

ALGORITHMIC POLYNOMIALS

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ABSTRACT. The *approximate degree* of a Boolean function $f(x_1, x_2, \dots, x_n)$ is the minimum degree of a real polynomial that approximates f pointwise within $1/3$. Upper bounds on approximate degree have a variety of applications in learning theory, differential privacy, and algorithm design in general. Nearly all known upper bounds on approximate degree arise in an existential manner from bounds on quantum query complexity.

We develop a first-principles, classical approach to the polynomial approximation of Boolean functions. We use it to give the first constructive upper bounds on the approximate degree of several fundamental problems:

- $O(n^{\frac{3}{4} - \frac{1}{4(2^k - 1)}})$ for the k -element distinctness problem;
- $O(n^{1 - \frac{1}{k+1}})$ for the k -subset sum problem;
- $O(n^{1 - \frac{1}{k+1}})$ for any k -DNF or k -CNF formula;
- $O(n^{3/4})$ for the surjectivity problem.

In all cases, we obtain explicit, closed-form approximating polynomials that are unrelated to the quantum arguments from previous work. Our first three results match the bounds from quantum query complexity. Our fourth result improves polynomially on the $\Theta(n)$ quantum query complexity of the problem and refutes the conjecture by several experts that surjectivity has approximate degree $\Omega(n)$. In particular, we exhibit the first *natural* problem with a polynomial gap between approximate degree and quantum query complexity.

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1. INTRODUCTION

Let $f: X \rightarrow \{0, 1\}$ be a given Boolean function, defined on a subset $X \subseteq \{0, 1\}^n$. The ϵ -approximate degree of f , denoted $\deg_\epsilon(f)$, is the minimum degree of a multivariate real polynomial p such that $|f(x) - p(x)| \leq \epsilon$ for all $x \in X$. The standard setting of the error parameter for most applications is $\epsilon = 1/3$, an aesthetically motivated constant that can be replaced by any other in $(0, 1/2)$ at the expense of a constant-factor increase in approximate degree. The notion of approximate degree originated 25 years ago in the pioneering work of Nisan and Szegedy [43] and has since proved to be a powerful and versatile tool in theoretical computer science. Lower bounds on approximate degree have complexity-theoretic applications, whereas upper bounds are a tool in algorithm design. In the former category, the notion of approximate degree has enabled spectacular progress in circuit complexity [46, 57, 12, 8, 35, 36, 52, 10], quantum query complexity [9, 15, 3, 1, 4, 32, 20], and communication complexity [16, 47, 19, 52, 53, 48, 38, 23, 50, 10, 56, 55]. On the algorithmic side, approximate degree underlies many of the strongest results obtained to date in computational learning [58, 34, 33, 31, 44, 7], differentially private data release [59, 22], and algorithm design in general [39, 30, 51].

Despite these applications, progress in understanding approximate degree as a complexity measure has been slow and difficult. With very few exceptions [43, 30, 51, 54], all known upper bounds on approximate degree arise from *quantum query algorithms*. The connection between approximate degree and quantum query complexity was discovered by Beals et al. [9], who proved that the acceptance probability of an algorithm that makes T queries is representable by a real polynomial of degree $2T$. Put another way, every quantum algorithm implies an approximating polynomial of comparable complexity for the problem in question. Since the seminal work of Beals et al., essentially all upper bounds on approximate degree have come from quantum query algorithms, e.g., [15, 60, 6, 28, 7, 27, 26, 13, 40]. An illustrative example is the problem of determining the approximate degree of Boolean formulas of size n , posed in 2003 by O'Donnell and Servedio [44]. Progress on this question was stalled for a long time until it was finally resolved by Ambainis et al. [7], who built on the work of Farhi et al. [28] to give a near-optimal quantum query algorithm for any Boolean formula.

While quantum query complexity has been a fruitful source of approximate degree upper bounds, the exclusive reliance on quantum techniques for the polynomial approximation of Boolean functions is problematic. For one thing, a quantum query algorithm generally does not give any information about the approximating polynomial apart from its existence. For example, converting the quantum algorithms of [6, 7, 13] to polynomials results in expressions so large and complicated that they are no longer meaningful. More importantly, quantum query algorithms are more constrained objects than real polynomials, and an optimal query algorithm for a given problem may be far less efficient than a polynomial constructed from scratch. Given the many unresolved questions on approximate degree, there is a compelling need for polynomial approximation techniques that go beyond quantum query complexity.

In this paper, we take a fresh look at several breakthrough upper bounds for approximate degree, obtained over the years by sophisticated quantum query algorithms. In each case, we are able to construct an approximating polynomial from first principles that matches or improves on the complexity of the best quantum algorithm. All of our constructions produce explicit, closed-form polynomials that are unrelated to the corresponding quantum algorithms and are in the author's

opinion substantially simpler. In one notable instance, our construction achieves a polynomial improvement on the complexity of the best possible quantum algorithm, refuting a conjecture [21] on the approximate degree of that problem and exhibiting the first *natural* example of a polynomial gap between approximate degree and quantum query complexity. Our proofs, discussed shortly, contribute novel techniques to the area.

1.1. k -Element distinctness. The starting point in our work is the *element distinctness problem* [17, 3, 6, 4, 37, 13], which is one of the most studied questions in quantum query complexity and a major success story of the field. The input to the problem is a list of n elements from a given range of size r , and the objective is to determine if the elements are pairwise distinct. A well-studied generalization of this problem is *k -element distinctness*, where k is an arbitrary constant and the objective is to determine if some k -tuple of the elements are identical. Formally, the input to element distinctness and k -element distinctness is represented by a Boolean matrix $x \in \{0, 1\}^{n \times r}$ in which every row i has precisely one “1” entry, corresponding to the value of the i th element.¹ Aaronson and Shi [3], Ambainis [4], and Kutin [37] showed that element distinctness has quantum query complexity $\Omega(n^{2/3})$. In follow-up work, Ambainis [6] gave a quantum algorithm for element distinctness with $O(n^{2/3})$ queries, matching the lower bound in [3, 4, 37]. For the more general problem of k -element distinctness, Ambainis’s algorithm [6] requires $O(n^{k/(k+1)})$ queries. Using a different approach, Belovs [13] gave a polynomially faster algorithm for k -element distinctness, with query complexity $O(n^{\frac{3}{4} - \frac{1}{4(2^k - 1)}})$. Belovs’s algorithm is currently the fastest known.

The algorithms of Ambainis [6] and Belovs [13] are highly nontrivial. The former is based on a quantum walk on the Johnson graph, whereas the latter uses the framework of learning graphs. We give an elementary, closed-form construction of an approximating polynomial for k -element distinctness that bypasses the quantum work. Formally, let $\text{ED}_{n,r,k}: \{0, 1\}_{\leq n}^{n \times r} \rightarrow \{0, 1\}$ be given by

$$\text{ED}_{n,r,k}(x) = \begin{cases} 1 & \text{if } x_{1,j} + x_{2,j} + \cdots + x_{n,j} < k \text{ for each } j, \\ 0 & \text{otherwise.} \end{cases}$$

The notation $\{0, 1\}_{\leq n}^{n \times r}$ for the domain of this function indicates that we allow *arbitrary* input matrices $x \in \{0, 1\}^{n \times r}$ of Hamming weight at most n , with no restriction on the placement of the “1” bits. This is of course a problem more general than k -element distinctness. We prove:

THEOREM 1.1 (k -element distinctness). *Let $k \geq 1$ be a fixed integer. Then for all $n, r \geq 1$,*

$$\deg_{1/3}(\text{ED}_{n,r,k}) = O\left(\sqrt{n} \min\{n, r\}^{\frac{1}{2} - \frac{1}{4(1-2^{-k})}}\right).$$

Moreover, the approximating polynomial is given explicitly in each case.

¹Alternately, the input can be represented by a string of $n \lceil \log r \rceil$ bits. Switching to this more compact representation changes the complexity of the problem by a factor of at most $\lceil \log r \rceil$, which is negligible in all settings of interest.

Theorem 1.1 matches the quantum query bound of $O(n^{\frac{3}{4} - \frac{1}{4(2^k-1)}}) \equiv O(n^{1 - \frac{1}{4(1-2^{-k})}})$ due to Belovs [13] and further generalizes it to every $r \geq 1$.

1.2. k -Subset sum, k -DNF and k -CNF formulas. Another well-studied problem in quantum query complexity is k -subset sum [25, 14]. The input to this problem is a list of n elements from a given finite Abelian group G , and the objective is to determine whether there is a k -tuple of elements that sum to 0. Formally, the input is represented by a matrix $x \in \{0, 1\}^{n \times |G|}$ with precisely one “1” entry in every row. Childs and Eisenberg [25] contributed an alternate analysis of Ambainis’s algorithm for k -element distinctness [6] and showed how to adapt it to compute k -subset sum or any other function property with 1-certificate complexity at most k . In particular, any such problem has an approximating polynomial of degree $O(n^{k/(k+1)})$. We give a first-principles construction of an approximating polynomial for any problem in this class, using techniques that are elementary and unrelated to the quantum work of Ambainis [6] and Childs and Eisenberg [25]. Our result is more general:

THEOREM 1.2 (k -DNF and k -CNF formulas). *Let $k \geq 0$ be a fixed integer. Let $f: \{0, 1\}_{\leq n}^N \rightarrow \{0, 1\}$ be representable on its domain by a k -DNF or k -CNF formula. Then*

$$\deg_{1/3}(f) = O(n^{\frac{k}{k+1}}).$$

Moreover, the approximating polynomial is given explicitly in each case.

Recall that a k -DNF formula in Boolean variables x_1, x_2, \dots, x_N is the disjunction of an arbitrary number of terms, where each term is the conjunction of at most k literals from among $x_1, \overline{x_1}, x_2, \overline{x_2}, \dots, x_N, \overline{x_N}$. An essential aspect of Theorem 1.2 is that the approximate degree upper bound depends only on the Hamming weight $x_1 + x_2 + \dots + x_N$ of the input and does not depend at all on the number of variables N , which can be arbitrarily large. Several special cases of Theorem 1.2 are worth noting. The theorem clearly applies to k -subset sum, which is by definition representable on its domain by a k -DNF formula. Moreover, in the terminology of Childs and Eisenberg [25], Theorem 1.2 applies to any function property with 1-certificate complexity at most k . Finally, taking $N = n$ shows that Theorem 1.2 applies to any function $f: \{0, 1\}^n \rightarrow \{0, 1\}$ representable by a k -DNF or k -CNF formula.

1.3. Surjectivity. While our proofs of Theorems 1.1 and 1.2 are significantly simpler than their quantum query counterparts, they do not give a quantitative improvement on previous work. This brings us to our next result. In the *surjectivity problem* [11], the input is a list of n elements from a given range of size r , where $r \leq n$. The objective is to determine whether the input features all r elements of the range. In function terminology, the input represents a mapping $\{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, r\}$, and the objective is to determine whether the mapping is surjective. As usual in the quantum query literature, the input is represented by a Boolean matrix $x \in \{0, 1\}^{n \times r}$ in which every row has precisely one “1” entry. Beame and Machmouchi [11] proved that for $r = \lfloor n/2 \rfloor + 1$, the surjectivity problem has the maximum possible quantum query complexity, namely, $\Theta(n)$. This led several experts to conjecture that the approximate degree of surjectivity is also $\Theta(n)$; see, e.g., [21]. The conjecture was significant because its resolution would give the first AC^0 circuit with approximate degree $\Theta(n)$, closing a long line of research [43, 3, 4, 21].

Surprisingly, we are able to show that surjectivity has an approximating polynomial of substantially lower degree, regardless of the range parameter r . Formally, let $\text{SURJ}_{n,r}: \{0,1\}_{\leq n}^{n \times r} \rightarrow \{0,1\}$ be given by

$$\text{SURJ}_{n,r}(x) = \bigwedge_{j=1}^r \bigvee_{i=1}^n x_{i,j}.$$

In keeping with our other results, our definition of $\text{SURJ}_{n,r}$ allows arbitrary input matrices $\{0,1\}^{n \times r}$ of Hamming weight at most n . In this generalization of the surjectivity problem, the input can be thought of as an arbitrary relation rather than a function. We prove:

THEOREM 1.3 (Surjectivity). *For all positive integers n and r ,*

$$\deg_{1/3}(\text{SURJ}_{n,r}) = \begin{cases} O(\sqrt{n} \cdot r^{1/4}) & \text{if } r \leq n, \\ 0 & \text{if } r > n. \end{cases}$$

Moreover, the approximating polynomial is given explicitly in each case.

In particular, the theorem gives an approximating polynomial of degree $O(n^{3/4})$ for all r . This upper bound is polynomially smaller than the problem's quantum query complexity $\Theta(n)$ for $r = \lfloor n/2 \rfloor + 1$. While explicit functions with a polynomial gap between approximate degree and quantum query complexity have long been known [5, 2], Theorem 1.3 exhibits the first *natural* function with this property. The functions in previous work [5, 2] were constructed with the specific purpose of separating complexity measures.

1.4. Symmetric functions. Key building blocks in our proofs are symmetric functions $f: \{0,1\}^n \rightarrow \{0,1\}$. A classic result due to Paturi [45] states that the $1/3$ -approximate degree of any such function f is $\Theta(\sqrt{n\ell})$, where $\ell \in \{0,1,2,\dots,n\}$ is the smallest number such that f is constant on inputs of Hamming weight in $[\ell, n-\ell]$. When a symmetric function is used in an auxiliary role as part of a larger construction, it becomes important to have approximating polynomials for every possible setting of the error parameter, $1/2^n \leq \epsilon \leq 1/3$. A complete characterization of the ϵ -approximate degree of symmetric functions for all ϵ was obtained by de Wolf [60], who sharpened previous bounds [30, 15, 51] using an elegant quantum query algorithm. Prior to our work, no classical, first-principles proof was known for de Wolf's characterization, which is telling in view of the basic role that $\text{AND}_n, \text{OR}_n$, and other symmetric functions play in the area. We are able to give such a first-principles proof—in fact, *three* of them.

THEOREM 1.4 (Symmetric functions). *Let $f: \{0,1\}^n \rightarrow \{0,1\}$ be a symmetric function. Let $\ell \in \{0,1,2,\dots,n\}$ be an integer such that f is constant on inputs of Hamming weight in $(\ell, n-\ell)$. Then for $1/2^n \leq \epsilon \leq 1/3$,*

$$\deg_{\epsilon}(f) = O\left(\sqrt{n\ell} + \sqrt{n \log \frac{1}{\epsilon}}\right).$$

Moreover, the approximating polynomial is given explicitly in each case.

Theorem 1.4 matches de Wolf’s quantum query result, tightly characterizing the ϵ -approximate degree of every nonconstant symmetric function.

1.5. Our techniques. Our proofs use only basic tools from approximation theory, such as Chebyshev polynomials. Our constructions additionally incorporate elements of classic algorithm design, e.g., the divide-and-conquer paradigm, the inclusion-exclusion principle, and probabilistic reasoning. The title of our paper, “Algorithmic Polynomials,” is a reference to this combination of classic algorithmic methodology and approximation theory. The informal message of our work is that algorithmic polynomials are not only more powerful than quantum algorithms but also easier to construct. A detailed discussion of Theorems 1.1–1.4 follows.

Extension theorem. As our starting point, we prove an *extension theorem* for polynomial approximation. This theorem allows one to construct an approximant for a given function F using an approximant for a restriction f of F . In more detail, let $f: \{0, 1\}_{\leq m}^N \rightarrow [-1, 1]$ be an arbitrary function, defined on inputs $x \in \{0, 1\}^N$ of Hamming weight at most m . Let $F_n: \{0, 1\}_{\leq n}^N \rightarrow [-1, 1]$ be the natural extension of f to inputs of Hamming weight at most n , defined by $F_n = 0$ outside the domain of f . From an approximation-theoretic point of view, a fundamental question to ask is how to efficiently “extend” any approximant for f to an approximant for F_n . Unfortunately, this naïve formulation of the extension problem has no efficient solution; we describe a counterexample in Section 3. We are able to show, however, that the extension problem becomes meaningful if one works with F_{2m} instead of f . In other words, we give an efficient, explicit, black-box transformation of any approximant for the extension F_{2m} into an approximant for the extension F_n , for any $n \geq 2m$. This result is essentially as satisfying as the “ideal” extension theorem in that the domains of f and F_{2m} almost coincide and can be arbitrarily smaller than the domain of F_n . Our proof makes use of extrapolation bounds, extremal properties of Chebyshev polynomials, and ideas from rational approximation theory.

Symmetric functions. As mentioned earlier, we give three proofs of Theorem 1.4 on the ϵ -approximate degree of symmetric functions. Each of the three proofs is fully constructive. Our simplest proof uses the extension theorem and is only half-a-page long. Here, we use brute-force interpolation to compute the function f of interest on inputs of small Hamming weight, and then apply the extension theorem to effortlessly extend the interpolant to the full domain of f . Our second proof of Theorem 1.4 is an explicit, closed-form construction that uses Chebyshev polynomials as its only ingredient. This proof is a refinement of previous, suboptimal approximants for the AND function [30, 51]. We eliminate the inefficiency in previous work by using Chebyshev polynomials to achieve improved control at every point of the domain. Finally, our third proof of Theorem 1.4 is inspired by combinatorics rather than approximation theory. Here, we use a *sampling experiment* to construct an approximating polynomial for any symmetric function f from an approximating polynomial for AND. In more detail, the experiment allows us to interpret f as a linear combination of conjunctions of arbitrary degree, where the sum of the absolute values of the coefficients is reasonably small. Once such a representation is available, we simply replace every conjunction with its approximating polynomial. These substitutions increase the error of the approximation by a factor bounded by the sum of the absolute values of the coefficients in the original linear combination, which is negligible.

k-Element distinctness, k-DNF and k-CNF formulas. We first establish an auxiliary result on the approximate degree of composed Boolean functions. Specifically, let $F: X \times \{0, 1\}_{\leq n}^N \rightarrow \{0, 1\}$ be given by $F(x, y) = \bigvee_{i=1}^N y_i \wedge f_i(x)$ for some set X and some functions $f_1, f_2, \dots, f_N: X \rightarrow \{0, 1\}$. We bound the ϵ -approximate degree of F in terms of the approximate degree of $\bigvee_{i \in S} f_i$, maximized over all sets $S \subseteq \{1, 2, \dots, N\}$ of certain size. Crucially for our applications, the bound that we derive has no dependence on N . The proof uses Chebyshev polynomials and the inclusion-exclusion principle. Armed with this *composition theorem*, we give a short proof of Theorem 1.2 on the approximate degree of k -DNF and k -CNF formulas. The argument proceeds by induction on k , with the composition theorem invoked to implement the inductive step. The proof of Theorem 1.1 on the approximate degree of k -element distinctness is more subtle. It too proceeds by induction, with the composition theorem playing a central role. This time, however, the induction is with respect to both k and the range parameter r , and the extension theorem is required to complete the inductive step. We note that we are able to bound the ϵ -approximate degree of k -DNF formulas and k -element distinctness for every setting of the error parameter ϵ , rather than just $\epsilon = 1/3$ in Theorems 1.1 and 1.2.

Surjectivity. Our proof of Theorem 1.3 is surprisingly short, given how improbable the statement was believed to be. As one can see from the defining equation for $\text{SURJ}_{n,r}$, this function is the componentwise composition $\text{AND}_r \circ \text{OR}_n$ restricted to inputs of Hamming weight at most n . With this in mind, we start with a degree- $O(\sqrt{r})$ polynomial $\widetilde{\text{AND}}_r$ that approximates AND_r pointwise within $1/4$. The approximant in question is simply a scaled and shifted Chebyshev polynomial. It follows that the componentwise composition $\widetilde{\text{AND}}_r \circ \text{OR}_n$, restricted to inputs of Hamming weight at most n , approximates $\text{SURJ}_{n,r}$ pointwise within $1/4$. We are not finished, however, because the degree of $\widetilde{\text{AND}}_r \circ \text{OR}_n$ is unacceptably large. Moving on, a few lines of algebra reveal that $\widetilde{\text{AND}}_r \circ \text{OR}_n$ is a linear combination of conjunctions in which the absolute values of the coefficients sum to $2^{O(\sqrt{r})}$. It remains to approximate each of these conjunctions pointwise within $2^{-\Omega(\sqrt{r})}$ by a polynomial of degree $O(\sqrt{n\sqrt{r}}) = O(\sqrt{n} \cdot r^{1/4})$, for which we use our explicit approximant from Theorem 1.4 along with the guarantee that the input has Hamming weight at most n . The proof of Theorem 1.3 is particularly emblematic of our work in its interplay of approximation-theoretic methodology (Chebyshev polynomials, linear combinations) and algorithmic thinking (reduction of the problem to the approximation of individual conjunctions).

We are pleased to report that our $O(n^{3/4})$ upper bound for the surjectivity problem has just sparked further progress in the area by Bun, Kothari, and Thaler [20], who prove tight or nearly tight lower bounds on the approximate degree of several key problems in quantum query complexity. In particular, the authors of [20] prove that our upper bound for surjectivity is tight. We are confident that the ideas of our work will inform future research as well.

2. PRELIMINARIES

We start with a review of the technical preliminaries. The purpose of this section is to make the paper as self-contained as possible, and comfortably readable by a broad audience. The expert reader may wish to skim it for the notation or skip it altogether.

2.1. Notation. We view Boolean functions as mappings $X \rightarrow \{0, 1\}$ for some finite set X . This arithmetization of the Boolean values “true” and “false” makes it possible to use Boolean operations in arithmetic expressions, as in $1 - 2\prod_{i=1}^n x_i$. The familiar functions $\text{OR}_n: \{0, 1\}^n \rightarrow \{0, 1\}$ and $\text{AND}_n: \{0, 1\}^n \rightarrow \{0, 1\}$ are given by $\text{OR}_n(x) = \bigvee_{i=1}^n x_i$ and $\text{AND}_n(x) = \bigwedge_{i=1}^n x_i = \prod_{i=1}^n x_i$. The negation of a Boolean function f is denoted as usual by $\bar{f} = 1 - f$. The composition of f and g is denoted $f \circ g$, with $(f \circ g)(x) = f(g(x))$.

