1}, u_{\lambda,1} \in [0, \pi/2]$ be such that $\sin \hat{u}_{\lambda,1} = \sin \hat{u}_\lambda$ and $\sin u_{\lambda,1} = \sin u_\lambda$. Then $0 \leq \hat{u}_{\lambda,1} \leq u_{\lambda,1} \leq \pi$. Moreover, by the convexity of the function s^2 with $s \geq 0$, we further have

$$\hat{u}_{\lambda,1}^2 \leq u_{\lambda,1}^2 \leq \lambda u^2 + (1 - \lambda)\hat{u}^2.$$

Consequently, we obtain

$$\begin{aligned} & \lambda \left(u^2 |f_1(x)|_g^2 + v |f_2(x)|_g \right) + (1 - \lambda) \left(\hat{u} |f_1(x)|_g^2 + \hat{v} |f_2(x)|_g \right) \\ & \geq \hat{u}_{\lambda,1}^2 |f_1(x)|_g^2 + v_\lambda |f_2(x)|_g, \end{aligned}$$

with

$$\lambda (\sin u f_1(x) + v f_2(x)) + (1 - \lambda) (\sin \hat{u} f_1(x) + \hat{v} f_2(x)) = \sin \hat{u}_{\lambda,1} f_1(x) + v_\lambda f_2(x).$$

Hence, we derive

$$\lambda \begin{pmatrix} z_0 \\ \sin u f_1(x) + v f_2(x) \end{pmatrix} + (1 - \lambda) \begin{pmatrix} \xi_0 \\ \sin \hat{u} f_1(x) + \hat{v} f_2(x) \end{pmatrix} \in \mathcal{E}(t, x),$$

which implies $\mathcal{E}(t, x)$ is convex.

Remark 2.2. *There are two possible ways to solve Example 2.1. The first one is to use the existing results for the setting of Euclidean spaces. [3, Theorem 4.2, p. 58] is about the problem where the state is constrained to a compact set at any time. However, this result is not applicable to Example 2.1, because the pointwise state constraint set H^3 is not compact. Another way is to eliminate the pointwise state constraints in form by transforming the problem to the one evolved on a manifold and using the results for the case of manifolds. In Example 2.1 we adopt the latter one. For the existence of optimal pairs for optimal control problems on differentiable manifolds, [1, Theorem 3.43, p.89] concerns on an affine control system with a special cost functional. However, the optimal control problem in Example 2.1 does not fulfill the assumptions required in [1, Theorem 3.43, p.89].*

In Example 2.1, the state space H^3 has a unique coordinate chart. In what follows, we exhibit another example, in which the state space contains more than one coordinate chart.

Example 2.2. *For $S^2 \triangleq \{(x_1, x_2, x_3)^\top \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1\}$, $N = (0, 0, 1)^\top$ and $C > 0$ being big enough, we consider the following control system*

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^3 u_i(t) f_i(x(t)), & \text{a.e. } t \in (0, T), \\ x(0) = N, \quad |x(T) - N|_{\mathbb{R}^3} \geq \sqrt{2}, \\ u(t) = (u_1(t), u_2(t), u_3(t))^\top \in \mathbb{R}^3, \quad |u(t)|_{\mathbb{R}^3} \leq C, & \text{a.e. } t \in (0, T), \\ x(t) \in S^2, \quad \forall t \in [0, T], \end{cases} \quad (2.6)$$

where the control function $u(\cdot) = (u_1(\cdot), u_2(\cdot), u_3(\cdot))^\top$ is required to be measurable, and f_1, f_2, f_3 are maps from \mathbb{R}^3 to \mathbb{R}^3 given by

$$f_1(x) = (x_2, -x_1, 0)^\top, \quad f_2(x) = (0, x_3, -x_2)^\top, \quad f_3(x) = (-x_3, 0, x_1)^\top,$$

for all $x = (x_1, x_2, x_3)^\top \in \mathbb{R}^3$. Let $f^0(= f^0(t, x, u)) : [0, T] \times \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}$ be a map satisfying: 1) $f^0 : [0, T] \times \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}$ is measurable in $t \in [0, T]$ and of class C^1 in $(x, u) \in \mathbb{R}^3 \times \mathbb{R}^3$; 2) There exists a positive constant K such that $\inf\{f^0(t, x, u) \mid x \in S^2, |u|_{\mathbb{R}^3} \leq C, \text{ a.e. } t \in (0, T)\} \geq -K$; 3) For a.e. $t \in (0, T)$ and each $x \in S^2$, the map $f^0(t, x, \cdot)$ is convex. Then, we shall show below the existence of optimal pairs for the following optimal control problem: Minimize $J(u(\cdot)) \triangleq \int_0^T f^0(t, x(t), u(t)) dt$ over the pairs $(u(\cdot), x(\cdot))$ satisfying (2.6).

First, the control system (2.6) is a system evolved on the submanifold S^2 of \mathbb{R}^3 . In fact, it is obvious that, for each $x \in S^2$, it holds that x is orthogonal to $f_i(x)$ (with $i = 1, 2, 3$). Thus $\{f_1(x), f_2(x), f_3(x)\} \subset T_x S^2$.

