

Sufficient criteria and sharp geometric conditions for observability in Banach spaces

Dennis Gallaun¹, Christian Seifert¹, and Martin Tautenhahn²

¹*Technische Universität Hamburg, Institut für Mathematik, Am Schwarzenberg-Campus 3, 21073 Hamburg, Germany, {dennis.gallaun, christian.seifert}@tuhh.de*

²*Technische Universität Chemnitz, Fakultät für Mathematik, 09107 Chemnitz, Germany, martin.tautenhahn@mathematik.tu-chemnitz.de*

Abstract

Let X, Y be Banach spaces, $(S_t)_{t \geq 0}$ a C_0 -semigroup on X , $-A$ the corresponding infinitesimal generator on X , C a bounded linear operator from X to Y , and $T > 0$. We consider the system

$$\dot{x}(t) = -Ax(t), \quad y(t) = Cx(t), \quad t \in (0, T], \quad x(0) = x_0 \in X.$$

We provide sufficient conditions such that this system satisfies a final state observability estimate in $L_r((0, T); Y)$, $r \in [1, \infty]$. These sufficient conditions are given by an uncertainty relation and a dissipation estimate. Our approach unifies and generalizes the respective advantages from earlier results obtained in the context of Hilbert spaces. As an application we consider the example where A is an elliptic operator in $L_p(\mathbb{R}^d)$ for $1 < p < \infty$ and where $C = \mathbf{1}_E$ is the restriction onto a thick set $E \subset \mathbb{R}^d$. In this case, we show that the above system satisfies a final state observability estimate if and only if $E \subset \mathbb{R}^d$ is a thick set. Finally, we make use of the well-known relation between observability and null-controllability of the predual system and investigate bounds on the corresponding control costs.

Mathematics Subject Classification (2010). 47D06, 35Q93, 47N70, 93B05, 93B07.

Keywords. Observability estimate, Banach space, C_0 -semigroups, elliptic operators, null-controllability, control costs

1 Introduction

Let X, Y be Banach spaces, $(S_t)_{t \geq 0}$ a C_0 -semigroup on X , $-A$ the corresponding infinitesimal generator on X , and C a bounded operator from X to Y . We consider systems of the form

$$\begin{aligned} \dot{x}(t) &= -Ax(t), & t \in (0, T], & \quad x(0) = x_0 \in X, \\ y(t) &= Cx(t), & t \in [0, T], & \end{aligned} \tag{1}$$

where $T > 0$ can be thought of as a final time for the system. One interpretation of the second equation in (1) is that we cannot measure the state $x(t)$ at time t directly, but just some $y(t) = Cx(t)$ from the range of C . The focus of this paper relates to the question whether

the system (1) satisfies a final state observability estimate in $L_r((0, T); Y)$ with $r \in [1, \infty]$, that is, there exists $C_{\text{obs}} > 0$ such that for all $x_0 \in X$ we have $\|x(T)\|_X \leq C_{\text{obs}} \|y\|_{L_r((0, T); Y)}$. A final state observability estimate thus allows one to recover information on the final state $x(T)$ from suitable measurements $y(t)$ for $t \in (0, T)$.

The most studied example of the system (1) is the heat equation with heat generation term in $L_2(\Omega)$ with $\Omega \subset \mathbb{R}^d$ open and some nonempty observability set, i.e. $A = \Delta - V$ is a self-adjoint Schrödinger operator in $L_2(\Omega)$ with bounded potential V , and $C = \mathbf{1}_E$ is the projection onto some non-empty measurable set $E \subset \Omega$. For bounded domains $\Omega \subset \mathbb{R}^d$ the observability problem for the heat equation is well understood since the seminal works by Lebeau and Robbiano [LR95] and Fursikov and Imanuvilov [FI96]. For unbounded domains this problem has been studied, e.g., in [Mil05a, Mil05b, Gd07, Bar14]. While for bounded domains it is sufficient that E is open and nonempty, or even measurable with positive Lebesgue measure [AEWZ14, EMZ15], this is of course not true for unbounded domains. On unbounded domains a sufficient geometric condition for observability is given in [LM16]. In addition to that, the papers [EV18, WWZZ19] show that the free heat equation in $L_2(\mathbb{R}^d)$ satisfies a final state observability estimate if and only if E is a thick set.

Since the observability constant C_{obs} can be interpreted (by duality) as the cost for the corresponding null-controllability problem, the problem of obtaining explicit bounds on C_{obs} attracted particular attention in the literature. The (optimal) dependence of C_{obs} on the model parameter T is investigated in [Güi85, FZ00, Mil04b, Phu04, Mil06a, Mil06b, TT07, Mil10, LL12, BPS18], while [Mil04b, TT11, EZ11, NTTV18, EV18, Phu18, Egi, LL, NTTV20a] also study the dependence on the geometry of the control set E . Moreover, [Güi85, Mil06a, TT07, Lis12, Lis15, DE19] concern one-dimensional problems and boundary control.

One possible approach to show an observability estimate has been described in the papers [LR95, LZ98, JL99], that is, to prove a quantitative uncertainty relation for spectral projectors. This is an inequality of the type

$$\forall \lambda > 0 \forall \psi \in L_2(\Omega): \quad \|P(\lambda)\psi\|_{L_2(\Omega)} \leq d_0 e^{d_1 \lambda^\gamma} \|\mathbf{1}_E P(\lambda)\psi\|_{L_2(E)},$$

where $\gamma \in (0, 1)$, $d_0, d_1 > 0$, and where $P(\lambda)$ denotes the projector to the spectral subspace of $-\Delta + V$ below λ . Subsequently, this strategy is generalized to (contraction) semigroups in abstract Hilbert spaces with (possibly self-adjoint) generators $-A$, to name those which are closest related to our result; see [Mil10, TT11, WZ17, BPS18, NTTV20a]. In particular, the papers [Mil10, WZ17, BPS18] allow for the $P(\lambda)$ to be arbitrary projectors (onto semigroup invariant subspaces) by assuming additionally a so-called dissipation estimate, that is, a decay estimate of the semigroup on the orthogonal complement of the range of $P(\lambda)$. This can be rephrased in a scheme in which an uncertainty relation together with a dissipation estimate implies an observability estimate. Since the constants appearing in the uncertainty relation (and the dissipation estimate) transfer into the observability constant C_{obs} , it is important to achieve its dependence on d_0 , d_1 , and γ , on the set E , and on the coefficients of the operator A as explicitly as possible. Uncertainty relations with an explicit dependence on the geometry of E are provided by the Logvinenko–Sereda theorem for the free heat equation observed on thick sets [LS74, Kov00, Kov01, EV20]. For Schrödinger operators such uncertainty relations have, for instance, been proven in [NTTV18, NTTV20b] for a certain class of equidistributed observation sets and bounded potentials and in [LM] for thick observation sets and analytic potentials.

So far, the discussion has been restricted to Hilbert spaces only. However, a natural setup to ask for observability estimates is the context of Banach spaces and C_0 -semigroups, since there are various applications of the above concepts in this situation. In this paper, we extend (some of) the above-mentioned results to the Banach space setting. In particular, in Section 2 we show in the general framework of Banach spaces that an uncertainty relation together with a dissipation estimate implies that the system (1) satisfies a final state observability estimate. Our observability constant C_{obs} is given explicitly with respect to the parameters coming from the uncertainty relation and the dissipation estimate and, in addition, is sharp in the dependence on T . Let us stress that, besides the fact that this result holds in its natural Banach space setting, our approach unifies and generalizes the respective advantages from earlier results even in the context of Hilbert spaces; cf. Remark 2.2 for more details. In Section 3 we verify these sufficient conditions in L_p -spaces for a class of elliptic operators A and observation operators $C = \mathbf{1}_E$. This way we obtain an observability estimate with an explicit dependence on the coefficients of the elliptic operator A , the final time T , and the geometry of the thick set E . Furthermore, we show that this result is sharp in the sense that the system (1) satisfies a final state observability estimate if and only if E is a thick set. Finally, in Section 4 we make use of the well-known relation between observability and null-controllability of the predual system to (1) and investigate bounds on the corresponding control costs.

2 Sufficient criteria for observability in Banach spaces

For normed spaces V and W we denote by $\mathcal{L}(V, W)$ the space of bounded linear operators from V to W . Let X, Y be Banach spaces, $(S_t)_{t \geq 0}$ a C_0 -semigroup on X , $-A$ the corresponding infinitesimal generator on X with domain $\mathcal{D}(-A)$, and $C \in \mathcal{L}(X, Y)$. For $T > 0$ we consider the system

$$\begin{aligned} \dot{x}(t) &= -Ax(t), & t \in (0, T], & \quad x(0) = x_0 \in X, \\ y(t) &= Cx(t), & t \in [0, T]. \end{aligned} \tag{2}$$

The mild solution of (2) is given by

$$x(t) = S_t x_0, \quad y(t) = CS_t x_0, \quad t \in [0, T].$$

In particular, if $x_0 \in \mathcal{D}(A)$ we may differentiate $x(\cdot) = S_{(\cdot)} x_0$ to obtain (2). Let $r \in [1, \infty]$. We say that the system (2) satisfies a *final state observability estimate in $L_r((0, T); Y)$* if there exists $C_{\text{obs}} > 0$ such that for all $x_0 \in X$ we have $\|x(T)\|_X \leq C_{\text{obs}} \|y\|_{L_r((0, T); Y)}$ or, equivalently, if for all $x_0 \in X$ we have

$$\begin{aligned} \|S_T x_0\|_X &\leq C_{\text{obs}} \left(\int_0^T \|CS_\tau x_0\|_Y^r d\tau \right)^{1/r} && \text{if } 1 \leq r < \infty \text{ or} \\ \|S_T x_0\|_X &\leq C_{\text{obs}} \operatorname{ess\,sup}_{\tau \in [0, T]} \|CS_\tau x_0\|_Y && \text{if } r = \infty. \end{aligned}$$

One motivation to study final state observability estimates is their relation to null-controllability of the predual system to (2) and its control cost. This is discussed in more detail in Section 4.

The following theorem provides sufficient conditions such that the system (2) satisfies a final state observability estimate.

