A STABILIZER FREE WEAK GALERKIN METHOD FOR THE BIHARMONIC EQUATION ON POLYTOPAL MESHES

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Abstract. A new stabilizer free weak Galerkin (WG) method is introduced and analyzed for the biharmonic equation. Stabilizing/penalty terms are often necessary in the finite element formulations with discontinuous approximations to ensure the stability of the methods. Removal of stabilizers will simplify finite element formulations and reduce programming complexity. This stabilizer free WG method has an ultra simple formulation and can work on general partitions with polygons/polyhedra. Optimal order error estimates in a discrete H^2 for $k \geq 2$ and in L^2 norm for $k > 2$ are established for the corresponding weak Galerkin finite element solutions. Numerical results are provided to confirm the theories.

Key words. weak Galerkin, finite element methods, weak Laplacian, biharmonic equations, polytopal meshes

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1. Introduction.

We consider the biharmonic equation of the form

$$\Delta^2 u = f \quad \text{in } \Omega, \tag{1.1}$$

$$u = g \quad \text{on } \partial\Omega, \tag{1.2}$$

$$\frac{\partial u}{\partial n} = \phi \quad \text{on } \partial\Omega, \tag{1.3}$$

where Ω is a bounded polytopal domain in $\mathbb{R}^d$.

For the biharmonic problem (1.1) with Dirichlet and Neumann boundary conditions (1.2) and (1.3), the corresponding weak form is given by seeking $u \in H^2(\Omega)$ satisfying $u|_{\partial\Omega} = g$ and $\frac{\partial u}{\partial n}|_{\partial\Omega} = \phi$ such that

$$(\Delta u, \Delta v) = (f, v) \qquad \forall v \in H_0^2(\Omega), \tag{1.4}$$

where $H_0^2(\Omega)$ is the subspace of $H^2(\Omega)$ consisting of functions with vanishing value and normal derivative on $\partial\Omega$.

It is known that H^2-conforming methods require C^1-continuous piecewise polynomials on a simplicial meshes, which imposes difficulty in practical computation. Due to the complexity in the construction of C^1-continuous elements, H^2-conforming finite element methods are rarely used in practice for solving the biharmonic equation.

As an alternative approach, nonconforming and discontinuous finite element methods have been developed for solving the biharmonic equation over the last several decades. The Morley element [2] is a well-known example of nonconforming

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element for the biharmonic equation by using piecewise quadratic polynomials. The weak Galerkin finite element methods use discontinuous approximations on general polytopal meshes introduced first in [9]. Many WG finite element methods have been developed for forth order problems [3, 4, 5, 6, 7, 10, 12]. These weak Galerkin finite element methods for (1.1)-(1.3) have the following symmetric, positive definite and parameter independent formulation:

$$(\Delta_w u_h, \Delta_w v) + s(u_h, v) = (f, v). \tag{1.5}$$

The stabilizer $s(\cdot,\cdot)$ in (1.5) is necessary to guarantee the well posedness and the convergence of the methods.

The purpose of the work is to further simplify the WG formulation (1.5) by removing the stabilizer to obtain an ultra simple formulation for the biharmonic equation:

$$(\Delta_w u_h, \Delta_w v) = (f, v). \tag{1.6}$$

We can obtain a stabilizer free WG method (1.6) by appropriately designing the weak Laplacian Δ_w. The idea is to raise the degree of polynomials used to compute weak Laplacian Δ_w. Using higher degree polynomials in computation of weak Laplacian will not change the size, neither the global sparsity of the stiffness matrix.

This new stabilizer free WG method for the forth order problem (1.2)-(1.3) has an ultra simple symmetric positive definite formulation (1.6) and can work on general polytopal meshes. For second order elliptic problems, stabilizer free WG methods have been studied in [1, 9, 11]. However for forth order problems, to the best of our knowledge, this is the first finite element method without any stabilizers for totally discontinuous approximations. Optimal order error estimates in a discrete H^2 norm is established for the corresponding WG finite element solutions. Error estimates in the L^2 norm are also derived with a sub-optimal order of convergence for the lowest order element and an optimal order of convergence for all high order of elements. Numerical results are presented to confirm the theory of convergence.

2. Weak Galerkin Finite Element Methods. Let $\mathcal{T}_h$ be a partition of the domain Ω consisting of polygons in two dimension or polyhedra in three dimension satisfying a set of conditions defined in [8] and additional conditions specified in Lemma 3.1 and Lemma 3.2. Denote by $\mathcal{E}_h$ the set of all edges or flat faces in $\mathcal{T}_h$, and let $\mathcal{E}_h^0 = \mathcal{E}_h \backslash \partial\Omega$ be the set of all interior edges or flat faces.

For simplicity, we adopt the following notations,

$$(v, w)_{\mathcal{T}_h} = \sum_{T\in\mathcal{T}_h} (v, w)_T = \sum_{T\in\mathcal{T}_h} \int_T vw d\mathbf{x},$$
$$\langle v, w\rangle_{\partial\mathcal{T}_h} = \sum_{T\in\mathcal{T}_h} \langle v, w\rangle_{\partial T} = \sum_{T\in\mathcal{T}_h} \int_{\partial T} vw ds.$$

Let $P_k(K)$ consist all the polynomials degree less or equal to k defined on K.

First we introduce a set of normal directions on $\mathcal{E}_h$ as follows

$$\mathcal{D}_h = \{\mathbf{n}_e : \ \mathbf{n}_e \text{ is unit and normal to } e, \ e \in \mathcal{E}_h\}. \tag{2.1}$$

Then, we can define a weak Galerkin finite element space V_h for $k \geq 2$ as follows

$$V_h = \{v = \{v_0, v_b, v_n\mathbf{n}_e\} : \ v_0 \in P_k(T), v_b \in P_k(e), v_n \in P_{k-1}(e), e \subset \partial T\}, \tag{2.2}$$

where v_n can be viewed as an approximation of $\nabla v_0 \cdot \mathbf{n}_e$.

Denote by V_h^0 a subspace of V_h with vanishing traces,

$$V_h^0 = \{v = \{v_0, v_b, v_n \mathbf{n}_e\} \in V_h, v_b|_e = 0,\ v_n \mathbf{n}_e \cdot \mathbf{n}|_e = 0,\ e \subset \partial T \cap \partial \Omega\}.$$

A weak Laplacian operator, denoted by Δ_w, is defined as the unique polynomial $\Delta_w v \in P_j(T)$ for $j > k$ that satisfies the following equation

$$(\Delta_w v,\ \varphi)_T = (v_0,\ \Delta\varphi)_T - \langle v_b,\ \nabla\varphi \cdot \mathbf{n}\rangle_{\partial T} + \langle v_n \mathbf{n}_e \cdot \mathbf{n},\ \varphi\rangle_{\partial T}, \quad \forall \varphi \in P_j(T). \tag{2.3}$$

Let Q_0, Q_b and Q_n be the locally defined L^2 projections onto $P_k(T)$, $P_k(e)$ and $P_{k-1}(e)$ accordingly on each element $T \in \mathcal{T}_h$ and $e \subset \partial T$. For the true solution u of (1.1)-(1.3), we define $Q_h u$ as

$$Q_h u = \{Q_0 u, Q_b u, Q_n(\nabla u \cdot \mathbf{n}_e)\mathbf{n}_e\} \in V_h.$$

Weak Galerkin Algorithm 1. *A numerical approximation for (1.1)-(1.3) can be obtained by seeking $u_h = \{u_0,\ u_b,\ u_n \mathbf{n}_e\} \in V_h$ satisfying $u_b = Q_b g$ and $u_n \mathbf{n}_e \cdot \mathbf{n} = Q_n \phi$ on $\partial\Omega$ and the following equation:*

$$(\Delta_w u_h,\ \Delta_w v)_{\mathcal{T}_h} = (f,\ v_0) \quad \forall\ v = \{v_0,\ v_b,\ v_n \mathbf{n}_e\} \in V_h^0. \tag{2.4}$$

Lemma 2.1. *Let $\phi \in H^2(\Omega)$, then on any $T \in \mathcal{T}_h$,*

$$\Delta_w \phi = \mathbb{Q}_h(\Delta\phi), \tag{2.5}$$

where $\mathbb{Q}_h$ is a locally defined L^2 projections onto $P_j(T)$ on each element $T \in \mathcal{T}_h$.