Second, we show the existence of admissible pairs. By the same argument as that will be used in “Step 4” of the “Proof of Theorem 2.1”, we only have to show that, for each $x \in S^2$, $\text{span}\{f_1(x), f_2(x), f_3(x)\} = T_x S^2$, where $\text{span}\{f_1(x), f_2(x), f_3(x)\}$ is the set of all linear combinations of $f_1(x), f_2(x)$ and $f_3(x)$. We will prove this via coordinate charts. The manifold S^2 has a family of coordinate charts: (O_+, φ_+) and (O_-, φ_-) , where $O_+ = S^2 \setminus \{(0, 0, -1)^\top\}$, $O_- = S^2 \setminus \{N\}$ and maps $\varphi_+ = (\xi_1, \xi_2)^\top : O_+ \rightarrow \mathbb{R}^2$ and $\varphi_- = (\eta_1, \eta_2)^\top : O_- \rightarrow \mathbb{R}^2$ are respectively defined by

$$\begin{aligned}\xi(x) &= (\xi_1(x), \xi_2(x))^\top = \varphi_+(x) = \left(\frac{x_1}{1+x_3}, \frac{x_2}{1+x_3} \right)^\top, \quad \forall x \in O_+, \\ \eta(x) &= (\eta_1(x), \eta_2(x))^\top = \varphi_-(x) = \left(\frac{x_1}{1-x_3}, \frac{x_2}{1-x_3} \right)^\top, \quad \forall x \in O_-.\end{aligned}\tag{2.7}$$

The inverse of φ_+ and φ_- are as follows:

$$\begin{aligned}\varphi_+^{-1}(\xi) &= (x_1(\xi), x_2(\xi), x_3(\xi))^\top = \left(\frac{2\xi_1}{a(\xi)}, \frac{2\xi_2}{a(\xi)}, \frac{1-\xi_1^2-\xi_2^2}{a(\xi)} \right)^\top, \\ \varphi_-^{-1}(\eta) &= (x_1(\eta), x_2(\eta), x_3(\eta))^\top = \left(\frac{2\eta_1}{a(\eta)}, \frac{2\eta_2}{a(\eta)}, \frac{\eta_1^2+\eta_2^2-1}{a(\eta)} \right)^\top,\end{aligned}\tag{2.8}$$

for all $\xi = (\xi_1, \xi_2)^\top, \eta = (\eta_1, \eta_2)^\top \in \mathbb{R}^2$, where $a : \mathbb{R}^2 \rightarrow \mathbb{R}$ is given by $a(\xi) = 1 + \xi_1^2 + \xi_2^2$.

For $x \in O_+$, $\left\{ \frac{\partial}{\partial \xi_1} \Big|_x, \frac{\partial}{\partial \xi_2} \Big|_x \right\}$ is the basis for $T_x S^2$. By means of (2.7) and (2.8), we obtain the expressions for f_1, f_2 and f_3 in the chart (O_+, φ_+) :

$$\begin{aligned}f_1(\xi) &= \xi_2 \frac{\partial}{\partial \xi_1} - \xi_1 \frac{\partial}{\partial \xi_2}, \\ f_2(\xi) &= \xi_1 \xi_2 \frac{\partial}{\partial \xi_1} + \frac{1-\xi_1^2+\xi_2^2}{2} \frac{\partial}{\partial \xi_2}, \\ f_3(\xi) &= \frac{-1-\xi_1^2+\xi_2^2}{2} \frac{\partial}{\partial \xi_1} - \xi_1 \xi_2 \frac{\partial}{\partial \xi_2},\end{aligned}$$

which immediately implies $\text{span}\{f_1(x), f_2(x), f_3(x)\} = T_x S^2$. When $x \in O_-$, we can obtain this relation similarly.

Third, we show that the condition (A6) holds by means of Proposition 2.1. For a.e. $t \in (0, T)$ and each $x \in S^2$, set

$$\begin{aligned}\mathcal{E}(t, x) &= \{(\zeta, X) \in \mathbb{R} \times T_x S^2 \mid \exists u = (u_1, u_2, u_3)^\top \in \mathbb{R}^3 \text{ s.t. } |u|_{\mathbb{R}^3} \leq C \\ &\text{and } X = \sum_{i=1}^3 u_i f_i(x), \zeta \geq f^0(t, x, u)\}.\end{aligned}$$

We obtain from the convexity of $f^0(t, x, \cdot)$ that $\mathcal{E}(t, x)$ is convex. It is obvious that $\mathcal{E}(t, x)$ is closed. Thus, according to Proposition 2.1, we derive that the condition (A6) is satisfied.

For the present optimal control problem, the conditions (A1), (A3), (A4) and (A5) hold obviously. Finally, we show that the condition (A2) holds. We view S^2 as a submanifold

of \mathbb{R}^3 with induced metric g . Since f_1, f_2, f_3 are smooth, S^2 is compact and the control set is bounded, there exists a positive constant C_1 such that $|\nabla \sum_{i=1}^3 u_i f_i(x)|_g \leq C_1$ holds for all $x \in S^2$ and $(u_1, u_2, u_3)^\top \in \mathbb{R}^3$ with $|(u_1, u_2, u_3)^\top|_{\mathbb{R}^3} \leq C$ (Recall that ∇ and $|\cdot|_g$ stand for respectively the Levi-Civita connection and the norm of a tensor with respect to the metric g). Thus, according to [7, Lemma 4.1], we obtain that, for the present problem (2.1) holds. Relation (2.2) also holds for this problem, since S^2 is compact and the control set is bounded.