For a string $x \in \{0, 1\}^n$, we denote its Hamming weight by $|x| = x_1 + x_2 + \dots + x_n$. We use the following notation for strings of Hamming weight at most k , greater than k , and exactly k :

$$\begin{aligned} \{0, 1\}_{\leq k}^n &= \{x \in \{0, 1\}^n : |x| \leq k\}, \\ \{0, 1\}_{> k}^n &= \{x \in \{0, 1\}^n : |x| > k\}, \\ \{0, 1\}_k^n &= \{x \in \{0, 1\}^n : |x| = k\}. \end{aligned}$$

For a string $x \in \{0, 1\}^n$ and a set $S \subseteq \{1, 2, \dots, n\}$, we let $x|_S$ denote the restriction of x to the indices in S . In other words, $x|_S = x_{i_1}x_{i_2}\dots x_{i_{|S|}}$, where $i_1 < i_2 < \dots < i_{|S|}$ are the elements of S . The characteristic vector of a subset $S \subseteq \{1, 2, \dots, n\}$ is denoted $\mathbf{1}_S$.

We let $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ and $[n] = \{1, 2, \dots, n\}$. For a set S and a real number k , we define

$$\begin{aligned} \binom{S}{k} &= \{A \subseteq S : |A| = k\}, \\ \binom{S}{\leq k} &= \{A \subseteq S : |A| \leq k\}. \end{aligned}$$

We analogously define $\binom{S}{\geq k}$, $\binom{S}{< k}$, and $\binom{S}{> k}$. We let $\ln x$ and $\log x$ stand for the natural logarithm of x and the logarithm of x to base 2, respectively. The following bound is well known [29, Proposition 1.4]:

$$\sum_{i=0}^k \binom{n}{i} \leq \left(\frac{en}{k}\right)^k, \quad k = 0, 1, 2, \dots, n, \quad (2.1)$$

where $e = 2.7182\dots$ denotes Euler’s number. For a logical condition C , we use the Iverson bracket notation

$$\mathbf{I}[C] = \begin{cases} 1 & \text{if } C \text{ holds,} \\ 0 & \text{otherwise.} \end{cases}$$

For a function $f: X \rightarrow \mathbb{R}$ on a finite set X , we use the standard norms

$$\begin{aligned} \|f\|_\infty &= \max_{x \in X} |f(x)|, \\ \|f\|_1 &= \sum_{x \in X} |f(x)|. \end{aligned}$$

2.2. Approximate degree. Recall that the *total degree* of a multivariate real polynomial $p: \mathbb{R}^n \rightarrow \mathbb{R}$, denoted $\deg p$, is the largest degree of any monomial of p . We use the terms “degree” and “total degree” interchangeably in this paper. This paper studies the approximate representation of functions of interest by polynomials. Specifically, let $f: X \rightarrow \mathbb{R}$ be a given function, for a finite subset $X \subset \mathbb{R}^n$. Define

$$E(f, d) = \min_{p: \deg p \leq d} \|f - p\|_\infty,$$

where the minimum is over polynomials of degree at most d . In words, $E(f, d)$ is the least error to which f can be approximated by a real polynomial of degree at most d . For a real number $\epsilon \geq 0$, the ϵ -*approximate degree* of f is defined as

$$\deg_\epsilon(f) = \min\{d : E(f, d) \leq \epsilon\}.$$

Thus, $\deg_\epsilon(f)$ is the least degree of a real polynomial that approximates f pointwise to within ϵ . We refer to any such polynomial as a *uniform approximant for f with error ϵ* . In the study of Boolean functions f , the standard setting of the error parameter is $\epsilon = 1/3$. This constant is chosen mostly for aesthetic reasons and can be replaced by any other constant in $(0, 1/2)$ at the expense of a constant-factor increase in approximate degree. The following fact on the *exact* representation of functions by polynomials is well known.

FACT 2.1. *For every function $f: \{0, 1\}_{\leq n}^N \rightarrow \mathbb{R}$,*

$$\deg_0(f) \leq n.$$

Proof. The proof is by induction on n . The base case $n = 0$ is trivial since f is then a constant function. For the inductive step, let $n \geq 1$ be arbitrary. By the inductive hypothesis, there is a polynomial $p_{n-1}(x)$ of degree at most $n - 1$ such that $f(x) = p_{n-1}(x)$ for inputs $x \in \{0, 1\}^N$ of Hamming weight at most $n - 1$. Define

$$p_n(x) = p_{n-1}(x) + \sum_{a \in \{0, 1\}_{\leq n}^N} (f(a) - p_{n-1}(a)) \prod_{i: a_i = 1} x_i.$$

For any fixed input x with $|x| \leq n - 1$, every term in the summation over a evaluates to zero and therefore $p_n(x) = p_{n-1}(x) = f(x)$. For any fixed input x with $|x| = n$, on the other hand, the summation over a contributes precisely one nonzero term, corresponding to $a = x$. As a result, $p_n(x) = p_{n-1}(x) + (f(x) - p_{n-1}(x)) = f(x)$ in that case. \square

2.3. Inclusion-exclusion. All Boolean, arithmetic, and relational operations on functions in this paper are to be interpreted pointwise. For example, $\bigvee_{i=1}^n f_i$ refers to the mapping $x \mapsto \bigvee_{i=1}^n f_i(x)$. Similarly, $\prod_{i=1}^n f_i$ is the pointwise product of f_1, f_2, \dots, f_n . Recall that in the case of Boolean functions, we have $\bigwedge_{i=1}^n f_i = \prod_{i=1}^n f_i$. The well-known *inclusion-exclusion principle*, stated in terms of Boolean

functions f_1, f_2, \dots, f_n , asserts that

$$\bigvee_{i=1}^n f_i = \sum_{\substack{S \subseteq \{1,2,\dots,n\} \\ S \neq \emptyset}} (-1)^{|S|+1} \prod_{i \in S} f_i.$$

We will need the following less common form of the inclusion-exclusion principle, where the AND and OR operators are interchanged.

FACT 2.2. *For any $n \geq 1$ and any Boolean functions $f_1, f_2, \dots, f_n: X \rightarrow \{0, 1\}$,*

$$\prod_{i=1}^n f_i = \sum_{\substack{S \subseteq \{1,2,\dots,n\} \\ S \neq \emptyset}} (-1)^{|S|+1} \bigvee_{i \in S} f_i.$$

Proof. We have

$$\begin{aligned} \prod_{i=1}^n f_i &= \prod_{i=1}^n (1 - \overline{f_i}) \\ &= \sum_{S \subseteq \{1,2,\dots,n\}} (-1)^{|S|} \prod_{i \in S} \overline{f_i} \\ &= \sum_{S \subseteq \{1,2,\dots,n\}} (-1)^{|S|} \left(\prod_{i \in S} \overline{f_i} - 1 \right) \\ &= \sum_{S \subseteq \{1,2,\dots,n\}} (-1)^{|S|} \left(- \bigvee_{i \in S} f_i \right) \\ &= \sum_{\substack{S \subseteq \{1,2,\dots,n\} \\ S \neq \emptyset}} (-1)^{|S|+1} \bigvee_{i \in S} f_i, \end{aligned}$$

where the third step uses the fact that half of the subsets of $\{1, 2, \dots, n\}$ have odd cardinality and the other half have even cardinality. \square

2.4. Symmetrization. Let S_n denote the symmetric group on n elements. For a permutation $\sigma \in S_n$ and a string $x = (x_1, x_2, \dots, x_n)$, we adopt the shorthand $\sigma x = (x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)})$. A function $f(x_1, x_2, \dots, x_n)$ is called *symmetric* if it is invariant under permutations of the input variables: $f(x_1, x_2, \dots, x_n) \equiv f(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)})$ for all x and σ . Symmetric functions on $\{0, 1\}^n$ are intimately related to univariate polynomials, as borne out by Minsky and Papert's *symmetrization argument* [42].

PROPOSITION 2.3 (Minsky and Papert). *Let $p: \{0, 1\}^n \rightarrow \mathbb{R}$ be a polynomial of degree d . Then there is a univariate polynomial p^* of degree at most d such that for all $x \in \{0, 1\}^n$,*

$$\mathbf{E}_{\sigma \in S_n} p(\sigma x) = p^*(|x|).$$

Minsky and Papert's result generalizes to block-symmetric functions, as pointed out in [48, Prop. 2.3]:

PROPOSITION 2.4. *Let n_1, \dots, n_k be positive integers. Let $p: \{0, 1\}^{n_1} \times \dots \times \{0, 1\}^{n_k} \rightarrow \mathbb{R}$ be a polynomial of degree d . Then there is a polynomial $p^*: \mathbb{R}^k \rightarrow \mathbb{R}$ of degree at most d such that for all $x_1 \in \{0, 1\}^{n_1}, \dots, x_k \in \{0, 1\}^{n_k}$,*

$$\mathbf{E}_{\sigma_1 \in S_{n_1}, \dots, \sigma_k \in S_{n_k}} p(\sigma_1 x_1, \dots, \sigma_k x_k) = p^*(|x_1|, \dots, |x_k|).$$

Proposition 2.4 follows in a straightforward manner from Proposition 2.3 by induction on the number of blocks, k .

2.5. Chebyshev polynomials. Recall from Euler's identity that

$$(\cos x + \mathbf{i} \sin x)^d = \cos dx + \mathbf{i} \sin dx, \quad d = 0, 1, 2, \dots, \quad (2.2)$$

where \mathbf{i} denotes the imaginary unit. Multiplying out the left-hand side and using $\sin^2 x = 1 - \cos^2 x$, we obtain a univariate polynomial T_d of degree d such that

$$T_d(\cos x) = \cos dx. \quad (2.3)$$

This unique polynomial is the *Chebyshev polynomial of degree d* . The representation (2.3) immediately reveals all the roots of T_d , and all the extrema of T_d in the interval $[-1, 1]$:

$$T_d\left(\cos\left(\frac{2i-1}{2d}\pi\right)\right) = 0, \quad i = 1, 2, \dots, d, \quad (2.4)$$

$$T_d\left(\cos\left(\frac{i}{d}\pi\right)\right) = (-1)^i, \quad i = 0, 1, \dots, d, \quad (2.5)$$

$$|T_d(t)| \leq 1, \quad t \in [-1, 1]. \quad (2.6)$$

The extremum at 1 is of particular significance, and we note it separately:

$$T_d(1) = 1. \quad (2.7)$$

In view of (2.2), the defining equation (2.3) implies that

$$\begin{aligned} T_d(\cos x) &= \sum_{i=0}^{\lfloor d/2 \rfloor} \binom{d}{2i} (-1)^i (\sin x)^{2i} (\cos x)^{d-2i} \\ &= \sum_{i=0}^{\lfloor d/2 \rfloor} \binom{d}{2i} (\cos^2 x - 1)^i (\cos x)^{d-2i}, \end{aligned}$$

so that the leading coefficient of T_d for $d \geq 1$ is given by $\sum_{i=0}^{\lfloor d/2 \rfloor} \binom{d}{2i} = 2^{d-1}$. As a result, we have the factored representation

$$T_d(t) = 2^{d-1} \prod_{i=1}^d \left(t - \cos \left(\frac{2i-1}{2d} \pi \right) \right), \quad d \geq 1. \quad (2.8)$$

By (2.2) and (2.3),

$$\begin{aligned} T_d(\cos x) &= \cos dx \\ &= \frac{1}{2}(\cos x - \mathbf{i} \sin x)^d + \frac{1}{2}(\cos x + \mathbf{i} \sin x)^d \\ &= \frac{1}{2}(\cos x - \mathbf{i} \sqrt{1 - \cos^2 x})^d + \frac{1}{2}(\cos x + \mathbf{i} \sqrt{1 - \cos^2 x})^d, \end{aligned}$$

whence

$$T_d(t) = \frac{1}{2}(t - \sqrt{t^2 - 1})^d + \frac{1}{2}(t + \sqrt{t^2 - 1})^d, \quad |t| \geq 1. \quad (2.9)$$

The following fundamental fact follows from (2.9) by elementary calculus.

FACT 2.5 (Derivative of Chebyshev polynomials). *For any integer $d \geq 0$ and real $t \geq 1$,*

$$T'_d(t) \geq d^2.$$

Together, (2.9) and Fact 2.5 give the following useful lower bound for Chebyshev polynomials on $[1, \infty)$.

PROPOSITION 2.6. *For any integer $d \geq 1$,*

$$\begin{aligned} T_d(1 + \delta) &\geq 1 + d^2 \delta, & 0 \leq \delta < \infty, \\ T_d(1 + \delta) &\geq 2^{d\sqrt{\delta}-1} & 0 \leq \delta \leq 1. \end{aligned}$$

Proof. The first bound follows from the intermediate value theorem in view of (2.7) and Fact 2.5. For the second bound, use (2.9) to write

$$\begin{aligned} T_d(1 + \delta) &\geq \frac{1}{2}(1 + \delta + \sqrt{(1 + \delta)^2 - 1})^d \\ &\geq \frac{1}{2}(1 + \sqrt{\delta})^d \\ &\geq \frac{1}{2} \cdot 2^{d\sqrt{\delta}}, \end{aligned}$$

where the last step uses $1 + x \geq 2^x$ for $x \in [0, 1]$. □

2.6. Coefficient bounds for univariate polynomials. We let P_d stand for the set of univariate polynomials of degree at most d . For a univariate polynomial $p(t) = a_d t^d + a_{d-1} t^{d-1} + \cdots + a_1 t + a_0$, we let $\|p\| = \sum_{i=0}^d |a_i|$ denote the sum of

the absolute values of the coefficients of p . Then $\|\cdot\|$ is a norm on the real linear space of polynomials, and it is in addition submultiplicative:

FACT 2.7. *For any polynomials p and q ,*

- (i) $\|p\| \geq 0$, with equality if and only if $p = 0$;
- (ii) $\|\lambda p\| = |\lambda| \cdot \|p\|$ for any real λ ;
- (iii) $\|p + q\| \leq \|p\| + \|q\|$;
- (iv) $\|p \cdot q\| \leq \|p\| \cdot \|q\|$.

Proof. All four properties follow directly from the definition. \square

We will need a bound on the coefficients of a univariate polynomial in terms of its degree d and its maximum absolute value on the interval $[0, 1]$. This fundamental problem was solved in the nineteenth century by V. A. Markov [41, p. 81], who proved an upper bound of

$$O\left(\frac{(1 + \sqrt{2})^d}{\sqrt{d}}\right) \tag{2.10}$$

on the size of the coefficients of any degree- d polynomial that is bounded on $[-1, 1]$ in absolute value by 1. Markov further showed that (2.10) is tight. Rather than appeal to this deep result in approximation theory, we will use the following weaker bound that suffices for our purposes.

LEMMA 2.8. *Let p be a univariate polynomial of degree d . Then*

$$\|p\| \leq 8^d \max_{i=0,1,\dots,d} \left| p\left(\frac{i}{d}\right) \right|. \tag{2.11}$$

Lemma 2.8 is a cosmetic modification of a lemma from [54], which in our notation states that $\|p\| \leq 4^d \max_{i=0,1,\dots,d} |p(1 - \frac{2i}{d})|$ for $p \in P_d$. We include a detailed proof for the reader's convenience.

Proof of Lemma 2.8. We use a common approximation-theoretic technique [24, 49] whereby one expresses p as a linear combination of more structured polynomials and analyzes the latter objects. For this, define $q_0, q_1, \dots, q_d \in P_d$ by

$$q_j(t) = \frac{(-1)^{d-j} d^d}{d!} \binom{d}{j} \prod_{\substack{i=0 \\ i \neq j}}^d \left(t - \frac{i}{d}\right), \quad j = 0, 1, \dots, d.$$

One easily verifies that these polynomials behave like delta functions, in the sense that for $i, j = 0, 1, 2, \dots, d$,

$$q_j\left(\frac{i}{d}\right) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Lagrange interpolation gives

$$p = \sum_{j=0}^d p\left(\frac{j}{d}\right) q_j. \quad (2.12)$$

By Fact 2.7,

$$\begin{aligned} \|q_j\| &\leq \frac{d^d}{d!} \binom{d}{j} \prod_{\substack{i=0 \\ i \neq j}}^d \left(1 + \frac{i}{d}\right) \\ &\leq \frac{d^d}{d!} \binom{d}{j} \prod_{i=1}^d \left(1 + \frac{i}{d}\right) \\ &= \frac{1}{d!} \binom{d}{j} \frac{(2d)!}{d!} \\ &= \binom{d}{j} \binom{2d}{d} \\ &\leq 4^d \binom{d}{j}, \end{aligned} \quad j = 0, 1, 2, \dots, d. \quad (2.13)$$

Now

$$\begin{aligned} \|p\| &\leq \left(\max_{j=0,1,\dots,d} \left| p\left(\frac{j}{d}\right) \right| \right) \sum_{j=0}^d 4^d \binom{d}{j} \\ &= 8^d \max_{j=0,1,\dots,d} \left| p\left(\frac{j}{d}\right) \right|, \end{aligned}$$

where the first step uses (2.12), (2.13), and Fact 2.7. \square

2.7. Coefficient bounds for multivariate polynomials. Let $\phi: \mathbb{R}^n \rightarrow \mathbb{R}$ be a multivariate polynomial. Analogous to the univariate case, we let $\|\phi\|$ denote the sum of the absolute values of the coefficients of ϕ . Fact 2.7 is clearly valid in this multivariate setting as well. Recall that a multivariate polynomial ϕ is *multilinear* if it has degree at most 1 in each variable. The following result is an analogue of Lemma 2.8.

LEMMA 2.9. *Let $\phi: \mathbb{R}^n \rightarrow \mathbb{R}$ be a symmetric multilinear polynomial. Then*

$$\|\phi\| \leq 8^{\deg \phi} \max_{x \in \{0,1\}^n} |\phi(x)|.$$

Proof. Abbreviate $d = \deg \phi$ and write

$$\phi(x) = \sum_{i=0}^d a_i \sum_{S \in \binom{[n]}{i}} \prod_{j \in S} x_j,$$

where a_0, a_1, \dots, a_d are real coefficients. For $0 \leq t \leq 1$, let $B(t)$ denote the Bernoulli distribution with success probability t . Then

$$\begin{aligned} \|\phi\| &= \sum_{i=0}^d |a_i| \binom{n}{i} \\ &\leq 8^d \max_{0 \leq t \leq 1} \left| \sum_{i=0}^d a_i \binom{n}{i} t^i \right| \\ &= 8^d \max_{0 \leq t \leq 1} \left| \mathbf{E}_{x_1, x_2, \dots, x_n \sim B(t)} \phi(x) \right| \\ &\leq 8^d \max_{x \in \{0,1\}^n} |\phi(x)|, \end{aligned}$$

where the second and third steps use Lemma 2.8 and multilinearity, respectively. \square

The following lemma, due to Razborov and Sherstov [48, Lemma 3.2], bounds the value of a polynomial p at a point of large Hamming weight in terms of p 's values at points of low Hamming weight.

LEMMA 2.10 (Extrapolation lemma). *Let d be an integer, $0 \leq d \leq n - 1$. Let $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$ be a polynomial of degree at most d . Then*

$$|\phi(1^n)| \leq 2^d \binom{n}{d} \max_{x \in \{0,1\}_{\leq d}^n} |\phi(x)|.$$

As one would expect, one can sharpen the bound of Lemma 2.10 by maximizing over a larger neighborhood of the Boolean hypercube than $\{0,1\}_{\leq d}^n$. The resulting bound is as follows.

LEMMA 2.11 (Generalized extrapolation lemma). *Fix positive integers $N > m \geq d$. Let $\phi : \mathbb{R}^N \rightarrow \mathbb{R}$ be a polynomial of degree at most d . Then*

$$|\phi(x^*)| \leq 2^d \binom{\lceil |x^*| / \lfloor m/d \rfloor \rceil}{d} \max_{x \in \{0,1\}_{\leq m}^N} |\phi(x)|, \quad x^* \in \{0,1\}_{> m}^N.$$

One recovers Lemma 2.10 as a special case by taking $N = n$, $m = d$, and $x^* = 1^n$.