Proof. It is not hard to see that for any $\tau \in P_j(T)$ we have

$$\begin{aligned}(\Delta_w \phi,\ \tau)_T &= (\phi,\ \Delta\tau)_T + \langle(\nabla\phi \cdot \mathbf{n}_e)\mathbf{n}_e \cdot \mathbf{n},\ \tau\rangle_{\partial T} - \langle\phi,\ \nabla\tau \cdot \mathbf{n}\rangle_{\partial T} \\ &= (\phi, \Delta\tau)_T + \langle\nabla\phi \cdot \mathbf{n},\ \tau\rangle_{\partial T} - \langle\phi,\ \nabla\tau \cdot \mathbf{n}\rangle_{\partial T} \\ &= (\Delta\phi,\ \tau)_T = (\mathbb{Q}_h \Delta\phi,\ \tau)_T,\end{aligned}$$

which implies

$$\Delta_w \phi = \mathbb{Q}_h(\Delta\phi). \tag{2.6}$$

It completes the proof. □

3. Well Posedness. For any $v \in V_h + H^2(\Omega)$, let

$$|||v|||^2 = (\Delta_w v, \Delta_w v)_{\mathcal{T}_h}. \tag{3.1}$$

We introduce a discrete H^2 norm as follows:

$$\|v\|_{2,h} = \left(\sum_{T \in \mathcal{T}_h} \left(\|\Delta v_0\|_T^2 + h_T^{-3}\|v_0 - v_b\|_{\partial T}^2 + h_T^{-1}\|(\nabla v_0 - v_n \mathbf{n}_e) \cdot \mathbf{n}\|_{\partial T}^2\right)\right)^{\frac{1}{2}}. \tag{3.2}$$

For any function $\varphi \in H^1(T)$, the following trace inequality holds true [8],

$$\|\varphi\|_e^2 \le C\left(h_T^{-1}\|\varphi\|_T^2 + h_T\|\nabla\varphi\|_T^2\right). \tag{3.3}$$

The main goal of this section is to obtain the equivalence of the two norms $\|\cdot\|_{2,h}$ and $|\!|\!|\cdot|\!|\!|$. To do so, we need the two following lemmas.

LEMMA 3.1. *Let T be a convex polygon/polyhedron of size h_T with edges/faces e_1, e_2, $\dots e_n$. Let $\lambda_1 \in P_1(T)$, $\lambda_1|e_1 = 0$ and $\max_T \lambda_1 = 1$. Let $\lambda_i \in P_1(T)$, $i > 1$, $\lambda_i|e_i = 0$ and $\lambda_i(\mathbf{m}_1) = 1$ where $\mathbf{m}_1$ is the barycenter of e_1. For any $f \in P_k(e_1)$, there is a unique polynomial $q = \lambda_1\lambda_2^2\cdots\lambda_n^2 q_k$ for some $q_k \in P_k(T)$ such that*

$$(q,p)_T = 0 \qquad \forall p \in P_{k-1}(T), \tag{3.4}$$

$$\langle \nabla q\cdot\mathbf{n} - f, p\rangle_{e_1} = 0 \qquad \forall p \in P_k(e_1), \tag{3.5}$$

$$\|q\|_T \le C h_T^{3/2}\|f\|_{e_1}, \tag{3.6}$$

where C depends on the minimum angle and the smallest ratio h_{e_i}/h_T, and C is defined in (3.14) *below.*

Proof. We prove q is uniquely defined by (3.4)–(3.5). Let $f = 0$ in (3.5). As T is convex, $\lambda_i > 0$ in the interior of e_1 for all $i > 1$. Because of the positive weight, the vanishing weighted $L^2(e_1)$ inner-product forces $\nabla q\cdot\mathbf{n} = 0$ on e_1:

$$\langle \nabla q\cdot\mathbf{n}, p\rangle_{e_1} = -\frac{1}{h_T}\langle \lambda_2^2\cdots\lambda_n^2 q_k, p\rangle_{e_1} = 0 \quad \forall p \in P_k(e_1). \tag{3.7}$$

Thus the vanishing weighted $L^2(T)$ inner-product forces $q = 0$ on T:

$$(q,p)T = (\lambda_1^2\cdots\lambda_n^2 q_{k-1}, p)_T = 0 \quad \forall p \in P_{k-1}(T), \tag{3.8}$$

where $q_{k-1}\lambda_1 = q_k$.

We find some upper bounds and lower bounds of these weight functions λ_i.

Let e_i, $1 < i \le m$ ($m = 3$ in 2D), be a neighboring edge/face of e_1. Using the distance as its variable, we have

$$\lambda_i|_{e_1} = \frac{2}{h_{e_1}}x,$$

where x is the distance, along e_1, of the point to the intersection of e_1 and e_i. Here in 3D, we assume the size of e_1 is roughly twice the distance from the barycenter $\mathbf{m}_1$ to the intersection edge $e_1 \cap e_i$. To avoid too many constants, we simply assume reasonably $h_{e_i} \ge h_T/4$. We compute the maximum as

$$\max_T \lambda_i = \frac{h_{\perp e_i}(T)}{(h_{e_1}/2)\sin\alpha_i} \le \frac{2h_T}{h_{e_1}\sin\alpha_i} \le \frac{8}{\sin\alpha_i} \le \frac{8}{\sin\alpha_0}, \tag{3.9}$$

where $\pi - \alpha_i$ ($\alpha_i \ge \alpha_0 > 0$, and $\alpha_i \le \pi - \alpha_0$) is the angle between e_1 and e_i, and $h_{\perp e_i}(T)$ is the maximal distance of a point on T to e_i. For a lower bound, we have

$$\lambda_i|_{T_0} \ge \begin{cases} \frac{15}{16} & \text{if } \alpha_i \le \pi/2, \\ 1 - \frac{\sqrt{d}}{16\sin\alpha_i} \ge \frac{1}{2} & \text{if } \alpha_i > \pi/2, \end{cases} \tag{3.10}$$

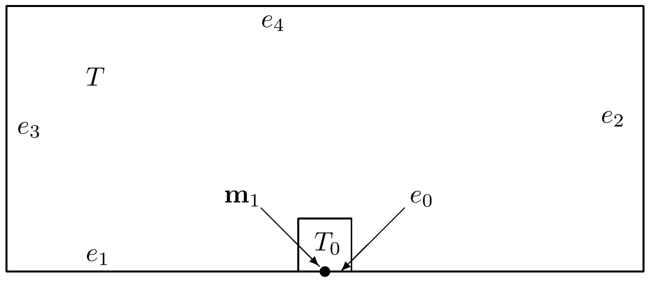

FIG. 3.1. *Size $h_{e_0} = h_{e_1}/16$, and T_0 is square/cube of size h_{e_0} at $\mathbf{m}_1$.*

where T_0 is a square/cube at middle of e_1 with size $h_{e_1}/16$, cf. Figure 3. We note that other than triangles, $\alpha_i \le \pi/2$ for most other polygons. Here in (3.10), we assumed $\sin\alpha_0 \ge \sqrt{d}/8$, where d is the space dimension, 2 or 3.