Remark 2.3. We point out that, neither [3, Theorem 4.2, p. 58] nor [1, Theorem 3.43, p. 89] can be used directly to show the existence result in Example 2.2. Indeed, for the optimal control problem discussed in Example 2.2, even though the pointwise state constraint set S^2 is compact, the existence of the corresponding minimizing sequence of admissible pairs (with an additional property of equi-absolutely continuous trajectories, required as an assumption in [3, Theorem 4.2, p. 58]) is unclear. On the other hand, in [1, Theorem 3.43, p. 89], the state of the control system is constrained to a fixed point at the terminal time, while in control system (2.6), the state at the terminal time is constrained to a subset of S^2 .

3. Proof of the main results

This section is addressing to proving Theorem 2.1 and Proposition 2.1.

First, write

$$C([0, T]; M) \triangleq \{y : [0, T] \rightarrow M \mid y(\cdot) \text{ is continuous in } M\}.$$

We show below some properties of trajectories of the control system (1.1).

Lemma 3.1. *Assume the conditions (A1) and (A2) hold. Then, for any $x \in M$ and $u(\cdot) \in \mathcal{U}$, the system (1.1) admits a unique solution $y(\cdot) \in C([0, T]; M)$ with $y(0) = x$ and*

$$\begin{aligned} \rho(y(s), y(t)) &\leq \rho(y(t), x_0)(e^{K(s-t)} - 1) + K \int_t^s \ell(\tau) d\tau e^{K(s-t)}, \\ &\forall t, s \in [0, T] \text{ with } 0 \leq t < s \leq T, \end{aligned} \quad (3.1)$$

where the positive constant K is given in (A2).

Proof. First, we claim that

$$|f(t, y, v)| \leq K\rho(x_0, y) + \ell(t), \quad \forall (y, v) \in M \times U \text{ and a.e. } t \in [0, T]. \quad (3.2)$$

In fact, since (M, g) is complete, it follows from [9, Theorem 2.8, p. 146] that, for any $y \in M$, there exists a geodesic $\gamma : [0, 1] \rightarrow M$ satisfying $\gamma(0) = x_0$, $\gamma(1) = y$, and

the length of $\gamma(\cdot)$ (i.e., $\int_0^1 |\dot{\gamma}(t)| dt$) equals to $\rho(x_0, y)$. Moreover, there exist $N \in \mathbb{N}$ and $\tau_0, \tau_1, \dots, \tau_N \in [0, 1]$ with $0 = \tau_0 < \tau_1 < \dots < \tau_{N-1} < \tau_N = 1$ such that $\rho(\gamma(\tau_j), \gamma(\tau_{j+1})) < \min\{i(\gamma(\tau_j)), i(\gamma(\tau_{j+1}))\}$ for $j = 0, 1, \dots, N-1$. For a.e. $t \in [0, T]$ and every $v \in U$, we obtain from [7, (2.6)] (or [9, Definition 2.5, p. 52 & Proposition 3.2, p. 53]), the triangle inequality of the norm $|\cdot|$ and the condition (A2) that

$$\begin{aligned}
|f(t, y, v)| &\leq |f(t, \gamma(\tau_N), v) - L_{\gamma(\tau_{N-1})\gamma(\tau_N)} f(t, \gamma(\tau_{N-1}), v)| + |f(t, \gamma(\tau_{N-1}), v)| \\
&\leq |f(t, \gamma(\tau_N), v) - L_{\gamma(\tau_{N-1})\gamma(\tau_N)} f(t, \gamma(\tau_{N-1}), v)| \\
&\quad + |f(t, \gamma(\tau_{N-1}), v) - L_{\gamma(\tau_{N-2})\gamma(\tau_{N-1})} f(t, \gamma(\tau_{N-2}), v)| + |f(t, \gamma(\tau_{N-2}), v)| \\
&\leq \dots \\
&\leq \sum_{j=1}^N |f(t, \gamma(\tau_j), v) - L_{\gamma(\tau_{j-1})\gamma(\tau_j)} f(t, \gamma(\tau_{j-1}), v)| + |f(t, x_0, v)| \\
&\leq K \sum_{j=1}^N \rho(\gamma(\tau_j), \gamma(\tau_{j-1})) + \ell(t) \\
&= K\rho(x_0, y) + \ell(t),
\end{aligned}$$

which implies (3.2).

Next, take any $x \in M$ and $u(\cdot) \in \mathcal{U}$. By employing a local coordinate transformation near x , we can transform the system (1.1) with initial state $y(0) = x$ into an ordinary differential equation with its state valued in \mathbb{R}^n . Applying [14, Proposition 5.3, p. 66] to this equation, we obtain that this equation admits a local solution. We now use the contradiction argument to show that such a solution can be extended globally. For this, assume that the system (1.1) admitted a solution $y(\cdot)$ on $[0, \tau)$ for some $\tau > 0$ and

$$\lim_{s \rightarrow \tau^-} \rho(y(s), x) = +\infty. \quad (3.3)$$

Take any $t \in [0, \tau)$. For any $s \in [t, \tau)$, we could obtain from (3.2) and [9, Proposition 2.5, p. 146] that

$$\begin{aligned}
&\int_t^s |f(\zeta, y(\zeta), u(\zeta))| d\zeta \\
&\leq K \int_t^s (\rho(y(\zeta), x_0) + \ell(\zeta)) d\zeta \\
&\leq K \int_t^s \rho(y(\zeta), y(t)) d\zeta + K\rho(x_0, y(t))(s-t) + K \int_t^s \ell(\zeta) d\zeta.
\end{aligned}$$