Theorem 2.1. *Let X and Y be Banach spaces, $C \in \mathcal{L}(X, Y)$, $(S_t)_{t \geq 0}$ be a C_0 -semigroup on X , $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\|S_t\| \leq Me^{\omega t}$ for all $t \geq 0$, $\lambda^* \geq 0$ and $(P_\lambda)_{\lambda > \lambda^*}$ be a family of bounded linear operators in X . Assume further that there exist $d_0, d_1, \gamma_1 > 0$ such that*

$$\forall x \in X \quad \forall \lambda > \lambda^*: \quad \|P_\lambda x\|_X \leq d_0 e^{d_1 \lambda^{\gamma_1}} \|CP_\lambda x\|_Y \quad (3)$$

and that there exist $d_2 \geq 1$, $d_3, \gamma_2, \gamma_3, T > 0$ with $\gamma_1 < \gamma_2$ such that

$$\forall x \in X \quad \forall \lambda > \lambda^* \quad \forall t \in (0, T/2]: \quad \|(\text{Id} - P_\lambda)S_t x\|_X \leq d_2 e^{-d_3 \lambda^{\gamma_2} t^{\gamma_3}} \|x\|_X. \quad (4)$$

Then we have for all $r \in [1, \infty]$ and $x \in X$

$$\|S_T x\|_X \leq C_{\text{obs}} \|CS_{(\cdot)} x\|_{L_r((0, T); Y)} \quad \text{with} \quad C_{\text{obs}} = \frac{C_1}{T^{1/r}} \exp\left(\frac{C_2}{T^{\frac{\gamma_1 \gamma_3}{\gamma_2 - \gamma_1}}} + C_3 T\right),$$

where $T^{1/r} = 1$ if $r = \infty$, and

$$\begin{aligned} C_1 &= (4Md_0) \max\left\{((4d_2 M^2)(d_0 \|C\|_{\mathcal{L}(X, Y)} + 1))^{8/(\text{e} \ln 2)}, e^{4d_1 (2\lambda^*)^{\gamma_1}}\right\}, \\ C_2 &= 4(2^{\gamma_1} (2 \cdot 4^{\gamma_3})^{\frac{\gamma_1 \gamma_2}{\gamma_2 - \gamma_1}} d_1^{\gamma_2} / d_3^{\gamma_1})^{\frac{1}{\gamma_2 - \gamma_1}}, \\ C_3 &= \max\{\omega, 0\}(1 + 10/(\text{e} \ln 2)). \end{aligned}$$

Remark 2.2. (a) Assumption (3) is called *uncertainty relation*, since a state $P_\lambda x \neq 0$ in the range of P_λ cannot be in the kernel of C . In particular, if $X = Y$ is a Hilbert space and P_λ and C are orthogonal projections, assumption (3) can be rewritten as

$$P_\lambda \leq d_0 e^{d_1 \lambda^{\gamma_1}} P_\lambda C P_\lambda, \quad (5)$$

where the inequality is understood in the quadratic form sense. If $X = Y = L_2(\Omega)$ with $\Omega \subset \mathbb{R}^d$ open, A is a Schrödinger operator, $C = \mathbf{1}_E: X \rightarrow Y$ is the restriction operator (i.e. the multiplication operator with $\mathbf{1}_E$) on some measurable set $E \subset \Omega$, and if P_λ is the spectral projector of a self-adjoint operator onto the interval $(-\infty, \lambda]$, then the spectral projector corresponds to a restriction in momentum-space and enforces delocalization in direct space, i.e., an uncertainty relation. Inequality (5) is sometimes also called *gain of positive definiteness*, since the restriction $P_\lambda C P_\lambda$ of C is strictly positive on the subspace $\text{Ran } P_\lambda$. In control theory inequalities of the type (3) are often called *spectral inequality*. We omit this notation, since the operators P_λ are in our setting not necessarily spectral projectors of some self-adjoint operator in a Hilbert space.

Assumption (4) is called *dissipation estimate*, as it assumes an exponential decay of $(\text{Id} - P_\lambda)S_t$ with respect to λ and t . In particular, it implies that $P_\lambda \rightarrow \text{Id}$ strongly as $\lambda \rightarrow \infty$.

(b) The dependence of C_{obs} on T is optimal for large and small T . In [Sei84] Seidman showed for one-dimensional controlled heat systems that C_{obs} blows up at most exponentially for small T . This result was extended to arbitrary dimension by Fursikov and Imanuvilov in [FI96]. That the exponential blow-up has to occur for small T was first shown by Güichal [Güi85] for one-dimensional systems and by Miller [Mil04a] in arbitrary dimension. It is folklore that in the large time regime, the decay rate $T^{-1/r}$ is optimal; for a proof see, e.g., [NTTV20a, Theorem 2.13].

(c) Let us discuss the novel aspects of Theorem 2.1 compared to earlier results in the literature. We restrict our discussion to the case where X and Y are Hilbert spaces and $r = 2$, since to the best of our knowledge, sufficient conditions for observability in Banach spaces as in Theorem 2.1 have not been obtained before.

That uncertainty relations imply observability estimates was first shown in the seminal papers [LR95, LZ98, JL99]. Subsequently, there is a huge amount of literature concerning abstract theorems which turn uncertainty relations into observability estimates in Hilbert spaces; to name a few, see [Mil10, TT11, WZ17, BPS18, NTTV20a]. The paper [Mil10] considered general C_0 -semigroups and the operators $(P_\lambda)_{\lambda>0}$ as projections onto a nondecreasing family of semigroup invariant subspaces. The obtained observability constant C_{obs} is of the form $C \exp(C/T^{\gamma_1\gamma_3/(\gamma_2-\gamma_1)})$ and hence misses the factor $T^{-1/2}$. A similar result has been obtained in [BPS18] for contraction semigroups and orthogonal projections $(P_\lambda)_{\lambda>0}$ onto semigroup invariant subspaces. The papers [TT11, NTTV20a] considered nonnegative and self-adjoint operators A , and the operators $(P_\lambda)_{\lambda>0}$ are assumed to be spectral projections of A onto the interval $[0, \lambda]$. In this setting, the dissipation estimate is automatically satisfied with $\gamma_2 = \gamma_3 = 1$. While both papers obtain the “optimal” bound $CT^{-1/2} \exp(C/T^{\gamma_1/(1-\gamma_1)})$ (including the factor $T^{-1/2}$), the paper [TT11] assumed additionally that A has purely discrete spectrum with an orthogonal basis of eigenvectors. Moreover, [NTTV20a] slightly improved the dependence of C_{obs} on the parameters d_0 and d_1 which was essential for their application to certain homogenization regimes. Let us emphasize that our result recovers this dependence on d_0 and d_1 as well and hence allows also for homogenization.

To conclude, our result extends the earlier mentioned results into three directions.

- (1) We allow for an arbitrary family $(P_\lambda)_{\lambda>\lambda^*}$ of bounded linear operators, and obtain at the same time the factor $T^{-1/2}$ in C_{obs} . In particular, we do not require that P_λ is an orthogonal projection.
- (2) We allow for general C_0 -semigroups, possibly with exponential growth. We do not require contraction (or quasi-contraction, or bounded) semigroups.

As in [WZ17], our result combines the respective advantages from earlier results in Hilbert space setting, e.g., the factor $T^{-1/2}$ in C_{obs} , general C_0 -semigroups, and arbitrary family $(P_\lambda)_{\lambda>\lambda^*}$ of bounded linear operators at the same time.

- (3) We consider Banach spaces X and Y instead of Hilbert spaces and $r \in [1, \infty]$ instead of $r = 2$.

Indeed, since the theory of strongly continuous semigroups essentially is a Banach space theory, our Theorem 2.1 now formulates the link from uncertainty relations and dissipation estimates to observability estimates in its natural setup. Let us stress that, in contrast to the above-mentioned references, we have no spectral calculus in the general framework of Banach spaces.

(d) Suppose we have a discrete sequence $(P_k)_{k \in \mathbb{N}}$ of bounded linear operators in X which satisfies the following discrete version of conditions (3) and (4):

$$\forall x \in X \ \forall k \in \mathbb{N}: \quad \|P_k x\|_X \leq \tilde{d}_0 e^{\tilde{d}_1 k^{\gamma_1}} \|CP_k x\|_Y$$

and

$$\forall x \in X \ \forall k \in \mathbb{N} \ \forall t \in (0, T/2]: \quad \|(1 - P_k)S_t x\|_X \leq \tilde{d}_2 e^{-\tilde{d}_3 k^{\gamma_2} t^{\gamma_3}} \|x\|_X$$

for constants $\tilde{d}_0, \tilde{d}_1, \tilde{d}_3 > 0$ and $\tilde{d}_2 \geq 1$, as in [BPS18]. Then we can apply Theorem 2.1 in the following way. Let $(P_\lambda)_{\lambda>0}$ be defined by $P_\lambda = P_k$ for $\lambda \in (k-1, k]$, $k \in \mathbb{N}$. Then $(P_\lambda)_{\lambda>0}$ fulfils the assumptions (3) and (4) of Theorem 2.1 with

$$d_0 := \tilde{d}_0 e^{\tilde{d}_1}, \quad d_1 := 2^{\gamma_1} \tilde{d}_1, \quad d_2 := \tilde{d}_2, \quad d_3 := \tilde{d}_3, \quad \text{and} \quad \lambda^* = 0.$$

(e) It is possible to extend the statement of Theorem 2.1 to the case of time-dependent observation operators $C: [0, T] \rightarrow \mathcal{L}(X, Y)$, as long as C is measurable (in a suitable sense) and essentially bounded. We refer to [NTTV20a, Theorem 2.11] for a similar extension in Hilbert spaces.

(f) It is also possible to prove an interpolation inequality as in [AEWZ14, Theorem 6] for our abstract context. This can then be used to obtain a final state observability estimate by only taking into account the observation function on a measurable subset of the time interval $[0, T]$; cf. [PW13, Theorem 1.1].

Proof of Theorem 2.1. Assume we have shown the statement of the theorem in the case $r = 1$, i.e., for all $x \in X$ we have

$$\|S_T x\|_X \leq C_{\text{obs}} \|CS_{(\cdot)} x\|_{L_1((0, T); Y)}, \quad \text{where} \quad C_{\text{obs}} = \frac{C_1}{T} \exp\left(\frac{C_2}{T^{\frac{\gamma_1 \gamma_3}{\gamma_2 - \gamma_1}}} + C_3 T\right).$$

Then, by Hölder's inequality we obtain for all $r \in [1, \infty]$ and all $x \in X$

$$\|S_T x\|_X \leq C_{\text{obs}} T^{1/r'} \|CS_{(\cdot)} x\|_{L_r((0, T); Y)},$$

where $r' \in [1, \infty]$ is such that $1/r + 1/r' = 1$. Since $T^{-1} T^{1/r'} = T^{-1/r}$, the statement of the theorem follows. Thus, it is sufficient to prove the theorem in the case $r = 1$.