For non-neighboring edges e_j, we have

$$\lambda_j|_{e_1} = \begin{cases} 1 & \text{if } e_j \parallel e_1, \\ \frac{2(x+x_j)}{h_{e_1}+x_j} & \text{otherwise,} \end{cases}$$

where x is the arc-length parametrization on e_1 toward the extended intersection of e_1 and e_i, x_j is the distance on e_1 from the an boundary point of e_1 to the intersection. Supposing e_i is the only edge/polygonal between e_1 and e_j, $x_j = h_{e_i}(\cos\alpha_i - \cos(\alpha_i + \alpha_j))$. Because $x_j \ge 0$, it follows that

$$\max_T \lambda_j = \frac{h_{\perp e_j}(T)}{(h_{e_1}/2)\sin\alpha_i} \le \frac{2h_T}{(h_{e_1}+x_j)\sin(\alpha_i+\alpha_j)} \le \frac{8}{\sin\alpha_0}. \tag{3.11}$$

For a lower bound, because $x_j > 0$ and e_i is an edge/polygon in between, we have

$$\lambda_j|_{T_0} \ge \lambda_i|_{T_0} \ge \frac{1}{2}. \tag{3.12}$$

Together, in (3.7) and (3.8), we have, noting $\lambda_1|_T \le 1$,

$$\lambda_2^2 \cdots \lambda_n^2|_{T_0} \ge \frac{1}{2^{2n-2}}, \quad \text{and} \quad \lambda_1^2\lambda_2^2 \cdots \lambda_n^2|_T \le \frac{8^{2n-2}}{\sin^{2n-2}\alpha_0}. \tag{3.13}$$

Let $\tilde{q}_k \in P_k(e_1)$ be the unique solution in (3.5), i.e., $\tilde{q}_k = q_k|_{e_1}$. Letting $p = -h_T\tilde{q}_k$ in (3.5), cf. (3.7), we get, by (3.13),

$$\begin{aligned} \frac{1}{16^{2k}}\frac{1}{2^{2n-2}}\|\tilde{q}_k\|_{e_1}^2 &\le \frac{1}{2^{2n-2}}\|\tilde{q}_k\|_{e_0}^2 \le \|\lambda_2\cdots\lambda_n\tilde{q}_k\|_{e_0}^2 \\ &\le \|\lambda_2\cdots\lambda_n\tilde{q}_k\|_{e_1}^2 = \langle\lambda_2^2\cdots\lambda_n^2\tilde{q}_k, \tilde{q}_k\rangle_{e_1} \\ &= -\frac{1}{h_T}\langle\lambda_2^2\cdots\lambda_n^2 q_k, p\rangle_{e_1} = \langle f, p\rangle_{e_1} = \langle f, -h_T\tilde{q}_k\rangle_{e_1} \\ &\le h_T\|f\|_{e_1}\|\tilde{q}_k\|_{e_1}, \end{aligned}$$

where in the first step we use the fact $\tilde{q}_k$ is a degree k polynomial. For the unique solution $\tilde{q}_k \in P_k(e_1)$, we view it as a polynomial on the whole line or whole plane containing e_1. We also extend it to $P_k(\mathbb{R}^d)$ by letting it be constant in the direction orthogonal to e_1. Let S_T be a square/cube of size h_T containing T, with one side S_{e_1} which contains e_1. It follows that, by (3.13),

$$\begin{aligned}\|\tilde{q}_k\|_T^2 &\le \|\tilde{q}_k\|_{S_T}^2 = h_T\|\tilde{q}_k\|_{S_{e_1}}^2 \le h_T(\frac{h_T}{h_{e_1}})^{2k}\|\tilde{q}_k\|_{e_1}^2 \\ &\le h_T 4^{2k}\|\tilde{q}_k\|_{e_1}^2 \le h_T 4^{2k}(2^{8k+2n-2}h_T\|f\|_{e_1})^2 \\ &= 2^{20k+4n-4}h_T^3\|f\|_{e_1}^2.\end{aligned}$$

We rewrite q in terms of this extended polynomial,

$$q = \lambda_1\lambda_2^2\cdots\lambda_n^2 q_k = \lambda_1\lambda_2^2\cdots\lambda_n^2(\lambda_1 q_{k-1}+\tilde{q}_k) \quad \text{for some} \quad q_{k-1}\in P_{k-1}(T).$$

Letting $p = q_{k-1}$ in (3.4), it follows that, by (3.13),

$$\begin{aligned}\|q_{k-1}\|_T^2 &\le (h_T/h_{e_0})^{2k-2}\|q_{k-1}\|_{T_0}^2 \le 64^{2k-2}2^{2n-2}(\lambda_2^2\cdots\lambda_n^2 q_{k-1}, q_{k-1})_{T_0} \\ &\le 2^{2n+12k-14}(2h_T/h_{e_0})^2(\lambda_1^2\cdots\lambda_n^2 q_{k-1}, q_{k-1})_{T_{0,0}} \\ &\le 2^{2n+12k}(2h_T/h_{e_0})^{2n+2k-2}(\lambda_1^2\cdots\lambda_n^2 q_{k-1}, q_{k-1})_T \\ &= 2^{16n+26k-14}(\lambda_1\lambda_2^2\cdots\lambda_n^2\tilde{q}_k, -q_{k-1})_T \\ &\le 2^{16n+26k-14}\frac{8^{2n-2}}{\sin^{2n-2}\alpha_0}\|\tilde{q}_k\|_T\|q_{k-1}\|_T,\end{aligned}$$

where $T_{0,0}$ is the top half of T_0, and we used the fact $\max_T \lambda_1 = 1$ and the fact that the integrant on $T_{0,0}$ is a degree $2n+2k-2$ polynomial. We estimate

$$\begin{aligned}\|q\|_T^2 &= (\lambda_1^2\lambda_2^4\cdots\lambda_n^4(\lambda_1 q_{k-1}+\tilde{q}_k), \lambda_1 q_{k-1}+\tilde{q}_k)_T \\ &\le \frac{8^{4n-4}}{\sin^{4n-4}\alpha_0}(\lambda_1 q_{k-1}+\tilde{q}_k, \lambda_1 q_{k-1}+\tilde{q}_k)_T \\ &\le \frac{8^{4n-4}}{\sin^{4n-4}\alpha_0}2(\|\lambda_1 q_{k-1}\|_T^2+\|\tilde{q}_k\|_T^2) \\ &\le \frac{2^{12n-11}}{\sin^{4n-4}\alpha_0}(\|q_{k-1}\|_T^2+\|\tilde{q}_k\|_T^2).\end{aligned}$$

Combining above three bounds, we get

$$\begin{aligned}\|q\|_T &\le \Big(\frac{2^{12n-11}}{\sin^{4n-4}\alpha_0}\Big)^{\frac12}\Big((\frac{2^{22n+26k-20}}{\sin^{2n-2}\alpha_0})^2+1\Big)^{\frac12}\|\tilde{q}_k\|_T \\ &\le \Big(\frac{2^{16n+20k-15}}{\sin^{4n-4}\alpha_0}\Big)^{\frac12}\Big((\frac{2^{22n+26k-20}}{\sin^{2n-2}\alpha_0})^2+1\Big)^{\frac12}h_T^{3/2}\|f\|_{e_1} =: Ch_T^{3/2}\|f\|_{e_1}.\end{aligned} \tag{3.14}$$

The proof is completed. ∎

Lemma 3.2. *Let the following notations be defined in Lemma 3.1. For any $g\in P_{k-1}(e_1)$, there is a unique polynomial $q=\lambda_2^2\cdots\lambda_n^2 q_{k+1}$ for some $q_{k+1}\in P_{k+1}(T)$ such that*

$$(q,p)_T = 0 \qquad \forall p\in P_{k-1}(T), \tag{3.15}$$

$$\langle\nabla q\cdot\mathbf{n}, p\rangle_{e_1} = 0 \qquad \forall p\in P_k(e_1), \tag{3.16}$$

$$\langle q-g, p\rangle_{e_1} = 0 \qquad \forall p\in P_{k+1}(e_1), \tag{3.17}$$

$$\|q\|_T \le Ch_T^{1/2}\|g\|_{e_1}, \tag{3.18}$$

where C depends on the minimum angle and the smallest ratio h_{e_i}/h_T, and C is defined in (3.19) *below.*

Proof. For unisovence, letting $g = 0$ in (3.17), we get $q = \lambda_1\lambda_2^2\cdots\lambda_n^2 q_k$ for some $q_k \in P_k(e_1)$, because the weights $\lambda_i > 0$. By (3.16), $\nabla q \cdot \mathbf{n}|_{e_1} = 0$ and thus $q = \lambda_1^2\lambda_2^2\cdots\lambda_n^2 q_{k-1}$ for some $q_{k-1} \in P_{k-1}(T)$. By (3.15), $q_{k-1}|_T = 0$ and thus $q = 0$.