By the definition of distance functions on Riemannian manifolds (See [9, Definition 2.4, p. 146]), it follows that

$$\rho(y(s), y(t)) \leq \int_t^s |\dot{y}(\zeta)| d\zeta = \int_t^s |f(\zeta, y(\zeta), u(\zeta))| d\zeta.$$

Applying Gronwall's inequality to the above, we obtain that the inequality (3.1) holds for $s \in [t, \tau)$. To conclude the proof, it suffices to show $T < \tau$. In fact, if this was not true, we would derive that

$$\lim_{s \rightarrow \tau^-} \rho(y(s), y(t)) \leq \rho(y(t), x_0)(e^{K(\tau-t)} - 1) + K \int_t^\tau \ell(\zeta) d\zeta e^{K(\tau-t)} < +\infty,$$

which contradicts (3.3). \square

Proof of Theorem 2.1 The proof is divided into five steps.

Step 1. Since $\mathcal{P}_{ad} \neq \emptyset$, there exists a sequence $\{(u_k(\cdot), y_k(\cdot))\} \subset \mathcal{P}_{ad}$ such that $\lim_{k \rightarrow +\infty} J(u_k(\cdot), y_k(\cdot)) = \inf_{(u(\cdot), y(\cdot)) \in \mathcal{P}_{ad}} J(u(\cdot), y(\cdot))$. We obtain from the assumption (A5), the completeness of M , and [9, Theorem 2.8, p. 146] that, there exists a subsequence of $\{y_k(\cdot)\}$ (we still denote it by $\{y_k(\cdot)\}$) and $(y_0, y_T) \in S$ such that $\lim_{k \rightarrow +\infty} (y_k(0), y_k(T)) = (y_0, y_T)$. Moreover, it follows from Lemma 3.1 and [9, Proposition 2.5, p. 146] that,

$$\begin{aligned} \rho(y_k(t), y_k(0)) &\leq \rho(y_k(0), x_0)(e^{Kt} - 1) + K \int_0^t \ell(\tau) d\tau e^{Kt} \\ &\leq \left(\rho(y_k(0), y_0) + \rho(y_0, x_0) \right) (e^{KT} - 1) + K \int_0^T \ell(\tau) d\tau e^{KT} \end{aligned}$$

holds for all $k \in \mathbb{N}$ and $t \in [0, T]$, which together with the convergence of $\{y_k(0)\}$, immediately implies $\{y_k(\cdot)\}$ is bounded in $C([0, T]; M)$. Applying Lemma 3.1 to $\{y_k(\cdot)\}$ again, we obtain that, for all $s \in [0, T]$, it holds that

$$\rho(y_k(t), y_k(s)) \leq \rho(y_k(s), x_0)(e^{K(t-s)} - 1) + K \int_s^t \ell(\tau) d\tau e^{K(t-s)} \quad \forall t \in [s, T].$$

Thus $\{y_k(\cdot)\}$ is equicontinuous in $C([0, T]; M)$. Applying Ascoli's theorem (ref. [16, Theorem 47.1, P. 290]) to $\{y_k(\cdot)\}$, there exists a subsequence of $\{y_k(\cdot)\}$ (we still denote it by $\{y_k(\cdot)\}$) and $\bar{y}(\cdot) \in C([0, T]; M)$ such that $y_k(\cdot)$ tends to $\bar{y}(\cdot)$ in $C([0, T]; M)$ as $k \rightarrow \infty$. Recalling that $\{(y_k(0), y_k(T))\}$ converges to $(\bar{y}(0), \bar{y}(T))$, we have $(\bar{y}(0), \bar{y}(T)) = (y_0, y_T)$. Moreover, since M is complete, we obtain from [9, Theorem 2.8, p. 146] that, there exists $\bar{\delta} > 0$ such that

$$\inf \{i(\bar{y}(t)) \mid t \in [0, T]\} > \bar{\delta}.$$

Then, there exists $\bar{k} > 0$ such that

$$\rho(y_k(t), \bar{y}(t)) \leq \bar{\delta} < i(\bar{y}(t)), \quad \forall t \in [0, T] \text{ and } k \geq \bar{k}.$$

Thus, according to the definition of the parallel translation (e.g., [7, Section 2.2]), we can define

$$L_{y_k(t)\bar{y}(t)} f(t, y_k(t), u_k(t)) \in T_{\bar{y}(t)} M \quad \text{for all } k \geq \bar{k} \text{ and a.e. } t \in [0, T].$$