For the first part of the proof, we adapt the strategy in [TT11, NTTV20a] with a slight modification in order to deal with general P_λ 's instead of spectral projectors. Fix $x \in X$ arbitrary, and introduce for $t > 0$ and $\lambda > \lambda^*$ the notation

$$\begin{aligned} F(t) &= \|S_t x\|_X, & F_\lambda(t) &= \|P_\lambda S_t x\|_X, & F_\lambda^\perp(t) &= \|(\text{Id} - P_\lambda) S_t x\|_X, \\ G(t) &= \|CS_t x\|_Y, & G_\lambda(t) &= \|CP_\lambda S_t x\|_Y, & G_\lambda^\perp(t) &= \|C(\text{Id} - P_\lambda) S_t x\|_Y. \end{aligned}$$

Then for $0 \leq \tau \leq t$ we obtain

$$F(t) = \|S_t x\|_X = \|S_{t-\tau} S_\tau x\|_X \leq M e^{\omega_+ t} \|S_\tau x\|_X = M e^{\omega_+ t} F(\tau),$$

where $\omega_+ = \max\{\omega, 0\}$. Integrating this inequality, we obtain

$$F(t) \leq M e^{\omega_+ t} \frac{2}{t} \int_{t/2}^t F(\tau) d\tau.$$

We now use the uncertainty relation (3) to obtain for all $t > 0$ and $\lambda > \lambda^*$

$$F(t) \leq M e^{\omega_+ t} \frac{2}{t} \int_{t/2}^t F(\tau) d\tau \leq M e^{\omega_+ t} \frac{2}{t} \int_{t/2}^t \left(F_\lambda(\tau) + F_\lambda^\perp(\tau)\right) d\tau$$

$$\leq Me^{\omega+t} \frac{2}{t} \int_{t/2}^t \left(d_0 e^{d_1 \lambda \gamma_1} G_\lambda(\tau) + F_\lambda^\perp(\tau) \right) d\tau.$$

By the semigroup property and the dissipation estimate (4) we have for all $\tau \in (0, T]$ the estimate

$$F_\lambda^\perp(\tau) = \|(\text{Id} - P_\lambda)S_{\tau/2}S_{\tau/2}x\|_X \leq d_2 e^{-d_3 \lambda \gamma_2 (\tau/2)^{\gamma_3}} F(\tau/2). \quad (6)$$

Since $F(\tau/2) \leq Me^{\omega+t/4} F(t/4)$ for $t > 0$ and $\tau \in [t/2, t]$, we obtain for all $t \in (0, T]$ and $\lambda > \lambda^*$

$$F(t) \leq \frac{2Me^{\omega+t} d_0 e^{d_1 \lambda \gamma_1}}{t} \int_{t/2}^t G_\lambda(\tau) d\tau + d_2 M^2 e^{5\omega+t/4} e^{-d_3 \lambda \gamma_2 (t/4)^{\gamma_3}} F(t/4). \quad (7)$$

Using $G_\lambda(\tau) \leq G(\tau) + G_\lambda^\perp(\tau) \leq G(\tau) + \|C\|_{\mathcal{L}(X,Y)} F_\lambda^\perp(\tau)$ and (6) again, we obtain for all $t \in (0, T]$ and $\lambda > \lambda^*$

$$\int_{t/2}^t G_\lambda(\tau) d\tau \leq \int_{t/2}^t G(\tau) d\tau + \|C\|_{\mathcal{L}(X,Y)} d_2 e^{-d_3 \lambda \gamma_2 (t/4)^{\gamma_3}} \int_{t/2}^t F(\tau/2) d\tau. \quad (8)$$

Since $F(\tau/2) \leq Me^{\omega+t/4} F(t/4)$ for $t > 0$ and $\tau \in [t/2, t]$ and $1 \leq e^{d_1 \lambda \gamma_1}$, we conclude from (7) and (8) for all $t \in (0, T]$ and $\lambda > \lambda^*$

$$F(t) \leq \frac{2Md_0 e^{d_1 \lambda \gamma_1}}{te^{-\omega+T}} \int_{t/2}^t G(\tau) d\tau + \frac{d_2 M^2 e^{5\omega+T/4} e^{d_1 \lambda \gamma_1}}{e^{d_3 \lambda \gamma_2 (t/4)^{\gamma_3}}} (d_0 \|C\|_{\mathcal{L}(X,Y)} + 1) F(t/4).$$

With the short hand notation

$$D_1(t, \lambda) = \frac{2Me^{\omega+T} d_0 e^{d_1 \lambda \gamma_1}}{t} \int_{t/2}^t G(\tau) d\tau, \quad D_2(t, \lambda) = K_1 e^{d_1 \lambda \gamma_1 - d_3 \lambda \gamma_2 (t/4)^{\gamma_3}},$$

where $K_1 = (d_0 \|C\|_{\mathcal{L}(X,Y)} + 1) d_2 M^2 e^{5\omega+T/4}$, this can be rewritten as

$$F(t) \leq D_1(t, \lambda) + D_2(t, \lambda) F(t/4). \quad (9)$$

This inequality can be iterated. Let $(\lambda_k)_{k \in \mathbb{N}_0}$ be a sequence with $\lambda_k > \lambda^*$ for $k \in \mathbb{N}_0$. First we apply inequality (9) with $t = T$ and $\lambda = \lambda_0$. The term $F(4^{-1}T)$ on the right-hand side is then estimated by inequality (9) with $t = 4^{-1}T$ and $\lambda = \lambda_1$. This way, we obtain after two steps

$$\begin{aligned} F(T) &\leq D_1(T, \lambda_0) + D_2(T, \lambda_0) (D_1(4^{-1}T, \lambda_1) + D_2(4^{-1}T, \lambda_1) F(4^{-2}T)) \\ &= D_1(T, \lambda_0) + D_1(4^{-1}T, \lambda_1) D_2(T, \lambda_0) + D_2(T, \lambda_0) D_2(4^{-1}T, \lambda_1) F(4^{-2}T). \end{aligned}$$

After $N + 1$ steps of this type we obtain

$$\begin{aligned} F(T) &\leq D_1(T, \lambda_0) + \sum_{k=1}^N D_1(4^{-k}T, \lambda_k) \prod_{l=0}^{k-1} D_2(4^{-l}T, \lambda_l) \\ &\quad + F(4^{-N-1}T) \prod_{k=0}^N D_2(4^{-k}T, \lambda_k). \quad (10) \end{aligned}$$

We now choose the sequence $(\lambda_k)_{k \in \mathbb{N}_0}$ given by $\lambda_k = \nu \alpha^k$ with

$$\alpha = \begin{cases} \alpha_0 & \text{if } T \leq T_0, \\ \alpha_0 \left(\frac{T}{T_0}\right)^{\frac{\gamma_3}{\gamma_2}} & \text{if } T > T_0, \end{cases} \quad \text{and} \quad \nu = \begin{cases} \nu_0 \left(\frac{T_0}{T}\right)^{\frac{\gamma_3}{\gamma_2 - \gamma_1}} & \text{if } T \leq T_0, \\ \nu_0 & \text{if } T > T_0, \end{cases}$$

where

$$\alpha_0 := (2 \cdot 4^{\gamma_3})^{\frac{1}{\gamma_2 - \gamma_1}}, \quad \nu_0 := \max \left\{ \left(\frac{2 \ln(4K_1)}{e \ln(2)d_1} \right)^{\frac{1}{\gamma_1}}, 2\lambda^* \right\}, \quad T_0 := \left(\frac{2d_1 \alpha_0^{\gamma_2}}{d_3 \nu_0^{\gamma_2 - \gamma_1}} \right)^{\frac{1}{\gamma_3}}.$$

With this notation we have (in both cases $T \leq T_0$ and $T > T_0$) the equality

$$d_3 T^{\gamma_3} \nu^{\gamma_2 - \gamma_1} = 2d_1 \alpha^{\gamma_2}, \quad (11)$$

which we will use frequently in the following. Moreover, the choice of α and ν ensures that the constants

$$K_2 := d_3 \left(\frac{T}{4}\right)^{\gamma_3} \nu^{\gamma_2} - d_1 \nu^{\gamma_1} \quad \text{and} \quad K_3 := \frac{K_2}{\alpha^{\gamma_2}/4^{\gamma_3} - 1} - d_1 \nu^{\gamma_1}$$

are positive. Indeed, using (11) we find

$$K_3 = \frac{d_3 (T/4)^{\gamma_3} \nu^{\gamma_2} - \alpha^{\gamma_2} d_1 \nu^{\gamma_1} / 4^{\gamma_3}}{\alpha^{\gamma_2}/4^{\gamma_3} - 1} = \nu^{\gamma_1} \frac{d_1 \alpha^{\gamma_2} / 4^{\gamma_3}}{\alpha^{\gamma_2}/4^{\gamma_3} - 1}.$$

Since $\alpha^{\gamma_2} > 2 \cdot 4^{\gamma_3}$, we conclude that K_3 is positive. Note that $K_2 > K_3$; hence K_2 is positive as well. Let us now show that the right-hand side in (10) converges for $N \rightarrow \infty$. Since $\alpha^{\gamma_1} \leq \alpha^{\gamma_2}/4^{\gamma_3}$, we have

$$\begin{aligned} \prod_{k=0}^N D_2(4^{-k}T, \lambda_k) &= \prod_{k=0}^N K_1 \exp \left(d_1 \nu^{\gamma_1} \alpha^{\gamma_1 k} - d_3 \left(\frac{T}{4}\right)^{\gamma_3} \nu^{\gamma_2} \left(\frac{\alpha^{\gamma_2}}{4^{\gamma_3}}\right)^k \right) \\ &\leq K_1^{N+1} \prod_{k=0}^N \exp \left(-K_2 \left(\frac{\alpha^{\gamma_2}}{4^{\gamma_3}}\right)^k \right). \end{aligned} \quad (12)$$

Since $K_1, K_2 > 0$ and $\alpha^{\gamma_2}/4^{\gamma_3} > 1$, this tends to zero as N tends to infinity. Moreover, using (12) and $\alpha^{\gamma_1} \leq \alpha^{\gamma_2}/4^{\gamma_3}$, we infer that the middle term of the right-hand side of (10) satisfies

$$\begin{aligned} &\sum_{k=1}^N D_1(4^{-k}T, \lambda_k) \prod_{\ell=0}^{k-1} D_2(4^{-\ell}T, \lambda_\ell) \\ &\leq 2M e^{\omega+T} d_0 \frac{1}{T} \int_0^T G(\tau) d\tau \sum_{k=1}^N (4K_1)^k \exp \left(-K_2 \frac{(\alpha^{\gamma_2}/4^{\gamma_3})^k - 1}{\alpha^{\gamma_2}/4^{\gamma_3} - 1} + d_1 \nu^{\gamma_1} \alpha^{\gamma_1 k} \right) \\ &\leq 2M e^{\omega+T} d_0 \frac{1}{T} \int_0^T G(\tau) d\tau \exp \left(\frac{K_2}{\alpha^{\gamma_2}/4^{\gamma_3} - 1} \right) \sum_{k=1}^N (4K_1)^k \exp \left(-K_3 \left(\frac{\alpha^{\gamma_2}}{4^{\gamma_3}}\right)^k \right). \end{aligned}$$

Since $K_3 > 0$, the right-hand side converges as N tends to infinity, and we obtain from (10) that

$$\|S_T x\|_X \leq \tilde{C}_{\text{obs}} \int_0^T \|CS_t x\|_Y dt,$$

where

$$\tilde{C}_{\text{obs}} = \frac{2Md_0}{Te^{-\omega_+ T}} \left(e^{d_1 \nu^{\gamma_1}} + \exp \left(\frac{K_2}{\alpha^{\gamma_2}/4^{\gamma_3} - 1} \right) \sum_{k=1}^{\infty} (4K_1)^k \exp \left(-K_3 \left(\frac{\alpha^{\gamma_2}}{4^{\gamma_3}} \right)^k \right) \right).$$

It remains to show the upper bound $\tilde{C}_{\text{obs}} \leq C_{\text{obs}}$ with C_{obs} as in the theorem. To this end, we note that for all $A > 1$ and $B > 0$ we have

$$\sum_{k=1}^{\infty} A^k e^{-B2^k} \leq \sup_{x \geq 1} A^x e^{-\frac{B}{2}2^x} \sum_{k=1}^{\infty} e^{-\frac{B}{2}2^k} = \left(\frac{2 \ln(A)}{B e \ln(2)} \right)^{\frac{\ln(A)}{\ln(2)}} \sum_{k=1}^{\infty} e^{-\frac{B}{2}2^k},$$

where the last identity follows from elementary calculus. Using $2^k \geq 2k$ and $e^B - 1 \geq B$, we further estimate

$$\sum_{k=1}^{\infty} e^{-\frac{B}{2}2^k} \leq \sum_{k=1}^{\infty} e^{-kB} = \frac{e^{-B}}{1 - e^{-B}} \leq \frac{1}{B}.$$