The upper and lower bounds for λ_i are same as that in Lemma 3.1.

Let $\tilde{q}_{k+1} \in P_{k+1}(e_1)$ be the unique solution in (3.17), i.e., $q|_{e_1} = \lambda_2^2\cdots\lambda_n^2\tilde{q}_{k+1}$. Letting $p = \tilde{q}_{k+1}$ in (3.17), we get, by (3.13),

$$\begin{aligned}\frac{1}{16^{2k+2}}\frac{1}{2^{2n-2}}\|\tilde{q}_{k+1}\|_{e_1}^2 &\le \frac{1}{2^{2n-2}}\|\tilde{q}_{k+1}\|_{e_0}^2 \le \|\lambda_2\cdots\lambda_n\tilde{q}_{k+1}\|_{e_0}^2\\ &\le \|\lambda_2\cdots\lambda_n\tilde{q}_{k+1}\|_{e_1}^2 = \langle\lambda_2^2\cdots\lambda_n^2\tilde{q}_{k+1}, \tilde{q}_{k+1}\rangle_{e_1}\\ &= \langle g, \tilde{q}_{k+1}\rangle_{e_1} \le \|g\|_{e_1}\|\tilde{q}_{k+1}\|_{e_1},\end{aligned}$$

where in the first step we use the fact $\tilde{q}_{k+1}$ is a degree $k+1$ polynomial. For the unique solution $\tilde{q}_{k+1} \in P_{k+1}(e_1)$, we view it as a polynomial on the whole line or whole plane containing e_1. We also extend it to $P_{k+1}(\mathbb{R}^d)$ by letting it be constant in the direction orthogonal to e_1. Let S_T be a square/cube of size h_T containing T, with one side S_{e_1} which contains e_1. It follows that, by (3.13),

$$\begin{aligned}\|\tilde{q}_{k+1}\|_T^2 &\le \|\tilde{q}_{k+1}\|_{S_T}^2 = h_T\|\tilde{q}_{k+1}\|_{S_{e_1}}^2 \le h_T(\frac{h_T}{h_{e_1}})^{2k+2}\|\tilde{q}_{k+1}\|_{e_1}^2\\ &\le h_T 4^{2k+2}\|\tilde{q}_{k+1}\|_{e_1}^2 \le h_T 4^{2k+2}(2^{8k+2n+6}\|g\|_{e_1})^2\\ &= 2^{20k+4n+16}h_T\|g\|_{e_1}^2.\end{aligned}$$

We rewrite q in terms of this extended polynomial,

$$q = \lambda_2^2\cdots\lambda_n^2 q_{k+1} = \lambda_2^2\cdots\lambda_n^2(\lambda_1 q_k + \tilde{q}_{k+1}) \quad \text{for some} \quad q_k \in P_k(T).$$

By (3.16), we have further, because $\nabla\tilde{q}_{k+1}\cdot\mathbf{n}|_{e_1} = 0$,

$$q = \lambda_2^2\cdots\lambda_n^2(\lambda_1^2 q_{k-1} + \tilde{q}_{k+1}) \quad \text{for some} \quad q_{k-1} \in P_{k-1}(T).$$

Letting $p = q_{k-1}$ in (3.15), it follows that, by (3.13),

$$\begin{aligned}\|q_{k-1}\|_T^2 &\le (h_T/h_{e_0})^{2k-2}\|q_{k-1}\|_{T_0}^2 \le 64^{2k-2}2^{2n-2}(\lambda_2^2\cdots\lambda_n^2 q_{k-1}, q_{k-1})_{T_0}\\ &\le 2^{2n+12k-14}(2h_T/h_{e_0})^2(\lambda_1^2\cdots\lambda_n^2 q_{k-1}, q_{k-1})_{T_{0,0}}\\ &\le 2^{2n+12k}(2h_T/h_{e_0})^{2n+2k-2}(\lambda_1^2\cdots\lambda_n^2 q_{k-1}, q_{k-1})_T\\ &= 2^{16n+26k-14}(\lambda_2^2\cdots\lambda_n^2\tilde{q}_{k+1}, -q_{k-1})_T\\ &\le 2^{16n+26k-14}\frac{8^{2n-2}}{\sin^{2n-2}\alpha_0}\|\tilde{q}_{k+1}\|_T\|q_{k-1}\|_T,\end{aligned}$$

where $T_{0,0}$ is the top half of T_0, and we used the fact $\max_T \lambda_1 = 1$ and the fact that the integrant on $T_{0,0}$ is a degree $2n+2k-2$ polynomial. We estimate

$$\begin{aligned}
\|q\|_T^2 &= (\lambda_2^4\cdots\lambda_n^4(\lambda_1^2 q_{k-1}+\tilde q_{k+1}), \lambda_1^2 q_{k-1}+\tilde q_{k+1})_T\\
&\le \frac{8^{4n-4}}{\sin^{4n-4}\alpha_0}(\lambda_1^2 q_{k-1}+\tilde q_{k+1}, \lambda_1^2 q_{k-1}+\tilde q_{k+1})_T\\
&\le \frac{8^{4n-4}}{\sin^{4n-4}\alpha_0}2(\|\lambda_1^2 q_{k-1}\|_T^2+\|\tilde q_{k+1}\|_T^2)\\
&\le \frac{2^{12n-11}}{\sin^{4n-4}\alpha_0}(\|q_{k-1}\|_T^2+\|\tilde q_{k+1}\|_T^2).
\end{aligned}$$

Combining above three bounds, we get

$$\begin{aligned}
\|q\|_T &\le \Big(\frac{2^{12n-11}}{\sin^{4n-4}\alpha_0}\Big)^{\frac12}\Big((\frac{2^{16n+26k-14}}{\sin^{2n-2}\alpha_0})^2+1\Big)^{\frac12}\|\tilde q_k\|_T\\
&\le \Big(\frac{2^{16n+20k+5}}{\sin^{4n-4}\alpha_0}\Big)^{\frac12}\Big((\frac{2^{16n+26k-14}}{\sin^{2n-2}\alpha_0})^2+1\Big)^{\frac12}h_T\|g\|_{e_1} =: Ch_T\|g\|_{e_1}.
\end{aligned} \tag{3.19}$$

The proof is completed. □

Lemma 3.3. *There exist two positive constants C_1 and C_2 such that for any $v=\{v_0,v_b,v_n\mathbf{n}_e\}\in V_h$, we have*

$$C_1\|v\|_{2,h} \le |\!|\!|v|\!|\!| \le C_2\|v\|_{2,h}. \tag{3.20}$$

Proof. For any $v=\{v_0,v_b,v_n\mathbf{n}_e\}\in V_h$, it follows from the definition of weak Laplacian (2.3) and integration by parts that

$$\begin{aligned}
(\Delta_w v,\ \varphi)_T &= (v_0,\ \Delta\varphi)_T - \langle v_b,\ \nabla\varphi\cdot\mathbf{n}\rangle_{\partial T} + \langle v_n\mathbf{n}_e\cdot\mathbf{n},\ \varphi\rangle_{\partial T}\\
&= -(\nabla v_0,\ \nabla\varphi)_T + \langle v_0-v_b,\ \nabla\varphi\cdot\mathbf{n}\rangle_{\partial T} + \langle v_n\mathbf{n}_e\cdot\mathbf{n},\ \varphi\rangle_{\partial T}\\
&= (\Delta v_0,\ \varphi)_T + \langle v_0-v_b,\ \nabla\varphi\cdot\mathbf{n}\rangle_{\partial T} + \langle (v_n\mathbf{n}_e-\nabla v_0)\cdot\mathbf{n},\ \varphi\rangle_{\partial T}.
\end{aligned} \tag{3.21}$$

By letting $\varphi=\Delta_w v$ in (3.21) we arrive at

$$\|\Delta_w v\|_T^2 = (\Delta v_0,\ \Delta_w v)_T + \langle v_0-v_b,\ \nabla(\Delta_w v)\cdot\mathbf{n}\rangle_{\partial T} + \langle (v_n\mathbf{n}_e-\nabla v_0)\cdot\mathbf{n},\ \Delta_w v\rangle_{\partial T}$$