Step 2. For $k \geq \bar{k}$, we express $L_{y_k(\cdot)\bar{y}(\cdot)}f(\cdot, y_k(\cdot), u_k(\cdot))$ by a frame along $\bar{y}(\cdot)$. For this purpose, let $\{e_1, \dots, e_n\} \subset T_{y_0}M$ be an orthonormal basis for $T_{y_0}M$, i.e. it is a basis for $T_{y_0}M$ with $\langle e_i, e_j \rangle = \delta_{ij}$ for $i, j = 1, \dots, n$, where δ_{ij} is the Kronecker delta symbol. Denote by $e_i(t) = L_{y_0\bar{y}(t)}^{\bar{y}(\cdot)} e_i$ for $i = 1, \dots, n$ and $t \in (0, T]$, which is the parallel translation of e_i from y_0 to $\bar{y}(t)$ along the curve $\bar{y}(\cdot)$ (See [7, Section 2.2]). We obtain from [7, (2.6)] that $\{e_1(t), \dots, e_n(t)\}$ forms an orthonormal basis for $T_{\bar{y}(t)}M$, i.e. $\langle e_i(t), e_j(t) \rangle = \delta_{ij}$ for all $i, j \in \{1, \dots, n\}$. Thus, we can write

$$L_{y_k(t)\bar{y}(t)}f(t, y_k(t), u_k(t)) = \sum_{i=1}^n f_k^i(t) e_i(t), \quad \forall k \geq \bar{k} \text{ and a.e. } t \in [0, T], \quad (3.4)$$

where $f_k^i(t) \triangleq \langle L_{y_k(t)\bar{y}(t)}f(t, y_k(t), u_k(t)), e_i(t) \rangle$. Set $\mathbb{F}_k(t) = (f_k^1(t), \dots, f_k^n(t))^\top$, a.e. $t \in [0, T]$. It follows from [7, (2.6)] that

$$|f(t, y_k(t), u_k(t))|^2 = |L_{y_k(t)\bar{y}(t)}f(t, y_k(t), u_k(t))|^2 = \left| \sum_{i=1}^n f_k^i(t) e_i(t) \right|^2 = \sum_{i=1}^n f_k^i(t)^2 \quad (3.5)$$

holds for $k \geq \bar{k}$ and a.e. $t \in [0, T]$.

Step 3. We now analyze the convergence of the sequence $\{L_{y_k(\cdot)\bar{y}(\cdot)}f(\cdot, y_k(\cdot), u_k(\cdot))\}$ in a suitable sense. By means of (3.5) and (3.2), we derive that

$$\begin{aligned} \int_0^T |\mathbb{F}_k(t)|^p dt &= \int_0^T \left(\sum_{i=1}^n f_k^i(t)^2 \right)^{p/2} dt \\ &= \int_0^T |f(t, y_k(t), u_k(t))|^p dt \\ &\leq \int_0^T (K\rho(y_k(t), x_0) + \ell(t))^p dt. \end{aligned}$$

Recalling that $\{y_k(\cdot)\}$ is bounded in $C([0, T]; M)$, we conclude that $\{\mathbb{F}_k(\cdot)\}_{k \geq \bar{k}}$ is bounded in $L^p(0, T; \mathbb{R}^n)$. Thus, there exists a subsequence of $\{\mathbb{F}_k(\cdot)\}_{k \geq \bar{k}}$ (we still denote it by $\{\mathbb{F}_k(\cdot)\}_{k \geq \bar{k}}$) and $\bar{\mathbb{F}}(\cdot) = (\bar{f}^1(\cdot), \dots, \bar{f}^n(\cdot))^\top \in L^p(0, T; \mathbb{R}^n)$ such that $\{\mathbb{F}_k(\cdot)\}_{k \geq \bar{k}}$ converges weakly to $\bar{\mathbb{F}}(\cdot)$ in $L^p(0, T; \mathbb{R}^n)$. Applying Mazur's theorem to $\{\mathbb{F}_k(\cdot)\}_{k \geq \bar{k}}$, we obtain that, for each $k \geq \bar{k}$, there exist $\beta_{k1}, \dots, \beta_{kn_k} \in [0, 1]$ for some $n_k \in \mathbb{N}$ such that $\sum_{i=1}^{n_k} \beta_{ki} = 1$ and the sequence $\{\sum_{i=1}^{n_k} \beta_{ki} \mathbb{F}_{k+i}(\cdot)\}_{k \geq \bar{k}}$ converges to $\bar{\mathbb{F}}(\cdot)$ strongly in $L^p(0, T; \mathbb{R}^n)$. Hence there exists a subsequence of $\{\sum_{i=1}^{n_k} \beta_{ki} \mathbb{F}_{k+i}(\cdot)\}_{k \geq \bar{k}}$ (we still denote it by $\{\sum_{i=1}^{n_k} \beta_{ki} \mathbb{F}_{k+i}(\cdot)\}_{k \geq \bar{k}}$) such that

$$\lim_{k \rightarrow +\infty} \sum_{j=1}^{n_k} \beta_{kj} \mathbb{F}_{k+j}(t) = \bar{\mathbb{F}}(t), \quad \text{a.e. } t \in [0, T].$$

Set

$$\bar{f}(t) = \sum_{i=1}^n \bar{f}^i(t) e_i(t), \quad \bar{f}^0(t) = \liminf_{k \rightarrow +\infty} \sum_{j=1}^{n_k} \beta_{kj} f^0(t, y_{k+j}(t), u_{k+j}(t)), \quad \text{a.e. } t \in [0, T].$$

Then, for a.e. $t \in [0, T]$,

$$\begin{aligned}
\bar{f}(t) &= \sum_{i=1}^n \lim_{k \rightarrow +\infty} \sum_{j=1}^{n_k} \beta_{kj} f_{k+j}^i(t) e_i(t) \\
&= \lim_{k \rightarrow +\infty} \sum_{j=1}^{n_k} \beta_{kj} \sum_{i=1}^n f_{k+j}^i(t) e_i(t) \\
&= \lim_{k \rightarrow +\infty} \sum_{j=1}^{n_k} \beta_{kj} L_{y_{k+j}(t)\bar{y}(t)} f(t, y_{k+j}(t), u_{k+j}(t)).
\end{aligned}$$