Proof of Lemma 2.11. Consider an arbitrary vector $x^* \in \{0,1\}^N$ of Hamming weight $|x^*| > m$, and abbreviate $n = \lceil |x^*| / \lfloor m/d \rfloor \rceil$. Let S_1, S_2, \dots, S_n be a partition of $\{i : x_i^* = 1\}$ such that $|S_i| \leq \lfloor m/d \rfloor$ for all i . Observe that

$$n > d. \tag{2.14}$$

Define $L : \{0,1\}^n \rightarrow \{0,1\}^N$ by

$$L(z) = \sum_{i=1}^n z_i \mathbf{1}_{S_i}.$$

Then clearly

$$L(1^n) = x^*, \quad (2.15)$$

$$|L(z)| \leq |z| \cdot \left\lfloor \frac{m}{d} \right\rfloor. \quad (2.16)$$

Moreover, the mapping $z \mapsto \phi(L(z))$ is a real polynomial on $\{0, 1\}^n$ of degree at most $\deg \phi \leq d$. As a result,

$$\begin{aligned} |\phi(x^*)| &= |\phi(L(1^n))| \\ &\leq 2^d \binom{n}{d} \max_{|z| \leq d} |\phi(L(z))| \\ &\leq 2^d \binom{n}{d} \max_{|x| \leq d \lfloor m/d \rfloor} |\phi(x)| \\ &\leq 2^d \binom{n}{d} \max_{|x| \leq m} |\phi(x)|, \end{aligned}$$

where the first step uses (2.15); the second step follows by (2.14) and Lemma 2.10; and the third step is valid by (2.16). \square

2.8. The conjunction norm. Recall that a *conjunction* in Boolean variables x_1, x_2, \dots, x_n is the AND of some subset of the literals $x_1, \overline{x_1}, x_2, \overline{x_2}, \dots, x_n, \overline{x_n}$. Analogously, a *disjunction* is the OR of some subset of $x_1, \overline{x_1}, x_2, \overline{x_2}, \dots, x_n, \overline{x_n}$. We regard conjunctions and disjunctions as Boolean functions $\{0, 1\}^n \rightarrow \{0, 1\}$ and in particular as a special case of real functions $\{0, 1\}^n \rightarrow \mathbb{R}$. For a subset $X \subseteq \{0, 1\}^n$ and a function $f: X \rightarrow \mathbb{R}$, we define the *conjunction norm* $\Pi(f)$ to be the minimum $\Lambda \geq 0$ such that

$$f(x) = \lambda_1 C_1(x) + \lambda_2 C_2(x) + \dots + \lambda_N C_N(x) \quad (x \in X)$$

for some integer N , some conjunctions C_1, C_2, \dots, C_N , and some real coefficients $\lambda_1, \lambda_2, \dots, \lambda_N$ with $|\lambda_1| + |\lambda_2| + \dots + |\lambda_N| \leq \Lambda$. Our choice of the symbol Π , for “product,” is motivated by the view of conjunctions as products of literals. In particular, we have $\Pi(\phi) \leq \|\phi\|$ for any multivariate polynomial $\phi: \{0, 1\}^n \rightarrow \mathbb{R}$. The next proposition shows that Π is a norm on the space of multivariate real functions and establishes other useful properties of this complexity measure.

PROPOSITION 2.12 (Conjunction norm). *Let $f, g: X \rightarrow \mathbb{R}$ be given functions, for a nonempty set $X \subseteq \{0, 1\}^n$. Then:*

- (i) $\Pi(f) \geq 0$, with equality if and only if $f = 0$;
- (ii) $\Pi(\lambda f) = |\lambda| \Pi(f)$ for any real λ ;
- (iii) $\Pi(f + g) \leq \Pi(f) + \Pi(g)$;
- (iv) $\Pi(f \cdot g) \leq \Pi(f) \Pi(g)$;
- (v) $\Pi(f) \leq \|f\|_1$;
- (vi) $\Pi(f) \leq 2$ if f is a disjunction;
- (vii) $\Pi(p \circ f) \leq \max\{1, \Pi(f)\}^d \|p\|$ for any polynomial $p \in P_d$.

Proof. (i)–(iii) Immediate from the definitions.

(iv) Express f and g individually as a linear combination of conjunctions with real coefficients whose absolute values sum to $\Pi(f)$ and $\Pi(g)$, respectively. Then,

multiply these two linear combinations. Since the product of conjunctions is again a conjunction, the resulting representation is a linear combination of conjunctions with real coefficients whose absolute values sum to at most $\Pi(f)\Pi(g)$.

(v) By the homogeneity (ii) and triangle inequality (iii), we have

$$\begin{aligned}\Pi(f) &= \Pi\left(\sum_{a \in X} f(a)C_a\right) \\ &\leq \sum_{a \in X} |f(a)| \Pi(C_a) \\ &\leq \sum_{a \in X} |f(a)| \\ &= \|f\|_1,\end{aligned}$$

where C_a denotes the conjunction that evaluates to true on a and to false on all other inputs in $\{0, 1\}^n$.

(vi) We have $\Pi(f) \leq \Pi(f - 1) + \Pi(1) = \Pi(1 - f) + \Pi(1) \leq 2$, where the first step applies the triangle inequality (iii), the second step uses the homogeneity (ii), and the third step uses the fact that 1 and $1 - f$ are conjunctions.

(vii) Let $p(t) = a_d t^d + a_{d-1} t^{d-1} + \dots + a_1 t + a_0$ be a given polynomial. Then

$$\begin{aligned}\Pi(p \circ f) &= \Pi\left(\sum_{i=0}^d a_i \underbrace{f \cdot f \cdots f}_i\right) \\ &\leq \sum_{i=0}^d |a_i| \Pi(\underbrace{f \cdot f \cdots f}_i) \\ &\leq \sum_{i=0}^d |a_i| \Pi(f)^i \\ &\leq \max\{1, \Pi(f)^d\} \sum_{i=0}^d |a_i| \\ &= \max\{1, \Pi(f)\}^d \|p\|,\end{aligned}$$

where the second step uses (ii) and (iii), and the third step applies (iv). \square

3. THE EXTENSION THEOREM

This section establishes an approximation-theoretic result of independent interest, the *extension theorem*, that we use several times in the rest of the paper to construct approximating polynomials. To set the stage for this result, let $f: \{0, 1\}_{\leq m}^N \rightarrow [-1, 1]$ be a given function, defined on inputs of Hamming weight up to m . For any integer $n > m$, consider the extension F_n of f to inputs of Hamming weight up to n , given by

$$F_n(x) = \begin{cases} f(x) & \text{if } |x| \leq m, \\ 0 & \text{otherwise.} \end{cases}$$

From the point of view of approximation theory, a fundamental question to ask is how to “extend” any approximant for f to an approximant for F_n , without degrading the quality of the approximation or significantly increasing the approximant’s degree. Ideally, we would like the approximant for the extension F_n to have degree within a small factor of the original degree, e.g., a factor of $O(n/m)^\alpha$ for some constant $0 < \alpha < 1$.

Unfortunately, the extension problem is hopeless as stated. Indeed, consider the special case of the constant function $f = 1$, so that

$$F_n(x) = \begin{cases} 1 & \text{if } 0 \leq |x| \leq m, \\ 0 & \text{if } m < |x| \leq n. \end{cases}$$

In this example, $\deg_{1/3}(f) = 0$ but $\deg_{1/3}(F_n) = \Omega(\sqrt{n})$ by a well-known result of Nisan and Szegedy [43]. In particular, there is no efficient way to transform an approximant for a general function f into an approximant for the extension F_n . Our contribution is to show that the extension problem becomes meaningful and efficiently solvable if one’s starting point is an approximant for F_{2m} rather than for f . In other words, we give an efficient, black-box transformation of an approximant for F_{2m} into an approximant for any extension F_n , where $n \geq 2m$. The formal statement of our result is as follows.

THEOREM 3.1 (Extension theorem). *Let $f: \{0, 1\}_{\leq m}^N \rightarrow [-1, 1]$ be given, where $N \geq m \geq 0$ are integers. For integers $n \geq m$, define $F_n: \{0, 1\}_{\leq n}^N \rightarrow [-1, 1]$ by*

$$F_n(x) = \begin{cases} f(x) & \text{if } |x| \leq m, \\ 0 & \text{otherwise.} \end{cases}$$

Then for some absolute constant $C > 1$ and all $\epsilon, \delta \in (0, 1/2)$ and $n \geq m$,

$$\deg_{\epsilon+\delta}(F_n) \leq C \sqrt{\frac{n}{m+1}} \cdot \left(\deg_\epsilon(F_{2m}) + \log \frac{1}{\delta} \right). \quad (3.1)$$

Theorem 3.1 solves the extension problem with only a factor- $\sqrt{n/m}$ increase in degree. The approximation quality of the new approximant can be made arbitrarily close to that of the original at a small additive cost in degree. This overhead in degree and error is optimal, as we will discover in applications later in this paper. We also note that the constant 2 in this result was chosen exclusively for aesthetic reasons, and (3.1) holds with F_{2m} replaced by $F_{\lceil cm \rceil}$ for any constant $c > 1$. The rest of this section is devoted to the proof of Theorem 3.1.

3.1. Proof strategy. In the notation of Theorem 3.1, let $p_{2m}(x)$ be an approximant for $F_{2m}(x)$. Then clearly

$$F_n(x) \approx p_{2m}(x) \cdot \mathbf{I}[|x| \leq 2m] \quad (3.2)$$

on the domain of F_n , where $\mathbf{I}[|x| \leq 2m]$ is the characteristic function of the set of inputs of Hamming weight at most $2m$. While $p_{2m}(x)$ can grow rapidly as the Hamming weight $|x|$ increases beyond $2m$, that growth is not entirely arbitrary. Specifically, the generalized extrapolation lemma (Lemma 2.11) bounds $|p_{2m}(x)|$

in terms of the Hamming weight $|x|$ and the degree of p_{2m} . In particular, the approximate equality (3.2) is preserved if $\mathbf{I}[|x| \leq 2m]$ is replaced by a low-degree approximant. The construction of such an approximant is the crux of our proof. More precisely, we construct a low-degree *univariate* approximant to the characteristic function of any interval. To crystallize our approach, we first consider the degenerate interval $[0, 0] = \{0\}$.

PROPOSITION 3.2. *For any positive integers n and d , there is a polynomial p with*

$$p(0) = 1, \tag{3.3}$$

$$|p(t)| \leq \frac{1}{t^d}, \quad t \in [1, n], \tag{3.4}$$

$$\deg p \leq 7d\sqrt{n}. \tag{3.5}$$

The key property here is (3.4), whereby the approximating polynomial gets smaller as one moves farther away from the point of interest, 0. Reproducing this behavior in the context of a general interval is much more subtle and is the subject of Sections 3.2–3.4.

Proof. Define

$$T(t) = \left(\prod_{i=0}^{\lceil \log n \rceil} T_{\lceil \sqrt{n/2^i} \rceil} \left(1 + \frac{2^i - t}{n} \right) \right)^d.$$

Fix an arbitrary point $t \in [1, n]$, and let j be the integer such that $t \in [2^j, 2^{j+1})$. Then

$$\begin{aligned} |T(t)| &= \prod_{i=0}^{\lceil \log n \rceil} \left| T_{\lceil \sqrt{n/2^i} \rceil} \left(1 + \frac{2^i - t}{n} \right) \right|^d \\ &\leq \prod_{i=j+1}^{\lceil \log n \rceil} \left| T_{\lceil \sqrt{n/2^i} \rceil} \left(1 + \frac{2^i - t}{n} \right) \right|^d \\ &\leq \prod_{i=j+1}^{\lceil \log n \rceil} \left| T_{\lceil \sqrt{n/2^i} \rceil} \left(1 + \frac{2^i}{n} \right) \right|^d \\ &= |T(0)| \prod_{i=0}^j \left| T_{\lceil \sqrt{n/2^i} \rceil} \left(1 + \frac{2^i}{n} \right) \right|^{-d} \\ &\leq |T(0)| \prod_{i=0}^j 2^{-d} \\ &\leq \frac{|T(0)|}{t^d}, \end{aligned}$$

where the second step uses (2.6), the third step follows from (2.7) and Fact 2.5, and the next-to-last step applies Proposition 2.6. Moreover,

$$\begin{aligned} \deg T &= d \sum_{i=0}^{\lceil \log n \rceil} \left\lceil \sqrt{\frac{n}{2^i}} \right\rceil \\ &\leq d \sum_{i=0}^{\infty} \sqrt{\frac{n}{2^i}} \cdot 2 \\ &\leq 7d\sqrt{n}. \end{aligned}$$

As a result, (3.3)–(3.5) hold for $p(t) = T(t)/T(0)$. \square

3.2. Approximating $1/t$. To handle actual intervals rather than singleton points, we need to develop a number of auxiliary results. To start with, we construct an approximant for the reciprocal function $1/t$ on $[1, n]$. We are specifically interested in approximation within a *multiplicative* factor close to 1, which is a more demanding regime than pointwise approximation.

LEMMA 3.3. *For any integer $d \geq 0$ and real $n > 1$, there is an (explicitly given) polynomial $p \in P_d$ such that*

$$\frac{1 - \epsilon}{t} \leq p(t) \leq \frac{1 + \epsilon}{t}, \quad 1 \leq t \leq n, \quad (3.6)$$

where

$$\epsilon = \frac{1}{T_{d+1}\left(\frac{n+1}{n-1}\right)}.$$

Proof. Property (3.6) can be restated as $\max_{1 \leq t \leq n} |1 - tp(t)| \leq \epsilon$. Thus, the existence of $p \in P_d$ that obeys (3.6) is equivalent to the existence of $q \in P_{d+1}$ that obeys $q(0) = 1$ and $\max_{1 \leq t \leq n} |q(t)| \leq \epsilon$. Now, define $q \in P_{d+1}$ by

$$q(t) = \frac{T_{d+1}\left(1 - 2 \cdot \frac{t-1}{n-1}\right)}{T_{d+1}\left(\frac{n+1}{n-1}\right)}.$$

Then $q(0) = 1$ by definition. Moreover,

$$\begin{aligned} \max_{1 \leq t \leq n} |q(t)| &\leq \max_{-1 \leq t \leq 1} \frac{|T_{d+1}(t)|}{\left|T_{d+1}\left(\frac{n+1}{n-1}\right)\right|} \\ &\leq \frac{1}{\left|T_{d+1}\left(\frac{n+1}{n-1}\right)\right|} \\ &= \frac{1}{T_{d+1}\left(\frac{n+1}{n-1}\right)}, \end{aligned}$$

where the last two steps use (2.6) and (2.9), respectively. \square

It is well known [49, Theorem 1.10] that among all polynomials of degree at most d that are bounded on $[-1, 1]$ in absolute value by 1, the Chebyshev polynomial T_d takes on the largest possible value at every point of $[1, \infty)$. Using this fact, it is straightforward to verify that Lemma 3.3 gives the best possible bound on ϵ in terms of n and d .

COROLLARY 3.4. *For any real $n > 1$, there is an (explicitly given) univariate polynomial p of degree at most $\sqrt{2(n-1)}$ such that*

$$\frac{1}{2t} \leq p(t) \leq \frac{1}{t}, \quad 1 \leq t \leq n.$$

Proof. By Proposition 2.6,

$$T_{\lfloor \sqrt{2(n-1)} \rfloor + 1} \left(\frac{n+1}{n-1} \right) \geq 5.$$

As a result, it suffices to invoke Lemma 3.3 with $d = \lfloor \sqrt{2(n-1)} \rfloor$. \square

3.3. Approximating $1/t^i$. We now construct approximants for powers of the reciprocal function, focusing this time on absolute rather than relative error. Here, we are interested only in approximation in the neighborhood of 1. In the following construction, increasing the approximant's degree makes the neighborhood larger and the approximation more accurate.

LEMMA 3.5. *Let $d \geq 1$ be a given integer. Then for every integer $D \geq 0$, there is an (explicitly given) polynomial p with*

$$\left| \frac{1}{t^d} - p(t) \right| \leq |1-t|^{D+1} \binom{D+d}{d} d, \quad t \in \left[\frac{d}{d+D}, 2 - \frac{d}{d+D} \right], \quad (3.7)$$

$$|p(t)| \leq \binom{D+d}{d}, \quad t \in [0, 2], \quad (3.8)$$

$$\deg p \leq D. \quad (3.9)$$

Proof. Define

$$p(t) = \sum_{i=0}^D \binom{i+d-1}{i} (1-t)^i.$$

Then (3.9) is immediate. For (3.8), it suffices to observe that

$$\begin{aligned} \sum_{i=0}^D \binom{i+d-1}{i} &= \binom{D+d}{D} \\ &= \binom{D+d}{d}, \end{aligned}$$

where the first equality is well-known and can be verified by using Pascal's triangle or by interpreting the left-hand side as the number of ways to distribute at most D identical balls into d distinct bins.

It remains to settle (3.7). For $0 < t < 2$, we have the Maclaurin expansion

$$\begin{aligned} \frac{1}{t^d} &= \left(\sum_{i=0}^{\infty} (1-t)^i \right)^d \\ &= \sum_{i=0}^{\infty} \binom{i+d-1}{i} (1-t)^i. \end{aligned}$$

Therefore,

$$\begin{aligned} \left| \frac{1}{t^d} - p(t) \right| &= \left| \sum_{i=D+1}^{\infty} \binom{i+d-1}{i} (1-t)^i \right| \\ &\leq \sum_{i=D+1}^{\infty} \binom{i+d-1}{i} |1-t|^i \\ &\leq |1-t|^{D+1} \binom{D+d}{D+1} \sum_{i=0}^{\infty} \left(\frac{D+d}{D+1} \right)^i |1-t|^i \\ &\leq |1-t|^{D+1} \binom{D+d}{D+1} \sum_{i=0}^{\infty} \left(\frac{D}{D+1} \right)^i \\ &= |1-t|^{D+1} \binom{D+d}{d} d, \end{aligned}$$

where the fourth step is legitimate in view of the range of t in (3.7). \square

3.4. Approximating the characteristic function of an interval. The following lemma is the last prerequisite to our construction of a low-degree approximant for the characteristic function of an interval. Without loss of generality, it suffices to consider the interval $[0, 1]$. The lemma below *almost* solves our problem except that it gives a flat bound on the approximant's value outside the interval, not taking into account how far one is from the interval.

LEMMA 3.6. *For any reals $n \geq 1$ and $0 < \epsilon < 1/2$, there is an (explicitly given) univariate polynomial p such that*

$$|p(t) - 1| \leq \epsilon, \quad t \in [0, 1], \quad (3.10)$$

$$|p(t)| \leq 1, \quad t \in (1, 2], \quad (3.11)$$

$$|p(t)| \leq \epsilon, \quad t \in (2, n], \quad (3.12)$$

$$\deg p = O\left(\sqrt{n} \log \frac{1}{\epsilon}\right). \quad (3.13)$$

Proof. For $n < 2$, the lemma holds trivially with $p = 1$. In what follows, we treat the complementary case $n \geq 2$. Consider the univariate polynomial

$$q(t) = T_{\lceil \sqrt{n} \rceil} \left(1 + \frac{2-t}{n} \right).$$

Using $n \geq 2$, we obtain

$$\begin{aligned} q([0, n]) &\subseteq \left[-1, T_{\lceil \sqrt{n} \rceil} \left(1 + \frac{2}{n} \right) \right] \\ &\subseteq \left[-1, \left(1 + \frac{2}{n} + \sqrt{\left(1 + \frac{2}{n} \right)^2 - 1} \right)^{\sqrt{n}+1} \right] \\ &\subseteq \left[-1, \left(1 + \frac{2}{n} + \sqrt{\frac{6}{n}} \right)^{\sqrt{n}+1} \right] \\ &\subset \left[-1, \exp \left(\left(\frac{2}{n} + \sqrt{\frac{6}{n}} \right) (\sqrt{n} + 1) \right) \right] \\ &\subset [-1, e^7 - 1], \end{aligned} \tag{3.14}$$

where the first step is legitimate in view of (2.6), (2.7), and Fact 2.5; and the second step uses (2.9). By Proposition 2.6,

$$\begin{aligned} \min_{0 \leq t \leq 1} q(t) &= \min_{1 \leq t \leq 2} T_{\lceil \sqrt{n} \rceil} \left(1 + \frac{t}{n} \right) \\ &\geq 2. \end{aligned} \tag{3.15}$$

By (2.6),

$$\begin{aligned} \max_{2 \leq t \leq n} |q(t)| &\leq \max_{0 \leq t \leq 1} |T_{\lceil \sqrt{n} \rceil}(t)| \\ &\leq 1. \end{aligned} \tag{3.16}$$

In view of (3.14)–(3.16), the normalized polynomial $q^*(t) = (q(t) + 1)/e^7$ obeys

$$q^*([0, n]) \subseteq [0, 1], \tag{3.17}$$

$$q^*([0, 1]) \subseteq [3e^{-7}, 1], \tag{3.18}$$

$$q^*([2, n]) \subseteq [0, 2e^{-7}]. \tag{3.19}$$

To complete the proof, we use a technique due to Buhrman et al. [18]. Consider the univariate polynomial

$$B_d(t) = \sum_{i=\lceil 2.5e^{-7}d \rceil}^d \binom{d}{i} t^i (1-t)^i.$$

In words, $B_d(t)$ is the probability of observing at least $2.5e^{-7}d$ heads in a sequence of d independent coin flips, each coming up heads with probability t . For large enough $d = O(\log(1/\epsilon))$, the Chernoff bound guarantees that

$$B_d([0, 1]) \subseteq [0, 1], \quad (3.20)$$

$$B_d([0, 2e^{-7}]) \subseteq [0, \epsilon], \quad (3.21)$$

$$B_d([3e^{-7}, 1]) \subseteq [1 - \epsilon, 1]. \quad (3.22)$$

Now define $p(t) = B_d(q^*(t))$. Then the degree bound (3.13) is immediate, whereas the remaining properties (3.10)–(3.12) follow from (3.17)–(3.22). \square

Finally, we are now in a position to construct the desired approximant for the characteristic function of an interval. As mentioned above, we may without loss of generality focus on the interval $[0, 1]$.

THEOREM 3.7. *For all integers $n, d \geq 0$ and all $0 < \epsilon < 1/2$, there is an (explicitly given) univariate polynomial p such that*

$$|p(t) - 1| \leq \epsilon, \quad t \in [0, 1], \quad (3.23)$$

$$|p(t)| \leq 1 + \epsilon, \quad t \in (1, 2], \quad (3.24)$$

$$|p(t)| \leq \frac{\epsilon}{t^d}, \quad t \in (2, n], \quad (3.25)$$

$$\deg p = O\left(\sqrt{n} \left(d + \log \frac{1}{\epsilon}\right)\right). \quad (3.26)$$

Proof. For $n < 2$, the theorem holds trivially by taking $p = 1$. In what follows, we focus on the complementary case $n \geq 2$.