From the trace inequality (3.3) and the inverse inequality we have

$$\begin{aligned}
\|\Delta_w v\|_T^2 &\le \|\Delta v_0\|_T\|\Delta_w v\|_T + \|v_0-v_b\|_{\partial T}\|\nabla(\Delta_w v)\|_{\partial T}\\
&\quad + \|(v_n\mathbf{n}_e-\nabla v_0)\cdot\mathbf{n}\|_{\partial T}\|\Delta_w v\|_{\partial T}\\
&\le C(\|\Delta v_0\|_T + h_T^{-3/2}\|v_0-v_b\|_{\partial T}\\
&\quad + h_T^{-1/2}\|(v_n\mathbf{n}_e-\nabla v_0)\cdot\mathbf{n}\|_{\partial T})\|\Delta_w v\|_T,
\end{aligned}$$

which implies

$$\|\Delta_w v\|_T \le C\left(\|\Delta v_0\|_T + h_T^{-3/2}\|v_0-v_b\|_{\partial T} + h_T^{-1/2}\|(v_n\mathbf{n}_e-\nabla v_0)\cdot\mathbf{n}\|_{\partial T}\right),$$

and consequently

$$|\!|\!|v|\!|\!| \le C_2\|v\|_{2,h}.$$

Next we will prove

$$\sum_{T\in\mathcal{T}_h} h_T^{-3}\|v_0-v_b\|_{\partial T}^2 \le C|\!|\!|v|\!|\!|^2.$$

It follows from (3.21) that for any $\varphi\in P_j(T)$,

$$\begin{aligned}(\Delta_w v,\ \varphi)_T &= (\Delta v_0,\ \varphi)_T + \langle v_0-v_b,\ \nabla\varphi\cdot\mathbf{n}\rangle_{\partial T}\\ &\quad + \langle (v_n\mathbf{n}_e-\nabla v_0)\cdot\mathbf{n},\ \varphi\rangle_{\partial T}.\end{aligned} \tag{3.22}$$

By Lemma 3.1, there exist a φ_0 such that for $e\subset\partial T$,

$$\begin{aligned}&(\Delta v_0,\varphi_0)_T=0,\ \ \langle (v_n\mathbf{n}_e-\nabla v_0)\cdot\mathbf{n},\ \varphi_0\rangle_{\partial T}=0,\\ &\langle v_0-v_b,\nabla\varphi_0\cdot\mathbf{n}\rangle_{\partial T\setminus e}=0,\ \ \langle v_0-v_b,\nabla\varphi_0\cdot\mathbf{n}\rangle_{\partial T}=\|v_0-v_b\|_e^2.\end{aligned} \tag{3.23}$$

and

$$\|\varphi_0\|_T\le Ch_T^{3/2}\|v_0-v_b\|_e. \tag{3.24}$$

Letting $\varphi=\varphi_0$ in (3.22) yields

$$\|v_0-v_b\|_e^2=(\Delta_w v,\ \varphi_0)_T\le\|\Delta_w v\|_T\|\varphi_0\|_T\le Ch_T^{3/2}\|\Delta_w v\|_T\|v_0-v_b\|_e, \tag{3.25}$$

which implies

$$\sum_{T\in\mathcal{T}_h} h_T^{-3}\|v_0-v_b\|_{\partial T}^2 \le C|\!|\!|v|\!|\!|^2. \tag{3.26}$$

Similarly, by Lemma 3.2, we can have

$$\sum_{T\in\mathcal{T}_h} h_T^{-1}\|(\nabla v_0-v_n\mathbf{n}_e)\cdot\mathbf{n}\|_{\partial T}^2 \le C|\!|\!|v|\!|\!|^2. \tag{3.27}$$

Finally, by letting $\varphi=\Delta_w v$ in (3.22) we arrive at

$$\begin{aligned}\|\Delta v_0\|_T^2 &= (\Delta v_0,\ \Delta_w v)_T-\langle v_0-v_b,\ \nabla(\Delta_w v)\cdot\mathbf{n}\rangle_{\partial T}\\ &\quad -\langle (v_n\mathbf{n}_e-\nabla v_0)\cdot\mathbf{n},\ \Delta_w v\rangle_{\partial T}.\end{aligned}$$

Using the trace inequality (3.3), the inverse inequality and (3.26)-(3.27), one has

$$\|\Delta v_0\|_T^2\le C\|\Delta_w v\|_T\|\Delta v_0\|_T,$$

which gives

$$\sum_{T\in\mathcal{T}_h}\|\Delta v_0\|_T^2\le C|\!|\!|v|\!|\!|^2.$$

We complete the proof. □

LEMMA 3.4. *The weak Galerkin finite element scheme (2.4) has a unique solution.*

Proof. It suffices to show that the solution of (2.4) is trivial if $f=g=\phi=0$. Take $v=u_h$ in (2.4). It follows that

$$(\Delta_w u_h,\Delta_w u_h)_{\mathcal{T}_h}=0.$$

Then the norm equivalence (3.20) implies $\|u_h\|_{2,h}=0$. Consequently, we have $\Delta u_0=0$, $u_0=u_b$, $\nabla u_0\cdot\mathbf{n}_e=u_n$ on ∂T. Thus u_0 is a smooth harmonic function on Ω. The boundary condition of $u_b=0$ implies that $u_0\equiv 0$ on Ω. We have $u_0=0$, then $u_b=u_n=0$, which completes the proof. □

4. An Error Equation. Let $e_h = u - u_h$. The goal of this section is to obtain an error equation that e_h satisfies.

LEMMA 4.1. *For any $v \in V_h^0$, we have*

$$(\Delta_w e_h, \Delta_w v)_{\mathcal{T}_h} = \ell_1(u,v) + \ell_2(u,v), \tag{4.1}$$

where

$$\ell_1(u,v) = \langle \nabla(\mathbb{Q}_h \Delta u - \Delta u)\cdot \mathbf{n}, v_0 - v_b\rangle_{\partial T_h},$$
$$\ell_2(u,v) = \langle \Delta u - \mathbb{Q}_h\Delta u, (\nabla v_0 - v_n \mathbf{n}_e)\cdot \mathbf{n}\rangle_{\partial T}.$$

Proof. For $v = \{v_0, v_b, v_n\mathbf{n}_e\} \in V_h^0$, testing (1.1) by v_0 and using the fact that $\sum_{T\in\mathcal{T}_h}\langle\nabla(\Delta u)\cdot\mathbf{n}, v_b\rangle_{\partial T} = 0$ and $\sum_{T\in\mathcal{T}_h}\langle \Delta u, v_n\mathbf{n}_e\cdot\mathbf{n}\rangle_{\partial T} = 0$ and integration by parts, we arrive at

$$\begin{aligned}(f, v_0) &= (\Delta^2 u, v_0)_{\mathcal{T}_h}\\ &= (\Delta u, \Delta v_0)_{\mathcal{T}_h} - \langle \Delta u, \nabla v_0\cdot\mathbf{n}\rangle_{\partial T_h} + \langle\nabla(\Delta u)\cdot\mathbf{n}, v_0\rangle_{\partial T_h}\\ &= (\Delta u, \Delta v_0)_{\mathcal{T}_h} - \langle \Delta u, (\nabla v_0 - v_n\mathbf{n}_e)\cdot\mathbf{n}\rangle_{\partial T_h}\\ &+ \langle\nabla(\Delta u)\cdot\mathbf{n}, v_0 - v_b\rangle_{\partial T_h}.\end{aligned} \tag{4.2}$$