Fix any $\delta \in (0, \bar{\delta})$. By the convergence of $\{y_k(\cdot)\}$, we obtain that, there exists $k_\delta \geq \bar{k}$ such that $\rho(\bar{y}(t), y_k(t)) < \delta$ holds for every $t \in [0, T]$ and $k \geq k_\delta$. Thus,

$$(\bar{f}^0(t), \bar{f}(t)) \in \text{cl co} \bigcup_{\rho(z, \bar{y}(t)) < \delta} L_{z\bar{y}(t)} \mathcal{E}(t, z), \quad \text{a.e. } t \in [0, T].$$

Since $\delta \in (0, \bar{\delta})$ is arbitrarily chosen, by the assumption (A6), we obtain that

$$(\bar{f}^0(t), \bar{f}(t)) \in \bigcap_{0 < \delta < i(\bar{y}(t))} \text{cl co} \bigcup_{\rho(z, \bar{y}(t)) < \delta} L_{z\bar{y}(t)} \mathcal{E}(t, z) = \mathcal{E}(t, \bar{y}(t)), \quad \text{a.e. } t \in [0, T]. \quad (3.6)$$

Step 4. In the step, we shall find an admissible control $\bar{u}(\cdot)$ such that $(\bar{u}(\cdot), \bar{y}(\cdot))$ is an admissible pair.

First, we apply [2, Theorem 8.2.9] to maps $(t, u) \mapsto f(t, \bar{y}(t), u)$, $t \mapsto \Gamma(t, \bar{y}(t))$ and $t \mapsto \bar{f}(t)$. For every $u \in U$, we obtain from the assumption (A2) that the map $(t, y) \mapsto f(t, y, u)$ satisfies

- (i) For a.e. $t \in [0, T]$, the map $y \in M \mapsto f(t, y, u) \in TM$ is continuous;
- (ii) For every $y \in M$, the map $t \in [0, T] \mapsto f(t, y, u) \in TM$ is measurable

Applying [2, Lemma 8.2.3, p. 311] to the maps $(t, y) \mapsto f(t, y, u)$ and $t \mapsto \bar{y}(t)$, we obtain that the map $t \mapsto f(t, \bar{y}(t), u)$ is measurable. On the other hand, for a.e. $t \in [0, T]$, we obtain from (A2) that $u \mapsto f(t, \bar{y}(t), u)$ is continuous.

Applying [2, Theorem 8.2.8, p. 314] to $\Gamma(\cdot, \cdot)$ and $\bar{y}(\cdot)$, we obtain from (A4) that, the map $t \mapsto \Gamma(t, \bar{y}(t))$ is measurable. By means of [2, Theorem 8.2.9, p. 315], there exists $\bar{u}(\cdot) \in \mathcal{U}$ such that

$$\bar{u}(t) \in \Gamma(t, \bar{y}(t)) \quad \text{and} \quad \bar{f}(t) = f(t, \bar{y}(t), \bar{u}(t)), \quad \text{a.e. } t \in [0, T]. \quad (3.7)$$

Consequently,

$$\bar{f}^0(t) \geq f^0(t, \bar{y}(t), \bar{u}(t)), \quad \text{a.e. } t \in [0, T]. \quad (3.8)$$

Next, we claim that

$$\dot{\bar{y}}(t) = f(t, \bar{y}(t), \bar{u}(t)), \quad \text{a.e. } t \in [0, T], \quad (3.9)$$

which immediately implies $(\bar{u}(\cdot), \bar{y}(\cdot)) \in \mathcal{P}_{ad}$. In fact, by the compactness of $\{\bar{y}(t) \mid t \in [0, T]\}$, there exist $N \in \mathbb{N}$ and $t_0, t_1, \dots, t_N \in [0, T]$ with $0 = t_0 < t_1 < \dots < t_N = T$ such that

$$\rho(\bar{y}(t_i), \bar{y}(t)) < \bar{\delta}/2, \quad \forall t \in [t_i, t_{i+1}] \text{ with } i \in \{0, 1, \dots, N-1\}.$$

Fix any $i \in \{0, 1, \dots, N-1\}$. By the above relation we can define $\exp_{\bar{y}(t_i)}^{-1} \bar{y}(t)$ and $\exp_{\bar{y}(t_i)}^{-1} y_k(t)$ for $t \in [t_i, t_{i+1}]$, when k is large enough. Take any $X \in T_{\bar{y}(t_i)}M$. For any $t \in [t_i, t_{i+1}]$, by means of [21, Proposition 5.5.1, p. 187], [8, (5.13) and (5.15)] and integration by parts over $[t_i, t]$, we derive that