Hence, we find

$$\sum_{k=1}^{\infty} A^k e^{-B2^k} \leq \left(\frac{2 \ln(A)}{B e \ln(2)} \right)^{\frac{\ln(A)}{\ln(2)}} \frac{1}{B}. \quad (13)$$

We now apply inequality (13) with $A = 4K_1$ and $B = K_3$ and obtain by using $\alpha^{\gamma_2}/4^{\gamma_3} \geq 2$

$$\tilde{C}_{\text{obs}} \leq \frac{2Me^{\omega_+ T} d_0}{T} \left(e^{d_1 \nu^{\gamma_1}} + \exp \left(\frac{K_2}{\alpha^{\gamma_2}/4^{\gamma_3} - 1} \right) \left(\frac{2 \ln(4K_1)}{K_3 e \ln(2)} \right)^{\frac{\ln(4K_1)}{\ln(2)}} \frac{1}{K_3} \right).$$

For K_3 we have the lower bound

$$K_3 = \nu^{\gamma_1} \frac{d_1 \alpha^{\gamma_2}/4^{\gamma_3}}{\alpha^{\gamma_2}/4^{\gamma_3} - 1} \geq \nu^{\gamma_1} d_1 \geq \nu_0^{\gamma_1} d_1 \geq \frac{2 \ln(4K_1)}{e \ln(2)}. \quad (14)$$

Since $Me^{\omega_+ T} d_2 \geq 1$ we have $K_1 \geq 1$; hence $\ln(4K_1) \geq \ln 4$ and $K_3 > 1$. From this, inequality (14), and $d_1 \nu^{\gamma_1} \leq d_1 \nu^{\gamma_1} + K_3 = K_2/(\alpha^{\gamma_2}/4^{\gamma_3} - 1)$, we conclude

$$\tilde{C}_{\text{obs}} \leq \frac{2Md_0}{Te^{-\omega_+ T}} \left(e^{d_1 \nu^{\gamma_1}} + \exp \left(\frac{K_2}{\alpha^{\gamma_2}/4^{\gamma_3} - 1} \right) \right) \leq \frac{4Md_0}{Te^{-\omega_+ T}} \exp \left(\frac{K_2}{\alpha^{\gamma_2}/4^{\gamma_3} - 1} \right). \quad (15)$$

For the constant K_2 we calculate, using (11) and $\alpha^{\gamma_2}/4^{\gamma_3} - 1 \geq (1/2)\alpha^{\gamma_2}/4^{\gamma_3}$,

$$\frac{K_2}{\alpha^{\gamma_2}/4^{\gamma_3} - 1} = \frac{d_3(T/4)^{\gamma_3} \nu^{\gamma_2} - d_1 \nu^{\gamma_1}}{\alpha^{\gamma_2}/4^{\gamma_3} - 1} \leq \frac{d_3(T/4)^{\gamma_3} \nu^{\gamma_2}}{\alpha^{\gamma_2}/4^{\gamma_3} - 1} = \nu^{\gamma_1} \frac{2d_1 \alpha^{\gamma_2}/4^{\gamma_3}}{\alpha^{\gamma_2}/4^{\gamma_3} - 1} \leq 4d_1 \nu^{\gamma_1}.$$

By our choice of ν we have

$$\nu^{\gamma_1} = \begin{cases} (2d_1 \alpha_0^{\gamma_2} d_3^{-1})^{\gamma_1/(\gamma_2 - \gamma_1)} \left(\frac{1}{T} \right)^{\gamma_1 \gamma_3/(\gamma_2 - \gamma_1)} & \text{if } T \leq T_0, \\ \max \left\{ \frac{2 \ln(4K_1)}{e \ln(2) d_1}, (2\lambda^*)^{\gamma_1} \right\} & \text{if } T > T_0, \end{cases}$$

and hence

$$\frac{K_2}{\alpha^{\gamma_2}/4^{\gamma_3} - 1} \leq 4 \left(\frac{2^{\gamma_1} \alpha_0^{\gamma_1 \gamma_2} d_1^{\gamma_2} / d_3^{\gamma_1}}{T^{\gamma_1 \gamma_3}} \right)^{\frac{1}{\gamma_2 - \gamma_1}} + \max \left\{ \frac{8 \ln(4K_1)}{e \ln(2)}, 4d_1 (2\lambda^*)^{\gamma_1} \right\}. \quad (16)$$

From inequalities (15) and (16) we conclude

$$\tilde{C}_{\text{obs}} \leq \frac{4Md_0}{Te^{-\omega_+ T}} \exp \left(4 \left(\frac{2^{\gamma_1} \alpha_0^{\gamma_1 \gamma_2} d_1^{\gamma_2} / d_3^{\gamma_1}}{T^{\gamma_1 \gamma_3}} \right)^{\frac{1}{\gamma_2 - \gamma_1}} \right) \max \left\{ (4K_1)^{8/(e \ln 2)}, e^{4d_1 (2\lambda^*)^{\gamma_1}} \right\}.$$

Finally, we insert the values of α_0 and K_1 and factor out $e^{10\omega_+ T/(e \ln 2)}$ from the maximum to obtain the assertion. \square

3 Sharp geometric conditions for observability of elliptic operators in $L_p(\mathbb{R}^d)$

In this section we consider the case where $X = L_p(\mathbb{R}^d)$, $Y = L_p(E)$ with $1 < p < \infty$, A_p is an elliptic operator in $L_p(\mathbb{R}^d)$ associated with a strongly elliptic polynomial in \mathbb{R}^d of degree $m \geq 2$, $(S_t)_{t \geq 0}$ the C_0 -semigroup on $L_p(\mathbb{R}^d)$ generated by $-A_p$, and $C = \mathbf{1}_E$ is the restriction operator of a function in $L_p(\mathbb{R}^d)$ to some measurable subset $E \subset \mathbb{R}^d$, i.e., $\mathbf{1}_E \in \mathcal{L}(L_p(\mathbb{R}^d), L_p(E))$ and $\mathbf{1}_E f = f$ on E . Let $T > 0$. Our goal is to show that the system

$$\begin{aligned} \dot{x}(t) &= -A_p x(t), & t \in (0, T], & \quad x(0) = x_0 \in L_p(\mathbb{R}^d), \\ y(t) &= \mathbf{1}_E x(t), & t \in [0, T], & \end{aligned} \quad (17)$$

satisfies a final state observability estimate in $L_r((0, T); L_p(\mathbb{R}^d))$, $r \in [1, \infty]$ if and only if E is a so-called thick set; cf. Definition 3.2. In particular, if E is a thick set, we conclude an observability estimate with an explicit dependence of C_{obs} on T , the order m of the operator A_p , and the geometry of the set E .

We start by recalling the class of elliptic operators A_p which we consider. We denote by $\mathcal{S}(\mathbb{R}^d)$ the Schwartz space of rapidly decreasing functions, which is dense in $L_p(\mathbb{R}^d)$ for all $1 < p < \infty$. For $f \in \mathcal{S}(\mathbb{R}^d)$ let $\mathcal{F}f: \mathbb{R}^d \rightarrow \mathbb{C}$ be the Fourier transform of f defined by

$$\mathcal{F}f(\xi) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(x) e^{-i\xi \cdot x} dx.$$

Then $\mathcal{F}: \mathcal{S}(\mathbb{R}^d) \rightarrow \mathcal{S}(\mathbb{R}^d)$ is bijective and continuous and has a continuous inverse, given by

$$\mathcal{F}^{-1}f(x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(\xi) e^{ix \cdot \xi} d\xi$$

for all $f \in \mathcal{S}(\mathbb{R}^d)$. Let $a: \mathbb{R}^d \rightarrow \mathbb{C}$ be a homogeneous strongly elliptic polynomial of degree $m \geq 2$, that is, a is of the form

$$a(\xi) = \sum_{|\alpha|_1 = m} a_\alpha i^{|\alpha|_1} \xi^\alpha$$

for given $a_\alpha \in \mathbb{C}$, and there is $c > 0$ such that for all $\xi \in \mathbb{R}^d$ we have

$$\operatorname{Re} a(\xi) \geq c|\xi|^m.$$

Note that this implies that m is even. For $f \in \mathcal{S}(\mathbb{R}^d)$ define $Af \in \mathcal{S}(\mathbb{R}^d)$ by

$$Af := \sum_{|\alpha|=m} a_\alpha \partial^\alpha f = \mathcal{F}^{-1}(a\mathcal{F}f).$$

Then, for every $1 < p < \infty$, A is closable in $L_p(\mathbb{R}^d)$, and its closure A_p is a sectorial operator of angle $\omega_a < \pi/2$. As a consequence, $-A_p$ generates a bounded C_0 -semigroup $(S_t)_{t \geq 0}$ on $L_p(\mathbb{R}^d)$. We call A_p the *elliptic operator associated with a* . For details we refer, e.g., to the book [Haa06].

Example 3.1. Let $1 < p < \infty$ and $a: \mathbb{R}^d \rightarrow \mathbb{R}$ defined by $a(\xi) = |\xi|^2$. Then a is a homogeneous strongly elliptic polynomial of degree $m = 2$ and $A_p = -\Delta$ is the negative Laplacian in $L_p(\mathbb{R}^d)$.

More generally, let $(a_{i,j}) \in \mathbb{R}^{d \times d}$ be a symmetric and negative definite matrix, and define $a: \mathbb{R}^d \rightarrow \mathbb{R}$ by $a(\xi) = \xi^\top (a_{i,j}) \xi$ for all $\xi \in \mathbb{R}^d$. Then a is a homogeneous strongly elliptic polynomial of degree $m = 2$ and $A_p = -\operatorname{div}(a_{i,j}) \operatorname{grad}$ is the corresponding elliptic operator in $L_p(\mathbb{R}^d)$.

The following definition characterizes the class of subsets $E \subset \mathbb{R}^d$ which we consider.

Definition 3.2. Let $\rho \in (0, 1]$ and $L \in (0, \infty)^d$. A set $E \subset \mathbb{R}^d$ is called (ρ, L) -thick if E is measurable and for all $x \in \mathbb{R}^d$ we have

$$\left| E \cap \left(\bigtimes_{i=1}^d (0, L_i) + x \right) \right| \geq \rho \prod_{i=1}^d L_i.$$

Here, $|\cdot|$ denotes Lebesgue measure in \mathbb{R}^d . Moreover, $E \subset \mathbb{R}^d$ is called *thick* if there are $\rho \in (0, 1)$ and $L \in (0, \infty)^d$ such that E is (ρ, L) -thick.

We are now in position to state our main theorems of this section.