Next we investigate the term $(\Delta u, \Delta v_0)_{\mathcal{T}_h}$ in the above equation. Using (2.5), integration by parts and the definition of weak Laplacian, we have

$$\begin{aligned}&(\Delta u, \Delta v_0)_{\mathcal{T}_h} = (\mathbb{Q}_h\Delta u, \Delta v_0)_{\mathcal{T}_h}\\ &= (v_0, \Delta(\mathbb{Q}_h\Delta u))_{\mathcal{T}_h} + \langle\nabla v_0\cdot\mathbf{n},\ \mathbb{Q}_h\Delta u\rangle_{\partial T_h} - \langle v_0, \nabla(\mathbb{Q}_h\Delta u)\cdot\mathbf{n}\rangle_{\partial T_h}\\ &= (\Delta_w v,\ \mathbb{Q}_h\Delta u)_{\mathcal{T}_h} - \langle v_0 - v_b, \nabla(\mathbb{Q}_h\Delta u)\cdot\mathbf{n}\rangle_{\partial T_h}\\ &\quad+\langle(\nabla v_0 - v_n\mathbf{n}_e)\cdot\mathbf{n}, \mathbb{Q}_h\Delta u\rangle_{\partial T_h}\\ &= (\Delta_w u,\ \Delta_w v)_{\mathcal{T}_h} - \langle v_0 - v_b, \nabla(\mathbb{Q}_h\Delta u)\cdot\mathbf{n}\rangle_{\partial T_T}\\ &\quad+\langle(\nabla v_0 - v_n\mathbf{n}_e)\cdot\mathbf{n}, \mathbb{Q}_h\Delta u\rangle_{\partial T_h}.\end{aligned}$$

Combining the above equation with (4.2) gives

$$\begin{aligned}(f, v_0) &= (\Delta^2 u, v_0)_{\mathcal{T}_h}\\ &= (\Delta_w u,\ \Delta_w v)_{\mathcal{T}_h} - \langle v_0 - v_b, \nabla(\mathbb{Q}_h\Delta u - \Delta u)\cdot\mathbf{n}\rangle_{\partial T_h}\\ &- \langle(\nabla v_0 - v_n\mathbf{n}_e)\cdot\mathbf{n}, \Delta u - \mathbb{Q}_h\Delta u\rangle_{\partial T}.\end{aligned} \tag{4.3}$$

which implies that

$$(\Delta_w u,\ \Delta_w v)_{\mathcal{T}_h} = (f, v_0) + \ell_1(u,v) + \ell_2(u,v).$$

The error equation follows from subtracting (2.4) from the above equation,

$$(\Delta_w e_h,\ \Delta_w v)_{\mathcal{T}_h} = \ell_1(u,v) + \ell_2(u,v).$$

We have proved the lemma. ▯

5. An Error Estimate in H^2. We will obtain the optimal convergence rate for the solution u_h of the stabilizer free WG method in (2.4) in a discrete H^2 norm.

LEMMA 5.1. *Let $k \geq 2$ and $w \in H^{\max\{k+1,4\}}(\Omega)$. There exists a constant C such that the following estimates hold true:*

$$\left(\sum_{T\in\mathcal{T}_h} h_T\|\Delta w - \mathbb{Q}_h\Delta w\|_{\partial T}^2\right)^{\frac{1}{2}} \leq Ch^{k-1}\|w\|_{k+1}, \tag{5.1}$$

$$\left(\sum_{T\in\mathcal{T}_h} h_T^3\|\nabla(\Delta w - \mathbb{Q}_h\Delta w)\|_{\partial T}^2\right)^{\frac{1}{2}} \leq Ch^{k-1}(\|w\|_{k+1} + h\delta_{k,2}\|w\|_4). \tag{5.2}$$

Here $\delta_{i,j}$ is the usual Kronecker's delta with value 1 when $i = j$ and value 0 otherwise.

The above lemma can be proved by using the trace inequality (3.3) and the definition of $\mathbb{Q}_h$. The proof can also be found in [3].

LEMMA 5.2. *Let $w \in H^{\max\{k+1,4\}}(\Omega)$ for $k \geq 2$ and $v \in V_h$. There exists a constant C such that*

$$|\ell_1(w,v)| \leq Ch^{k-1}\left(\|w\|_{k+1} + h\delta_{k,2}\|w\|_4\right) |||v|||. \tag{5.3}$$

$$|\ell_2(w,v)| \leq Ch^{k-1}\|w\|_{k+1}|||v|||. \tag{5.4}$$

Proof. Using the Cauchy-Schwartz inequality, (5.1)-(5.2) and (3.20), we have

$$\begin{aligned}
\ell_1(w,v) &= \left|\sum_{T\in\mathcal{T}_h}\langle\nabla(\Delta w - \mathbb{Q}_h\Delta w)\cdot\mathbf{n}, v_0 - v_b\rangle_{\partial T}\right| \\
&\leq \left(\sum_{T\in\mathcal{T}_h} h_T^3\|\nabla(\Delta w - \mathbb{Q}_h\Delta w)\|_{\partial T}^2\right)^{\frac{1}{2}}\left(\sum_{T\in\mathcal{T}_h} h_T^{-3}\|v_0 - v_b\|_{\partial T}^2\right)^{\frac{1}{2}} \\
&\leq Ch^{k-1}(\|w\|_{k+1} + h\delta_{k,2}\|w\|_4)|||v|||, \qquad (5.5)
\end{aligned}$$

and

$$\begin{aligned}
\ell_2(w,v) &= \left|\sum_{T\in\mathcal{T}_h}\langle\Delta w - \mathbb{Q}_h\Delta w, (\nabla v_0 - v_n\mathbf{n}_e)\cdot\mathbf{n}\rangle_{\partial T}\right| \\
&\leq \left(\sum_{T\in\mathcal{T}_h} h_T\|\Delta w - \mathbb{Q}_h\Delta w\|_{\partial T}^2\right)^{\frac{1}{2}}\left(\sum_{T\in\mathcal{T}_h} h_T^{-1}\|(\nabla v_0 - v_n\mathbf{n}_e)\cdot\mathbf{n}\|_{\partial T}^2\right)^{\frac{1}{2}} \\
&\leq Ch^{k-1}\|w\|_{k+1}|||v|||. \qquad (5.6)
\end{aligned}$$

We have completed the proof. ◻

LEMMA 5.3. *Let $w \in H^{\max\{k+1,4\}}(\Omega)$, then*

$$|||w - Q_hw||| \leq Ch^{k-1}\|w\|_{k+1}. \tag{5.7}$$

Proof. For any $T \in \mathcal{T}_h$, it follows from (2.3), integration by parts, (3.3) and inverse inequality,

$$
\begin{aligned}
&\|\Delta_w(w - Q_h w)\|_T^2 \\
=&\ (\Delta_w(w - Q_h w), \Delta_w(w - Q_h w))_T \\
=&\ (w - Q_0 w, \Delta(\Delta_w(w - Q_h w)))_T - \langle w - Q_b w, \nabla(\Delta_w(w - Q_h w)) \cdot \mathbf{n}\rangle_{\partial T} \\
&\ + \langle(\nabla w \cdot \mathbf{n}_e - Q_n(\nabla w \cdot \mathbf{n}_e) \cdot \mathbf{n}, \Delta_w(w - Q_h w)\rangle_{\partial T} \\
\leq&\ C(h_T^{-2}\|w - Q_0 w\|_T + h_T^{-3/2}\|w - Q_b w\|_{\partial T} \\
&\ + h_T^{-1/2}\|\nabla w \cdot \mathbf{n}_e - Q_n(\nabla w \cdot \mathbf{n}_e)\|_{\partial T})\|\Delta_w(w - Q_h w)\|_T \\
\leq&\ Ch^{k-1}|w|_{k+1,T}\|\Delta_w(w - Q_h w)\|_T.
\end{aligned}
$$

Using the above inequality and taking the summation of it over T, we derive (5.7) and prove the lemma. ☐

THEOREM 5.4. *Let $u_h \in V_h$ be the weak Galerkin finite element solution arising from (2.4). Assume that the exact solution $u \in H^{\max\{k+1,4\}}(\Omega)$. Then, there exists a constant C such that*

$$
|||u - u_h||| \leq Ch^{k-1}\left(\|u\|_{k+1} + h\delta_{k,2}\|u\|_4\right). \tag{5.8}
$$