$$\begin{aligned} & \langle \exp_{\bar{y}(t_i)}^{-1} \bar{y}(t), X \rangle \\ &= \lim_{k \rightarrow +\infty} \langle \exp_{\bar{y}(t_i)}^{-1} y_k(t), X \rangle \\ &= -\frac{1}{2} \lim_{k \rightarrow +\infty} \left(\nabla_1 \rho^2(\bar{y}(t_i), y_k(t))(X) - \nabla_1 \rho^2(\bar{y}(t_i), y_k(t_i))(X) \right) \\ &= -\frac{1}{2} \lim_{k \rightarrow +\infty} \int_{t_i}^t \nabla_2 \nabla_1 \rho^2(\bar{y}(t_i), y_k(s))(X, f(s, y_k(s), u_k(s))) ds \\ &= -\frac{1}{2} \int_{t_i}^t \nabla_2 \nabla_1 \rho^2(\bar{y}(t_i), \bar{y}(s))(X, \bar{f}(s)) ds + r_1(t) + r_2(t) \\ &= \left\langle \int_{t_i}^t d \exp_{\bar{y}(t_i)}^{-1} |_{\bar{y}(s)} \bar{f}(s) ds, X \right\rangle + r_1(t) + r_2(t), \end{aligned} \quad (3.10)$$

where

$$\begin{aligned} r_1(t) &\triangleq -\frac{1}{2} \lim_{k \rightarrow +\infty} \int_{t_i}^t \nabla_2 \nabla_1 \rho^2(\bar{y}(t_i), \bar{y}(s))(X, L_{y_k(s)\bar{y}(s)} f(s, y_k(s), u_k(s)) - \bar{f}(s)) ds, \\ r_2(t) &\triangleq -\frac{1}{2} \lim_{k \rightarrow +\infty} \int_{t_i}^t \left(\nabla_2 \nabla_1 \rho^2(\bar{y}(t_i), y_k(s))(X, f(s, y_k(s), u_k(s))) \right. \\ &\quad \left. - \nabla_2 \nabla_1 \rho^2(\bar{y}(t_i), \bar{y}(s))(X, L_{y_k(s)\bar{y}(s)} f(s, y_k(s), u_k(s))) \right) ds, \end{aligned}$$

and the integral in the last line is the Lebesgue integral on the Euclidean space $T_{\bar{y}(t_i)}M$ with metric $\langle \cdot, \cdot \rangle$ evaluated at $\bar{y}(t_i)$. By (3.4), the definition of $\bar{f}(\cdot)$, and the weak convergence of $\{\mathbb{F}_k(\cdot)\}$ in $L^p(0, T; \mathbb{R}^n)$, we derive that

$$r_1(t) = -\frac{1}{2} \lim_{k \rightarrow +\infty} \int_{t_i}^t \sum_{j=1}^n (f_k^j(s) - \bar{f}^j(t)) \nabla_2 \nabla_1 \rho^2(\bar{y}(t_i), \bar{y}(s))(X, e_j(s)) ds = 0.$$

For the term $r_2(t)$, we obtain from [8, (5.5) and (5.15)], the convergence of $\{y_k(\cdot)\}$ in the space $C([0, T]; M)$, (3.5), the boundedness of $\{\mathbb{F}_k(\cdot)\}$ in $L^p(0, T; \mathbb{R}^n)$, and [8, Lemma 5.1]

that

$$\begin{aligned}
r_2(t) &= \lim_{k \rightarrow +\infty} \int_{t_i}^t \left(\langle d \exp_{y_k(s)}^{-1} |_{\bar{y}(t_i)} X, f(s, y_k(s), u_k(s)) \rangle \right. \\
&\quad \left. - \langle d \exp_{\bar{y}(s)}^{-1} |_{\bar{y}(t_i)} X, L_{y_k(s)\bar{y}(s)} f(s, y_k(s), u_k(s)) \rangle \right) ds \\
&= \lim_{k \rightarrow +\infty} \int_{t_i}^t \langle d \exp_{y_k(s)}^{-1} |_{\bar{y}(t_i)} X - L_{\bar{y}(s)y_k(s)} d \exp_{\bar{y}(s)}^{-1} |_{\bar{y}(t_i)} X, f(s, y_k(s), u_k(s)) \rangle ds \\
&= 0.
\end{aligned}$$

Note that $X \in T_{\bar{y}(t_i)}M$ is arbitrarily chosen, we obtain that

$$\exp_{\bar{y}(t_i)}^{-1} \bar{y}(t) = \int_{t_i}^t d \exp_{\bar{y}(s)}^{-1} |_{\bar{y}(s)} \bar{f}(s) ds, \quad \forall t \in [t_i, t_{i+1}],$$

which is a curve in $T_{\bar{y}(t_i)}M$. This curve is differentiable for a.e. $t \in [t_i, t_{i+1}]$. Thus, the curve

$$\bar{y}(t) = \exp_{\bar{y}(t_i)} \int_{t_i}^t d \exp_{\bar{y}(s)}^{-1} |_{\bar{y}(s)} \bar{f}(s) ds, \quad \forall t \in [t_i, t_{i+1}]$$

is differentiable for a.e. t , and

$$\dot{\bar{y}}(t) = d \exp_{\bar{y}(t_i)} |_{\exp_{\bar{y}(t_i)}^{-1} \bar{y}(t)} \circ d \exp_{\bar{y}(t)}^{-1} |_{\bar{y}(t)} \bar{f}(t), \quad \text{a.e. } t \in [t_i, t_{i+1}].$$

Applying [13, Proposition 3.6, p. 55] to $\exp_{\bar{y}(t_i)}$ and $\exp_{\bar{y}(t_i)}^{-1}$, we obtain that $\dot{\bar{y}}(t) = \bar{f}(t)$ for a.e. $t \in [t_i, t_{i+1}]$, which together with (3.7), confirms (3.9).