Corollary 3.4 gives an explicit univariate polynomial p_1 such that

$$\frac{1}{2(t+1)} \leq p_1(t) \leq \frac{1}{t+1}, \quad 0 \leq t \leq n, \quad (3.27)$$

$$\deg p_1 \leq \sqrt{2n}. \quad (3.28)$$

Let D be an integer parameter to be chosen later, $D > 5d$. Then Lemma 3.5 provides an explicit polynomial p_2 such that

$$\left|\frac{1}{t^d} - p_2(t)\right| \leq \left(\frac{5}{6}\right)^{D+1} \binom{D+d}{d} d, \quad t \in \left[\frac{1}{6}, 1\right], \quad (3.29)$$

$$|p_2(t)| \leq \binom{D+d}{d}, \quad t \in [0, 2], \quad (3.30)$$

$$\deg p_2 \leq D. \quad (3.31)$$

As our last building block, Lemma 3.6 constructs an explicit polynomial p_3 with

$$|p_3(t) - 1| \leq \epsilon 2^{-D-d}, \quad t \in [0, 1], \quad (3.32)$$

$$|p_3(t)| \leq 1, \quad t \in (1, 2], \quad (3.33)$$

$$|p_3(t)| \leq \epsilon 2^{-D-d}, \quad t \in (2, n], \quad (3.34)$$

$$\deg p_3 = O\left(\sqrt{n} \left(D + d + \log \frac{1}{\epsilon}\right)\right). \quad (3.35)$$

In the rest of the proof, we will show that the conclusion of the theorem holds for the polynomial

$$p(t) = p_1(t)^d p_2(p_1(t)) p_3(t).$$

To begin with,

$$\begin{aligned} \max_{0 \leq t \leq 1} |p(t) - 1| &\leq \max_{0 \leq t \leq 1} (1 + |p_1(t)^d p_2(p_1(t)) - 1|) \cdot (1 + |1 - p_3(t)|) - 1 \\ &\leq \left(1 + \frac{\epsilon}{2}\right) \max_{0 \leq t \leq 1} (1 + |p_1(t)^d p_2(p_1(t)) - 1|) - 1 \\ &\leq \left(1 + \frac{\epsilon}{2}\right) \left(1 + \max_{1/4 \leq t \leq 1} |t^d p_2(t) - 1|\right) - 1 \\ &\leq \left(1 + \frac{\epsilon}{2}\right) \left(1 + \max_{1/4 \leq t \leq 1} \left|p_2(t) - \frac{1}{t^d}\right|\right) - 1 \\ &\leq \left(1 + \frac{\epsilon}{2}\right) \left(1 + \left(\frac{5}{6}\right)^{D+1} \binom{D+d}{d} d\right) - 1, \end{aligned} \quad (3.36)$$

where the first step uses the inequality $|ab - 1| \leq (1 + |a - 1|)(1 + |b - 1|) - 1$ for any real a, b ; the second step is valid by (3.32); the third applies (3.27); and the final step is legitimate by (3.29). Continuing,

$$\begin{aligned} \max_{1 \leq t \leq 2} |p(t)| &= \max_{1 \leq t \leq 2} |p_1(t)^d p_2(p_1(t)) p_3(t)| \\ &\leq \max_{1 \leq t \leq 2} |p_1(t)^d p_2(p_1(t))| \\ &\leq \max_{1/6 \leq t \leq 1/2} |t^d p_2(t)| \\ &\leq \max_{1/6 \leq t \leq 1/2} |t^d p_2(t) - 1| + 1 \\ &\leq \max_{1/6 \leq t \leq 1/2} \left|p_2(t) - \frac{1}{t^d}\right| + 1 \\ &\leq 1 + \left(\frac{5}{6}\right)^{D+1} \binom{D+d}{d} d, \end{aligned} \quad (3.37)$$

where the second step uses (3.33), the third step applies (3.27), the fourth step is immediate from the triangle inequality, and the last step follows from (3.29).

Moreover,

$$\begin{aligned}
\max_{2 \leq t \leq n} |t^d p(t)| &\leq \max_{2 \leq t \leq n} |t^d p_1(t)^d| \cdot \max_{2 \leq t \leq n} |p_2(p_1(t))| \cdot \max_{2 \leq t \leq n} |p_3(t)| \\
&\leq \max_{2 \leq t \leq n} |t^d p_1(t)^d| \cdot \max_{2 \leq t \leq n} |p_2(p_1(t))| \cdot \epsilon 2^{-D-d} \\
&\leq \max_{2 \leq t \leq n} |p_2(p_1(t))| \cdot \epsilon 2^{-D-d} \\
&\leq \max_{0 \leq t \leq 1/3} |p_2(t)| \cdot \epsilon 2^{-D-d} \\
&\leq \binom{D+d}{d} \cdot \epsilon 2^{-D-d} \\
&\leq \epsilon,
\end{aligned} \tag{3.38}$$

where the second step is legitimate by (3.34), the third and fourth steps use (3.27), and the fifth step is immediate from (3.30). Finally, (3.28), (3.31), and (3.35) imply that

$$\deg p = O\left(\sqrt{n} \left(D + d + \log \frac{1}{\epsilon}\right)\right). \tag{3.39}$$

Now the claimed bounds (3.23)–(3.26) in the theorem statement follow immediately from (3.36)–(3.39) by taking

$$D = c \left\lceil d + \log \frac{1}{\epsilon} \right\rceil$$

for a sufficiently large absolute constant $c > 1$. \square

3.5. Proof of the extension theorem. Using the approximant constructed in Theorem 3.7, we now prove the extension theorem. We restate it below for the reader's convenience.

THEOREM (restatement of Theorem 3.1). *Let $f: \{0, 1\}_{\leq m}^N \rightarrow [-1, 1]$ be given, where $N \geq m \geq 0$ are integers. For integers $n \geq m$, define $F_n: \{0, 1\}_{\leq n}^N \rightarrow [-1, 1]$ by*

$$F_n(x) = \begin{cases} f(x) & \text{if } |x| \leq m, \\ 0 & \text{otherwise.} \end{cases}$$

Then for some absolute constant $C > 1$ and all $\epsilon, \delta \in (0, 1/2)$ and $n \geq m$,

$$\deg_{\epsilon+\delta}(F_n) \leq C \sqrt{\frac{n}{m+1}} \cdot \left(\deg_{\epsilon}(F_{2m}) + \log \frac{1}{\delta}\right). \tag{3.40}$$

Proof. To simplify the presentation, we first settle two degenerate cases. For $m = 0$, consider the polynomial

$$T(t) = \left(T_{\lceil \sqrt{n} \rceil} \left(1 + \frac{1-t}{n}\right)\right)^{\lceil \log \frac{1}{\delta} \rceil}.$$

Then $T(0) \geq 1/\delta$ by Proposition 2.6, and $\max_{1 \leq t \leq n} |T(t)| \leq 1$ by (2.6). Therefore, in this case F_n is approximated pointwise within δ by the degree- $O(\sqrt{n} \log(1/\delta))$ polynomial $F_n(0^N)T(|x|)/T(0)$. Another degenerate possibility is $n \leq 2m$, in which case $\deg_\epsilon(F_n) \leq \deg_\epsilon(F_{2m})$ and the theorem holds trivially. In what follows, we focus on the general case when

$$\begin{aligned} m &\geq 1, \\ n &> 2m. \end{aligned}$$

Abbreviate $d = \max\{\deg_\epsilon(F_{2m}), 1\}$. By Fact 2.1,

$$1 \leq d \leq 2m. \quad (3.41)$$

Fix a polynomial $\phi: \{0, 1\}^N \rightarrow \mathbb{R}$ such that

$$|F_{2m}(x) - \phi(x)| \leq \epsilon, \quad x \in \{0, 1\}_{\leq 2m}^N, \quad (3.42)$$

$$\deg \phi \leq d. \quad (3.43)$$

Let $0 < \alpha < 1/2$ be a parameter to be chosen later. Then Theorem 3.7 gives an explicit univariate polynomial p such that

$$|p(t) - 1| \leq \alpha, \quad t \in [0, 1], \quad (3.44)$$

$$|p(t)| \leq 1 + \alpha, \quad t \in (1, 2], \quad (3.45)$$

$$|p(t)| \leq \frac{\alpha}{t^d}, \quad t \in \left(2, \frac{n}{m}\right], \quad (3.46)$$

$$\deg p = O\left(\sqrt{\frac{n}{m}} \left(d + \log \frac{1}{\alpha}\right)\right). \quad (3.47)$$

Consider the polynomial $\Phi: \{0, 1\}^N \rightarrow \mathbb{R}$ given by

$$\Phi(x) = \phi(x) p\left(\frac{|x|}{m}\right).$$

By (3.43) and (3.47),

$$\deg \Phi = O\left(\sqrt{\frac{n}{m}} \left(d + \log \frac{1}{\alpha}\right)\right). \quad (3.48)$$

As the notation suggests, Φ is meant to be an extension of the approximant ϕ to inputs $x \in \{0, 1\}^N$ of Hamming weight up to n . To analyze the accuracy of this new approximant, we will examine three cases depending on the Hamming weight $|x|$.

To start with,

$$\begin{aligned}
\max_{|x| \leq m} |F_n(x) - \Phi(x)| &= \max_{|x| \leq m} |F_{2m}(x) - \Phi(x)| \\
&\leq \max_{|x| \leq m} \{|F_{2m}(x) - \phi(x)| + |\phi(x) - \Phi(x)|\} \\
&\leq \epsilon + \max_{|x| \leq m} |\phi(x) - \Phi(x)| \\
&\leq \epsilon + \max_{|x| \leq m} |\phi(x)| \max_{0 \leq t \leq 1} |1 - p(t)| \\
&\leq \epsilon + (1 + \epsilon) \max_{0 \leq t \leq 1} |1 - p(t)| \\
&\leq \epsilon + (1 + \epsilon) \cdot \alpha,
\end{aligned} \tag{3.49}$$

where the third and fifth steps use (3.42), and the last step uses (3.44). Continuing,

$$\begin{aligned}
\max_{m < |x| \leq 2m} |F_n(x) - \Phi(x)| &= \max_{m < |x| \leq 2m} |\Phi(x)| \\
&\leq \max_{m < |x| \leq 2m} |\phi(x)| \max_{1 < t \leq 2} |p(t)| \\
&\leq \max_{m < |x| \leq 2m} (|F_{2m}(x)| + \epsilon) \max_{1 < t \leq 2} |p(t)| \\
&\leq \epsilon \cdot (1 + \alpha),
\end{aligned} \tag{3.50}$$

where the last two steps use (3.42) and (3.45), respectively. Finally,

$$\begin{aligned}
\max_{2m < |x| \leq n} |F_n(x) - \Phi(x)| &= \max_{2m < |x| \leq n} |\Phi(x)| \\
&= \max_{2m < |x| \leq n} |\phi(x)| p\left(\frac{|x|}{m}\right) \\
&\leq \max_{2m < |x| \leq n} |\phi(x)| \cdot \alpha \cdot \left(\frac{m}{|x|}\right)^d \\
&\leq \max_{2m < |x| \leq n} \left\{ 2^d \binom{\lceil |x|/2m \rceil}{d} \max_{|x'| \leq 2m} |\phi(x')| \cdot \alpha \cdot \left(\frac{m}{|x|}\right)^d \right\} \\
&\leq \max_{2m < t \leq n} \left\{ 2^d \binom{\lceil t/2m \rceil}{d} (1 + \epsilon) \cdot \alpha \cdot \left(\frac{m}{t}\right)^d \right\} \\
&\leq (4e)^d (1 + \epsilon) \cdot \alpha,
\end{aligned} \tag{3.51}$$

where the third step uses (3.46), the fourth step applies (3.41) and the generalized extrapolation lemma (Lemma 2.11), the fifth step follows from (3.42), and the last step uses (2.1) and (3.41). Now (3.40) follows from (3.48)–(3.51) by taking $\alpha = \delta(4e)^{-d-1}$. \square

4. SYMMETRIC FUNCTIONS

In this section, we study the approximation of symmetric functions. This class includes AND_n and OR_n , which are fundamental building blocks of our constructions in the rest of the paper. Our result here is as follows.

THEOREM 4.1. *Let $f: \{0, 1\}^n \rightarrow [-1, 1]$ be an arbitrary symmetric function. Let k be a nonnegative integer such that f is constant on inputs of Hamming weight in $(k, n - k)$. Then for $0 < \epsilon < 1/2$,*

$$\deg_\epsilon(f) = O\left(\sqrt{nk} + \sqrt{n \log \frac{1}{\epsilon}}\right). \quad (4.1)$$

Moreover, the approximating polynomial is given explicitly in each case.

Theorem 4.1 is tight [51] for every $\epsilon \in [1/2^n, 1/3]$ and every symmetric function $f: \{0, 1\}^n \rightarrow \{0, 1\}$, with the obvious exception of the constant functions $f = 0$ and $f = 1$. Prior to our work, de Wolf [60] proved the upper bound (4.1) by giving an ϵ -error quantum query algorithm for any symmetric function f . The novelty of Theorem 4.1 is the construction of an explicit, closed-form approximating polynomial that achieves de Wolf's upper bound. We give three proofs of Theorem 4.1, corresponding to Sections 4.1–4.3 below.

4.1. Approximation using the extension theorem. Our first proof of Theorem 4.1 is based on the extension theorem, and is the shortest of the three. The centerpiece of the proof is the following technical lemma, in which we construct a closed-form approximant for any function supported on inputs of low Hamming weight.

LEMMA 4.2. *Let $f: \{0, 1\}^n \rightarrow [-1, 1]$ be given. Let k be a nonnegative integer such that $f(x) = 0$ for $|x| > k$. Then for $0 < \epsilon < 1/2$,*

$$\deg_\epsilon(f) = O\left(\sqrt{nk} + \sqrt{n \log \frac{1}{\epsilon}}\right).$$

Moreover, the approximating polynomial is given explicitly in each case.

Proof. Abbreviate

$$m = \left\lceil k + \log \frac{1}{\epsilon} \right\rceil.$$

If $m \geq n$, the bound in the theorem statement follows trivially from $\deg_0(f) \leq n$. In the rest of the proof, we focus on the complementary case $m < n$.

For $i \geq m$, define $F_i: \{0, 1\}_{\leq i}^n \rightarrow [-1, 1]$ by

$$F_i(x) = \begin{cases} f(x) & \text{if } |x| \leq m, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned}
\deg_\epsilon(f) &= \deg_\epsilon(F_n) \\
&\leq \sqrt{\frac{n}{m}} \cdot O\left(\deg_0(F_{2m}) + \log \frac{1}{\epsilon}\right) \\
&\leq \sqrt{\frac{n}{m}} \cdot O\left(2m + \log \frac{1}{\epsilon}\right) \\
&= O\left(\sqrt{nk} + \sqrt{n \log \frac{1}{\epsilon}}\right),
\end{aligned}$$

where the first step uses $f = F_n$, the second step applies the extension theorem (Theorem 3.1), and the third step is valid by Fact 2.1. Moreover, the approximating polynomial is given explicitly because the extension theorem and Fact 2.1 are fully constructive. \square

We are now in a position to prove the claimed result on the approximation of arbitrary symmetric functions.

THEOREM 4.3. *Let $f: \{0, 1\}^n \rightarrow [-1, 1]$ be given. Let k be a nonnegative integer such that f is constant on inputs of Hamming weight in $(k, n - k)$. Then for $0 < \epsilon < 1/2$,*

$$\deg_\epsilon(f) = O\left(\sqrt{nk} + \sqrt{n \log \frac{1}{\epsilon}}\right).$$

Moreover, the approximating polynomial is given explicitly in each case.

A powerful feature of Theorem 4.3 is that the function of interest is only assumed to be symmetric on inputs of Hamming weight in $(k, n - k)$. In particular, Theorem 4.3 is significantly more general than Theorem 4.1.

Proof of Theorem 4.3. If $k \geq n/2$, the theorem follows from the trivial bound $\deg_0(f) \leq n$. For the complementary case $k < n/2$, write

$$f(x_1, \dots, x_n) = \lambda + f'(x_1, \dots, x_n) + f''(\overline{x}_1, \dots, \overline{x}_n),$$

where $\lambda \in [-1, 1]$ and $f', f'': \{0, 1\}^n \rightarrow [-2, 2]$ are functions that vanish on $\{0, 1\}_{>k}^n$. Then

$$\begin{aligned}
\deg_\epsilon(f) &\leq \max\{\deg_{\epsilon/2}(f'), \deg_{\epsilon/2}(f'')\} \\
&\leq \max\left\{\deg_{\epsilon/4}\left(\frac{f'}{2}\right), \deg_{\epsilon/4}\left(\frac{f''}{2}\right)\right\} \\
&= O\left(\sqrt{nk} + \sqrt{n \log \frac{1}{\epsilon}}\right),
\end{aligned}$$

where the last step uses Lemma 4.2. \square

4.2. Approximation from first principles. We now present our second proof of Theorem 4.1. This proof proceeds from first principles, using Chebyshev polynomials as its only ingredient. To convey the construction as clearly as possible, we first present an approximant for the simplest and most important symmetric function, AND_n . For this, we adopt the strategy of previous constructions [30, 51], whereby one first zeroes out as many of the integer points $n-1, n-2, n-3, \dots$ as possible and then uses a Chebyshev polynomial to approximate AND_n on the remaining points of $\{0, 1, 2, \dots, n\}$. We depart from the previous work in the implementation of the first step. Specifically, we produce the zeroes using a product of Chebyshev polynomials, each of which is stretched and shifted so as to obtain an extremum at n and a root at one of the points $n-1, n-2, n-3, \dots$. The use of Chebyshev polynomials allows us to avoid explosive growth at the nonzeros, thereby eliminating a key source of inefficiency in [30, 51]. The lemma below shows how to produce a single zero, at any given point m .

LEMMA 4.4. *Let n and m be given integers, $0 \leq m < n$. Then there is a univariate polynomial $T_{n,m}$ such that*

$$T_{n,m}(n) = 1, \tag{4.2}$$

$$T_{n,m}(m) = 0, \tag{4.3}$$

$$|T_{n,m}(t)| \leq 1, \quad 0 \leq t \leq n, \tag{4.4}$$

$$\deg(T_{n,m}) \leq \left\lceil \frac{\pi}{4} \sqrt{\frac{n}{n-m}} \right\rceil. \tag{4.5}$$

Proof. As mentioned above, the construction involves starting with a Chebyshev polynomial and stretching and shifting it so as to move an extremum to n and a root to m . In more detail, let

$$d = \left\lceil \frac{\pi}{4} \sqrt{\frac{n}{n-m}} \right\rceil.$$

Consider the linear map L that sends

$$L(n) = 1, \tag{4.6}$$

$$L(m) = \cos\left(\frac{\pi}{2d}\right). \tag{4.7}$$

Observe that under L , the length of any given interval of the real line changes by a factor of

$$\begin{aligned} \frac{1}{n-m} \left(1 - \cos\left(\frac{\pi}{2d}\right)\right) &\leq \frac{1}{n-m} \left(1 - \left(1 - \frac{\pi^2}{8d^2}\right)\right) \\ &= \frac{\pi^2}{8d^2(n-m)} \\ &\leq \frac{2}{n}, \end{aligned}$$

where the first step uses $\cos x \geq 1 - \frac{1}{2}x^2$ for $x \in \mathbb{R}$. In particular,

$$\begin{aligned} L([0, n]) &\subseteq \left[L(n) - \frac{2}{n} \cdot n, L(n) \right] \\ &\subseteq [-1, 1]. \end{aligned} \tag{4.8}$$

We now show that the sought properties (4.2)–(4.5) hold for the polynomial $T_{n,m}(t) = T_d(L(t))$, where T_d denotes as usual the Chebyshev polynomial of degree d . To start with,

$$\begin{aligned} T_{n,m}(n) &= T_d(L(n)) \\ &= T_d(1) \\ &= 1, \end{aligned}$$

where the last two steps use (4.6) and (2.7), respectively. Similarly,

$$\begin{aligned} T_{n,m}(m) &= T_d(L(m)) \\ &= T_d\left(\cos\left(\frac{\pi}{2d}\right)\right) \\ &= \cos\left(\frac{\pi}{2}\right) \\ &= 0, \end{aligned}$$

where the second and third steps follow from (4.7) and (2.3), respectively. Continuing,

$$\begin{aligned} T_{n,m}([0, n]) &= T_d(L([0, n])) \\ &\subseteq T_d([-1, 1]) \\ &\subseteq [-1, 1], \end{aligned}$$

where the last two steps follow from (4.8) and (2.6), respectively. Finally, the degree bound (4.5) is immediate from the choice of d . \square

We now obtain the desired approximant for AND and OR, using the two-stage approach described earlier. The reader interested exclusively in the general case may wish to skip to Theorem 4.8.