Theorem 3.3. Let $1 < p < \infty$, $r \in [1, \infty]$, $a: \mathbb{R}^d \rightarrow \mathbb{C}$ a homogeneous strongly elliptic polynomial in \mathbb{R}^d of degree $m \geq 2$, A_p the associated elliptic operator in $L_p(\mathbb{R}^d)$, $(S_t)_{t \geq 0}$ the bounded C_0 -semigroup on $L_p(\mathbb{R}^d)$ generated by $-A_p$, $E \subset \mathbb{R}^d$ a (ρ, L) -thick set, and $T > 0$. Then the system (17) satisfies a final state observability estimate in $L_r((0, T); L_p(\mathbb{R}^d))$. In particular, we have for all $x_0 \in L_p(\mathbb{R}^d)$

$$\|S_T x_0\|_{L_p(\mathbb{R}^d)} \leq C_{\text{obs}} \|\mathbf{1}_E S_{(\cdot)} x_0\|_{L_r((0, T); L_p(E))}$$

with

$$C_{\text{obs}} = \frac{D_1 M^{16}}{T^{1/r}} \left(\frac{K^d}{\rho} \right)^{D_2} \exp \left(\frac{D_3 (|L|_1 \ln(K^d/\rho))^{m/(m-1)}}{(cT)^{\frac{1}{m-1}}} \right),$$

where $K \geq 1$ is a universal constant, $D_1, D_2 \geq 1$ depending on d , $D_3 \geq 1$ depending on d and p , $M = \sup_{t \geq 0} \|S_t\|$, and $c > 0$ is such that $\operatorname{Re} a(\xi) \geq c|\xi|^m$ for all $\xi \in \mathbb{R}^d$.

The universal constant K in Theorem 3.3 can be chosen to be the same as the constant K in the Logvinenko–Sereda theorem (Theorem 3.5). Theorem 3.3 shows that the system (17) satisfies a final state observability estimate if E is a thick set. Note that C_{obs} is optimal in T (see Remark 2.2(b)), as well as in the geometric parameters ρ and L by [NTTV20a, Remark 4.14]. The following theorem shows the converse: If the system (17) satisfies a final state observability estimate, then the set E is necessarily a thick set.

Theorem 3.4. *Let $1 < p < \infty$, $r \in [1, \infty]$, $a: \mathbb{R}^d \rightarrow \mathbb{C}$ a homogeneous strongly elliptic polynomial in \mathbb{R}^d of degree $m \geq 2$, A_p the associated elliptic operator in $L_p(\mathbb{R}^d)$, $(S_t)_{t \geq 0}$ the bounded C_0 -semigroup on $L_p(\mathbb{R}^d)$ generated by $-A_p$, $E \subset \mathbb{R}^d$ measurable, and $T > 0$, and assume there exists $C_{\text{obs}} > 0$ such that for all $x_0 \in L_p(\mathbb{R}^d)$ we have*

$$\|S_T x_0\|_{L_p(\mathbb{R}^d)} \leq C_{\text{obs}} \|\mathbf{1}_E S_{(\cdot)} x_0\|_{L_r((0, T); L_p(E))}.$$

Then E is a thick set.

In order to prove Theorem 3.3 we apply Theorem 2.1 in the case where $X = L_p(\mathbb{R}^d)$, $Y = L_p(E)$, $C \in \mathcal{L}(X, Y)$ is the restriction operator of functions from \mathbb{R}^d to E , and $A = A_p$. To this end we define a family $(P_\lambda)_{\lambda > 0}$ of operators in $L_p(\mathbb{R}^d)$ such that the assumptions of Theorem 2.1, i.e., the uncertainty relation (3) and the dissipation estimate (4), are satisfied. Concerning the dissipation estimate we first consider the case $p = 2$ and then apply the Riesz–Thorin interpolation theorem. For the uncertainty relation we shall need a so-called Logvinenko–Sereda theorem. It was originally proven by Logvinenko and Sereda in [LS74] and significantly improved by Kovrijkine in [Kov00, Kov01]. Recently, it has been adapted to functions on the torus instead of \mathbb{R}^d ; see [EV20]. We quote a special case of Theorem 1 from [Kov01].

Theorem 3.5 (Logvinenko–Sereda theorem). *There exists $K \geq 1$ such that for all $p \in [1, \infty]$, all $\lambda > 0$, all $\rho \in (0, 1]$, all $L \in (0, \infty)^d$, all (ρ, L) -thick sets $E \subset \mathbb{R}^d$, and all $f \in \mathcal{S}(\mathbb{R}^d)$ satisfying $\text{supp } \mathcal{F}f \subset [-\lambda, \lambda]^d$ we have*

$$\|f\|_{L_p(\mathbb{R}^d)} \leq \left(\frac{\rho}{K^d}\right)^{-K(d+2\lambda|L|_1)} \|f\|_{L_p(E)}.$$

We now proceed with the proofs of Theorems 3.3 and 3.4.

Proof of Theorem 3.3. We apply Theorem 2.1 in the case where $X = L_p(\mathbb{R}^d)$, $Y = L_p(E)$, $C \in \mathcal{L}(X, Y)$ is the restriction of functions from \mathbb{R}^d to E , and $A = A_p$. For this purpose we define a family $(P_\lambda)_{\lambda > 0}$ of operators in $L_p(\mathbb{R}^d)$ such that the assumptions of Theorem 2.1 are satisfied. Let $\eta \in C_c^\infty([0, \infty))$ with $0 \leq \eta \leq 1$ such that $\eta(r) = 1$ for $r \in [0, 1/2]$ and $\eta(r) = 0$ for $r \geq 1$. For $\lambda > 0$ we define $\chi_\lambda: \mathbb{R}^d \rightarrow \mathbb{R}$ by $\chi_\lambda(\xi) = \eta(|\xi|/\lambda)$. Since $\chi_\lambda \in \mathcal{S}(\mathbb{R}^d)$ for all $\lambda > 0$, we have $\mathcal{F}^{-1}\chi_\lambda \in \mathcal{S}(\mathbb{R}^d)$. For $\lambda > 0$ we define $P_\lambda: L_p(\mathbb{R}^d) \rightarrow L_p(\mathbb{R}^d)$ by $P_\lambda f = (2\pi)^{-d/2}(\mathcal{F}^{-1}\chi_\lambda) * f$. By Young’s inequality we have for all $f \in L_p(\mathbb{R}^d)$

$$\|P_\lambda f\|_{L_p(\mathbb{R}^d)} = \|(2\pi)^{-d/2}(\mathcal{F}^{-1}\chi_\lambda) * f\|_{L_p(\mathbb{R}^d)} \leq (2\pi)^{-d/2} \|\mathcal{F}^{-1}\chi_\lambda\|_{L_1(\mathbb{R}^d)} \|f\|_{L_p(\mathbb{R}^d)}.$$

Moreover, the norm $\|\mathcal{F}^{-1}\chi_\lambda\|_{L_1(\mathbb{R}^d)}$ is independent of $\lambda > 0$. Indeed, by the scaling property of the Fourier transform and by change of variables we have for all $\lambda > 0$

$$\|\mathcal{F}^{-1}\chi_\lambda\|_{L_1(\mathbb{R}^d)} = |\lambda|^d \|(\mathcal{F}^{-1}\chi_1)(\lambda \cdot)\|_{L_1(\mathbb{R}^d)} = \|\mathcal{F}^{-1}\chi_1\|_{L_1(\mathbb{R}^d)}.$$

Hence, for all $\lambda > 0$ the operator P_λ is a bounded linear operator, and the family $(P_\lambda)_{\lambda > 0}$ is uniformly bounded by $C_d := (2\pi)^{-d/2} \|\mathcal{F}^{-1}\chi_1\|_{L_1(\mathbb{R}^d)}$. For all $f \in \mathcal{S}(\mathbb{R}^d)$ we have by construction $P_\lambda f \in \mathcal{S}(\mathbb{R}^d)$, $\mathcal{F}P_\lambda f = \chi_\lambda \mathcal{F}f \in \mathcal{S}(\mathbb{R}^d)$, and $\text{supp } \mathcal{F}P_\lambda f \subset \{y \in \mathbb{R}^d: |y| \leq \lambda\} \subset [-\lambda, \lambda]^d$. By Theorem 3.5 we obtain for all $\lambda > 0$ and all $f \in \mathcal{S}(\mathbb{R}^d)$

$$\|P_\lambda f\|_{L_p(\mathbb{R}^d)} \leq d_0 e^{d_1 \lambda} \|P_\lambda f\|_{L_p(E)}, \quad (18a)$$

where

$$d_0 = e^{-Kd \ln(\rho/K^d)} \quad \text{and} \quad d_1 = -2K|L|_1 \ln\left(\frac{\rho}{K^d}\right). \quad (18b)$$

Since $\mathcal{S}(\mathbb{R}^d)$ is dense in $L_p(\mathbb{R}^d)$ and P_λ is bounded, inequality (18) holds for all $f \in L_p(\mathbb{R}^d)$. Thus, the uncertainty relation (3) of Theorem 2.1 is satisfied with d_0 and d_1 as in (18), $\gamma_1 = 1$, and $\lambda^* = 0$.

It remains to verify the dissipation estimate. Since $\mathcal{F}S_t\mathcal{F}^{-1}f = e^{-ta}f$ for $f \in \mathcal{S}(\mathbb{R}^d)$ by functional calculus arguments (see, e.g., section 8 in [Haa06]), and since the Fourier transform is an isometry in $L_2(\mathbb{R}^d)$, we obtain for all $f \in \mathcal{S}(\mathbb{R}^d)$ and all $\lambda > 0$

$$\begin{aligned} \|(1 - P_\lambda)S_t f\|_{L_2(\mathbb{R}^d)} &= \|\mathcal{F}^{-1}(1 - \chi_\lambda)e^{-ta}\mathcal{F}f\|_{L_2(\mathbb{R}^d)} \\ &\leq \|(1 - \chi_\lambda)e^{-ta}\|_{L_\infty(\mathbb{R}^d)} \|f\|_{L_2(\mathbb{R}^d)}. \end{aligned}$$

Since $1 - \chi_\lambda \leq 1 - \mathbf{1}_{B(0, \lambda/2)} \leq 1$, $\operatorname{Re} a(\xi) \geq c|\xi|^m$ for all $\xi \in \mathbb{R}^d$, and since $\mathcal{S}(\mathbb{R}^d)$ is dense in $L_2(\mathbb{R}^d)$, this yields for all $f \in L_2(\mathbb{R}^d)$

$$\|(1 - P_\lambda)S_t f\|_{L_2(\mathbb{R}^d)} \leq e^{-ct(\lambda/2)^m} \|f\|_{L_2(\mathbb{R}^d)}. \quad (19)$$

This shows that the dissipation estimate (4) of Theorem 2.1 is satisfied if $p = 2$. In order to treat the case $p \neq 2$ we apply the Riesz–Thorin interpolation theorem. Let

$$p_0 := \begin{cases} p^2 - 2p + 2 & \text{if } p \in (1, 2), \\ 2p & \text{if } p \in (2, \infty), \end{cases} \quad \text{and} \quad \theta := \begin{cases} \frac{-2p^2 + 6p - 4}{-p^3 + 2p^2} & \text{if } p \in (1, 2), \\ \frac{1}{p-1} & \text{if } p \in (2, \infty). \end{cases}$$