Further, we obtain from the assumption (A5) and the convergence of $\{y_k(\cdot)\}$ in $C([0, T]; M)$ that $(\bar{y}(0), \bar{y}(T)) \in S$ and $\bar{y}(t) \in Q$ for all $t \in [0, T]$. Thus, $(\bar{u}(\cdot), \bar{y}(\cdot)) \in \mathcal{P}_{ad}$.

Step 5. By means of the assumption (A3) and (3.8), one has

$$\begin{aligned}
J(\bar{u}(\cdot), \bar{y}(\cdot)) &= h(\bar{y}(0), \bar{y}(T)) + \int_0^T f^0(t, \bar{y}(t), \bar{u}(t)) dt \\
&\leq \liminf_{k \rightarrow +\infty} h(y_k(0), y_k(T)) + \int_0^T \bar{f}^0(t) dt \\
&= \liminf_{k \rightarrow +\infty} h(y_k(0), y_k(T)) + \int_0^T \liminf_{k \rightarrow +\infty} \sum_{j=1}^{n_k} \beta_{kj} f^0(t, y_{k+j}(t), u_{k+j}(t)) dt \\
&\leq \liminf_{k \rightarrow +\infty} h(y_k(0), y_k(T)) + \limsup_{k \rightarrow +\infty} \int_0^T f^0(t, y_k(t), u_k(t)) dt \\
&\leq \limsup_{k \rightarrow +\infty} \left(h(y_k(0), y_k(t)) + \int_0^T f^0(t, y_k(t), u_k(t)) dt \right) \\
&= \lim_{k \rightarrow +\infty} J(u_k(\cdot), y_k(\cdot)),
\end{aligned}$$

which implies $(\bar{u}(\cdot), \bar{y}(\cdot))$ is an optimal pair. The proof of Theorem 2.1 is completed. \square

Proof of Proposition 2.1 It suffices to prove that, $\mathcal{E}(t, \cdot)$ satisfies the Cesari-type property on M provided that $\mathcal{E}(t, x)$ is convex and closed for each $x \in M$.

For the case that (A3') holds, we fix any $\epsilon > 0$. It follows from the condition (A3') that, there exists $\sigma \in (0, i(x))$ such that, for any $(x', u, u') \in M \times U \times U$ with $\max\{\rho(x, x'), d(u, u')\} < \sigma$, one has

$$|L_{x'}f(t, x', u') - f(t, x, u)| < \frac{\epsilon}{\sqrt{2}} \quad \text{and} \quad f^0(t, x', u') > f^0(t, x, u) - \frac{\epsilon}{\sqrt{2}}. \quad (3.11)$$

Since $\Gamma(t, \cdot)$ is upper semicontinuous, it follows from [14, Proposition 2.2, p. 89] that, there exists $\delta' \in (0, \sigma)$ such that

$$\Gamma(t, B_x(\delta')) \subseteq B_\sigma(\Gamma(t, x)),$$

where $B_x(\delta') \triangleq \{z \in M \mid \rho(z, x) < \delta'\}$ and $B_\sigma(\Gamma(t, x)) \triangleq \{v \in U \mid \inf_{w \in \Gamma(t, x)} d(w, v) < \sigma\}$. For any $z \in B_x(\delta')$, take any $(\zeta^0, \zeta) \in \mathcal{E}(t, z)$. Choose $u_z \in \Gamma(t, z)$ and $u'_z \in \Gamma(t, x)$ such that

$$\zeta = f(t, z, u_z), \quad \zeta^0 \geq f^0(t, z, u_z) \quad \text{and} \quad d(u_z, u'_z) < \sigma.$$

It follows from (3.11) that $(\zeta^0, L_{zx}\zeta) \in B_\epsilon(\mathcal{E}(t, x))$, where $B_\epsilon(\mathcal{E}(t, x)) \triangleq \{(y^0, Y) \in \mathbb{R} \times T_x M \mid \inf_{(z^0, Z) \in \mathcal{E}(t, x)} |z^0 - y^0|^2 + |Z - Y|^2 < \epsilon^2\}$. Since both $z \in B_x(\delta')$ and $(\zeta^0, \zeta) \in \mathcal{E}(t, z)$ are arbitrarily chosen, one has $\bigcup_{\rho(y, x) < \delta'} L_{yx}\mathcal{E}(t, y) \subseteq B_\epsilon(\mathcal{E}(t, x))$, which implies that

$$\bigcap_{i(x) > \delta > 0} \text{cl co} \left(\bigcup_{\rho(y, x) < \delta} L_{yx}\mathcal{E}(t, y) \right) \subseteq \text{cl co } B_\epsilon(\mathcal{E}(t, x)) \subseteq \text{cl } B_\epsilon(\mathcal{E}(t, x)).$$

Here one has used the relation $\text{co } B_\epsilon(\mathcal{E}(t, x)) \subseteq B_\epsilon(\text{co } \mathcal{E}(t, x))$ and the assumption that $\mathcal{E}(t, x)$ is convex. Since $\epsilon > 0$ is arbitrarily chosen and $\mathcal{E}(t, x)$ is closed, (2.3) follows.

For the case that (A4') holds, the proof is analogous to the previous case. This completes the proof of Proposition 2.1. \square

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