THEOREM 4.5. *For some constant $c > 0$ and all integers $n \geq 1$ and $d \geq 0$, there is an (explicitly given) univariate polynomial p such that*

$$p(n) = 1, \tag{4.9}$$

$$|p(t)| \leq \exp\left(-\frac{cd^2}{n}\right), \quad t = 0, 1, 2, \dots, n-1, \tag{4.10}$$

$$|p(t)| \leq 1, \quad t \in [0, n], \tag{4.11}$$

$$\deg p \leq d. \tag{4.12}$$

In particular,

$$E(\text{AND}_n, d) \leq \frac{1}{2} \exp\left(-\frac{c}{2} \cdot \frac{d^2}{n}\right), \quad d = 0, 1, 2, 3, \dots, \quad (4.13)$$

$$\text{deg}_\epsilon(\text{AND}_n) \leq O\left(\sqrt{n \log \frac{1}{\epsilon}}\right), \quad 0 < \epsilon < \frac{1}{2}, \quad (4.14)$$

and analogously

$$E(\text{OR}_n, d) \leq \frac{1}{2} \exp\left(-\frac{c}{2} \cdot \frac{d^2}{n}\right), \quad d = 0, 1, 2, 3, \dots \quad (4.15)$$

$$\text{deg}_\epsilon(\text{OR}_n) \leq O\left(\sqrt{n \log \frac{1}{\epsilon}}\right), \quad 0 < \epsilon < \frac{1}{2}. \quad (4.16)$$

Proof. For $d \geq n$, we may simply take $p(t) = t(t-1)(t-2)\cdots(t-n+1)/n!$. In what follows, we focus on the construction of p for $d < n$. Let ℓ, r be integer parameters to be chosen later, where $1 \leq \ell \leq n-1$ and $1 \leq r \leq n$. We define

$$p(t) = \frac{T_r(t/(n-\ell))}{T_r(n/(n-\ell))} \prod_{i=n-\ell+1}^{n-1} T_{n,i}(t),$$

where $T_{n,i}$ is as constructed in Lemma 4.4, and T_r stands as usual for the Chebyshev polynomial of degree r . By (4.2) and (4.3),

$$p(n) = 1, \quad (4.17)$$

$$p(t) = 0, \quad t = n - \ell + 1, \dots, n - 1. \quad (4.18)$$

Moreover,

$$\begin{aligned} \max_{0 \leq t \leq n-\ell} |p(t)| &= \max_{0 \leq t \leq n-\ell} \frac{|T_r(t/(n-\ell))|}{|T_r(n/(n-\ell))|} \prod_{i=n-\ell+1}^{n-1} |T_{n,i}(t)| \\ &\leq \frac{1}{|T_r(n/(n-\ell))|} \\ &\leq \frac{1}{\max\left\{1 + \frac{r^2\ell}{n}, 2^r \sqrt{\ell/n-1}\right\}} \\ &\leq \frac{1}{\min\left\{\exp\left(\frac{r^2\ell}{3n}\right), \exp\left(\frac{r\sqrt{\ell}}{3\sqrt{n}}\right)\right\}}, \end{aligned} \quad (4.19)$$

where the second step uses (2.6) and (4.4); the third step follows from Proposition 2.6; and the last step uses $1 + x \geq \exp(x/3)$ for $0 \leq x \leq 4$, and $2^{\sqrt{x-1}} \geq$

$\exp(\sqrt{x}/3)$ for $x \geq 4$. Next,

$$\begin{aligned} \max_{0 \leq t \leq n} |p(t)| &= \max_{0 \leq t \leq n} \frac{|T_r(t/(n-\ell))|}{|T_r(n/(n-\ell))|} \prod_{i=n-\ell+1}^{n-1} |T_{n,i}(t)| \\ &\leq \max_{0 \leq t \leq n} \frac{|T_r(t/(n-\ell))|}{|T_r(n/(n-\ell))|} \\ &\leq 1, \end{aligned} \tag{4.20}$$

where the second inequality uses (4.4), and the third inequality follows from (2.6), (2.7), and Fact 2.5. Finally,

$$\begin{aligned} \deg p &\leq r + \sum_{i=n-\ell+1}^{n-1} \deg(T_{n,i}) \\ &\leq r + \sum_{i=1}^{\ell-1} \left(\frac{\pi}{4} \sqrt{\frac{n}{i}} + 1 \right) \\ &\leq r + \ell - 1 + \frac{\pi\sqrt{n}}{4} \int_0^{\ell-1} \frac{dt}{\sqrt{t}} \\ &= r + \ell - 1 + \frac{\pi\sqrt{n(\ell-1)}}{2} \\ &\leq r + 3\sqrt{n(\ell-1)}, \end{aligned} \tag{4.21}$$

where the second step uses (4.5). Now (4.9)–(4.12) follow from (4.17)–(4.21) by setting $r = \lceil d/2 \rceil$ and $\ell = \lfloor d^2/(36n) \rfloor + 1$.

The remaining claims in the theorem statement follow in a straightforward manner from (4.9)–(4.12). For (4.13), we have

$$\begin{aligned} E(\text{AND}_n, d) &\leq \max_{x \in \{0,1\}^n} \left| \text{AND}_n(x) - \frac{p(\sum_{i=1}^n x_i)}{1 + \exp(-cd^2/n)} \right| \\ &\leq \frac{\exp(-cd^2/n)}{1 + \exp(-cd^2/n)} \\ &\leq \frac{1}{2} \exp\left(-\frac{c}{2} \cdot \frac{d^2}{n}\right), \end{aligned}$$

where the last step uses $a/(1+a) \leq \sqrt{a}/2$ for any $a \geq 0$. This in turn settles (4.15) since $\text{OR}_n(x) = 1 - \text{AND}_n(1 - x_1, \dots, 1 - x_n)$. Finally, (4.14) and (4.16) are immediate from (4.13) and (4.15), respectively. \square

To generalize Theorem 4.5 to an arbitrary symmetric function f , it is helpful to think of f as a linear combination of the characteristic functions of individual levels of the Boolean hypercube. Specifically, define $\text{EXACT}_{n,k} : \{0,1\}^n \rightarrow \{0,1\}$ by

$$\text{EXACT}_{n,k}(x) = \begin{cases} 1 & \text{if } |x| = k, \\ 0 & \text{otherwise.} \end{cases}$$

In this notation, Theorem 4.5 treats the special case $\text{AND}_n = \text{EXACT}_{n,n}$. The technique of that theorem is easily adapted to yield the following more general result.

THEOREM 4.6. *For any $0 < \epsilon < 1/2$ and any integers $n \geq m \geq k \geq 0$, there is a univariate polynomial p such that*

$$\begin{aligned} p(|x|) &= \text{EXACT}_{n,n-k}(x), & |x| &\leq m, \\ p(|x|) &= \text{EXACT}_{n,n-k}(x), & |x| &\geq n - m, \\ |p(|x|) - \text{EXACT}_{n,n-k}(x)| &\leq \epsilon, & x &\in \{0, 1\}^n, \\ \deg p &= O\left(\sqrt{nm} + \sqrt{n \log \frac{1}{\epsilon}}\right). \end{aligned}$$

Proof. Define

$$\ell = \left\lceil m + \log \frac{2}{\epsilon} \right\rceil, \quad (4.22)$$

$$r = \left\lceil \sqrt{n \log \frac{2}{\epsilon}} \right\rceil. \quad (4.23)$$

If $\ell \geq n/2$, the theorem holds trivially for the degree- n polynomial

$$p(t) = \prod_{\substack{i=0 \\ i \neq n-k}}^n \frac{t-i}{n-k-i}.$$

In the complementary case $\ell < n/2$, define

$$\begin{aligned} p(t) &= \frac{T_r(t/(n-\ell))}{T_r((n-k)/(n-\ell))} \cdot \prod_{i=0}^{\ell} T_{n-k,i}(t) \cdot \prod_{i=n-\ell}^{n-k-1} T_{n-k,i}(t) \\ &\quad \times \prod_{i=n-k+1}^n (1 - T_{i,n-k}(t)^2), \end{aligned}$$

where $T_{n-k,i}$ and $T_{i,n-k}$ are as constructed in Lemma 4.4, and T_r denotes as usual the Chebyshev polynomial of degree r . Then (4.2) and (4.3) imply that

$$p(t) = 0, \quad t \in \{0, 1, \dots, \ell\}, \quad (4.24)$$

$$p(t) = 0, \quad t \in \{n-\ell, \dots, n-1, n\} \setminus \{n-k\}, \quad (4.25)$$

and

$$\begin{aligned}
p(n-k) &= \prod_{i=0}^{\ell} T_{n-k,i}(n-k) \cdot \prod_{i=n-\ell}^{n-k-1} T_{n-k,i}(n-k) \\
&\quad \times \prod_{i=n-k+1}^n (1 - T_{i,n-k}(n-k)^2) \\
&= \prod_{i=0}^{\ell} 1 \cdot \prod_{i=n-\ell}^{n-k-1} 1 \cdot \prod_{i=n-k+1}^n (1 - 0^2) \\
&= 1.
\end{aligned} \tag{4.26}$$

Moreover,

$$\begin{aligned}
\max_{0 \leq t \leq n-\ell} |p(t)| &= \max_{0 \leq t \leq n-\ell} \left| \frac{T_r(t/(n-\ell))}{T_r((n-k)/(n-\ell))} \cdot \prod_{i=0}^{\ell} T_{n-k,i}(t) \right. \\
&\quad \times \left. \prod_{i=n-\ell}^{n-k-1} T_{n-k,i}(t) \cdot \prod_{i=n-k+1}^n (1 - T_{i,n-k}(t)^2) \right| \\
&\leq \max_{0 \leq t \leq n-\ell} \left| \frac{T_r(t/(n-\ell))}{T_r((n-k)/(n-\ell))} \right| \\
&\leq \frac{1}{|T_r((n-k)/(n-\ell))|} \\
&\leq 2^{-r\sqrt{(\ell-k)/n+1}} \\
&\leq \epsilon,
\end{aligned} \tag{4.27}$$

where the second step uses (4.4), the third step uses (2.6), the fourth step applies Proposition 2.6, and the final step substitutes the parameters (4.22) and (4.23). Finally,

$$\begin{aligned}
\deg p &\leq r + \sum_{i=0}^{\ell} \deg(T_{n-k,i}) + \sum_{i=n-\ell}^{n-k-1} \deg(T_{n-k,i}) + 2 \sum_{i=n-k+1}^n \deg(T_{i,n-k}) \\
&\leq r + \sum_{i=0}^{\ell} \left(\frac{\pi}{4} \sqrt{\frac{n-k}{n-k-i}} + 1 \right) + \sum_{i=1}^{\ell-k} \left(\frac{\pi}{4} \sqrt{\frac{n-k}{i}} + 1 \right) \\
&\quad + 2 \sum_{i=1}^k \left(\frac{\pi}{4} \sqrt{\frac{n-k+i}{i}} + 1 \right) \\
&\leq r + 3\ell + \pi \sum_{i=1}^{\ell+1} \sqrt{\frac{n}{i}}
\end{aligned}$$

$$\begin{aligned}
&\leq r + 3\ell + \pi\sqrt{n} \int_0^{\ell+1} \frac{dt}{\sqrt{t}} \\
&= r + 3\ell + 2\pi\sqrt{n(\ell+1)} \\
&= O\left(\sqrt{nm} + \sqrt{n \log \frac{1}{\epsilon}}\right), \tag{4.28}
\end{aligned}$$

where the second and third steps use (4.4) and $k < \ell < n - k$, respectively. In view of (4.24)–(4.28), the proof is complete. \square

We are now in a position to handle arbitrary symmetric functions by expressing them as a linear combination of $\text{EXACT}_{n,i}$ for $i = 0, 1, 2, \dots, n$. This result provides a new proof of Theorem 4.1.

THEOREM 4.7. *Let $f: \{0, 1\}^n \rightarrow [-1, 1]$ be an arbitrary symmetric function. Let k be a nonnegative integer such that f is constant on inputs of Hamming weight in $(k, n - k)$. Then for $0 < \epsilon < 1/2$,*

$$\deg_\epsilon(f) = O\left(\sqrt{nk} + \sqrt{n \log \frac{1}{\epsilon}}\right). \tag{4.29}$$

More precisely, there is an (explicitly given) polynomial $\tilde{f}: \{0, 1\}^n \rightarrow \mathbb{R}$ such that

$$f(x) = \tilde{f}(x), \quad |x| \leq k, \tag{4.30}$$

$$f(x) = \tilde{f}(x), \quad |x| \geq n - k, \tag{4.31}$$

$$|f(x) - \tilde{f}(x)| \leq \epsilon, \quad x \in \{0, 1\}^n, \tag{4.32}$$

$$\deg \tilde{f} = O\left(\sqrt{nk} + \sqrt{n \log \frac{1}{\epsilon}}\right). \tag{4.33}$$

Proof. If $k \geq n/2$, the theorem follows from the trivial bound $\deg_0(f) \leq n$. For the complementary case $k < n/2$, write

$$\begin{aligned}
f(x) &= \lambda + \sum_{i=0}^k \lambda'_i \cdot \text{EXACT}_{n,i}(x) + \sum_{i=0}^k \lambda''_i \cdot \text{EXACT}_{n,n-i}(x) \\
&= \lambda + \sum_{i=0}^k \lambda'_i \cdot \text{EXACT}_{n,n-i}(\bar{x}_1, \dots, \bar{x}_n) + \sum_{i=0}^k \lambda''_i \cdot \text{EXACT}_{n,n-i}(x),
\end{aligned}$$

where $\lambda, \lambda'_0, \lambda''_0, \dots, \lambda'_k, \lambda''_k \in [-2, 2]$ are fixed reals. By Theorem 4.6, each of the functions $\text{EXACT}_{n,n-i}$ in this linear combination can be approximated pointwise to within $\epsilon/(2k+2)$ by a polynomial of degree $O(\sqrt{nk} + \sqrt{n \log(1/\epsilon)})$. Moreover, the lemma guarantees that in each case, the approximation is exact on $\{0, 1\}_{\leq k}^n$ and $\{0, 1\}_{\geq n-k}^n$. Now (4.29)–(4.32) are immediate. \square

4.3. Approximation using a sampling argument. We now give a third proof of Theorem 4.1, inspired by combinatorics rather than approximation theory. Here,

we show how to approximate an arbitrary symmetric function f using an approximant for AND (cf. Theorem 4.5) and a sampling argument. Suppose for the sake of concreteness that f is supported on inputs of Hamming weight at most k . Given a string $x \in \{0, 1\}^n$, consider the experiment whereby one chooses $\lfloor n/k \rfloor$ bits of x independently and uniformly at random, and outputs the disjunction of those bits. To approximate f , we feed the expected value of the sampling experiment to a suitable univariate polynomial constructed by Lagrange interpolation. The expected value of the experiment as a function of x has Π -norm at most 2, which by Proposition 2.12 means that the overall composition has small Π -norm as well. The complete details of this construction are provided in Lemma 4.8. To finish the proof, we expand the composition as a linear combination of conjunctions and replace each conjunction by a corresponding approximant from Theorem 4.5.

LEMMA 4.8. *Let $k \geq 0$ be a given integer. Let $f: \{0, 1\}^n \rightarrow [-1, 1]$ be a symmetric function that vanishes on $\{0, 1\}_{>k}^n$. Then for every $0 < \epsilon < 1/2$, there exists an (explicitly given) function $\tilde{f}: \{0, 1\}^n \rightarrow \mathbb{R}$ such that*

$$f(x) = \tilde{f}(x), \quad |x| \leq k, \quad (4.34)$$

$$f(x) = \tilde{f}(x), \quad |x| \geq n - k, \quad (4.35)$$

$$|f(x) - \tilde{f}(x)| \leq \epsilon, \quad x \in \{0, 1\}^n, \quad (4.36)$$

$$\Pi(\tilde{f}) \leq C^{k+\log(1/\epsilon)}, \quad (4.37)$$

where $C > 1$ is an absolute constant independent of f, n, k, ϵ .

Proof. If $k = 0$, the only possibilities are $f(x) \equiv 0$ and $f(x) = \bigwedge \overline{x_i}$, and therefore we may take $\tilde{f} = f$. If $k \geq n/4$, we again may take $\tilde{f} = f$ since $\Pi(f) \leq 2^n$ by Proposition 2.12(v). In what follows, we treat the remaining case

$$1 \leq k < \frac{n}{4}. \quad (4.38)$$

Consider the points $0 = t_0 \leq t_1 \leq t_2 \leq \dots \leq t_n = 1$, where

$$t_i = 1 - \left(1 - \frac{i}{n}\right)^{\lfloor \frac{n}{2k} \rfloor}, \quad i = 0, 1, 2, \dots, n.$$

The derivative of $t \mapsto 1 - (1 - \frac{t}{n})^{\lfloor n/(2k) \rfloor}$ on $[0, 2k]$ ranges in $[\frac{1}{6k}, \frac{1}{2k}]$. Therefore, the mean value theorem gives

$$\frac{|i-j|}{6k} \leq |t_i - t_j| \leq \frac{|i-j|}{2k}, \quad i, j = 0, 1, 2, \dots, 2k. \quad (4.39)$$

In particular,

$$\frac{i}{6k} \leq t_i \leq \frac{i}{2k}, \quad i = 0, 1, 2, \dots, 2k. \quad (4.40)$$

Consider the univariate polynomials

$$p(t) = (1-t)^d \prod_{i=n-k}^n (t-t_i),$$

$$q(t) = \sum_{i=0}^k \frac{f(1^i 0^{n-i})}{p(t_i)} \prod_{\substack{j=0 \\ j \neq i}}^{2k} \frac{t-t_j}{t_i-t_j},$$

where

$$d = 5 \left\lceil 8k + \ln \frac{1}{\epsilon} \right\rceil. \quad (4.41)$$

Our definitions ensure that $p(t_i)q(t_i) = f(1^i 0^{n-i})$ for $i = 0, 1, 2, \dots, k$. Moreover, we have $p(t_i)q(t_i) = 0$ for $i = \{k+1, k+2, \dots, 2k\} \cup \{n-k, n-k+1, \dots, n\}$. Since f vanishes on inputs of Hamming weight greater than k , we conclude that

$$p(t_i)q(t_i) = f(1^i 0^{n-i}),$$

$$i = \{0, 1, \dots, 2k\} \cup \{n-k, n-k+1, \dots, n\}. \quad (4.42)$$

A routine calculation reveals the following additional properties of p and q .

CLAIM 4.9. $|p(t_i)q(t_i) - f(1^i 0^{n-i})| \leq \epsilon$ for $i \geq 2k$.

CLAIM 4.10. $\|p \cdot q\| = 2^{O(k + \log(1/\epsilon))}$.

We will settle these claims once we complete the main proof. Define $\tilde{f}: \{0, 1\}^n \rightarrow \mathbb{R}$ by $\tilde{f}(x) = p(t_{|x|})q(t_{|x|})$. Then (4.34)–(4.36) follow directly from (4.42) and Claim 4.9. For (4.37), observe that

$$\tilde{f}(x) = p \left(\mathbf{E}_S \bigvee_{i \in S} x_i \right) q \left(\mathbf{E}_S \bigvee_{i \in S} x_i \right)$$

where the expectation is over a multiset S of $\lfloor \frac{n}{2k} \rfloor$ elements that are chosen independently and uniformly at random from $\{1, 2, \dots, n\}$. As a result, (4.37) follows from Claim 4.10 and Proposition 2.12 (ii), (iii), (vi), (vii). \square

Proof of Claim 4.9. Fix an arbitrary point $t \in [t_{2k}, t_n] = [t_{2k}, 1]$. Recall from (4.38) that $k < n/4$. As a result,

$$\begin{aligned}
|p(t)q(t)| &\leq |p(t_{2k})q(t)| \\
&\leq |p(t_{2k})| \sum_{i=0}^k \frac{1}{\min\{|p(t_0)|, \dots, |p(t_k)|\}} \prod_{\substack{j=0 \\ j \neq i}}^{2k} \frac{|1-t_j|}{|t_i-t_j|} \\
&\leq \frac{|p(t_{2k})|}{|p(t_k)|} \sum_{i=0}^k \prod_{\substack{j=0 \\ j \neq i}}^{2k} \frac{|1-t_j|}{|t_i-t_j|} \\
&\leq \left(\frac{1-t_{2k}}{1-t_k}\right)^d \sum_{i=0}^k \prod_{\substack{j=0 \\ j \neq i}}^{2k} \frac{|1-t_j|}{|t_i-t_j|} \\
&= \left(1 - \frac{t_{2k}-t_k}{1-t_k}\right)^d \sum_{i=0}^k \prod_{\substack{j=0 \\ j \neq i}}^{2k} \frac{|1-t_j|}{|t_i-t_j|} \\
&\leq \exp\left(-\frac{t_{2k}-t_k}{1-t_k} \cdot d\right) \sum_{i=0}^k \prod_{\substack{j=0 \\ j \neq i}}^{2k} \frac{|1-t_j|}{|t_i-t_j|}.
\end{aligned}$$

Using the lower bounds in (4.39) and (4.40), we obtain

$$\begin{aligned}
|p(t)q(t)| &\leq \exp\left(-\frac{(2k-k)/6k}{1-(k/6k)} \cdot d\right) \sum_{i=0}^k \prod_{\substack{j=0 \\ j \neq i}}^{2k} \frac{1-(j/6k)}{|i-j|/6k} \\
&= \exp\left(-\frac{d}{5}\right) \sum_{i=0}^k \prod_{\substack{j=0 \\ j \neq i}}^{2k} \frac{6k-j}{|i-j|} \\
&\leq \exp\left(-\frac{d}{5}\right) \sum_{i=0}^k \frac{(6k)!/(4k)!}{i!(2k-i)!} \\
&= \exp\left(-\frac{d}{5}\right) \sum_{i=0}^k \binom{6k}{4k} \binom{2k}{i} \\
&\leq \exp\left(-\frac{d}{5}\right) \binom{6k}{4k} \cdot 2^{2k} \\
&\leq \exp\left(-\frac{d}{5}\right) \cdot 2^{8k} \\
&\leq \epsilon,
\end{aligned}$$

where the last step follows from the definition of d in (4.41). Hence, $|p(t_i)q(t_i) - f(1^i 0^{n-i})| = |p(t_i)q(t_i)| \leq \epsilon$ for $i \geq 2k$. \square

Proof of Claim 4.10. Recall from (4.38) that $k < n/4$. As a result,

$$\begin{aligned}
\min_{i=0,1,\dots,k} |p(t_i)| &= |p(t_k)| \\
&= |1 - t_k|^d \prod_{i=n-k}^n |t_k - t_i| \\
&\geq |1 - t_k|^d \cdot |t_k - t_{2k}|^{k+1} \\
&\geq \frac{1}{2^d \cdot 6^{k+1}},
\end{aligned} \tag{4.43}$$

where the last step uses the estimates in (4.39) and (4.40). As a result,

$$\begin{aligned}
\|p \cdot q\| &\leq (1+1)^d \prod_{i=n-k}^n (1+t_i) \cdot \sum_{i=0}^k \frac{|f(1^i 0^{n-i})|}{|p(t_i)|} \prod_{\substack{j=0 \\ j \neq i}}^{2k} \frac{1+t_j}{|t_i - t_j|} \\
&\leq 2^d \cdot 2^{k+1} \sum_{i=0}^k \frac{1}{2^{-d} \cdot 6^{-k-1}} \prod_{\substack{j=0 \\ j \neq i}}^{2k} \frac{2}{|t_i - t_j|} \\
&\leq 4^d \cdot 12^{k+1} \sum_{i=0}^k \prod_{\substack{j=0 \\ j \neq i}}^{2k} \frac{2 \cdot 6k}{|i - j|} \\
&= 4^d \cdot 12^{k+1} \cdot 6^{2k} \cdot \frac{(2k)^{2k}}{(2k)!} \sum_{i=0}^k \binom{2k}{i} \\
&\leq 4^d \cdot 12^{k+1} \cdot 6^{2k} \cdot \frac{(2k)^{2k}}{(2k)!} \cdot 2^{2k} \\
&= 2^{O(d+k)},
\end{aligned}$$

where the first step is valid by Fact 2.7; the second step uses $0 \leq t_i \leq 1$ and (4.43); the third step follows from the lower bound in (4.39); and the last step is legitimate by Stirling's approximation. In view of (4.41), the proof is complete. \square

We have reached the promised construction of an approximating polynomial for any symmetric function.