If $p = 2$ we set $p_0 := 2$ and $\theta := 1$ for convenience. Then $p_0 \in (1, \infty)$, $\theta \in (0, 1]$, and

$$\frac{1}{p} = \frac{1 - \theta}{p_0} + \frac{\theta}{2}.$$

Since the family $(P_\lambda)_{\lambda > 0}$ is uniformly bounded by C_d , we have for all $f \in L_{p_0}(\mathbb{R}^d)$ and all $\lambda > 0$

$$\|(1 - P_\lambda)S_t f\|_{L_{p_0}(\mathbb{R}^d)} \leq (1 + C_d)M \|f\|_{L_{p_0}(\mathbb{R}^d)},$$

where $M = \sup_{t \geq 0} \|S_t\|$. Interpolation between $L_2(\mathbb{R}^d)$ and $L_{p_0}(\mathbb{R}^d)$ if $p \neq 2$, and inequality (19) if $p = 2$, now yields for all $f \in L_p(\mathbb{R}^d)$

$$\|(1 - P_\lambda)S_t f\|_{L_p(\mathbb{R}^d)} \leq d_2 e^{-d_3 t \lambda^m} \|f\|_{L_p(\mathbb{R}^d)}, \quad (20a)$$

where

$$d_2 = (1 + C_d)^{1-\theta} M^{1-\theta} \quad \text{and} \quad d_3 = c\theta/2^m. \quad (20b)$$

Thus, the dissipation estimate (3) of Theorem 2.1 is satisfied with d_2 and d_3 as in (20), $\gamma_2 = m$, $\gamma_3 = 1$, and $\lambda^* = 0$. Since $\gamma_2 = m > 1 = \gamma_1$, we conclude from the uncertainty relation (18), the dissipation estimate (20), and Theorem 2.1 that the statement of the theorem holds with

$$\tilde{C}_{\text{obs}} = \frac{4Md_0(4K_1)^{\frac{8}{e \ln 2}}}{T^{1/r}} \exp\left(\frac{4(2 \cdot 8^{\frac{m}{m-1}} d_1^m / d_3)^{\frac{1}{m-1}}}{T^{\frac{1}{m-1}}}\right),$$

where $K_1 = (d_0 + 1)M^2 d_2$, and $T^{1/r} = 1$ if $r = \infty$. From the definitions of d_i , $i \in \{0, 1, 2, 3\}$ and straightforward estimates we obtain $\tilde{C}_{\text{obs}} \leq C_{\text{obs}}$ with C_{obs} as in the theorem. \square

Remark 3.6. As the proof shows we can obtain an explicit dependence of C_{obs} on p . Then, it turns out that $C_{\text{obs}} \rightarrow \infty$ as $p \rightarrow 1$ and as $p \rightarrow \infty$. This shows that our method of proof is only valid for $p \in (1, \infty)$.

Proof of Theorem 3.4. We improve the strategy developed in [EV18]. We show the contraposition. Assume that E is not thick. Then there exists a sequence $(x_n)_{n \in \mathbb{N}}$ in \mathbb{R}^d such that for all $n \in \mathbb{N}$ we have

$$|E \cap B(x_n, n)| < \frac{1}{n}. \quad (21)$$

Note that $e^{-ta} \in \mathcal{S}(\mathbb{R}^d)$ for $t > 0$. Thus, for $t > 0$, the operator S_t is given as a convolution operator with convolution kernel $p_t := (1/(2\pi)^{d/2})\mathcal{F}^{-1}e^{-ta} \in \mathcal{S}(\mathbb{R}^d)$ for $t > 0$. Indeed, for $f \in \mathcal{S}(\mathbb{R}^d)$ and $t > 0$ we have $e^{-ta}\mathcal{F}f \in \mathcal{S}(\mathbb{R}^d)$ and

$$S_t f = \mathcal{F}^{-1}e^{-ta}\mathcal{F}f = (2\pi)^{d/2}\mathcal{F}^{-1}(\mathcal{F}p_t\mathcal{F}f) = p_t * f,$$

and the claim follows by density. For $n \in \mathbb{N}$ we define $f_n := p_1(\cdot - x_n)$. As a consequence, we observe for all $t > 0$ and $n \in \mathbb{N}$

$$S_t f_n = p_t * f_n = p_t * p_1(\cdot - x_n) = p_{t+1}(\cdot - x_n), \quad (22)$$

and hence by translation invariance of the Lebesgue measure

$$\|S_t f_n\|_{L_p(\mathbb{R}^d)} = \|p_{t+1}(\cdot - x_n)\|_{L_p(\mathbb{R}^d)} = \|p_{t+1}\|_{L_p(\mathbb{R}^d)}. \quad (23)$$

For $n \in \mathbb{N}$ we now shift the set E by x_n and consider the set $E - x_n = \{y \in \mathbb{R}^d : y + x_n \in E\}$. Note that (21) is equivalent to $|(E - x_n) \cap B(0, n)| < 1/n$ for all $n \in \mathbb{N}$. From the latter fact, (22), and substitution we obtain for all $t > 0$ and $n \in \mathbb{N}$

$$\begin{aligned} \|\mathbf{1}_E S_t f_n\|_{L_p(\mathbb{R}^d)}^p &= \|\mathbf{1}_{(E-x_n) \cap B(0,n)} p_{t+1}\|_{L_p(\mathbb{R}^d)}^p + \|\mathbf{1}_{(E-x_n)}(1 - \mathbf{1}_{B(0,n)})p_{t+1}\|_{L_p(\mathbb{R}^d)}^p \\ &\leq \|p_{t+1}\|_{L_\infty(\mathbb{R}^d)}^p |(E - x_n) \cap B(0, n)| + \|(1 - \mathbf{1}_{B(0,n)})p_{t+1}\|_{L_p(\mathbb{R}^d)}^p \\ &< \frac{1}{n} \|p_{t+1}\|_{L_\infty(\mathbb{R}^d)}^p + \|(1 - \mathbf{1}_{B(0,n)})p_{t+1}\|_{L_p(\mathbb{R}^d)}^p. \end{aligned} \quad (24)$$

Since a is a homogeneous polynomial, we have by substitution $p_t(x) = t^{-d/m}p_1(x/t^{1/m})$. Hence, we find for all $t > 0$

$$\|p_{t+1}\|_{L_\infty(\mathbb{R}^d)} = \frac{1}{(t+1)^{d/m}} \|p_1\|_{L_\infty(\mathbb{R}^d)} \leq \|p_1\|_{L_\infty(\mathbb{R}^d)}. \quad (25)$$

Moreover, it follows for all $t \in (0, T]$ that

$$\begin{aligned} \|(1 - \mathbf{1}_{B(0,n)})p_{t+1}\|_{L_p(\mathbb{R}^d)}^p &= \int_{\mathbb{R}^d} (1 - \mathbf{1}_{B(0,n)}(x)) \left| \frac{p_1(x/(t+1)^{1/m})}{(t+1)^{d/m}} \right|^p dx \\ &= \int_{\mathbb{R}^d} (1 - \mathbf{1}_{B(0,n/(t+1)^{1/m})}(x)) |p_1(x)|^p \frac{1}{(t+1)^{(p-1)d/m}} dx \\ &\leq \int_{\mathbb{R}^d} (1 - \mathbf{1}_{B(0,n/(T+1)^{1/m})}(x)) |p_1(x)|^p dx. \end{aligned} \quad (26)$$

From (24), (25), and (26) (and since p_1 is a Schwartz function and hence integrable), we obtain that

$$\|\mathbf{1}_E S(\cdot) f_n\|_{L_r((0,T);L_p(\mathbb{R}^d))} \rightarrow 0 \quad (27)$$

as n tends to infinity. From (23) and (27) we conclude that for all $C_{\text{obs}} > 0$ there exists $x_0 \in L_p(\mathbb{R}^d)$ such that

$$\|S_T x_0\|_{L_p(\mathbb{R}^d)} > C_{\text{obs}} \|\mathbf{1}_E S(\cdot) x_0\|_{L_r((0,T);L_p(E))}.$$

This proves the contraposition of the theorem. \square

4 Null-controllability and control costs

Let X and U be Banach spaces, $(S_t)_{t \geq 0}$ be a C_0 -semigroup on X , $-A$ the corresponding infinitesimal generator on X , $B \in \mathcal{L}(U, X)$, and $T > 0$. We consider the linear control system

$$\dot{x}(t) = -Ax(t) + Bu(t), \quad t \in (0, T], \quad x(0) = x_0 \in X, \quad (28)$$

where $u \in L_r((0, T); U)$ with $1 \leq r \leq \infty$. The function x is called *state function*, and u is called *control function*. The unique mild solution of (28) is given by Duhamel's formula

$$x(t) = S_t x_0 + \int_0^t S_{t-\tau} B u(\tau) d\tau, \quad t \in [0, T].$$

We say that the system (28) is *null-controllable in time T via $L_r((0, T); U)$* if for all $x_0 \in X$ there exists $u \in L_r((0, T); U)$ such that $x(T) = 0$. The *controllability map* is given by $\mathcal{B}^T : L_r((0, T); U) \rightarrow X$,

$$\mathcal{B}^T u := \int_0^T S_{T-\tau} B u(\tau) d\tau. \quad (29)$$

Note that we suppress the dependence of \mathcal{B}^T on r . The system (28) is null-controllable in time T via $L_r((0, T); U)$ if and only if $\text{Ran } \mathcal{B}^T \supset \text{Ran } S_T$. This gives an alternative definition of null-controllability.

Denote by A' in X' the dual operator of A and by $B' \in \mathcal{L}(X', U')$ the dual operator of B . It is well known that null-controllability of the system (28) is in certain situations equivalent to final state observability of its adjoint or dual system

$$\begin{aligned} \dot{\varphi}(t) &= -A' \varphi(t), & t \in (0, T], & \quad \varphi(0) = \varphi_0 \in X', \\ \psi(t) &= B' \varphi(t), & t \in [0, T]. \end{aligned} \quad (30)$$

Recall that the system (30) satisfies a final state observability estimate in $L_{r'}((0, T); U')$, $r' \in [1, \infty]$ if there exists $C_{\text{obs}} > 0$ such that for all $\varphi_0 \in X'$ we have $\|\varphi(T)\|_{X'} \leq C_{\text{obs}} \|\psi\|_{L_{r'}((0, T); U')}$. This equivalence can be described in an abstract form due to Douglas [Dou66] and Dolecki and Russell [DR77]; see in particular Theorem 2.5 and Section 5 in [DR77].

Lemma 4.1 ([Dou66, DR77]). *Let V, W, Z be reflexive Banach spaces, and let $F \in \mathcal{L}(V, Z)$, $G \in \mathcal{L}(W, Z)$. Then the following are equivalent:*

(a) $\text{Ran } F \subset \text{Ran } G$.

(b) There exists $c_1 > 0$ such that $\|F'z'\|_{V'} \leq c_1\|G'z'\|_{W'}$ for all $z' \in Z'$.

(c) There exists $H: \overline{\text{Ran } G'} \rightarrow V'$ and $c_2 > 0$ such that $HG' = F'$ and $\|Hw'\|_{V'} \leq c_2\|w'\|_{W'}$ for all $w' \in \overline{\text{Ran } G'}$.

Moreover, in (b) and (c) we can choose $c_1 = c_2$.

In particular, if $V = Z = X$, $W = L_r((0, T); U)$, $F = S_T$ and $G = \mathcal{B}^T$, statement (a) of Lemma 4.1 is equivalent to the fact that the system (28) is null-controllable in time $T > 0$ via $L_r((0, T); U)$, while statement (b) of Lemma 4.1 is equivalent to the fact that the system (30) satisfies a final state observability estimate in $L_{r'}((0, T); U')$, where $1/r + 1/r' = 1$ provided $r \in (1, \infty)$. Thus, if X and U are reflexive and $r \in (1, \infty)$, Lemma 4.1 implies that null-controllability of the system (28) is equivalent to final state observability of the adjoint or dual system (30). More recently, in [Vie05] and [YLC06] it is shown that this equivalence holds true even if X is a general Banach space, U a reflexive Banach space, and $r \in (1, \infty]$.