Proof. Let $\epsilon_h = Q_h u - u_h \in V_h^0$. It is straightforward to obtain

$$
\begin{aligned}
|||e_h|||^2 &= (\Delta_w e_h, \Delta_w e_h)_{\mathcal{T}_h} \\
&= (\Delta_w e_h, \Delta_w(u - u_h))_{\mathcal{T}_h} \\
&= (\Delta_w e_h, \Delta_w(Q_h u - u_h))_{\mathcal{T}_h} + (\Delta_w e_h, \Delta_w(u - Q_h u))_{\mathcal{T}_h} \\
&= (\Delta_w e_h, \Delta_w \epsilon_h)_{\mathcal{T}_h} + (\Delta_w e_h, \Delta_w(u - Q_h u))_{\mathcal{T}_h}.
\end{aligned} \tag{5.9}
$$

Next, we bound the two terms on the right hand side in (5.9). Letting $v = \epsilon_h \in V_h^0$ in (4.1) and using (5.3)-(5.4) and (5.7), we have

$$
\begin{aligned}
|(\Delta_w e_h, \Delta_w \epsilon_h)_{\mathcal{T}_h}| &\leq |\ell_1(u, \epsilon_h)| + |\ell_2(u, \epsilon_h)| \\
&\leq Ch^{k-1}(\|u\|_{k+1} + h\delta_{k,2}\|u\|_4)|||\epsilon_h||| \\
&\leq Ch^{k-1}(\|u\|_{k+1} + h\delta_{k,2}\|u\|_4)(|||u - Q_h u||| + |||u - u_h|||) \\
&\leq Ch^{2(k-1)}(\|u\|_{k+1}^2 + h^2\delta_{k,2}\|u\|_4^2) + \frac{1}{4}|||e_h|||^2.
\end{aligned} \tag{5.10}
$$

The estimate (5.7) implies

$$
\begin{aligned}
|(\Delta_w e_h, \Delta_w(u - Q_h u))_{\mathcal{T}_h}| &\leq C|||u - Q_h u||| |||e_h||| \\
&\leq Ch^{2(k-1)}\|u\|_{k+1}^2 + \frac{1}{4}|||e_h|||^2.
\end{aligned} \tag{5.11}
$$

Combining the estimates (5.10) and (5.11) with (5.9), we arrive

$$
|||e_h||| \leq Ch^{k-1}\left(\|u\|_{k+1} + h\delta_{k,2}\|u\|_4\right),
$$

which completes the proof. ☐

6. Error Estimates in L^2 Norm. In this section, we will provide an estimate for the L^2 norm of the WG solution u_h.

Recall that $e_h = u - u_h$ and $\epsilon_h = Q_h u - u_h = \{\epsilon_0, \epsilon_b, \epsilon_n \mathbf{n}_e\} \in V_h^0$.

Let us consider the following dual problem

$$\Delta^2 w = \epsilon_0 \quad \text{in } \Omega, \tag{6.1}$$
$$w = 0 \quad \text{on } \partial\Omega, \tag{6.2}$$
$$\nabla w \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega. \tag{6.3}$$

The H^4 regularity assumption of the dual problem implies the existence of a constant C such that

$$\|w\|_4 \le C\|\epsilon_0\|. \tag{6.4}$$

THEOREM 6.1. *Let $u_h \in V_h$ be the weak Galerkin finite element solution arising from (2.4). Assume that the exact solution $u \in H^{k+1}(\Omega)$ and (6.4) holds true. Then, there exists a constant C such that*

$$\|Q_0 u - u_0\| \le C h^{k+1-\delta_{k,2}}(\|u\|_{k+1} + h\delta_{k,2}\|u\|_4). \tag{6.5}$$

Proof. Testing (6.1) by ϵ_0 and then using the equation (4.3) with $u = w$ and $v = \epsilon_h$, we obtain

$$\begin{aligned}
\|\epsilon_0\|^2 &= (\Delta^2 w, \epsilon_0) \\
&= (\Delta_w w,\ \Delta_w \epsilon_h)_{\mathcal{T}_h} - \langle \epsilon_0 - \epsilon_b, \nabla(\mathbb{Q}_h \Delta w - \Delta w)\cdot \mathbf{n}\rangle_{\partial T_h} \\
&\quad -\langle (\nabla \epsilon_0 - \epsilon_n \mathbf{n}_e)\cdot \mathbf{n}, \Delta w - \mathbb{Q}_h \Delta w\rangle_{\partial T} \\
&= (\Delta_w w,\ \Delta_w \epsilon_h)_{\mathcal{T}_h} - \ell_1(w, \epsilon_h) - \ell_2(w, \epsilon_h)
\end{aligned}$$

The error equation (4.1) gives

$$\begin{aligned}
(\Delta_w w,\ \Delta_w \epsilon_h)_{\mathcal{T}_h} &= (\Delta_w w,\ \Delta_w e_h)_{\mathcal{T}_h} + (\Delta_w w,\ \Delta_w (Q_h u - u))_{\mathcal{T}_h} \\
&= (\Delta_w e_h,\ \Delta_w Q_h w)_{\mathcal{T}_h} + (\Delta_w e_h,\ \Delta_w (w - Q_h w))_{\mathcal{T}_h} \\
&\quad +(\Delta_w w,\ \Delta_w (Q_h u - u))_{\mathcal{T}_h} \\
&= \ell_1(u, Q_h w) + \ell_2(u, Q_h w) + (\Delta_w e_h,\ \Delta_w (w - Q_h w))_{\mathcal{T}_h} \\
&\quad +(\Delta_w w,\ \Delta_w (Q_h u - u))_{\mathcal{T}_h}.
\end{aligned}$$

Combining the two equations above, we obtain

$$\begin{aligned}
\|\epsilon_0\|^2 &= \ell_1(u, Q_h w) + \ell_2(u, Q_h w) + (\Delta_w e_h,\ \Delta_w (w - Q_h w))_{\mathcal{T}_h} \\
&\quad +(\Delta_w w,\ \Delta_w (Q_h u - u))_{\mathcal{T}_h} - \ell_1(w, \epsilon_h) - \ell_2(w, \epsilon_h) \\
&= I_1 + I_2 + I_3 + I_4 + I_5 + I_6.
\end{aligned}$$

Next, we will estimate the all the terms on the right hand side of the above equation.

Using the Cauchy-Schwartz inequality, (3.3) and (5.2), we have

$$
\begin{aligned}
I_1 = |\ell_1(u, Q_h w)| &= \left| \sum_{T\in\mathcal{T}_h} \langle \nabla(\Delta u - \mathbb{Q}_h \Delta u)\cdot \mathbf{n}, Q_0 w - Q_b w\rangle_{\partial T} \right| \\
&\le \left(\sum_{T\in\mathcal{T}_h} h_T^3 \|\nabla(\Delta u - \mathbb{Q}_h \Delta u)\|_{\partial T}^2 \right)^{\frac12} \left(\sum_{T\in\mathcal{T}_h} h_T^{-3} \|Q_0 w - Q_b w\|_{\partial T}^2 \right)^{\frac12} \\
&\le \left(\sum_{T\in\mathcal{T}_h} h_T^3 \|\nabla(\Delta u - \mathbb{Q}_h \Delta u)\|_{\partial T}^2 \right)^{\frac12} \left(\sum_{T\in\mathcal{T}_h} h_T^{-3} \|Q_0 w - w\|_{\partial T}^2 \right)^{\frac12} \\
&\le C h^{k+1-\delta_{k,2}} \left(\|u\|_{k+1} + h\delta_{k,2} \|u\|_4 \right) \|w\|_4,
\end{aligned}
$$

Similarly, by the Cauchy-Schwartz inequality, (5.1) and (3.3), we have

$$
\begin{aligned}
I_2 = |\ell_2(u, Q_h w)| &= \left| \sum_{T\in\mathcal{T}_h} \langle \Delta u - \mathbb{Q}_h \Delta u, (\nabla Q_0 w \cdot \mathbf{n} - Q_n(\nabla w\cdot \mathbf{n}))\rangle_{\partial T} \right| \\
&\le \left(\sum_{T\in\mathcal{T}_h} h_T \|\Delta u - \mathbb{Q}_h \Delta u\|_{\partial T}^2 \right)^{\frac12} \times \\
&\left(\sum_{T\in\mathcal{T}_h} h_T^{-1} (\|\nabla Q_0 w\cdot \mathbf{n} - \nabla w \cdot \mathbf{n}\|_{\partial T}^2 + \|\nabla w\cdot \mathbf{n} - Q_n(\nabla w\cdot\mathbf{n})\|_{\partial T}^2) \right)^{\frac12} \\
&\le C h^{k+1-\delta_{k,2}} \|u\|_{k+1} \|w\|_4.
\end{aligned}
$$