THEOREM (restatement of Theorem 4.1). *Let $f: \{0, 1\}^n \rightarrow [-1, 1]$ be an arbitrary symmetric function. Let k be a nonnegative integer such that f is constant on inputs of Hamming weight in $(k, n - k)$. Then for $0 < \epsilon < 1/2$,*

$$\deg_\epsilon(f) = O\left(\sqrt{nk} + \sqrt{n \log \frac{1}{\epsilon}}\right). \tag{4.44}$$

Moreover, the approximating polynomial is given explicitly in each case.

Proof. If $k \geq n/2$, the theorem follows from the trivial bound $\deg_0(f) \leq n$. For the complementary case $k < n/2$, write

$$f(x_1, \dots, x_n) = \lambda + f'(x_1, \dots, x_n) + f''(\bar{x}_1, \dots, \bar{x}_n),$$

where $\lambda \in [-1, 1]$ and $f', f'' : \{0, 1\}^n \rightarrow [-2, 2]$ are symmetric functions that vanish on $\{0, 1\}_{>k}^n$. Lemma 4.8 shows that $f'/2$ and $f''/2$ are each approximated pointwise to within $\epsilon/5$ by a linear combination of conjunctions, with real coefficients whose absolute values sum to $2^{O(k+\log(1/\epsilon))}$. By Theorem 4.5, each such conjunction can in turn be approximated pointwise by a polynomial of degree d to within $2^{-\Theta(d^2/n)}$. Summarizing,

$$\begin{aligned} E(f, d) &\leq E(f', d) + E(f'', d) \\ &\leq 2E\left(\frac{f'}{2}, d\right) + 2E\left(\frac{f''}{2}, d\right) \\ &\leq 2\left(2 \cdot \frac{\epsilon}{5} + 2^{O(k+\log(1/\epsilon))} \cdot 2^{-\Theta(d^2/n)}\right), \end{aligned}$$

whence (4.44). Moreover, the approximating polynomial is given explicitly because Theorem 4.5 and Lemma 4.8 provide closed-form expressions for the approximants involved. \square

4.4. Generalizations. Theorem 4.5 on the approximation of AND and OR obviously generalizes to arbitrary conjunctions and disjunctions. Somewhat less obviously, it generalizes in an optimal manner to conjunctions and disjunctions whose domain of definition is restricted to the first few levels of the hypercube. We record this generalization for later use.

THEOREM 4.11. *Let $f : \{0, 1\}_{\leq n}^N \rightarrow \{0, 1\}$ be given by*

$$f(x) = \left(\bigvee_{i \in A} x_i\right) \vee \left(\bigvee_{i \in B} \bar{x}_i\right),$$

for some subsets $A, B \subseteq \{1, 2, \dots, N\}$. Then

$$E(f, d) \leq \frac{1}{2} \exp\left(-\frac{cd^2}{n}\right), \quad d = 0, 1, 2, \dots,$$

where $c > 0$ is an absolute constant. Moreover, the approximating polynomial is given explicitly in each case.

Proof. If $|B| > n$, then $f \equiv 1$ on its domain of definition and hence $E(f, 0) = 0$.

In the complementary case when $|B| \leq n$, we have

$$\sum_{i \in A} x_i + \sum_{i \in B} (1 - x_i) \in \{0, 1, 2, \dots, 2n\}, \quad x \in \{0, 1\}_{\leq n}^N. \quad (4.45)$$

Theorem 4.5 gives an explicit univariate polynomial p of degree d such that

$$p(2n) = 1, \quad (4.46)$$

$$|p(t)| \leq \exp\left(-\frac{Cd^2}{n}\right), \quad t = 0, 1, 2, \dots, 2n-1, \quad (4.47)$$

where $C > 0$ is an absolute constant. Define

$$P(x) = 1 - \frac{1}{1 + \exp(-Cd^2/n)} \cdot p\left(2n - \sum_{i \in A} x_i - \sum_{i \in B} (1 - x_i)\right).$$

Then

$$\begin{aligned} \max_{x \in \{0,1\}_{\leq n}^N} |f(x) - P(x)| &\leq \frac{\exp(-Cd^2/n)}{1 + \exp(-Cd^2/n)} \\ &\leq \frac{1}{2} \exp\left(-\frac{Cd^2}{2n}\right), \end{aligned}$$

where the first step follows from (4.45)–(4.47), and the second step uses $a/(1+a) \leq \sqrt{a}/2$ for any $a \geq 0$. \square

COROLLARY 4.12. *Let $f: \{0,1\}_{\leq n}^N \rightarrow \{0,1\}$ be given by*

$$f(x) = \left(\bigwedge_{i \in A} x_i\right) \wedge \left(\bigwedge_{i \in B} \overline{x_i}\right),$$

for some subsets $A, B \subseteq \{1, 2, \dots, N\}$. Then

$$E(f, d) \leq \frac{1}{2} \exp\left(-\frac{cd^2}{n}\right), \quad d = 0, 1, 2, \dots,$$

where $c > 0$ is an absolute constant. Moreover, the approximating polynomial is given explicitly in each case.

Proof. Apply Theorem 4.11 to $1 - f$. \square

5. k -DNF AND k -CNF FORMULAS

Recall that a k -DNF formula in Boolean variables x_1, x_2, \dots, x_N is the disjunction of zero or more *terms*, where each term is the conjunction of at most k literals from among $x_1, \overline{x_1}, x_2, \overline{x_2}, \dots, x_N, \overline{x_N}$. As a convention, we consider the constant functions 0 and 1 to be valid k -DNF formulas for every $k \geq 0$. Analogously, a k -CNF formula in Boolean variables x_1, x_2, \dots, x_N is the conjunction of zero or more *clauses*, where each clause is the disjunction of at most k literals from among $x_1, \overline{x_1}, x_2, \overline{x_2}, \dots, x_N, \overline{x_N}$. Again, we consider the constant functions 0 and 1 to be valid k -CNF formulas for all $k \geq 0$. Recall that a function f is representable by a k -DNF formula if and only if its negation \overline{f} is representable by a k -CNF formula. Note also that the definition of k -DNF formulas is hereditary in the sense that a

k -DNF formula is also a k' -DNF formula for any $k' \geq k$, and analogously for CNF formulas.

The contribution of this section is to settle Theorem 1.2 on the approximate degree of every k -DNF and k -CNF formula. We will in fact prove the following more precise result, for every setting of the error parameter.

THEOREM 5.1. *Let $f: \{0, 1\}_{\leq n}^N \rightarrow \{0, 1\}$ be representable on its domain by a k -DNF or k -CNF formula. Then*

$$\deg_{\epsilon}(f) \leq c \cdot (\sqrt{2})^k n^{\frac{k}{k+1}} \left(\log \frac{1}{\epsilon} \right)^{\frac{1}{k+1}} \quad (5.1)$$

for all $0 < \epsilon < 1/2$, where $c > 1$ is an absolute constant independent of f, N, n, k, ϵ . Moreover, the approximating polynomial is given explicitly in each case.

We present the proof of this theorem in Sections 5.1–5.4 below.

5.1. Key quantities. For nonnegative integers n and k and a real number $\Delta \geq 1$, we define

$$D(n, k, \Delta) = \max_f \deg_{2^{-\Delta}}(f),$$

where the maximum is over all functions $f: \{0, 1\}_{\leq n}^N \rightarrow \{0, 1\}$ for some $N \geq n$ that are representable by a k -DNF formula. Fact 2.1 gives the upper bound

$$D(n, k, \Delta) \leq n. \quad (5.2)$$

Since the only 0-DNF formulas are the constant functions 0 and 1, we obtain

$$D(n, 0, \Delta) = 0. \quad (5.3)$$

We will prove Theorem 5.1 by induction of k , with (5.3) serving as the base case.

5.2. A composition theorem for approximate degree. The inductive step in our analysis of $D(n, k, \Delta)$ relies on a certain general bound on approximate degree for a class of composed functions, as follows.

LEMMA 5.2. *Let $F: X \times \{0, 1\}_n^N \rightarrow \{0, 1\}$ be given by*

$$F(x, y) = \bigvee_{i=1}^N y_i \wedge f_i(x)$$

for some functions $f_1, f_2, \dots, f_N: X \rightarrow \{0, 1\}$. Let b be an integer with $b \mid n$ and $b \mid N$. Then

$$\deg_{\epsilon}(F) \leq C \sqrt{nb \log \frac{1}{\epsilon}} + \max_{\substack{S \subseteq \{1, \dots, N\} \\ |S| \leq C \sqrt{nb \log \frac{1}{\epsilon}}}} \deg_{\epsilon \exp(-C \sqrt{\frac{n}{b} \log \frac{1}{\epsilon}})} \left(\bigvee_{i \in S} f_i \right)$$

for all $0 < \epsilon \leq 1/2$, where $C > 1$ is an absolute constant independent of F, N, n, b, ϵ .

As we will see shortly, the bound of Lemma 5.2 generalizes to functions $F: X \times \{0, 1\}_{\leq n}^N \rightarrow \{0, 1\}$ and to arbitrary reals $b \geq 1$. It is this more general, and more natural, result on the approximate degree of composed functions that we need for our analysis of $D(n, k, \Delta)$. However, establishing Lemma 5.2 first considerably improves the readability and modularity of the proof. By way of notation, we remind the reader that the symbol $\bigvee_{i \in S} f_i$ denotes the mapping $x \mapsto \bigvee_{i \in S} f_i(x)$. The reader will also recall the shorthand $[n] = \{1, 2, \dots, n\}$. In particular, $\binom{[n]}{\leq d}$ denotes the family of subsets of $\{1, 2, \dots, n\}$ of cardinality at most d .

Proof of Lemma 5.2. The proof is constructive and uses as its building blocks two main components: an “outer” approximant (for the OR function) and “inner” approximants (for disjunctions of small sets of f_i). We first describe these components individually and then present the overall construction and error analysis.

STEP 1: OUTER APPROXIMANT. Theorem 4.5 provides a symmetric multilinear polynomial $\widetilde{\text{OR}}_{n/b}: \{0, 1\}^{n/b} \rightarrow [0, 1]$ of degree $d = O(\sqrt{n \log(1/\epsilon)/b})$ that approximates $\text{OR}_{n/b}$ pointwise to within $\epsilon/2$. More specifically, there are real coefficients a_0, a_1, a_2, \dots such that

$$\left| \bigvee_{i=1}^{n/b} z_i - \sum_{S \in \binom{[n/b]}{\leq d}} a_{|S|} \prod_{i \in S} z_i \right| \leq \frac{\epsilon}{2}, \quad z \in \{0, 1\}^{n/b}, \quad (5.4)$$

where

$$1 \leq d \leq c \sqrt{\frac{n}{b} \log \frac{1}{\epsilon}} \quad (5.5)$$

for some absolute constant $c > 1$. By Lemma 2.9,

$$\sum_{\ell=0}^d \binom{n/b}{\ell} |a_\ell| \leq 8^d. \quad (5.6)$$

STEP 2: INNER APPROXIMANTS. For a subset $S \subseteq \{1, 2, \dots, N\}$, define $f_S: X \rightarrow \{0, 1\}$ by

$$f_S(x) = \bigvee_{i \in S} f_i(x).$$

Fix a polynomial $\tilde{f}_S: X \rightarrow \mathbb{R}$ of the smallest possible degree such that

$$\|f_S - \tilde{f}_S\|_\infty \leq \frac{\epsilon}{2} \left(\sum_{\ell=0}^d \binom{n/b}{\ell} 2^\ell |a_\ell| \right)^{-1}. \quad (5.7)$$

To avoid notational clutter in the formulas below, we will frequently write f_S and \tilde{f}_S instead of $f_S(x)$ and $\tilde{f}_S(x)$, respectively, when referring to the value of these

functions at a given point $x \in X$. We have

$$\begin{aligned} \deg(\tilde{f}_S) &\leq \deg_{\epsilon/(2 \cdot 16^d)} \left(\bigvee_{i \in S} f_i \right) \\ &\leq \deg_{\epsilon \exp(-4c\sqrt{\frac{n}{b} \log \frac{1}{\epsilon}})} \left(\bigvee_{i \in S} f_i \right), \end{aligned} \quad (5.8)$$

where the first and second steps use (5.6) and (5.5), respectively.

STEP 3: OVERALL APPROXIMANT. By appropriately composing the outer approximant with the inner approximants, we obtain an approximant for the overall function F . Specifically, define $\tilde{F}: X \times \{0, 1\}_n^N \rightarrow \mathbb{R}$ by

$$\begin{aligned} \tilde{F}(x, y) &= a_0 + \sum_{\ell=1}^d a_\ell \binom{n/b}{\ell} \binom{N}{b\ell} \binom{n}{b\ell}^{-1} \\ &\quad \times \mathbf{E}_{B_1, \dots, B_{n/b}} \left[\left(\sum_{\substack{S \subseteq \{1, 2, \dots, \ell\} \\ S \neq \emptyset}} (-1)^{|S|+1} \tilde{f}_{\bigcup_{i \in S} B_i} \right) \prod_{i \in B_1 \cup \dots \cup B_\ell} y_i \right], \end{aligned} \quad (5.9)$$

where expectation is taken over a uniformly random tuple of sets $B_1, \dots, B_{n/b} \subseteq \{1, 2, \dots, N\}$ that are pairwise disjoint and have cardinality b each. Then

$$\begin{aligned} \deg(\tilde{F}) &\leq \max_{B_1, \dots, B_{n/b}} \max_{S \subseteq \{1, 2, \dots, d\}} \left\{ \deg(\tilde{f}_{\bigcup_{i \in S} B_i}) + \sum_{i=1}^d |B_i| \right\} \\ &\leq \max_{S \in \binom{[N]}{\leq db}} \left\{ \deg(\tilde{f}_S) \right\} + db \\ &\leq \max_{\substack{S \subseteq \{1, \dots, N\} \\ |S| \leq c\sqrt{nb \log \frac{1}{\epsilon}}}} \left\{ \deg_{\epsilon \exp(-4c\sqrt{\frac{n}{b} \log \frac{1}{\epsilon}})} \left(\bigvee_{i \in S} f_i \right) \right\} + c\sqrt{nb \log \frac{1}{\epsilon}}, \end{aligned} \quad (5.10)$$

where the final step uses (5.5) and (5.8).

STEP 4: ERROR ANALYSIS. For the rest of the proof, fix $y \in \{0, 1\}_n^N$ arbitrarily. Let $L = \{i : y_i = 1\}$. In the defining equation (5.9), the product $\prod_{i \in B_1 \cup \dots \cup B_\ell} y_i$ acts like an indicator random variable for the event that $B_1 \cup \dots \cup B_\ell \subseteq L$, which occurs with probability precisely

$$\binom{|L|}{b\ell} \binom{N}{b\ell}^{-1} = \binom{n}{b\ell} \binom{N}{b\ell}^{-1}.$$

Therefore,

$$\begin{aligned}
\tilde{F}(x, y) &= a_0 + \sum_{\ell=1}^d a_\ell \binom{n/b}{\ell} \mathbf{E}_{B_1, \dots, B_{n/b}} \left[\sum_{\substack{S \subseteq [\ell] \\ S \neq \emptyset}} (-1)^{|S|+1} \tilde{f}_{\bigcup_{i \in S} B_i} \middle| B_1, \dots, B_\ell \subseteq L \right] \\
&= a_0 + \sum_{\ell=1}^d a_\ell \binom{n/b}{\ell} \mathbf{E}_{B_1, \dots, B_{n/b}} \left[\sum_{\substack{S \subseteq [\ell] \\ S \neq \emptyset}} (-1)^{|S|+1} \tilde{f}_{\bigcup_{i \in S} B_i} \middle| \bigcup_{i=1}^{n/b} B_i = L \right] \\
&= a_0 + \sum_{\ell=1}^d a_\ell \mathbf{E}_{B_1, \dots, B_{n/b}} \left[\sum_{T \in \binom{[n/b]}{\ell}} \sum_{\substack{S \subseteq T \\ S \neq \emptyset}} (-1)^{|S|+1} \tilde{f}_{\bigcup_{i \in S} B_i} \middle| \bigcup_{i=1}^{n/b} B_i = L \right] \\
&= \mathbf{E}_{B_1, \dots, B_{n/b}} \left[a_0 + \sum_{\ell=1}^d a_\ell \sum_{T \in \binom{[n/b]}{\ell}} \sum_{\substack{S \subseteq T \\ S \neq \emptyset}} (-1)^{|S|+1} \tilde{f}_{\bigcup_{i \in S} B_i} \middle| \bigcup_{i=1}^{n/b} B_i = L \right], \tag{5.11}
\end{aligned}$$

where the second step is valid because a uniformly random tuple of pairwise disjoint sets $B_1, \dots, B_\ell \subseteq L$ of cardinality b each can be generated by partitioning L uniformly at random into parts of size b and using the first ℓ parts of that partition; the third step is valid in view of the symmetry of the distribution of $B_1, \dots, B_{n/b}$; and the last step uses the linearity of expectation. Analogously,

$$\begin{aligned}
F(x, y) &= \bigvee_{i=1}^N y_i \wedge f_i \\
&= \bigvee_{i \in L} f_i \\
&= f_L \\
&= \mathbf{E}_{B_1, \dots, B_{n/b}} \left[f_{B_1 \cup \dots \cup B_{n/b}} \middle| \bigcup_{i=1}^{n/b} B_i = L \right]. \tag{5.12}
\end{aligned}$$

As a result,

$$\begin{aligned}
& |F(x, y) - \tilde{F}(x, y)| \\
& \leq \max_{B_1, \dots, B_{n/b}} \left| f_{B_1 \cup \dots \cup B_{n/b}} - a_0 - \sum_{\ell=1}^d a_\ell \sum_{T \in \binom{[n/b]}{\ell}} \sum_{\substack{S \subseteq T \\ S \neq \emptyset}} (-1)^{|S|+1} \tilde{f}_{\cup_{i \in S} B_i} \right| \\
& \leq \max_{B_1, \dots, B_{n/b}} \left| f_{B_1 \cup \dots \cup B_{n/b}} - a_0 - \sum_{\ell=1}^d a_\ell \sum_{T \in \binom{[n/b]}{\ell}} \sum_{\substack{S \subseteq T \\ S \neq \emptyset}} (-1)^{|S|+1} f_{\cup_{i \in S} B_i} \right| \\
& \quad + \max_{B_1, \dots, B_{n/b}} \sum_{\ell=1}^d |a_\ell| \sum_{T \in \binom{[n/b]}{\ell}} \sum_{\substack{S \subseteq T \\ S \neq \emptyset}} \left| f_{\cup_{i \in S} B_i} - \tilde{f}_{\cup_{i \in S} B_i} \right| \\
& \leq \max_{B_1, \dots, B_{n/b}} \left| f_{B_1 \cup \dots \cup B_{n/b}} - a_0 - \sum_{\ell=1}^d a_\ell \sum_{T \in \binom{[n/b]}{\ell}} \sum_{\substack{S \subseteq T \\ S \neq \emptyset}} (-1)^{|S|+1} f_{\cup_{i \in S} B_i} \right| \\
& \quad + \frac{\epsilon}{2} \\
& = \max_{B_1, \dots, B_{n/b}} \left| \bigvee_{i=1}^{n/b} f_{B_i} - a_0 - \sum_{\ell=1}^d a_\ell \sum_{T \in \binom{[n/b]}{\ell}} \sum_{\substack{S \subseteq T \\ S \neq \emptyset}} (-1)^{|S|+1} \bigvee_{i \in S} f_{B_i} \right| + \frac{\epsilon}{2} \\
& = \max_{B_1, \dots, B_{n/b}} \left| \bigvee_{i=1}^{n/b} f_{B_i} - \sum_{\ell=0}^d a_\ell \sum_{T \in \binom{[n/b]}{\ell}} \prod_{i \in T} f_{B_i} \right| + \frac{\epsilon}{2} \\
& \leq \epsilon, \tag{5.13}
\end{aligned}$$

where the first step is immediate from (5.11) and (5.12), the second step applies the triangle inequality, the third step is valid by (5.7), the fourth step is a change of notation, the fifth step uses the inclusion-exclusion formula (Fact 2.2), and the last step is justified by (5.4). By (5.10) and (5.13), the proof of Lemma 5.2 is complete by taking $C = 4c$. \square

To remove the homogeneity and divisibility assumptions in Lemma 5.2, we now show how to reduce the approximation of any function on $\{0, 1\}_{\leq n}^N$ to the approximation of a closely related function on $\{0, 1\}_n^{N+n}$. This connection is surprising at first but has a short proof based on Minsky and Papert's symmetrization argument.