Theorem 4.2 ([Vie05, YLC06]). *Let X and U be Banach spaces, U reflexive, $(S_t)_{t \geq 0}$ a C_0 -semigroup on X , $-A$ the corresponding infinitesimal generator on X , $B \in \mathcal{L}(U, X)$, $r \in (1, \infty]$, and $r' \in [1, \infty)$ with $1/r + 1/r' = 1$. Then the system (28) is null-controllable in time $T > 0$ via $L_r((0, T); U)$ if and only if there exists $C_{\text{obs}} > 0$ such that*

$$\forall x' \in X': \quad \|S'_T x'\|_{X'} \leq C_{\text{obs}} \|B' S'_{(\cdot)} x'\|_{L_{r'}((0, T); U')}.$$

Note that $-A'$ in general does not generate a C_0 -semigroup on X' but is merely the weak* generator of the weak*-continuous semigroup $(S'_t)_{t \geq 0}$ on X' given by $S'_t := (S_t)'$ for all $t \geq 0$. However, if X is reflexive, then $(S'_t)_{t \geq 0}$ is strongly continuous and $-A'$ is the infinitesimal generator of $(S'_t)_{t \geq 0}$. If we assume that $(S'_t)_{t \geq 0}$ is strongly continuous, we can combine Theorem 2.1 and Theorem 4.2 and obtain sufficient conditions for null-controllability of the system (28).

Theorem 4.3. *Let X, U be Banach spaces, U reflexive, $(S_t)_{t \geq 0}$ a C_0 -semigroup on X , $-A$ the corresponding infinitesimal generator on X , $B \in \mathcal{L}(U, X)$, and assume that $(S'_t)_{t \geq 0}$ is strongly continuous. Let further $\lambda^* \geq 0$ and $(P'_\lambda)_{\lambda > \lambda^*}$ be a family of bounded linear operators in X' , $r \in (1, \infty]$, $d_0, d_1, d_3, \gamma_1, \gamma_2, \gamma_3, T > 0$ with $\gamma_1 < \gamma_2$, $d_2 \geq 1$, and assume that*

$$\forall x' \in X' \quad \forall \lambda > \lambda^*: \quad \|P'_\lambda x'\|_{X'} \leq d_0 e^{d_1 \lambda^{\gamma_1}} \|B' P'_\lambda x'\|_{U'}$$

and

$$\forall x' \in X' \quad \forall \lambda > \lambda^* \quad \forall t \in (0, T/2]: \quad \|(\text{Id} - P'_\lambda) S'_t x'\|_{X'} \leq d_2 e^{-d_3 \lambda^{\gamma_2} t^{\gamma_3}} \|x'\|_{X'}.$$

Then the system (28) is null-controllable in time T via $L_r((0, T); U)$.

Combining Theorem 4.2 with Theorems 3.3 and 3.4 we obtain a sharp geometric condition on null-controllability for linear systems governed by strongly elliptic operators with interior control. For $E \subset \mathbb{R}^d$ measurable we denote by $\mathbf{1}'_E \in \mathcal{L}(L_p(E), L_p(\mathbb{R}^d))$ the canonical embedding, i.e., $\mathbf{1}'_E f = f$ on E and $\mathbf{1}'_E f = 0$ on $\mathbb{R}^d \setminus E$.

Theorem 4.4. *Let $1 < p < \infty$, $a: \mathbb{R}^d \rightarrow \mathbb{C}$ a homogeneous strongly elliptic polynomial in \mathbb{R}^d of degree $m \geq 2$, A_p the associated elliptic operator in $L_p(\mathbb{R}^d)$, $E \subset \mathbb{R}^d$ measurable, $r \in (1, \infty]$, and $T > 0$. Then the system*

$$\dot{x}(t) = -A_p x(t) + \mathbf{1}'_E u(t), \quad t \in (0, T], \quad x(0) = x_0 \in L_p(\mathbb{R}^d)$$

is null-controllable in time T via $L_r((0, T); L_p(\mathbb{R}^d))$ if and only if E is a thick set.

Proof. Let $1 < p' < \infty$ be such that $1/p + 1/p' = 1$. Note that $L_p(\mathbb{R}^d)' \cong L_{p'}(\mathbb{R}^d)$, $(A_p)' = A_{p'}$ (note that m is even), and $A_{p'}$ is the generator of the bounded C_0 -semigroup $(S'_t)_{t \geq 0}$. If we set $B = \mathbf{1}'_E \in \mathcal{L}(L_p(E), L_p(\mathbb{R}^d))$ we have for the dual operator $B' = \mathbf{1}_E \in \mathcal{L}(L_{p'}(\mathbb{R}^d), L_{p'}(E))$, i.e., B' is the restriction operator of a function $x \in L_{p'}(\mathbb{R}^d)$ on E . Hence, combining Theorem 4.2 with Theorems 3.3 and 3.4 for the dual system we obtain the assertion. \square

We now turn to the discussion of the control costs. For $T > 0$ we call the quantity

$$C_T := \sup_{\substack{x_0 \in X \\ \|x_0\|_X = 1}} \inf \{ \|u\|_{L_r((0, T); U)} : u \in L_r((0, T); U), S_T x_0 + \mathcal{B}^T u = 0 \}$$

the *control cost in time T via $L_r((0, T); U)$* of the system (28). If X and U are Hilbert spaces and $r = 2$, it is well known that the control cost C_T equals the smallest constant C_{obs} such that the system (30) satisfies a final state observability estimate. This fact is a direct consequence of Lemma 4.1.

If U is not a Hilbert space, or $r \neq 2$, the construction above does not apply directly, since it is not clear how to extend the operator H to the whole space by keeping its relevant properties. It is an open question if control costs can be estimated by the observability constant in the general setting. Under some additional assumption we can formulate the following lemma.

Lemma 4.5. *Let X and U be Banach spaces, $(S_t)_{t \geq 0}$ a C_0 -semigroup on X , $-A$ the corresponding infinitesimal generator on X , $B \in \mathcal{L}(U, X)$, $T > 0$, $r, r' \in (1, \infty)$ with $1/r' + 1/r = 1$, and $C_{\text{obs}} > 0$. Assume that the system (30) satisfies the final state observability estimate*

$$\forall x' \in X': \quad \|S'_T x'\|_{X'} \leq C_{\text{obs}} \|B' S'_{(\cdot)} x'\|_{L_{r'}((0, T); U')}. \quad (31)$$

Then the system (28) is null-controllable in time T via $L_r((0, T); U)$. Moreover, there exists

$$H: \overline{\text{Ran } \mathcal{B}^{T'}} \rightarrow X' \quad \text{such that} \quad H \mathcal{B}^{T'} = S'_T \quad \text{and} \quad \|H u'\|_{X'} \leq C_{\text{obs}} \|u'\|_{L_{r'}((0, T); U')}$$

for all $u' \in \overline{\text{Ran } \mathcal{B}^{T'}}$, where $\mathcal{B}^T: L_r((0, T); U) \rightarrow X$ is as in (29). Suppose further that there is an extension $\tilde{H}: L_{r'}((0, T); U') \rightarrow X'$ of H with $\|\tilde{H} u'\| \leq C_{\text{obs}} \|u'\|_{L_{r'}((0, T); U')}$ for all $u' \in L_{r'}((0, T); U')$. Then the control cost in time T via $L_r((0, T); U)$ of the system (28) satisfies $C_T \leq C_{\text{obs}}$.

Proof of Lemma 4.5. Equation (31) is equivalent to statement (b) of Lemma 4.1 with $V = Z = X$, $W = L_r((0, T); U)$, $F = S_T$, and $G = \mathcal{B}^T \in \mathcal{L}(W, X)$ with \mathcal{B}^T as in (29). The implication (b) \Rightarrow (a) of Lemma 4.1 implies that $\text{Ran}(S_T) \subset \text{Ran}(\mathcal{B}^T)$, i.e., null-controllability

of the system (28). The implication (b) \Rightarrow (c) of Lemma 4.1 ensures the existence of the operator H with the desired properties, which proves the first assertion.

The dual operator of \mathcal{B}^T is given by $(\mathcal{B}^{T'} x')(t) = B' S'_t x'$ for $x' \in X'$. For an arbitrary initial state $x_0 \in X$ we choose the control function $u \in L_r((0, T); U)$, $u(t) = (-\tilde{H}' x_0)(T - t)$, with \tilde{H} as in the hypothesis of the lemma. Since $\tilde{H} \mathcal{B}^{T'} = S'_T$ by assumption, we obtain for all $x' \in X'$

$$\begin{aligned} \langle S_T x_0, x' \rangle_{X, X'} &= \langle \tilde{H}' x_0, \mathcal{B}^{T'} x' \rangle_{L_r((0, T); U), L_{r'}((0, T); U')} \\ &= - \int_0^T \langle u(T - t), B' S'_t x' \rangle_{U, U'} dt \\ &= - \int_0^T \langle S_t B u(T - t), x' \rangle_{X, X'} dt = - \langle \mathcal{B}^T u, x' \rangle_{X, X'}. \end{aligned}$$

Thus, the solution of (28) satisfies $x(T) = S_T x_0 + \mathcal{B}^T u = 0$. For the norm of the control function we have by assumption on \tilde{H}

$$\|u\|_{L_r((0, T); U)} \leq \|\tilde{H}\| \|x_0\|_X \leq C_{\text{obs}} \|x_0\|_X.$$

This shows that the control cost in time T via $L_r((0, T); U)$ of the system (28) satisfies $C_T \leq C_{\text{obs}}$. \square

From Lemma 4.5 and Theorem 3.3 we obtain the following corollary. It complements Theorem 4.4 and provides an explicit upper bound on the control cost for elliptic operators A and interior control on thick sets.