It follows from (5.8) and (5.7),

$$
I_3 = (\Delta_w e_h,\ \Delta_w(w - Q_h w))_{\mathcal{T}_h} \le C h^{k+1-\delta_{k,2}} \|u\|_{k+1} \|w\|_4.
$$

To bound I_4, we define a L^2 projection element-wise onto $P_1(T)$ denoted by R_h. Then it follows from the definition of weak Laplacian (2.3)

$$
\begin{aligned}
&(\Delta_w (Q_h u - u),\ R_h \Delta_w w)_T \\
&= (Q_0 u - u,\ \Delta(R_h \Delta_w w))_T - \langle Q_b u - u,\ \nabla(R_h \Delta_w w)\cdot \mathbf{n}\rangle_{\partial T} \\
&\quad + \langle (Q_n(\nabla u\cdot \mathbf{n}_e) - \nabla u \cdot \mathbf{n}_e)\cdot \mathbf{n},\ R_h \Delta_w w\rangle_{\partial T} = 0.
\end{aligned}
$$

Using the equation above and (5.7) and the definition of R_h, we have

$$
\begin{aligned}
I_4 &= |\Delta_w (Q_h u - u),\ \Delta_w w)_{\mathcal{T}_h}| \\
&= |(\Delta_w (Q_h u - u),\ \Delta_w w - R_h \Delta_w w)_{\mathcal{T}_h}| \\
&\le C h^{k+1} \|u\|_{k+1} \|w\|_4.
\end{aligned}
$$

Using (4.2), (5.3), (5.8) and (5.7), we have

$$
\begin{aligned}
I_5 = |\ell_1(w, \epsilon_h)| &\le C h^{2-\delta_{k,2}} \|w\|_4 |||\epsilon_h||| \le C h^{2-\delta_{k,2}} \|w\|_4 (|||Q_h u - u||| + |||e_h|||) \\
&\le C h^{k+1-\delta_{k,2}} \|u\|_{k+1} \|w\|_4.
\end{aligned}
$$

Similarly, we obtain

$$
I_6 = |\ell_2(w, \epsilon_h)| \le C h^{k+1-\delta_{k,2}} \|u\|_{k+1} \|w\|_4.
$$

Combining all the estimates above yields

$$\|\epsilon_0\|^2 \leq Ch^{k+1-\delta_{k,2}}(\|u\|_{k+1} + h\delta_{k,2}\|u\|_4)\|w\|_4.$$

It follows from the above inequality and the regularity assumption (6.4).

$$\|\epsilon_0\| \leq Ch^{k+1-\delta_{k,2}}(\|u\|_{k+1} + h\delta_{k,2}\|u\|_4).$$

We have completed the proof. □

7. Numerical Test. We solve the following 2D biharmonic equation on the unit square:

$$\Delta^2 u = f, \quad (x,y) \in \Omega = (0,1)^2, \tag{7.1}$$

with the boundary conditions $u = g_1$ and $\nabla u \cdot \mathbf{n} = g_2$ on $\partial\Omega$. Here f, g_1 and g_2 are chosen so that the exact solution is

$$u = e^{x+y}.$$

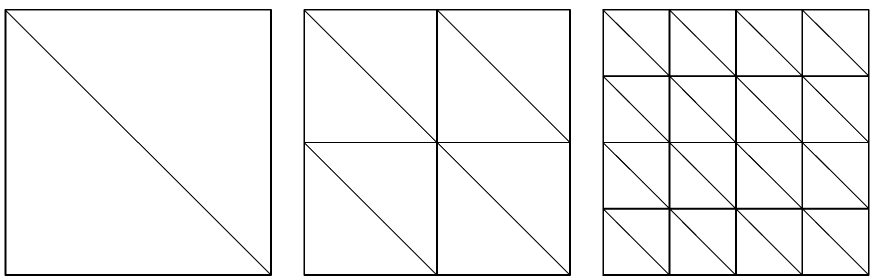

FIG. 7.1. *The first three levels of grids used in the computation of Table 7.1.*

In the first computation, the level one grid consists of two unit right triangles cutting from the unit square by a forward slash. The high level grids are the half-size refinements of the previous grid. The first three levels of grids are plotted in Figure 7.1. The error and the order of convergence for the method are shown in Tables 7.1. Here on triangular grids, we let $j = k + 2$ defined in (2.3) for computing the weak Laplacian $\Delta_w v$. The numerical results confirm the convergence theory.

In the next computation, we use a family of polygonal grids (with pentagons) shown in Figure 7.2. We let the polynomial degree $j = k+3$ for the weak Laplacian on such polygonal meshes. The rate of convergence is listed in Table 7.2. The convergence history confirms the theory.

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TABLE 7.1
Error profiles and convergence rates for (7.1) on triangular grids (Figure 7.1)

level	$\|u_h - u\|_0$	rate	$\|u_h - u\|_{1,h}$	rate	$\|\|\|u_h - u\|\|\|$	rate
	by the P_2 weak Galerkin finite element					
5	0.7913E-04	1.96	0.5596E-03	2.00	0.2764E+00	1.00
6	0.2016E-04	1.97	0.1412E-03	1.99	0.1383E+00	1.00
7	0.5049E-05	2.00	0.3547E-04	1.99	0.6912E-01	1.00
	by the P_3 weak Galerkin finite element					
3	0.3788E-05	4.20	0.1398E-03	3.09	0.2949E-01	2.00
4	0.2114E-06	4.16	0.1713E-04	3.03	0.7384E-02	2.00
5	0.1284E-07	4.04	0.2128E-05	3.01	0.1848E-02	2.00

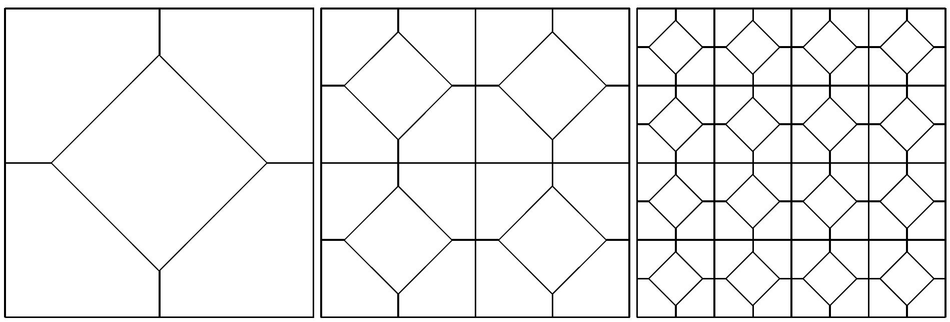

FIG. 7.2. *The first three levels of polygonal grids used in the computation of Table 7.2.*

TABLE 7.2
Error profiles and convergence rates for (7.1) on polygonal grids (Figure 7.2)

level	$\|u_h - u\|_0$	rate	$\|u_h - u\|_{1,h}$	rate	$\|\|\|u_h - u\|\|\|$	rate
	by the P_2 weak Galerkin finite element					
3	0.5699E-03	2.6	0.8766E-02	1.9	0.4895E+01	1.0
4	0.1035E-03	2.5	0.2346E-02	1.9	0.2445E+01	1.0
5	0.2477E-04	2.1	0.6175E-03	1.9	0.1222E+01	1.0
6	0.6835E-05	1.9	0.1598E-03	2.0	0.6112E+00	1.0
	by the P_3 weak Galerkin finite element					
1	0.1571E-02	0.0	0.1905E-01	0.0	0.3251E+01	0.0
2	0.9077E-04	4.1	0.2259E-02	3.1	0.7397E+00	2.1
3	0.5368E-05	4.1	0.2888E-03	3.0	0.1793E+00	2.0
4	0.3474E-06	3.9	0.3939E-04	2.9	0.4445E-01	2.0

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