LEMMA 5.3 (Homogenization lemma). *Let $f: X \times \{0, 1\}_{\leq n}^N \rightarrow \mathbb{R}$ be given. Define $f': X \times \{0, 1\}_n^{N+n} \rightarrow \mathbb{R}$ by*

$$f'(x, y_1 \dots y_{N+n}) = f(x, y_1 \dots y_N). \tag{5.14}$$

Then for all $\epsilon \geq 0$,

$$\deg_\epsilon(f') = \deg_\epsilon(f).$$

Proof. The upper bound $\deg_\epsilon(f') \leq \deg_\epsilon(f)$ is immediate from the defining equation (5.14). For a matching lower bound, fix a polynomial $\phi' : X \times \mathbb{R}^{N+n} \rightarrow \mathbb{R}$ such that

$$|f'(x, y) - \phi'(x, y)| \leq \epsilon, \quad x \in X, y \in \{0, 1\}_n^{N+n}, \quad (5.15)$$

$$\deg \phi' = \deg_\epsilon(f'). \quad (5.16)$$

Minsky and Papert's symmetrization argument (Proposition 2.4) yields a polynomial $\phi^* : X \times \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}$ such that for $t = 0, 1, 2, \dots, n$,

$$\phi^*(x, y, t) = \mathbf{E}_{z \in \{0, 1\}_t^n} \phi'(x, yz), \quad x \in X, y \in \{0, 1\}^N, \quad (5.17)$$

$$\deg \phi^* \leq \deg \phi'. \quad (5.18)$$

We are now in a position to construct the desired approximant for f . For any $x \in X$ and $y \in \{0, 1\}_{\leq n}^N$, we have

$$\begin{aligned} & \left| f(x, y) - \phi^* \left(x, y, n - \sum_{i=1}^N y_i \right) \right| \\ & \leq \left| f(x, y) - \mathbf{E}_{z \in \{0, 1\}_{n-|y|}^n} f'(x, yz) \right| \\ & \quad + \left| \mathbf{E}_{z \in \{0, 1\}_{n-|y|}^n} f'(x, yz) - \phi^* \left(x, y, n - \sum_{i=1}^N y_i \right) \right| \\ & = \left| \mathbf{E}_{z \in \{0, 1\}_{n-|y|}^n} f'(x, yz) - \phi^* \left(x, y, n - \sum_{i=1}^N y_i \right) \right| \\ & = \left| \mathbf{E}_{z \in \{0, 1\}_{n-|y|}^n} [f'(x, yz) - \phi'(x, yz)] \right| \\ & \leq \epsilon, \end{aligned}$$

where the first step applies the triangle inequality, the second step is immediate from the definition of f' , the third step uses (5.17), and the last step follows from (5.15). In summary, we have shown that $\deg_\epsilon(f) \leq \deg \phi^*$, which in view of (5.18) and (5.16) completes the proof. \square

We are now in a position to remove the divisibility assumption in Lemma 5.2 and additionally generalize it to the nonhomogeneous setting.

THEOREM 5.4. *Let $F : X \times \{0, 1\}_{\leq n}^N \rightarrow \{0, 1\}$ be given by*

$$F(x, y) = \bigvee_{i=1}^N y_i \wedge f_i(x)$$

for some functions $f_1, f_2, \dots, f_N: X \rightarrow \{0, 1\}$. Then

$$\begin{aligned} \deg_\epsilon(F) &\leq C\sqrt{nb \log \frac{1}{\epsilon}} \\ &\quad + \max_{\substack{S \subseteq \{1, \dots, N\} \\ |S| \leq C\sqrt{nb \log \frac{1}{\epsilon}}}} \deg_{\epsilon \exp(-C\sqrt{\frac{n}{b} \log \frac{1}{\epsilon}})} \left(\bigvee_{i \in S} f_i \right) \end{aligned} \quad (5.19)$$

for all reals $b \geq 1$ and $0 < \epsilon \leq 1/2$, where $C > 1$ is an absolute constant independent of F, N, n, b, ϵ .

Proof. We first examine the case $1 \leq b \leq n$. Consider the function $F': X \times \{0, 1\}_{n'}^{N'} \rightarrow \{0, 1\}$ given by

$$F'(x, y) = \bigvee_{i=1}^{N'} y_i \wedge f_i(x),$$

where

$$\begin{aligned} n' &= \lfloor b \rfloor \left\lceil \frac{n}{\lfloor b \rfloor} \right\rceil, \\ N' &= \lfloor b \rfloor \left\lceil \frac{N}{\lfloor b \rfloor} \right\rceil + \lfloor b \rfloor \left\lceil \frac{n}{\lfloor b \rfloor} \right\rceil, \\ f_{N+1} &= f_{N+2} = \dots = f_{N'} = 0. \end{aligned}$$

Then

$$\begin{aligned} \deg_\epsilon(F) &\leq \deg_\epsilon(F') \\ &\leq c\sqrt{n' \lfloor b \rfloor \log \frac{1}{\epsilon}} + \max_{\substack{S \subseteq \{1, \dots, N'\} \\ |S| \leq c\sqrt{n' \lfloor b \rfloor \log \frac{1}{\epsilon}}}} \deg_{\epsilon \exp(-c\sqrt{\frac{n'}{\lfloor b \rfloor} \log \frac{1}{\epsilon}})} \left(\bigvee_{i \in S} f_i \right), \end{aligned}$$

for some absolute constant $c \geq 1$, where the first step uses the homogenization lemma (Lemma 5.3) and the second step follows from Lemma 5.2. This settles (5.19) for $C = 2c$.

For the complementary case $b \geq n$, define $F': X \times \{0, 1\}_n^{N+n} \rightarrow \{0, 1\}$ by

$$F'(x, y) = \bigvee_{i=1}^{N+n} y_i \wedge f_i(x),$$

where $f_{N+1} = f_{N+2} = \dots = f_{N+n} = 0$. Then

$$\deg_\epsilon(F) = \deg_\epsilon(F') \quad (5.20)$$

by the homogenization lemma (Lemma 5.3). On the other hand,

$$F'(x, y) = \sum_{S \in \binom{[N+n]}{n}} \left(\bigvee_{i \in S} f_i(x) \right) \prod_{i \in S} y_i. \quad (5.21)$$

For any input y of Hamming weight n , every term in this summation vanishes except for the term corresponding to $S = \{i : y_i = 1\}$. This means that an approximant for F' with error ϵ can be obtained by replacing each disjunction in (5.21) with a polynomial that approximates that disjunction to within ϵ . As a result,

$$\deg_\epsilon(F') \leq n + \max_{S \in \binom{[N]}{\leq n}} \deg_\epsilon \left(\bigvee_{i \in S} f_i \right).$$

This upper bound along with (5.20) settles (5.19) for $b \geq n$. \square

5.3. A recursive bound. Using Theorem 5.4 as our main tool, we now derive the promised recurrence for $D(n, k, \Delta)$.

LEMMA 5.5. *There is a constant $C \geq 1$ such that for all integers $n, k \geq 1$ and reals $\Delta \geq 1$,*

$$D(n, k, \Delta) \leq \max_{b \geq 1} \left\{ C\sqrt{nb\Delta} + D\left(n, k-1, \Delta + C\sqrt{\frac{n\Delta}{b}}\right) \right\}. \quad (5.22)$$

Proof. Let $f: \{0, 1\}_{\leq n}^N \rightarrow \{0, 1\}$ be a k -DNF formula. Our objective is to bound $\deg_{2^{-\Delta}}(f)$ by the right-hand side of (5.22). We may assume that

$$f \neq 1, \quad (5.23)$$

since the bound holds trivially for the constant function $f = 1$.

Write $f = f' \vee f''$, where f' is a k -DNF formula in which every term has an unnegated variable, and f'' is a k -DNF formula whose terms feature only negated variables. Collecting like terms in f' , we immediately obtain

$$f'(x) = \bigvee_{i=1}^N x_i \wedge f'_i(x) \quad (5.24)$$

for some $(k-1)$ -DNF formulas f'_1, f'_2, \dots, f'_N .

We now turn to f'' . By (5.23), there exists $x^* \in \{0, 1\}_{\leq n}^N$ such that $f''(x^*) = 0$. Consider the subset $I = \{i : x_i^* = 1\}$, of cardinality

$$|I| \leq n. \quad (5.25)$$

Since every occurrence of a variable in $f''(x)$ is negated, we conclude that every term in $f''(x)$ features some literal \bar{x}_i with $i \in I$. Collecting like terms, we obtain

the representation

$$f''(x) = \bigvee_{i \in I} \overline{x_i} \wedge f_i''(x), \quad (5.26)$$

where each f_i'' is a $(k-1)$ -DNF formula.

To summarize (5.24)–(5.26), the function $f = f' \vee f''$ is a subfunction of some $F: \{0, 1\}_{\leq n}^N \times \{0, 1\}_{\leq 2n}^{N+n} \rightarrow \{0, 1\}$ of the form

$$F(x, y) = \bigvee_{i=1}^{N+n} y_i \wedge f_i(x),$$

where each f_i is a $(k-1)$ -DNF formula. Now

$$\begin{aligned} \deg_{2-\Delta}(f) &\leq \deg_{2-\Delta}(F) \\ &\leq \max_{b \geq 1} \left\{ c\sqrt{2nb\Delta} + \max_{S \subseteq \{1, 2, \dots, N+n\}} \deg_{2-\Delta} \exp(-c\sqrt{2n\Delta/b}) \left(\bigvee_{i \in S} f_i \right) \right\} \\ &\leq \max_{b \geq 1} \left\{ c\sqrt{2nb\Delta} + D \left(n, k-1, \Delta + \frac{c}{\ln 2} \sqrt{\frac{2n\Delta}{b}} \right) \right\}, \end{aligned}$$

where the second step follows from Theorem 5.4 for a suitable absolute constant $c \geq 1$, and the third step is justified by the fact that each $\bigvee_{i \in S} f_i: \{0, 1\}_{\leq n}^N \rightarrow \{0, 1\}$ is a $(k-1)$ -DNF formula. In conclusion, (5.22) holds with $C = c\sqrt{2}/\ln 2$. \square

5.4. Solving the recurrence. It remains to solve the recurrence for $D(n, k, \Delta)$ given by (5.3) and Lemma 5.5.

THEOREM 5.6. *There is a constant $c \geq 1$ such that for all integers $n, k \geq 0$ and reals $\Delta \geq 1$,*

$$D(n, k, \Delta) \leq c \cdot (\sqrt{2})^k n^{\frac{k}{k+1}} \Delta^{\frac{1}{k+1}}. \quad (5.27)$$

This result settles Theorem 5.1. Indeed, if $f: \{0, 1\}_{\leq n}^N \rightarrow \{0, 1\}$ is representable by a k -DNF formula, then (5.1) is immediate from (5.27). The same bound applies to k -CNF formulas because they are negations of k -DNF formulas, and $\deg_\epsilon(f) = \deg_\epsilon(1-f)$ for any f .

Proof of Theorem 5.6. We will prove (5.27) for $c = 2(C+1)^2$, where $C \geq 1$ is the absolute constant from Lemma 5.5. The proof is by induction on k . The base $k=0$ is valid due to (5.3). For the inductive step, let $k \geq 1$ be arbitrary. For $\Delta \geq n$, the claim is immediate from (5.2), and we focus on the complementary case

$$1 \leq \Delta \leq n. \quad (5.28)$$

For every $b \geq 1$,

$$\begin{aligned}
D(n, k, \Delta) &\leq \min \left\{ n, C\sqrt{nb\Delta} + D\left(n, k-1, \Delta + C\sqrt{\frac{n\Delta}{b}}\right) \right\} \\
&\leq \min \left\{ n, C\sqrt{nb\Delta} + 2(C+1)^2 2^{\frac{k-1}{2}} n^{\frac{k-1}{k}} \left(\Delta + C\sqrt{\frac{n\Delta}{b}} \right)^{\frac{1}{k}} \right\} \\
&\leq (C+1)\sqrt{nb\Delta} + (C+1)^2 2^{\frac{k+1}{2}} n^{\frac{k-1}{k}} \left((C+1)\sqrt{\frac{n\Delta}{b}} \right)^{\frac{1}{k}},
\end{aligned} \tag{5.29}$$

where the first step uses (5.2) and Lemma 5.5; the second step applies the inductive hypothesis; and the last step can be verified in a straightforward manner by examining the cases $\Delta \leq n/b$ and $\Delta \geq n/b$. Setting

$$b = (C+1)^2 2^k \left(\frac{n}{\Delta} \right)^{1 - \frac{2}{k+1}}$$

in (5.29) now yields (5.27), completing the inductive step. Note that our choice of parameter meets the requirement $b \geq 1$, as one can see from (5.28). \square

6. k -ELEMENT DISTINCTNESS

For an integer k , recall that the *threshold function* $\text{THR}_k: \{0, 1\}^* \rightarrow \{0, 1\}$ is given by

$$\text{THR}_k(x) = \begin{cases} 1 & \text{if } |x| \geq k, \\ 0 & \text{otherwise.} \end{cases}$$

As a generate case, we have

$$\text{THR}_0 \equiv 1. \tag{6.1}$$

In the *k -element distinctness problem*, the input is a list of n integers from some range of size r , and the objective is to determine whether some integer occurs at least k times. Traditionally, the input to k -element distinctness is represented by a Boolean matrix $x \in \{0, 1\}^{n \times r}$ with precisely one nonzero entry in each row. We depart from tradition by allowing the input $x \in \{0, 1\}^{n \times r}$ to be an arbitrary matrix with at most n ones. Formally, we define the k -element distinctness function $\text{ED}_{n,r,k}: \{0, 1\}_{\leq n}^{nr} \rightarrow \{0, 1\}$ by

$$\text{ED}_{n,r,k}(x) = \neg \bigvee_{i=1}^r \text{THR}_k(x_{1,i}x_{2,i} \dots x_{n,i}).$$

Since our focus is on upper bounds, working with the more general domain makes our results stronger. Our main result in this section is as follows.

THEOREM 6.1. *Let $k \geq 1$ be a fixed integer. Then for all integers $n, r \geq 1$ and all reals $0 < \epsilon \leq 1/2$,*

$$\deg_\epsilon(\text{ED}_{n,r,k}) = O\left(\sqrt{n} \min\{n, r\}^{\frac{1}{2} - \frac{1}{4(1-2^{-k})}} \left(\log \frac{1}{\epsilon}\right)^{\frac{1}{4(1-2^{-k})}}\right) + O\left(\sqrt{n \log \frac{1}{\epsilon}}\right).$$

Moreover, the approximating polynomial is given explicitly in each case.

Taking $\epsilon = 1/3$ in this result settles Theorem 1.1 from the introduction. To prove Theorem 6.1, we will need to consider a more general class of functions. For non-negative integers n, r, k and a real number $\Delta \geq 1$, we define

$$D(n, r, k, \Delta) = \max_F \deg_{2^{-\Delta}}(F),$$

where the maximum is over all functions $F: \{0, 1\}_{\leq n}^N \rightarrow \{0, 1\}$ for some N that are expressible as

$$F(x) = \bigvee_{i=1}^r \text{THR}_{k_i}(x|_{S_i})$$

for some pairwise disjoint sets $S_1, S_2, \dots, S_r \subseteq \{1, 2, \dots, N\}$ and some $k_1, k_2, \dots, k_r \in \{0, 1, 2, \dots, k\}$. The four-argument quantity D that we have just defined is unrelated to the three-argument quantity D from Section 5. We abbreviate

$$D(n, \infty, k, \Delta) = \max_{r \geq 1} D(n, r, k, \Delta).$$

By definition,

$$\deg_\epsilon(\text{ED}_{n,r,k}) \leq D\left(n, r, k, \log \frac{1}{\epsilon}\right), \quad 0 < \epsilon \leq \frac{1}{2}. \quad (6.2)$$

Our analysis of $D(n, r, k, \Delta)$ proceeds by induction on k . As the base cases, we have

$$D(n, \infty, 0, \Delta) = 0 \quad (6.3)$$

by (6.1), and

$$D(n, \infty, 1, \Delta) = C\sqrt{n\Delta} \quad (6.4)$$

by Theorem 4.11 for some constant $C \geq 1$. Also, Fact 2.1 implies that

$$D(n, \infty, k, \Delta) \leq n. \quad (6.5)$$

6.1. A recursive bound for small range. To implement the inductive step, we derive two complementary recursive bounds for $D(n, r, k, \Delta)$. The first of these bounds, presented below, is tailored to the case when $n \geq kr$.

LEMMA 6.2. *There is a constant $C \geq 1$ such that for all positive integers n, r, k and all reals $\Delta \geq 1$,*

$$D(n, r, k, \Delta) \leq C \cdot \sqrt{1 + \frac{n}{kr}} \cdot (D(2kr, r, k, \Delta + 1) + \Delta).$$

Proof. Since D is monotonically increasing in every argument, the lemma holds trivially for $n < kr$. In what follows, we consider the complementary case

$$n \geq kr. \tag{6.6}$$

Consider an arbitrary function $F: \{0, 1\}_{\leq n}^N \rightarrow \{0, 1\}$ of the form

$$F(x) = \bigvee_{i=1}^r \text{THR}_{k_i}(x|_{S_i}) \tag{6.7}$$

for some pairwise disjoint sets $S_1, S_2, \dots, S_r \subseteq \{1, 2, \dots, N\}$ and $k_1, k_2, \dots, k_r \in \{0, 1, 2, \dots, k\}$. By discarding any irrelevant variables among x_1, x_2, \dots, x_N , we may assume that $S_1 \cup S_2 \cup \dots \cup S_r = \{1, 2, \dots, N\}$. Then by the pigeonhole principle, any input x with Hamming weight at least kr satisfies at least one of the disjuncts in (6.7). Therefore,

$$F(x) = 1, \quad x \in \{0, 1\}_{\geq kr}^N. \tag{6.8}$$

For $i \geq kr$, define $F_i: \{0, 1\}_{\leq i}^N \rightarrow \{0, 1\}$ by

$$F_i(x) = \begin{cases} F(x) & \text{if } |x| \leq kr, \\ 1 & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} \deg_{2-\Delta}(F) &= \deg_{2-\Delta}(F_n) \\ &= \deg_{2-\Delta}(1 - F_n) \\ &\leq c \sqrt{\frac{n}{kr}} \cdot (\deg_{2-\Delta-1}(1 - F_{2kr}) + \Delta + 1) \\ &= c \sqrt{\frac{n}{kr}} \cdot (\deg_{2-\Delta-1}(F_{2kr}) + \Delta + 1) \\ &\leq c \sqrt{\frac{n}{kr}} \cdot (D(2kr, r, k, \Delta + 1) + \Delta + 1) \end{aligned}$$

for some absolute constant $c \geq 1$ and all $\Delta \geq 1$, where the first and last steps use (6.8), and the third step applies (6.6) and the extension theorem (Theorem 3.1) with $m = kr$ and $\epsilon = \delta = 2^{-\Delta-1}$. As a result, the lemma holds with $C = 2c$. \square

6.2. A recursive bound for large range. We now derive an alternate upper bound on $D(n, r, k, \Delta)$, with no dependence on the range parameter r . This result addresses the case of large r and complements Lemma 6.2.

LEMMA 6.3. *There is a constant $C \geq 1$ such that for all integers $n, k \geq 1$ and all reals $\Delta, b \geq 1$,*

$$D(n, \infty, k, \Delta) \leq C\sqrt{nb\Delta} + C \left(1 + \frac{1}{\sqrt{k}} \left(\frac{n}{b\Delta}\right)^{1/4}\right) \times \\ \times \left(D \left(\lfloor Ck\sqrt{nb\Delta} \rfloor, \infty, k-1, C\sqrt{\frac{n\Delta}{b}} + 1 \right) + \sqrt{\frac{n\Delta}{b}} \right). \quad (6.9)$$

Proof. Consider an arbitrary function $F: \{0, 1\}_{\leq n}^N \rightarrow \{0, 1\}$ of the form

$$F(x) = \bigvee_{i=1}^r \text{THR}_{k_i}(x|_{S_i}) \quad (6.10)$$

for some integer $r \geq 1$, some pairwise disjoint sets $S_1, S_2, \dots, S_r \subseteq \{1, 2, \dots, N\}$, and some $k_1, k_2, \dots, k_r \in \{0, 1, 2, \dots, k\}$. If $k_i = 0$ for some i , then the corresponding term in (6.10) is the constant function 1, resulting in $\deg_0(F) = 0$. In what follows, we treat the complementary case when $k_i \geq 1$ for each i .

Rewriting (6.10),

$$F(x) = \bigvee_{i=1}^r \bigvee_{j \in S_i} x_j \wedge \text{THR}_{k_i-1}(x|_{S_i \setminus \{j\}}). \quad (6.11)$$

As this representation suggests, our intention is to bound the approximate degree of F by appeal to Theorem 5.4.