Corollary 4.6. *Let $p, p', r, r' \in (1, \infty)$ such that $1/p' + 1/p = 1$ and $1/r' + 1/r = 1$, $a: \mathbb{R}^d \rightarrow \mathbb{C}$ a homogeneous strongly elliptic polynomial in \mathbb{R}^d of degree $m \geq 2$, A_p the associated elliptic operator in $L_p(\mathbb{R}^d)$, $(S_t)_{t \geq 0}$ the bounded C_0 -semigroup on $L_p(\mathbb{R}^d)$ generated by $-A_p$, $E \subset \mathbb{R}^d$ a (ρ, L) -thick set, and $T > 0$. Then the system*

$$\dot{x}(t) = -A_p x(t) + \mathbf{1}'_E u(t), \quad t \in (0, T], \quad x(0) = x_0 \in L_p(\mathbb{R}^d) \quad (32)$$

is null-controllable in time T via $L_r((0, T); L_p(E))$. Moreover, there exists

$$H: \overline{\text{Ran } \mathcal{B}^{T'}} \rightarrow L_{p'}(\mathbb{R}^d) \quad \text{such that} \quad H \mathcal{B}^{T'} = S'_T$$

and $\|Hu'\|_{L_{p'}(\mathbb{R}^d)} \leq C_{\text{obs}} \|u'\|_{L_{r'}((0, T); L_{p'}(E))}$ for all $u' \in \overline{\text{Ran } \mathcal{B}^{T'}}$, where as in (29) we have $\mathcal{B}^T: L_r((0, T); L_p(E)) \rightarrow L_p(\mathbb{R}^d)$ with $B = \mathbf{1}'_E$,

$$C_{\text{obs}} = \frac{D_1 M^{16}}{T^{1/r'}} \left(\frac{K^d}{\rho} \right)^{D_2} \exp \left(\frac{D_3 (|L|_1 \ln(K^d/\rho))^{m/(m-1)}}{(cT)^{\frac{1}{m-1}}} \right),$$

where $K \geq 1$ is a universal constant, $D_1, D_2 \geq 1$ depending on d , $D_3 \geq 1$ depending on d and p' , $M = \sup_{t \geq 0} \|S'_t\|$, and where $c > 0$ is such that $\text{Re} a(\xi) \geq c|\xi|^m$ for all $\xi \in \mathbb{R}^d$. Suppose further that there is an extension $\tilde{H}: L_{r'}((0, T); L_{p'}(E)) \rightarrow L_{p'}(\mathbb{R}^d)$ of H with $\|\tilde{H}u'\|_{L_{p'}(\mathbb{R}^d)} \leq C_{\text{obs}} \|u'\|_{L_{r'}((0, T); L_{p'}(E))}$ for all $u' \in L_{r'}((0, T); L_{p'}(E))$. Then the control cost in time T via $L_r((0, T); L_p(E))$ of the system (32) satisfies $C_T \leq C_{\text{obs}}$.

Remark 4.7. In general, it may be difficult to show the existence of an extension \tilde{H} of H as in Lemma 4.5 and Corollary 4.6. However, if U is a Hilbert space (or $p = 2$) and $r = 2$, the existence is trivial, since we can choose $\tilde{H} = HP$ with a suitable orthogonal projection P .

Acknowledgment

The authors thank Clemens Bombach for valuable comments which helped to significantly improve an earlier version of this manuscript.

References

- [AEWZ14] J. Apraiz, L. Escauriaza, G. Wang, and C. Zhang. Observability inequalities and measurable sets. *J. Eur. Math. Soc. (JEMS)*, 16(11):2433–2475, 2014.
- [Bar14] V. Barbu. Exact null internal controllability for the heat equation on unbounded convex domains. *ESAIM Control Optim. Calc. Var.*, 20(1):222–235, 2014.
- [BPS18] K. Beauchard and K. Pravda-Starov. Null-controllability of hypoelliptic quadratic differential equations. *J. Éc. polytech. Math.*, 5:1–43, 2018.
- [DE19] J. Dardé and S. Ervedoza. On the cost of observability in small times for the one-dimensional heat equation. *Anal. PDE*, 12(6):1455–1488, 2019.
- [Dou66] R. G. Douglas. On majorization, factorization, and range inclusion of operators on hilbert space. *Proc. Amer. Math. Soc.*, 2(17):413–415, 1966.
- [DR77] S. Dolecki and D. L. Russell. A general theory of observation and control. *SIAM J. Control Optim.*, 2(15):185–220, 1977.
- [Egi] M. Egidi. On null-controllability of the heat equation on infinite strips and control cost estimate. *Math. Nachr.*, to appear.
- [EMZ15] L. Escauriaza, S. Montaner, and C. Zhang. Observation from measurable sets for parabolic analytic evolutions and applications. *J. Math. Pures Appl.*, 104(5):837–867, 2015.
- [EV18] M. Egidi and I. Veselić. Sharp geometric condition for null-controllability of the heat equation on \mathbb{R}^d and consistent estimates on the control cost. *Arch. Math. (Basel)*, 111(1):1–15, 2018.
- [EV20] M. Egidi and I. Veselić. Scale-free unique continuation estimates and Logvinenko-Sereda Theorems on the torus. *Ann. Henri Poincaré*, 21(12):3757–3790, 2020.
- [EZ11] S. Ervedoza and E. Zuazua. Sharp observability estimates for heat equations. *Arch. Ration. Mech. Anal.*, 202(3):975–1017, 2011.
- [FI96] A. V. Fursikov and O. Y. Imanuvilov. *Controllability of Evolution Equations*, volume 34 of *Suhak kangyürok*. Seoul National University, Seoul, Korea, 1996.
- [FZ00] E. Fernández-Cara and E. Zuazua. The cost of approximate controllability for heat equations: The linear case. *Adv. Differential Equations*, 5(4–6):465–514, 2000.

- [Gd07] M. González-Burgos and L. de Teresa. Some results on controllability for linear and nonlinear heat equations in unbounded domains. *Adv. Differential Equations*, 12(11):1201–1240, 2007.
- [Güi85] E. N. Güichal. A lower bound of the norm of the control operator for the heat equation. *J. Math. Anal. Appl.*, 110(2):519–527, 1985.
- [Haa06] M. Haase. *The Functional Calculus for Sectorial Operators*, volume 169 of *Oper. Theory Adv. Appl.* Birkhäuser, Basel, 2006.
- [JL99] D. Jerison and G. Lebeau. Nodal sets of sums of eigenfunctions. In M. Christ, C. E. Kenig, and C. Sadosky, editors, *Harmonic Analysis and Partial Differential Equations*, Chicago Lectures in Mathematics, pages 223–239. University of Chicago Press, Chicago, IL, 1999.
- [Kov00] O. Kovrijkine. *Some Estimates of Fourier Transforms*. PhD thesis, California Institute of Technology, 2000.
- [Kov01] O. Kovrijkine. Some results related to the Logvinenko-Sereda Theorem. *Proc. Amer. Math. Soc.*, 129(10):3037–3047, 2001.
- [Lis12] P. Lissy. A link between the cost of fast controls for the 1-d heat equation and the uniform controllability of a 1-d transport-diffusion equation. *C. R. Math. Acad. Sci. Paris*, 350(11):591–595, 2012.
- [Lis15] P. Lissy. Explicit lower bounds for the cost of fast controls for some 1-D parabolic or dispersive equations, and a new lower bound concerning the uniform controllability of the 1-D transport-diffusion equation. *J. Differential Equations*, 259(10):5331–5352, 2015.
- [LL] C. Laurent and M. Léautaud. Observability of the heat equation, geometric constants in control theory, and a conjecture of Luc Miller. *Anal. PDE*, to appear.
- [LL12] J. Le Rousseau and G. Lebeau. On Carleman estimates for elliptic and parabolic operators. Applications to unique continuation and control of parabolic equations. *ESAIM Control Optim. Calc. Var.*, 18(3):712–747, 2012.
- [LM] G. Lebeau and I. Moyano. Spectral inequalities for the Schrödinger operator. 2019, arXiv:1901.03513.
- [LM16] J. Le Rousseau and I. Moyano. Null-controllability of the Kolmogorov equation in the whole phase space. *J. Differential Equations*, 260(4):3193–3233, 2016.
- [LR95] G. Lebeau and L. Robbiano. Contrôle exact de l'équation de la chaleur. *Comm. Partial Differential Equations*, 20(1–2):335–356, 1995.
- [LS74] V. N. Logvinenko and Ju. F. Sereda. Equivalent norms in spaces of entire functions of exponential type. *Teor. Funkts., Funkts. anal. Prilozh.*, 20:102–111, 1974.

- [LZ98] G. Lebeau and E. Zuazua. Null-controllability of a system of linear thermoelasticity. *Arch. Ration. Mech. Anal.*, 141(4):297–329, 1998.
- [Mil04a] L. Miller. Geometric bounds on the growth rate of null-controllability cost for the heat equation in small time. *J. Differential Equations*, 204(1):202–226, 2004.
- [Mil04b] L. Miller. How violent are fast controls for Schrödinger and plate vibrations? *Arch. Ration. Mech. Anal.*, 172(3):429–456, 2004.
- [Mil05a] L. Miller. On the null-controllability of the heat equation in unbounded domains. *Bull. Sci. Math.*, 129(2):175–185, 2005.
- [Mil05b] L. Miller. Unique continuation estimates for the laplacian and the heat equation on non-compact manifolds. *Math. Res. Lett.*, 12(1):37–47, 2005.
- [Mil06a] L. Miller. The control transmutation method and the cost of fast controls. *SIAM J. Control Optim.*, 45(2):762–772, 2006.
- [Mil06b] L. Miller. On exponential observability estimates for the heat semigroup with explicit rates. *Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl.*, 17(4):351–366, 2006.
- [Mil10] L. Miller. A direct Lebeau-Robbiano strategy for the observability of heat-like semigroups. *Discrete Contin. Dyn. Syst. Ser. B*, 14(4):1465–1485, 2010.
- [NTTV18] I. Nakić, M. Täufer, M. Tautenhahn, and I. Veselić. Scale-free unique continuation principle, eigenvalue lifting and Wegner estimates for random Schrödinger operators. *Anal. PDE*, 11(4):1049–1081, 2018.
- [NTTV20a] I. Nakić, M. Täufer, M. Tautenhahn, and I. Veselić. Sharp estimates and homogenization of the control cost of the heat equation on large domains. *ESAIM Control Optim. Calc. Var.*, 26(54):26 pages, 2020.
- [NTTV20b] I. Nakić, M. Täufer, M. Tautenhahn, and I. Veselić. Unique continuation and lifting of spectral band edges of Schrödinger operators on unbounded domains. *J. Spectr. Theory*, 10(3):843–885, 2020. With an appendix by Albrecht Seemann.
- [Phu04] K. D. Phung. Note on the cost of the approximate controllability for the heat equation with potential. *J. Math. Anal. Appl.*, 295(2):527–538, 2004.
- [Phu18] K. D. Phung. Carleman commutator approach in logarithmic convexity for parabolic equations. *Math. Control Relat. Fields*, 8(3&4):899–933, 2018.
- [PW13] Kim Dang Phung and Gengsheng Wang. An observability estimate for parabolic equations from a measurable set in time and its applications. *J. Eur. Math. Soc. (JEMS)*, 15(2):681–703, 2013.
- [Sei84] T. I. Seidman. Two results on exact boundary control of parabolic equations. *Appl. Math. Optim.*, 11(2):145–152, 1984.

- [TT07] G. Tenenbaum and M. Tucsnak. New blow-up rates for fast controls of Schrödinger and heat equations. *J. Differential Equations*, 243(1):70–100, 2007.
- [TT11] G. Tenenbaum and M. Tucsnak. On the null-controllability of diffusion equations. *ESAIM Control Optim. Calc. Var.*, 17(4):1088–1100, 2011.
- [Vie05] A. Vieru. On null controllability of linear systems in Banach spaces. *Systems Control Lett.*, 54(4):331–337, 2005.
- [WWZZ19] G. Wang, M. Wang, C. Zhang, and Y. Zhang. Observable set, observability, interpolation inequality and spectral inequality for the heat equation in \mathbb{R}^n . *J. Math. Pures Appl.*, 126:144–194, 2019.
- [WZ17] G. Wang and C. Zhang. Observability inequalities from measurable sets for some abstract evolution equations. *SIAM J. Control Optim.*, 55(3):1862–1886, 2017.
- [YLC06] X. Yu, K. Liu, and P. Chen. On null controllability of linear systems via bounded control functions. In *in Proceedings of the 2006 American Control Conference*, pages 1458–1461. IEEE, Piscataway, 2006.