CLAIM 6.4. *Fix a subset $S'_i \subseteq S_i$ for each $i = 1, 2, \dots, r$. Then for $\Delta \geq 1$,*

$$\deg_{2-\Delta} \left(\bigvee_{i=1}^r \bigvee_{j \in S'_i} \text{THR}_{k_i-1}(x|_{S_i \setminus \{j\}}) \right) \\ \leq D \left(n, \sum_{i=1}^r |S'_i|, k-1, \Delta \right) + \sum_{i=1}^r |S'_i|.$$

We will settle Claim 6.4 once we complete the main proof. In light of this claim, the representation (6.11) shows that

$$F(x) = \bigvee_{i=1}^N x_i \wedge f_i(x)$$

for some functions f_i such that

$$\deg_{2^{-\Delta}} \left(\bigvee_{i \in S} f_i \right) \leq D(n, |S|, k-1, \Delta) + |S| \quad (6.12)$$

for all $S \subseteq \{1, 2, \dots, N\}$ and all $\Delta \geq 1$. Then for some absolute constants $c', c'' \geq 1$ and all $\Delta \geq 1$ and $b \geq 1$, we have

$$\begin{aligned} \deg_{2^{-\Delta}}(F) &\leq c' \sqrt{nb\Delta} + \max_{\substack{S \subseteq \{1, \dots, N\} \\ |S| \leq c' \sqrt{nb\Delta}}} \deg_{2^{-\Delta} \exp(-c' \sqrt{n\Delta/b})} \left(\bigvee_{i \in S} f_i \right) \\ &\leq 2c' \sqrt{nb\Delta} + D \left(n, \lceil c' \sqrt{nb\Delta} \rceil, k-1, \Delta + \frac{c'}{\ln 2} \sqrt{\frac{n\Delta}{b}} \right) \\ &\leq 2c' \sqrt{nb\Delta} + c'' \cdot \sqrt{1 + \frac{n}{k \cdot c' \sqrt{nb\Delta}}} \times \\ &\quad \times \left(D \left(2k \lceil c' \sqrt{nb\Delta} \rceil, \infty, k-1, \Delta + \frac{c'}{\ln 2} \sqrt{\frac{n\Delta}{b}} + 1 \right) \right. \\ &\quad \left. + \Delta + \frac{c'}{\ln 2} \sqrt{\frac{n\Delta}{b}} \right), \end{aligned}$$

where the first step applies Theorem 5.4, the second step uses (6.12), and the final step follows from (6.3) for $k = 1$ and from Lemma 6.2 for $k \geq 2$. This directly implies (6.9) for $\Delta \leq n/b$. In the complementary case $\Delta > n/b$, the right-hand side of (6.9) exceeds n and therefore the bound follows trivially from (6.5). \square

Proof of Claim 6.4. To start with,

$$\begin{aligned} \bigvee_{i=1}^r \bigvee_{j \in S'_i} \text{THR}_{k_i-1}(x|_{S_i \setminus \{j\}}) &= \bigvee_{i: S'_i \neq \emptyset} \bigvee_{j \in S'_i} \text{THR}_{k_i-1}(x|_{S_i \setminus \{j\}}) \\ &= \bigvee_{i: S'_i \neq \emptyset} \text{THR}_{k_i-1-\min\{|x|_{S'_i}|, |S'_i|-1\}}(x|_{S_i \setminus S'_i}). \end{aligned}$$

Considering the possible values for the Hamming weight of each $x|_{S'_i}$, we arrive at the representation

$$\begin{aligned} \bigvee_{i=1}^r \bigvee_{j \in S'_i} \text{THR}_{k_i-1}(x|_{S_i \setminus \{j\}}) &= \sum_{\ell_1=0}^{|S'_1|} \cdots \sum_{\ell_r=0}^{|S'_r|} \mathbf{I}[|x|_{S'_i} = \ell_i \text{ for each } i] \\ &\quad \times \left(\bigvee_{i: S'_i \neq \emptyset} \text{THR}_{k_i-1-\min\{\ell_i, |S'_i|-1\}}(x|_{S_i \setminus S'_i}) \right). \quad (6.13) \end{aligned}$$

The indicator functions in this summation are mutually exclusive in that for any given value of x , precisely one of them is nonzero. As a result, the right-hand side of (6.13) can be approximated pointwise to within $2^{-\Delta}$ by replacing each parenthesized expression with its $2^{-\Delta}$ -error approximant, which by definition can be chosen to have degree at most $D(n, \sum |S'_i|, k-1, \Delta)$. This completes the proof since each indicator function in (6.13) depends on only $\sum |S'_i|$ Boolean variables and is therefore a polynomial of degree at most $\sum |S'_i|$. \square

6.3. Solving the recurrence. It remains to solve the newly obtained recurrences. We first solve the recurrence given by (6.4) and Lemma 6.3, corresponding to the infinite-range case.

THEOREM 6.5 (Range-independent bound). *There is a constant $c \geq 1$ such that for all positive integers n and k , and all reals $\Delta \geq 1$,*

$$D(n, \infty, k, \Delta) \leq c^k \sqrt{k!} \cdot n^{1 - \frac{1}{4(1-2^{-k})}} \Delta^{\frac{1}{4(1-2^{-k})}}. \quad (6.14)$$

Proof. We will prove (6.14) for $c = (4C)^2$, where $C \geq 1$ is the larger of the constants in (6.4) and Lemma 6.3. The proof is by induction on k . The base case $k = 1$ is immediate from (6.4). For the inductive step, let $k \geq 2$ be arbitrary. When $\Delta > n$, the right-hand side of (6.14) exceeds n and therefore the bound is immediate from (6.5). In what follows, we assume that

$$1 \leq \Delta \leq n. \quad (6.15)$$

Let $b \geq 1$ be a parameter to be fixed later. By Lemma 6.3,

$$\begin{aligned} D(n, \infty, k, \Delta) &\leq C\sqrt{nb\Delta} + C \left(1 + \frac{1}{\sqrt{k}} \left(\frac{n}{b\Delta} \right)^{\frac{1}{4}} \right) \left(\sqrt{\frac{n\Delta}{b}} \right. \\ &\quad \left. + D \left(\lfloor Ck\sqrt{nb\Delta} \rfloor, \infty, k-1, C\sqrt{\frac{n\Delta}{b}} + 1 \right) \right). \end{aligned}$$

It follows that

$$\begin{aligned} D(n, \infty, k, \Delta) &\leq C\sqrt{knb\Delta} + 2C \left(\frac{n}{kb\Delta} \right)^{\frac{1}{4}} \left(\sqrt{\frac{n\Delta}{b}} \right. \\ &\quad \left. + D \left(\lfloor Ck\sqrt{nb\Delta} \rfloor, \infty, k-1, C\sqrt{\frac{n\Delta}{b}} + 1 \right) \right), \end{aligned}$$

as one can verify from the previous step if $n \geq kb\Delta$ and from (6.5) if $n < kb\Delta$. Applying the inductive hypothesis,

$$D(n, \infty, k, \Delta) \leq C\sqrt{knb\Delta} + 2C \left(\frac{n}{kb\Delta} \right)^{\frac{1}{4}} \left(\sqrt{\frac{n\Delta}{b}} + c^{k-1} \sqrt{(k-1)!} \cdot (Ck\sqrt{nb\Delta})^{1-\frac{1}{4(1-2^{-k+1})}} \left(C\sqrt{\frac{n\Delta}{b}} + 1 \right)^{\frac{1}{4(1-2^{-k+1})}} \right).$$

Now the bound

$$D(n, \infty, k, \Delta) \leq C\sqrt{knb\Delta} + 2C \left(\frac{n}{kb\Delta} \right)^{\frac{1}{4}} \times 4c^{k-1} \sqrt{(k-1)!} \cdot (Ck\sqrt{nb\Delta})^{1-\frac{1}{4(1-2^{-k+1})}} \left(C\sqrt{\frac{n\Delta}{b}} \right)^{\frac{1}{4(1-2^{-k+1})}}$$

is immediate from the previous step if $n \geq b/\Delta$ and from (6.5) if $n < b/\Delta$. Rearranging, we find that

$$D(n, \infty, k, \Delta) \leq C\sqrt{knb\Delta} \left(1 + C \left(\frac{n}{\Delta} \right)^{\frac{1}{4}} \cdot 8c^{k-1} \sqrt{(k-1)!} b^{-\frac{1}{4}-\frac{1}{4(1-2^{-k+1})}} \right). \quad (6.16)$$

The right-hand side is minimized at

$$b = \left(C \left(\frac{n}{\Delta} \right)^{\frac{1}{4}} \cdot 8c^{k-1} \sqrt{(k-1)!} \right)^{\frac{2^{k+1}-4}{2^{k-1}}},$$

which in view of (6.15) is a real number in $[1, \infty)$ and therefore a legitimate parameter setting. Making this substitution in (6.16), we arrive at

$$\begin{aligned} D(n, \infty, k, \Delta) &\leq 2C\sqrt{kn\Delta} \left(C \left(\frac{n}{\Delta} \right)^{\frac{1}{4}} \cdot 8c^{k-1} \sqrt{(k-1)!} \right)^{\frac{2^k-2}{2^{k-1}}} \\ &\leq 2C^2 \cdot 8c^{k-1} \sqrt{k! n \Delta \left(\frac{n}{\Delta} \right)^{\frac{2^k-1}{2^{k-1}}}} \\ &= c^k \sqrt{k!} n^{1-\frac{1}{4(1-2^{-k})}} \Delta^{\frac{1}{4(1-2^{-k})}}. \end{aligned}$$

This completes the inductive step and settles (6.14). \square

By combining the previous result with an application of Lemma 6.2, we will now prove our main bound on $D(n, r, k, \Delta)$.

THEOREM 6.6 (Range-dependent bound). *There is a constant $c \geq 1$ such that for all positive integers n, r, k and all reals $\Delta \geq 1$,*

$$D(n, r, k, \Delta) \leq c^k \sqrt{k!} \left(\sqrt{n} \min\{n, kr\}^{\frac{1}{2} - \frac{1}{4(1-2^{-k})}} \Delta^{\frac{1}{4(1-2^{-k})}} + \sqrt{n\Delta} \right).$$

Proof. The bound follows from Theorem 6.5 if $kr \geq n$; and from (6.5) if $\Delta \geq n$. As a result, we may assume that

$$n > kr, \tag{6.17}$$

$$n > \Delta. \tag{6.18}$$

In what follows, let $C \geq 1$ denote the larger of the constants in Lemma 6.2 and Theorem 6.5. Then

$$\begin{aligned} D(n, r, k, \Delta) &\leq D\left(n, r + \left\lceil \frac{\Delta}{k} \right\rceil, k, \Delta\right) \\ &\leq C \cdot \sqrt{1 + \frac{n}{kr + k\lceil \Delta/k \rceil}} \cdot \left(D\left(2kr + 2k \left\lceil \frac{\Delta}{k} \right\rceil, \infty, k, \Delta + 1\right) + \Delta \right) \\ &\leq 2C \cdot \sqrt{\frac{n}{kr + k\lceil \Delta/k \rceil}} \cdot \left(D\left(2kr + 2k \left\lceil \frac{\Delta}{k} \right\rceil, \infty, k, \Delta + 1\right) + \Delta \right) \\ &\leq 2C \cdot \sqrt{\frac{n}{kr + k\lceil \Delta/k \rceil}} \\ &\quad \times \left(C^k \sqrt{k!} \left(2kr + 2k \left\lceil \frac{\Delta}{k} \right\rceil \right)^{1 - \frac{1}{4(1-2^{-k})}} (\Delta + 1)^{\frac{1}{4(1-2^{-k})}} + \Delta \right) \\ &\leq 2C \cdot \sqrt{\frac{n}{kr + k\lceil \Delta/k \rceil}} \\ &\quad \times 2C^k \sqrt{k!} \left(2kr + 2k \left\lceil \frac{\Delta}{k} \right\rceil \right)^{1 - \frac{1}{4(1-2^{-k})}} (\Delta + 1)^{\frac{1}{4(1-2^{-k})}} \\ &= 4C^{k+1} \sqrt{k!} \cdot \sqrt{2n} \left(2kr + 2k \left\lceil \frac{\Delta}{k} \right\rceil \right)^{\frac{1}{2} - \frac{1}{4(1-2^{-k})}} (\Delta + 1)^{\frac{1}{4(1-2^{-k})}}, \end{aligned}$$

where the first step is valid because D is monotonically increasing in every argument; the second step applies Lemma 6.2; the third step uses (6.17) and (6.18); and the fourth step applies Theorem 6.5. This completes the proof of the theorem for $n > kr$. \square

Equation (6.2) and Theorem 6.6 establish the main result of this section, Theorem 6.1. We note that with a more careful analysis, the multiplicative factor $\sqrt{k!}$ in Theorems 6.5 and 6.6 can be improved to a slightly smaller quantity, still of the order of $k^{O(k)}$.

7. SURJECTIVITY

For positive integers n and r , the *surjectivity problem* is to determine whether a given mapping $\{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, r\}$ is surjective. Traditionally, the input

to this problem is represented by a Boolean matrix $x \in \{0, 1\}^{n \times r}$ with precisely one nonzero entry in every row. Analogous to our work on element distinctness in the previous section, we depart from tradition by allowing arbitrary matrices $x \in \{0, 1\}^{n \times r}$ with at most n ones. Specifically, we define the surjectivity function $\text{SURJ}_{n,r} : \{0, 1\}_{\leq n}^{nr} \rightarrow \{0, 1\}$ by

$$\text{SURJ}_{n,r}(x) = \bigwedge_{j=1}^r \bigvee_{i=1}^n x_{i,j}.$$

This formalism corresponds to determining the surjectivity of arbitrary relations on $\{1, 2, \dots, n\} \times \{1, 2, \dots, r\}$, including functions $\{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, r\}$ as a special case. Since we are interested in upper bounds, working in this more general setting makes our results stronger.

7.1. Approximation to 1/3. For clarity of exposition, we first bound the approximate degree of surjectivity with the error parameter set to $\epsilon = 1/3$. This setting covers most applications of interest and allows for a shorter and simpler proof. Readers with an interest in general ϵ can skip directly to Section 7.2.

THEOREM (restatement of Theorem 1.3). *For all positive integers n and r ,*

$$\deg_{1/3}(\text{SURJ}_{n,r}) = O(\sqrt{n} \cdot r^{1/4}) \quad (r \leq n), \quad (7.1)$$

$$\deg_{1/3}(\text{SURJ}_{n,r}) = 0 \quad (r > n). \quad (7.2)$$

Moreover, the approximating polynomial is given explicitly in each case.

The theorem shows that $\deg_{1/3}(\text{SURJ}_{n,r}) = O(n^{3/4})$ for all r , disproving the conjecture of Bun and Thaler [21] that the 1/3-approximate degree of $\text{SURJ}_{n,\Omega(n)}$ is linear in n .

Proof. The identity $\text{SURJ}_{n,r} \equiv 0$ for $r > n$ implies (7.2) directly. The proof of (7.1) involves two steps. First, we construct an explicit real-valued function $\widetilde{\text{SURJ}}_{n,r}$ that approximates $\text{SURJ}_{n,r}$ pointwise and is representable by a linear combination of conjunctions with reasonably small coefficients. Then, we replace each conjunction in this linear combination by an approximating polynomial of low degree.

In more detail, let $m \geq 1$ be an integer parameter to be chosen later. Recall from (2.6) and Proposition 2.6 that the Chebyshev polynomial T_m obeys

$$\begin{aligned} |T_m(t)| &\leq 1, & -1 \leq t \leq 1, \\ T_m\left(1 + \frac{1}{r}\right) &\geq 1 + \frac{m^2}{r}. \end{aligned}$$

As a result, $\text{SURJ}_{n,r}$ is approximated pointwise within $1/(1 + \frac{m^2}{r})$ by

$$\widetilde{\text{SURJ}}_{n,r}(x) = \frac{1}{T_m(1 + \frac{1}{r})} \cdot T_m\left(\frac{1}{r} + \frac{1}{r} \sum_{j=1}^r \bigvee_{i=1}^n x_{i,j}\right).$$

Therefore,

$$E(\text{SURJ}_{n,r}, d) \leq \frac{1}{1 + \frac{m^2}{r}} + E(\widetilde{\text{SURJ}}_{n,r}, d), \quad d = 1, 2, 3, \dots \quad (7.3)$$

To estimate the rightmost term in (7.3), use the factored representation (2.8) to write

$$\begin{aligned} \widetilde{\text{SURJ}}_{n,r}(x) &= \frac{2^{m-1}}{T_m(1 + \frac{1}{r})} \cdot \prod_{i=1}^m \left(\frac{1}{r} + \frac{1}{r} \sum_{j=1}^r \left(\bigvee_{i=1}^n x_{i,j} \right) - \cos \frac{(2i-1)\pi}{2m} \right) \\ &= \frac{2^{m-1}}{T_m(1 + \frac{1}{r})} \cdot \prod_{i=1}^m \left(\frac{1}{r} + 1 - \frac{1}{r} \sum_{j=1}^r \prod_{i=1}^n x_{i,j} - \cos \frac{(2i-1)\pi}{2m} \right). \end{aligned}$$

Multiplying out shows that $\widetilde{\text{SURJ}}_{n,r}(x)$ is a linear combination of conjunctions with real coefficients whose absolute values sum to $2^{O(m)}$. By Corollary 4.12, each of these conjunctions can be approximated by a polynomial of degree d to within $2^{-\Theta(d^2/n)}$ pointwise. We conclude that

$$E(\widetilde{\text{SURJ}}_{n,r}, d) \leq 2^{O(m)} \cdot 2^{-\Theta(d^2/n)},$$

which along with (7.3) gives

$$E(\text{SURJ}_{n,r}, d) \leq \frac{1}{1 + \frac{m^2}{r}} + 2^{O(m)} \cdot 2^{-\Theta(d^2/n)}.$$

Now (7.1) follows by taking $m = \lceil \sqrt{3r} \rceil$ and $d = \Theta(\sqrt{n} \cdot r^{1/4})$. The approximating polynomial in question is given explicitly because every stage of our proof, including the appeal to Corollary 4.12, is constructive. \square

7.2. Approximation to arbitrary error. We now generalize the previous theorem to arbitrary ϵ . The proof closely mirrors the case of $\epsilon = 1/3$ but features additional ingredients, such as Lemma 2.8.

THEOREM 7.1. *For all positive integers n and r , and all reals $0 < \epsilon < 1/2$,*

$$\deg_{\epsilon}(\text{SURJ}_{n,r}) = O \left(\sqrt{n} \left(r \log \frac{1}{\epsilon} \right)^{1/4} + \sqrt{n \log \frac{1}{\epsilon}} \right) \quad (r \leq n), \quad (7.4)$$

$$\deg_{\epsilon}(\text{SURJ}_{n,r}) = 0 \quad (r > n). \quad (7.5)$$

Moreover, the approximating polynomial is given explicitly in each case.

Proof. As before, we need only prove (7.4) since $\text{SURJ}_{n,r} \equiv 0$ for $r > n$. Theorem 4.5 provides, after rescaling, an explicit univariate polynomial p such that

$$p(1) = 1, \tag{7.6}$$

$$|p(t)| \leq \frac{\epsilon}{2}, \quad t \in \left\{0, \frac{1}{r}, \frac{2}{r}, \dots, \frac{r-1}{r}\right\}, \tag{7.7}$$

$$|p(t)| \leq 1, \quad t \in [0, 1], \tag{7.8}$$

$$\deg p = O\left(\sqrt{r \log \frac{1}{\epsilon}}\right). \tag{7.9}$$

Now define $\widetilde{\text{SURJ}}_{n,r}: \{0, 1\}_{\leq n}^{nr} \rightarrow \mathbb{R}$ by

$$\widetilde{\text{SURJ}}_{n,r}(x) = p\left(\frac{1}{r} \sum_{j=1}^r \bigvee_{i=1}^n x_{i,j}\right).$$

This function clearly approximates $\text{SURJ}_{n,r}$ pointwise to $\epsilon/2$. It follows that for any d ,

$$\begin{aligned} E(\text{SURJ}_{n,r}, d) &\leq \|\text{SURJ}_{n,r} - \widetilde{\text{SURJ}}_{n,r}\|_{\infty} + E(\widetilde{\text{SURJ}}_{n,r}, d) \\ &\leq \frac{\epsilon}{2} + E(\widetilde{\text{SURJ}}_{n,r}, d). \end{aligned} \tag{7.10}$$

We have

$$\begin{aligned} \Pi(\widetilde{\text{SURJ}}_{n,r}) &\leq \max \left\{ 1, \Pi\left(\frac{1}{r} \sum_{j=1}^r \bigvee_{i=1}^n x_{i,j}\right) \right\}^{\deg p} \|p\| \\ &\leq 2^{\deg p} \|p\| \\ &\leq 16^{\deg p} \\ &\leq 2^{O(\sqrt{r \log(1/\epsilon)})}, \end{aligned} \tag{7.11}$$

where the first and second steps use Proposition 2.12 (vii), (vi); the third step follows from (7.8) and Lemma 2.8; and the final step is valid by (7.9).

To restate (7.11), we have shown that $\widetilde{\text{SURJ}}_{n,r}$ is a linear combination of conjunctions with real coefficients whose absolute values sum to $\exp(O(\sqrt{r \log(1/\epsilon)}))$. By Corollary 4.12, each of these conjunctions can be approximated by a polynomial of degree d to within $2^{-\Theta(d^2/n)}$ pointwise. We conclude that

$$E(\widetilde{\text{SURJ}}_{n,r}, d) \leq 2^{O(\sqrt{r \log(1/\epsilon)})} \cdot 2^{-\Theta(d^2/n)},$$

which along with (7.10) gives

$$E(\text{SURJ}_{n,r}, d) \leq \frac{\epsilon}{2} + 2^{O(\sqrt{r \log(1/\epsilon)})} \cdot 2^{-\Theta(d^2/n)}.$$

Now (7.4) follows by taking

$$d = \Theta \left(\sqrt{n} \left(r \log \frac{1}{\epsilon} \right)^{1/4} + \sqrt{n \log \frac{1}{\epsilon}} \right).$$

Finally, the approximating polynomial in question is given explicitly because every stage of our proof, including the appeal to Theorem 4.5 and Corollary 4.12, is constructive. \square

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